Section 9.7. Chordal Graphs

Note. In this section, we define a chordal graph, give a tree-like construction of the (called a simplicial decomposition), and give two classifications of chordal graphs (in Corollary 9.22 and Theorem 9.23).

Definition. A *chord* of a cycle C in a simple graph G is an edge in $E(G) \setminus E(C)$ both of those ends lie on C (see Exercise 2.2.19). A *chordal graph* is a simple graph in which every cycle of length greater than three has a chord.

Note 9.7.A. Notice that a subgraph of a chordal graph which is induced by some set of vertices is either cycle-free or contains a cycle of length three; it if contains a cycle of length greater than three then there is a chord of this cycle and hence a cycle of shorter length, which in turn has a chord, etc. That is, no induced cycle of a chordal graph is a cycle of length four or more (though an induced subgraph can include such a cycle, but must also include the chords of such a cycle). Hence, every induced subgraph of a chordal graph is chordal. Complete graphs are trivially chordal and trees are vacuously chordal. Figure 9.14 gives an example of a chordal graph.

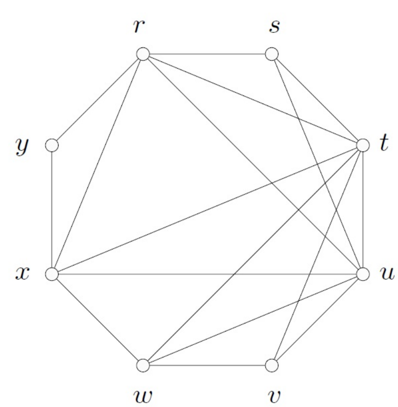


Fig. 9.14. A chordal graph

Definition. A *clique cut* of graph G is a vertex cut (that is, some set of vertices S such that some pair of nonadjacent vertices of G appear in different components of $G - S$) which is also a clique.

Theorem 9.19. Let G be a connected chordal graph which is not complete, and let S be a minimal vertex cut of G. Then S is a clique of G.

Note. Similar to the idea of a block tree in Section 5.2. Separations and Blocks, Theorem 9.19 let's us build up a connected chordal graph by pasting together its cliques in a "treelike fashion."

Theorem 9.20. Let G be a connected chordal graph, and let V_1 be a maximal clique of G. Then the maximal cliques of G can be arranged in a sequence $(V_1, v_2, \ldots, V_k)$ such that $V_j \cap \left(\cup_{i=1}^{j-1} V_j\right)$ is a clique of G for $2 \leq j \leq k$.

Definition. For chordal graph G, a sequence $(V_1, V_2, \ldots, V_k)$ of maximal cliques such that $V_j \cap \left(\cup_{i=1}^{j-1} V_j\right)$ is a clique of G for $2 \leq j \leq k$ is called a *simplicial decomposition* of G.

Note. For the chordal graph of Figure 9.14, the simplicial decomposition is given in Figure 9.15 where V_1, V_2, V_3, V_4, V_5 are given (with bold edges giving the complete graphs corresponding to the cliques; each occurrence of edge tu is bold though it's hard to tell) from left to right. We have $V_1 = \{r, x, y\}$, $V_2 = \{r, t, u, x\}$, $V_3 = \{r, s, t, u\}$, $V_4 = \{t, u, w, x\}$, and $V_5 = \{t, u, v, w\}$, so that $V_2 \cap V_1 = \{r, s\}$, $V_3 \cap (V_2 \cup V_3) = \{r, t, u\}$, $V_4 \cap (V_1 \cup V_2 \cup V_3) = \{t, u, x\}$, and $V_5 \cap (V_1 \cup V_2 \cup V_3 \cup V_4) = \{t, u, w\}$. In Exercise 9.7.2 it is to be shown that a graph is chordal if and only if it has a simplicial decomposition. We give another example at the end of this section which illustrates all of the ideas we introduce.

