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Physics in Perspective

Paul A.M. Dirac's *The Principles of Quantum Mechanics**

Laurie M. Brown**

Paul A.M. Dirac's book, *The Principles of Quantum Mechanics*, summarized the foundations of a new science, much of which was his own creation. It expressed the spirit of the new quantum mechanics, creating a descriptive language that we still use. I discuss the successive editions of Dirac's book and their critical reception, noting changes, especially in the formulation of the general theory and in its treatment of relativistic quantum theory and quantum electrodynamics. In the case of the later editions, I discuss Dirac's negative attitude toward renormalized quantum electrodynamics.

Key words: Paul A.M. Dirac; Dirac equation; quantum mechanics; quantum electrodynamics; bracket notation.

Introduction

Paul A.M. Dirac's great treatise, *The Principles of Quantum Mechanics*, which set the stage, the tone, and much of the language of the quantum-mechanical revolution, was published three-quarters of a century ago in 1930.[1] Abdus Salam and Eugene P. Wigner declared in their preface of a book commemorating Dirac's seventieth birthday that:

> Posterity will rate Dirac [figure 1] as one of the greatest physicists of all time. The present generation values him as one of its great teachers – teaching both through his lucid lectures as well as through his book *The Principles of Quantum Mechanics*. This exhibits a clarity and a spirit similar to those of the *Principia* written by a predecessor of his in the Lucasian Chair in Cambridge…. Dirac has left his mark, not only by his observations … but even more by his human greatness.… He is a legend in his own lifetime and rightly so.[2]

Abundant praise has been heaped upon Dirac's *Principles*. I will give only two brief quotations to illustrate its impact on students. The first is by Harish-Chandra:

> As a young undergraduate in 1940, I came across a copy of Dirac's book… – the first edition of 1930 – in the library of Allahabad University in India and was immedi-

* This article is based upon a talk I gave at the Baylor University Dirac Centennial Conference, September 30-October 2, 2003, organized by Bruce Gordon.

** Laurie M. Brown is Professor Emeritus of Physics and Astronomy at Northwestern University in Evanston, Illinois, USA.

Fig. 1. Paul A.M. Dirac (1902–1984) in the middle of the 1920s. *Credit*: Photograph by A. Börtzells Tryckeri; courtesy of American Institute of Physics Emilio Segré Visual Archives, E. Scott Barr and Weber Collections.

> ately fascinated by it. The exposition was so lucid and elegant that it gave me the illusion of having understood most of it and prompted in me a strong desire to devote my life to theoretical physics.[3]

The second is by Nicholas Kemmer, speaking of his copy of *Principles* acquired in Zurich in 1934:

> I had heard from [Gregor] Wentzel of the "transformation theory" of quantum mechanics, which somehow unified the different possible approaches to the subject, but even so, the bold, sweeping exposition of this unification I found in the book astonished and thrilled me. Let me dwell for a moment on just one feature of Dirac's approach – the "delta function." My earlier studies had been centered on mathematics, mainly under instruction from purists and formalists. Reading of the delta

> function, I felt I was treading on forbidden ground, to be avoided by all good little mathematicians.[4]

And having introduced this forbidden subject, I cannot refrain from giving Laurent Schwartz's tribute. After discovering distribution theory in 1945, he realized that it could solve the difficulties of the Dirac delta function, and he noted:

> I then looked at a certain number of works of theoretical physics, and became aware with awe [*avec effroi*] of the enormous breakthrough [*percée*] that had been made by physicists in the manipulation of distributions, without the mathematicians having "given them the right."… It was not only the Dirac function itself that Dirac put forward, but likewise for all singular functions, he had the idea of distributions as kernels.[5]

