Real Analysis 1, MATH 5210, Fall 2018

Homework 6, Outer and Inner Approximation of Lebesgue Measurable Sets and Countable Additivity, Continuity, and the Borel-Cantelli Lemma, Solutions

2.18. (REVISED from the text's version.) Let $m^*(E) < \infty$. Then if there exists F_σ set F and G_δ set G with $F \subseteq E \subseteq G$ and $m^*(F) = m^*(E) = m^*(G)$, then E is measurable. NOTE: The text's statement is incorrect since it does not assume that E is measurable. We know (from page 3 of the class notes for Section 2.3) that there exist F_σ set F and G_δ set G, called the inner approximation and outer approximation, such that $\lambda_*(F) = \lambda_*(E) \le \lambda^*(E) = \lambda^*(G)$ (in terms of inner measure λ_* and outer measure λ^*). So the text's conclusion holds only if set E is measurable (which it did not assume). HINT: You may assume this behavior of inner and outer measure. Use the definition of Lebesgue measurable in terms of inner and outer measure.

Proof. We know that for E of finite outer measure, such a set G exists by Exercise 2.7. Similarly, F_σ set F exists with $F \subseteq E$ and $m_*(F) = m_*(E)$ where m_* is inner measure (as given in the hint). Now an F_σ set is measurable and so $m_*(F) = m^*(F)$. So, in general, $m^*(F) = m_*(F) = m_*(E) \le m^*(E) = m^*(G)$. But since we hypothesized $m^*(F) = m^*(G)$, then it must be that $m_*(E) = m^*(E)$ and hence E is measurable. ∎

Note. If $m^*(E) = \infty$, then we can take $G = \mathbb{R}$ and still have $m^*(E) = m^*(G)$. If $m_*(E) = \infty$ then we can construct F_σ set F with $F \subseteq E$ and $m_*(F) = m_*(E)$ (by an argument similar to the proof of the existence of G for $m^*(E) < \infty$). So the (corrected) result of this exercise also holds for $m^*(E) = \infty$.

2.19. Let E have finite outer measure. Prove that if E is <u>not</u> measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \setminus E) > m^*(\mathcal{O}) - m^*(E).$$

NOTE: This is our first encounter with the behavior of a (Lebesgue) non-measurable set. It will get weirder.

Proof. We consider the contrapositive. That is, suppose E has finite outer measure and that for any open $\mathcal{O}$ containing E where the outer measure of $\mathcal{O}$ is finite, that we have

$$m^*(\mathcal{O} \setminus E) \le m^*(\mathcal{O}) - m^*(E). \qquad (1)$$

Now, $\mathcal{O} = (\mathcal{O} \setminus E) \cup E$ and so by subadditivity,

$$m^*(\mathcal{O}) \leq m^*(\mathcal{O} \setminus E) + m^*(E),$$

or (since $m^*(E) < \infty$)

$$m^*(\mathcal{O}) - m^*(E) \leq m^*(\mathcal{O} \setminus E).$$

So by (1)

$$m^*(\mathcal{O}) - m^*(E) = m^*(\mathcal{O} \setminus E) \qquad (2)$$

for all open $\mathcal{O} \supset E$. (Notice that you cannot use the Excision Property since E is not measurable.)

Next, by the definition of outer measure and Theorem 0.3(b), we know that for all $\varepsilon > 0$, there exists open $\mathcal{O}_\varepsilon \supset E$ such that (since $m^*(E) < \infty$): $m^*(E) + \varepsilon > m^*(\mathcal{O}_\varepsilon)$ —in fact, $\mathcal{O}$ can be written as a countable union of bounded open intervals. So for open $\mathcal{O}_\varepsilon$, we have $m^*(\mathcal{O}_\varepsilon) - m^*(E) < \varepsilon$. Since (2) holds for all open sets $\mathcal{O}$, we have that for all $\varepsilon > 0$, there exists open $\mathcal{O}_\varepsilon \supset E$ such that

$$m^*(\mathcal{O}_\varepsilon \setminus E) = m^*(\mathcal{O}_\varepsilon) - m^*(E) < \varepsilon.$$

But then by Theorem 2.11(i), $E \in \mathcal{M}$. Thus the contrapositive of the claim holds and hence the claim holds. ∎

2.28. Prove that continuity of measure together with finite additivity of measure implies countable additivity.

Proof. Let $\{E_k\}_{k=1}^\infty$ be a countable collection of disjoint measurable sets. Define $A_n = \cup_{k=1}^n E_k$. Notice that $\cup_{n=1}^\infty A_n = \cup_{k=1}^\infty E_k$. Then $\{A_n\}_{n=1}^\infty$ is an ascending collection of measurable sets (measurable since $\mathcal{M}$ is closed under finite unions by Proposition 2.5). So

$$\begin{aligned}
m\left(\cup_{k=1}^\infty E_k\right) &= m\left(\cup_{n=1}^\infty A_n\right) \\
&= \lim_{n\to\infty} m(A_n) \text{ by Continuity of Measure (Theorem 2.15(ii))} \\
&= \lim_{n\to\infty} m\left(\cup_{k=1}^n E_k\right) \text{ since } A_n = \cup_{k=1}^n E_k \\
&= \lim_{n\to\infty} \sum_{k=1}^n m(E_k) \text{ by finite additivity (Proposition 2.6)} \\
&= \sum_{k=1}^\infty m(E_k).
\end{aligned}$$

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