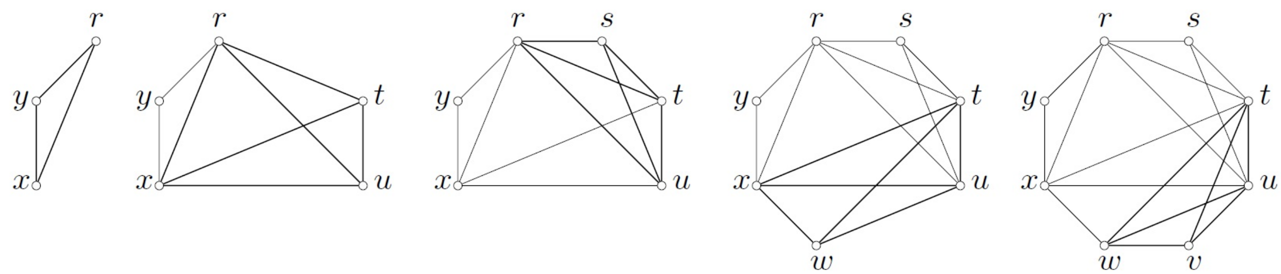


Fig. 9.15. A simplicial decomposition of the chordal graph of Figure 9.14

Definition. A *simplicial vertex* of a graph is a vertex whose neighbors induce a clique.

Note. For chordal graph G of Figure 9.14, the simplicial vertices are s, v, and y. Notice that $N_G(s) = \{r, t, u\}$, $N_G(v) = \{t, u, w\}$, and $N_G(y) = \{r, x\}$.

Theorem 9.21. Every chordal graph which is not complete has two nonadjacent simplicial vertices.

Note. G. A. Dirac in "On Rigid Circuit Graphs," Âbhanhlungen aus dem Mathematischen Seminar der Universität Hamburg, **25**, 71-76 (1961) proved that a graph is chordal if and only if it has a simplicial decomposition (i.e., Exercise 9.7.2) and proved Theorem 9.21.

Definition. A *simplicial order* of a graph G is an enumeration $v_1, v_2, \ldots, v_n$ of its vertices such that v_i is a simplicial vertex of the induced subgraph $G[\{v_1, v_{i+1}, \ldots, v_n\}]$ for $1 \le i \le n$.

Note. By Note 9.7.A, an induced subgraph of a chordal graph G is chordal, so the induced subgraphs $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ for $1 \le i \le n$ are chordal. Then by Theorem 9.21 each of these subgraphs has a simplicial vertex, say v_i. Then $v_1, v_2, \ldots, v_n$ is a simplicial order of G. Therefore, every chordal graph has a simplicial order. It is to be shown in Exercise 9.7.3 that every graph with a simplicial order is chordal. Hence, we have the following.

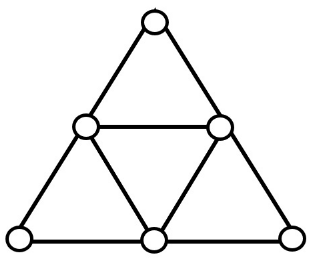
Corollary 9.22. A graph is chordal if and only if it has a simplicial order.

Note. We now have that a graph being chordal is equivalent to it having a simplicial decomposition, and equivalent to it have a simplicial order. The next theorem gives another classification of chordal graphs, this time in terms of intersection graphs of subtrees of a tree.

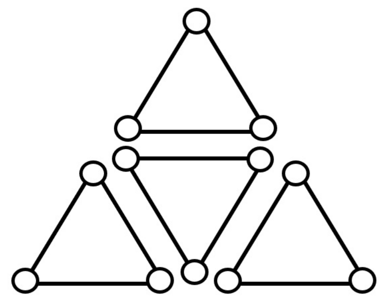
Theorem 9.23. A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.

Note. The tree and collection of subtrees $(T, \mathcal{T})$ in the proof of Theorem 9.23 for which chordal graph G is the intersection graph is a *tree representation* of the chordal graph G.