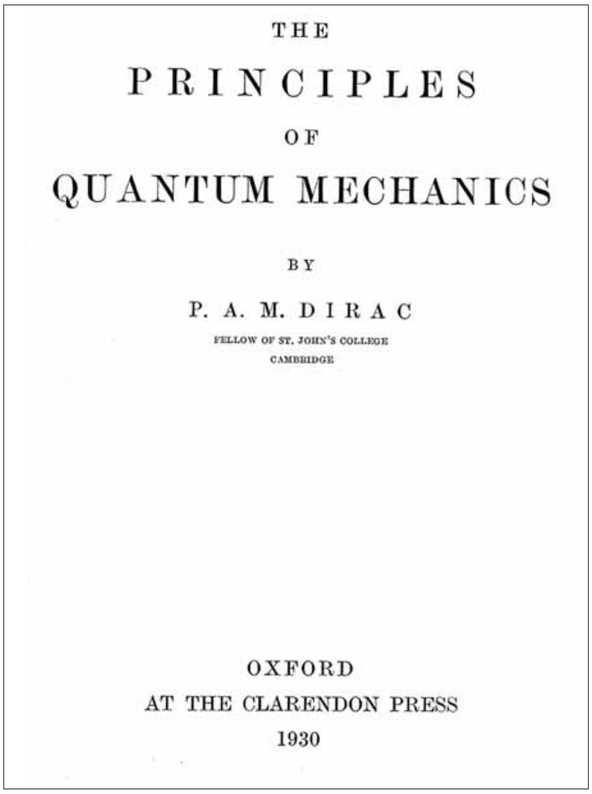
Principles as discussed in Dirac's *Collected Works*

Dirac's *Collected Works*, published in 1995, does not include his books, but in the case of *Principles*, Richard H. Dalitz, the editor, provided the front matter of each of its editions and gave a brief analysis of its structure and successive changes. He noted in his preface that:

> The greatest change in this book was that from its first edition in 1930 [figure 2] to its second edition in 1935, the latter being almost completely rewritten. The third edition (1947) was considerably changed in appearance and detail since Dirac adopted in this edition the "bra" and "ket" notation which he had developed and advocated in 1939.… The fourth edition was printed in 1958.… The fourth edition (revised) was printed in 1967, with reprintings of it in 1971 and 1974, but is now out-of-print as a hardback. This last edition was printed as a paperback in 1981 and has been reprinted in almost every year since then, up to 1993.… The later editions, from the third (1947) to the fourth revised (1968) differ mostly in the last chapter, where quantum field theory and quantum electrodynamics are discussed, since there was a great increase in our knowledge of these topics, both experimental and theoretical, over those two decades, whereas Dirac's opinions hardened concerning the inacceptability of the renormalization programme and its lack of rigor beyond the perturbative regime.[6]

In Dirac's *Collected Works* there is a list of the translations of *Principles*, and it also contains prefaces (translated into English) of the Russian editions, written by the Russian editors and publishers. For the third Russian edition, Dirac supplied an additional preface of his own. Dalitz explained:

> Dirac's book *The Principles of Quantum Mechanics* has been translated and published in many languages. Indeed, nobody knows how many, not even Oxford University Press. Why have we selected for discussion only the Russian editions among all those foreign editions? This is because of Dirac's close relationship with Russian physicists [notably Peter Kapitza and Igor Tamm].…


THE

PRINCIPLES

OF

QUANTUM MECHANICS

BY

P. A. M. DIRAC

FELLOW OF ST. JOHN'S COLLEGE

CAMBRIDGE

OXFORD

AT THE CLARENDON PRESS

1930

Fig. 2. Title page of the first edition of Paul A.M. Dirac's *Principles of Quantum Mechanics*.

> There were a number of difficulties about Dirac's book…. [One] difficulty was local; Dirac's philosophical approach to his physics was rather idealistic, not along Marxist lines, so that the appearance of the book in Russia could be dangerous for those taking responsibility for its translation and publication.[7]

Japanese translations of *Principles*

In a letter dated April 25, 1936, Yoshio Nishina, who is regarded as the father of Japanese nuclear and elementary-particle physics, wrote to Dirac from the Institute for Physical and Chemical Research (Japanese acronym *Riken*) in Tokyo:

> I thank you very much for your kind letter of the 25th March, a short "Preface to the Japanese Edition" of your book, and the manuscript about "Approximate Method" which is to be included in the Japanese Edition.* We have just sent our translation

* This may be the manuscript, "Approximate methods," in Dirac, *Collected Works* (ref. 6), pp. 481–496, which became Chapter XIa of the first Russian edition of *Principles*.

