The Sieve of Eratosthenes

1. The idea

Two is the only even prime.
Every even number > 2 is composite because it is divisible by 2.
Similarly, all multiples of $3 \geq 6$ are composite;
all multiples of $5 \geq 10$ are composite;
and so on.

2. Sieve of Eratosthenes

Eratosthenes (an old Greek guy) figured out a way to use these facts to make a list of the primes.
His method uses a technique called "Calculatus Eliminatus" (coined by Dr. Seuss) which means "to discover where something is, discover where it's not."
His idea is very simple.
Write down all the numbers from 1 to 100 in a list.
Ignore 1 because it is a unit—it's neither prime nor composite.

3. The prime 2

After 1 the first number we come to is 2, which must be prime because there are no smaller numbers which could divide it.
Using the fact that all multiples of 2 are composite, mark an 'X' through every other number starting with 4.

4. The prime 3

The prime 3 has not been crossed out, so it is the next prime after 2. Cross out every third number, starting with 6; these are all composite because they are divisible by 3.
Note that 6 was already crossed out as a multiple of 2, but that's okay. It's like having a second opinion from another doctor.
Four has been crossed out, so we know it isn't prime.

5. The prime 5

The next number we come to which has not been crossed out is 5.
So we procced to cross out every multiple of 5, starting with 10.
(Actually we could have started with 25.)
Six has been crossed out (twice).

6. The prime 7

So we come to 7, the fourth prime on our list.
Cross out every seventh number starting with 14.
At this point the numbers on the list up to 100 which have not been crossed out are all primes.

7. Sieve in Action

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

8. Remove Multiples of 2

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

9. Remove Multiples of 3

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

10. Remove Multiples of 5

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			

11. Remove Multiples of 7

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

12. The Sieve Bound Theorem

The 25 numbers that remain are all primes.

The reason why we don't need to cross out the multiples of 11, 13, etc, is given by the following theorem:

Sieve Bound Theorem. If n is composite then some prime number p which is $\leq \sqrt{n}$ divides n.

If we are using the Sieve of Erastothenes to find all primes up to n, then we can stop our sieve as soon as we reach a prime bigger than $\sqrt{n}$.
Prove the Sieve Bound Theorem.

Use the Sieve of Erastothenes to construct a list of the primes up to 200.
By the Sieve Bound Theorem, what is the last prime you need to use?