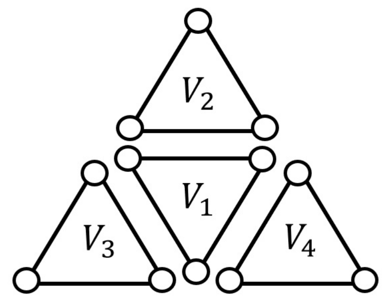
Note. Reinhardt Diestel authored "Simplicial Decompositions of Graphs: A Survey of Applications," *Discrete Mathematics*, **75**, 121–144 (1989); a copy is online on the Science Direct website. In it, he gave the graph G and maximal cliques given in the following figure. We use this to illustrate the ideas of this section.



G

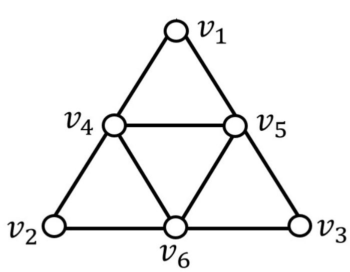


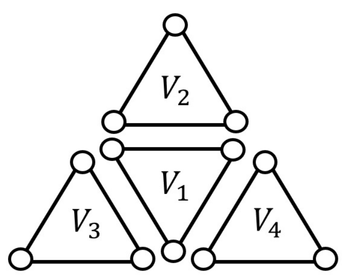
Maximal cliques



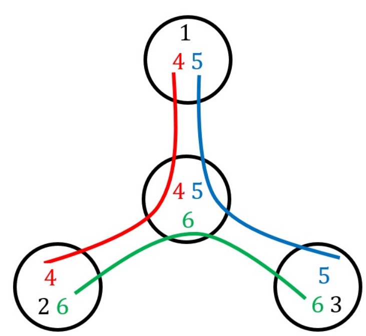
Simplicial Decomposition

Notice that in the simplicial decomposition, each of the vertex sets $V_1 \cap V_2$, $V_2 \cap (V_1 \cup V_2)$, and $V_4 \cap (V_1 \cup V_2 \cup V_3)$ induce a K_2 of G (and hence a clique, as required). The simplicial vertices are the three "corner" vertices. The next figure gives a simplicial order of the vertices of G:



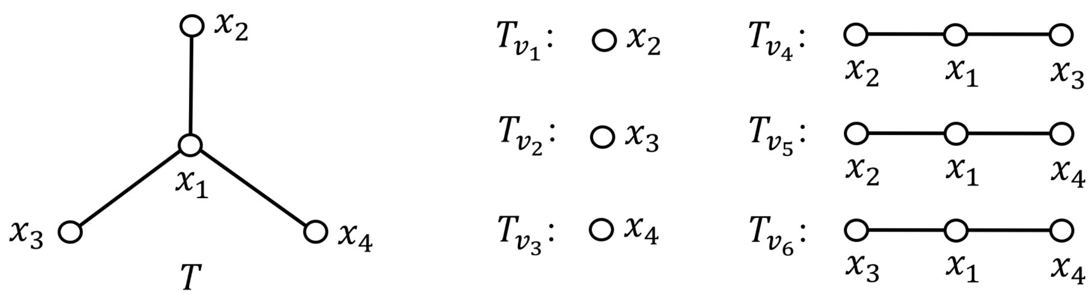


Simplicial Decomposition



Plan for tree T

Notice that in $G[v_1, v_2, v_3, v_4, v_5, v_6] = G$ the neighbors of vertex v_1, $\{v_4, v_5\}$, form a clique. In $G[v_2, v_3, v_4, v_5, v_6]$ the neighbors of vertex v_2, $\{v_4, v_6\}$, form a clique. In $G[v_3, v_4, v_5, v_6]$ the neighbors of vertex v_3, $\{v_5, v_6\}$, form a clique. In $G[v_4, v_5, v_6]$ the neighbors of vertex v_4, $\{v_5, v_6\}$, form a clique. In $G[v_5, v_6]$ the neighbor of vertex v_5, $\{v_6\}$, form a clique. The figure above also gives a plan for constructing tree T for which chordal graph G is the intersection graph of subtrees of T. The next figure gives tree T and the trees in $\mathcal{T} = \{T_v \mid v \in V(G)\}$.



The intersection graph of the six subtrees then yields graph G with the vertices as given in the simplicial order above (illustrating Theorem 9.23).

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