> to the press without the Preface and the Approximate Method, both of which will be added later....[8]

Ten months later, on February 15, 1937, Nishina wrote again to Dirac: "I suppose that you have received a copy of Japanese translation of your book, which appeared December last."[9] His letter ends: "P.S. The names of the translators of your book are as follows: Yoshio Nishina, Shin-Ichiro Tomonaga, Minoru Kobayashi, and Hidehiko Tamaki." Tamaki described how this translation was prepared:

> Dr. Nishina had been thinking about translating Dirac's *Principles of Quantum Mechanics* since 1931... [and] Heisenberg's *Die Physikalischen Prinzipien der Quantentheorie*. Back then, Dr. Nishina had written to the authors and the publishers of those books, and he had secured the translation rights for both books.... [He] was so busy at the time that the work progressed at a snail's pace. Before too long, Dirac sent him a letter, saying, "The second edition will soon be published. It may be worthwhile to delay publication of your translation until the second edition becomes available." To this Dr. Nishina agreed.
>
> In 1935, on the way home from the United States, Dirac visited Japan for the second time. A little before that, Dr. Nishina had suggested that we translate the second edition of his book. According to Dr. Nishina, the author himself had told him that the time was right to bring out a Japanese version, and that the second edition was written in a style far easier to comprehend than the first one.
>
> We set to work after Dirac left Japan. Led by Dr. Nishina, the translation project started in the form of a seminar, but progress was slow, and before we had gotten very far, it was summer. We decided to lodge together in some cool place to complete the work and borrowed a mountain villa at the so-called Hosei University Village in Kita-Karuizawa.... The villa which the three of us, Tomonaga, Kobayasi, and I, rented was a snug little one.... Dr. Nishina and his family rented a little larger house on the outskirts of the village....
>
> Although the daily work of translation was hard, we had no lack of delights. From July to early August, the weather was fine, and little cuckoos, bush warblers, and many other birds which we could not name seemed to vie with each other to see who would sing the most tunefully. Beautiful flowers bloomed everywhere.[10]

Tamaki concluded his romantic recollections by noting that by the end of 1935 Tomonaga had put the finishing touches on the translation and by December 1936 had completed the proofreading. "Dirac's book went through third and fourth editions. Succeeding Dr. Nishina, Tomonaga became the representative for the Japanese edition."[11]

Introductory Material in the First and Second Editions of *Principles*

I now turn to a comparison of the different editions of Dirac's *Principles*. For the most part, the book is based upon his reformulation of quantum theory during 1925-1927. I will deal with his guiding philosophy only as he expressed it in *Principles*.[12]

Dalitz remarked on the relationship of Dirac's *Principles* to his doctoral thesis: "It is sometimes stated that its chapters are simply an expansion of the chapters of his Ph.D.

dissertation, written in 1926, but this is a misunderstanding, as any reader will realize at once."[13] Dirac worked out some of the material in *Principles*, such as his relativistic theory of the electron, in 1928, while he had published the *q*-number and *c*-number terminology in 1926 and had used it in his thesis.[14] His fundamental discussions in *Principles* are close in spirit to his article of 1927, "The Physical Interpretation of the Quantum Dynamics,"[15] in which he described how to make coordinate-free calculations in terms of *q*-numbers and then interpret the results in terms of real *c*-numbers that can be compared with experimental measurements. He also defined in it the δ-function and its derivatives, described the transformations of functions of *q*-numbers, and gave their equations of motion.

In each edition of *Principles*, Dirac established the basic theory in the first half of the book and dealt with applications in the second half. I will focus on its early chapters as well as its later chapters that treat relativistic theory and radiation.

Dirac expressed his philosophy in his preface to the first edition of *Principles*, which he included in each succeeding edition along with a new preface pointing out the differences between that edition and the preceding one. He emphasized the "vast change" that had taken place since the classical tradition, in which one could "form a mental picture in space and time of the whole scheme."[16] Instead, the fundamental laws now "control a substratum of which we cannot form a mental picture without introducing irrelevancies." We are obliged to rely on the "mathematics of transformations," in which the "important things in the world appear as the invariants (or more generally the nearly invariants, or quantities with simple transformation properties) of these transformations."

Dirac noted that the required mathematics was not essentially different from that currently used by physicists. Instead of the usual method of coordinates or representations that Werner Heisenberg and Erwin Schrödinger used for instance, he preferred the symbolic method, which "deals directly in an abstract way with the quantities of fundamental importance." Dirac's transformation theory, which he used to formulate the foundations of quantum theory, is really group theory, which physicists had used to treat particular problems in quantum mechanics, such as those of angular momentum and atomic spectra.[17] He cautioned that, although *Principles* was very mathematical,

> All the same the mathematics is only a tool and one should learn to hold the physical ideas in one's mind without reference to the mathematical form. In this book I have tried to keep the physics to the forefront, by beginning with an entirely physical chapter and in the later work examining the physical meaning underlying the formalism wherever possible.[18]

Thus, in the first chapter, 'The Principle of Superposition,' Dirac discusses its physics without using any equations. He begins by stating that "it is quite hopeless on the basis of classical ideas to try to account for the remarkable stability of atoms and molecules." "Classical electrodynamics forms a self-consistent and very elegant theory," but quantum mechanics is "even more elegant and pleasing than the classical theory...."[19] We see here immediately the high value that Dirac placed on mathematical beauty.

Besides the stability of matter, Dirac considers the nature of light as requiring a departure from classical mechanics and electrodynamics (he appears to use these terms interchangeably). Since light exhibits interference and diffraction as well as causes the emission of photoelectrons, it consists of both waves and particles, which "should be regarded as two abstractions which are useful for describing the same physical reality."[20] Interference is a special case of the principle of superposition, which he illustrates in more detail in the case of polarization. Consider a plane-polarized beam of light whose intensity is so low that we can consider it to consist of one photon, and imagine that it passes through a polarimeter set at an angle α to its direction of polarization. Then the "result predicted by quantum mechanics is that sometimes one would find the whole of the energy in one component [at either α or $\alpha + \pi/2$] and the other times one would find the whole in the other component." "Thus the individuality of the photon is preserved in all cases, but only at the expense of determinacy."[21]

A photon of polarization zero (state zero) also can be considered to be partly in the state specified by the angle α and partly in the state specified by the angle $\alpha + \pi/2$. Dirac recognized that his readers might feel that he had

> not really solved the difficulty of the conflict between the waves and the corpuscles, but have merely talked about it in a certain way and, by using some of the concepts of waves and some of corpuscles, have arrived at a formal account of the phenomena, which does not really tell us anything that we did not know before. The difficulty of the conflict between the waves and corpuscles is, however, actually solved as soon as one can give an unambiguous answer to any experimental question. *The only object of theoretical physics is to calculate results that can be compared with experiment*....[22]

Dirac continues in Chapter I to discuss and generalize the concepts of superposition and indeterminacy. He defines the term "state" as a "condition [that exists] throughout an indefinite period of time...." "A system when once prepared in a given state, remains in that state so long as it remains undisturbed."[23] His definition introduces some arbitrariness, since in the case of the polarized photon, for example, the polariscope can be considered to be either a part of the system or a "disturbance." Similar arbitrariness is involved in the definitions of *preparation* and *observation*. Although superposition also occurs in classical wave theory, "*the superposition that occurs in quantum mechanics is of an essentially different nature from that occurring in the classical theory*. The analogies are therefore very misleading."[24]

Dirac finally gives a general statement of the principle of superposition:

> *We say that a state A may be formed by a superposition of states B and C when, if any observation is made on the system in the state A leading to any result, there is a finite probability for the same result being obtained when the same observation is made on the system in one (at least) of the two states B and C.* The Principle of Superposition says that any two states *B* and *C* may be superposed in accordance with this definition to form a state *A* and indeed an infinite number of different states *A* may be formed by superposing *B* and *C* in different ways. This principle forms the foundation of quantum mechanics. It is completely opposed to classical ideas, according to which the result of any observation is certain and for any two states there exists an observation that will certainly lead to two different results.[25]

I now turn to the second edition of Dirac's *Principles*, published in 1935. As he explained in its preface:

> The book has been mostly rewritten. I have tried by carefully overhauling the method of presentation to give the development of the theory in a rather less abstract form, without making any sacrifices in exactness of expression or in the logical character of the development. This should make the work suitable for a wider circle of readers, although the reader who likes abstractness for its own sake may prefer the style of the first edition.[26]

Apart from stylistic changes, the main difference between the second and first editions of *Principles* is that Dirac now uses the term "state" to denote the condition of a system at a given time, not for all time, noting that this usage "contributes so essentially to the possibilities of clear exposition as to lead one to suspect that the fundamental ideas of the present quantum mechanics are in need of serious alteration at just this point...."[27] Dirac is probably expressing here the discomfort that he and many other theorists were feeling about the contradictions that were by then apparent in quantum field theory.

The physics community welcomed Dirac's less abstract presentation in the second edition of *Principles*. Heisenberg, for example, wrote to Dirac:

> Many heartfelt thanks for the second edition of your book, which I received a few days ago. I have studied it with great pleasure and hope especially to learn from the new section on radiation theory. On the whole, I find it very beautiful that your book is now rather more human [*menschlicher*] than earlier, and that one still observes nevertheless on every page that it is you.[28]

As in the first edition, Dirac also presents the general theory in the first half of the second edition. He begins Chapter I by discussing the need for a quantum theory, adding that any classical treatment of either atomic systems or the electromagnetic field predicts specific heats of solids that are much too large, since the degrees of freedom of their internal motions do not contribute to them.

Dirac now also gave a general philosophical reason why classical ideas could never account for the ultimate structure of matter: Classically speaking, "big" and "small" are only relative concepts. He continued:

> In order to give an absolute meaning to size, such as is required for any theory of the ultimate structure of matter, it becomes necessary to assume that *there is a limit to the fineness of our powers of observation and the smallness of the accompanying disturbance – a limit which is inherent in the nature of things and can never be surpassed by improved technique or increased skill on the part of the observer*.[29]

This meant, Dirac argued, that we must revise our concept of causality. If a system is small, we necessarily introduce indeterminacy when observing it, and therefore we are able to calculate only the probability of obtaining a particular result. He concluded:

> The lack of determinacy in the quantum theory should not be considered as a thing to be regretted. It is necessary for a rational theory of the ultimate structure of mat-

> ter. One of the most satisfactory features of the present quantum theory is that the differential equations that express the causality of classical mechanics do not get lost, but are all retained in symbolic form, and indeterminacy appears only in the application of these equations to the results of observations.[30]

Dirac continues in Chapter I by repeating his arguments about the stability of matter and the wave-particle duality, citing again his example of a plane-polarized photon passing through a polarimeter – now described concretely as a tourmaline crystal that passes only light polarized perpendicular to its optical axis. A photon obliquely polarized to the optical axis of the crystal will make "a sudden jump from being partly in each of these two states [parallel and perpendicular to the optical axis] to being entirely in one or the other of them."[31] He then gives a new example of the principle of superposition in which he considers a single photon passing through an interferometer. In this case, a photon in a definite state of motion need not be associated with a single beam of light, "but may be associated with two or more beams of light which are the components into which an original beam has been split."[32] In an interferometer, therefore, each photon interferes only with itself. He concludes Chapter I by giving the mathematical formulation of the principle of superposition, introducing state vectors and using a few simple equations to show how they may be combined to form new state vectors.

Reviews of the First Two Editions of *Principles*

Heisenberg began his review of the first edition of Dirac's *Principles* of 1930 by saying that it was directed toward "theoreticians who are studying the mathematical apparatus of the new quantum theory and want to investigate it thoroughly." Generally praising the book, especially its treatment of perturbation theory and collision problems, he issued a small caveat:

> On the whole, the book gives an outstanding overview of the – known very important – part of quantum mechanics, whose originator is Dirac himself. At several points, the reviewer has the impression that Dirac presents quantum mechanics, especially its physical content, perhaps somewhat "more symbolically" than is necessary. This situation, however, is characteristic of the method with which Dirac has obtained such great success, and the physicist will enjoy being able to once become acquainted with this method in context.[33]

An unsigned review of *Principles* appeared in *Nature* in 1931, grouped with reviews of related books by Heisenberg and Léon Brillouin. Regarding Dirac's *Principles*, the reviewer wrote:

> The original writings of the author have prepared us for a logical and original mode of approach to the difficult problems of atomic physics. His method has the character of a new physical principle. He bids us throw aside preconceived ideas regarding the nature of phenomena and admit the existence of a substratum of which it is impossible to form a picture. We may describe this as the application of "pure

> thought" to physics, and it is this which makes Dirac's method more profound than that of other writers.[34]

Wolfgang Pauli reviewed the first English and first German editions of *Principles* in 1931.[35] He noted that Dirac's abstract presentation had many advantages, especially in giving a unified view of wave functions, operators, and matrices as different representations of the algebraic symbols by systems of numbers. Pauli continued:

> Also it should be mentioned that the author treats the cases of both the discrete and the continuous eigenvalues of the observables … and if in the latter case the full mathematical rigor is not generally observed, this will be regarded by the physically oriented reader as more an advantage than a blemish, since the lucidity of the exposition is appreciably raised.[36]

Pauli pointed out that Dirac's symbolic method also has serious drawbacks, since the reader does not learn whether measurements of a general observable can really be made. The observable he chose as an example was "an arbitrary function of the momentum and position of a particle, satisfying only the condition of reality."

> Be that as it may, it is so that the symbolic method, aside from questions of this kind, deserves a legitimate and important position in the present state of the theory, owing to its elegance and generality.[37]

Pauli then listed various applications, noting such features as "the beautiful derivation of the selection rules for angular momentum," but commented on Dirac's treatment of radiation theory that:

> At this point the writer cannot refrain from warning that this radiation theory leads in its application to free electrons to an infinitely large self-energy. This difficulty must probably be noted as most characteristic and fundamental for the present form of the quantum theory. The last Chapter 13 contains finally the author's relativistic theory of the spinning electron and closes with a section that argues about the not yet satisfactorily clear questions concerning the solutions with negative energy.[38]

Pauli concluded:

> To summarize let it be said that the learned can find in this book, besides the fundamental principles of today's quantum mechanics, all the important methods and results in completely reliable form, and it will soon be an essential standard work for all independent workers in this field. The German translation is unobjectionable [*einwandfrei*].[39]

Paul S. Epstein reviewed the second edition of Dirac's *Principles* in *Science* in June 1935. He began by describing the first edition in words almost taken from Pauli's review of the first edition ("absolutely reliable," an "indispensable aid to independent workers in this field") and then continued:

> It had, however, one serious drawback: the highly abstract character of the introductory chapters. In the first place, the notion of observable … was introduced in a

> manner so detached from experiment that the reader may have remained unconvinced that their measurement is in all cases possible. In the second place, a rather unusual meaning was assigned to the term "state."… It was mainly due to these two features that the first part of the old edition made difficult reading, overtaxing the powers of abstraction of the less experienced student and making the book unsuitable as a classroom text.
>
> Both flaws are completely eliminated from the second edition. The author does not forget for a moment to stress the experimental point of view and lives up in his exposition to the principle stated by him on page 5: "Only questions about the results of experiments have real significance and it is only such questions that theoretical physicists consider."[40]

After praising the clarity and simplicity of the second edition, Epstein said that "there is no longer any reason why it should not prove of excellent service as a text in advanced courses."* He pointed out that the "subject matter is not materially changed in the new edition," even though the positron had been discovered in the meantime, which provided an interpretation of the troublesome negative-energy states of the electron in Dirac's relativistic theory. The formation of electron pairs was a subject of current theoretical interest. According to Epstein:

> The author must have felt, however, that these theories have not yet crystallized into a consistent system and are not secure enough to be included in a treatise of the character of a text- and hand-book; only one brief section is devoted to the positron. On the other hand, there is attached a new chapter on the electromagnetic field which has attained in the last years a formally satisfactory character … (although some deeper problems connected with the structure of the electron remain unresolved).[41]

Illustrations of Dirac's Style in the First Two Editions of *Principles*

I now illustrate with two examples how the first and second editions of Dirac's *Principles* differed in their styles. To get a feeling for his style in the first edition, I quote the first paragraph of Chapter II, Symbolic Algebra of States and Observables:

> We introduce certain symbols which we say denote physical things such as states of a system or dynamical variables. These symbols we shall use in algebraic analysis in accordance with certain axioms which will be laid down. To complete the theory we require laws by which any physical conditions may be expressed by equations between the symbols and by which, conversely, physical results may be inferred from

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* As a personal aside, I might comment that I began to read the still more accessible third edition of Dirac's *Principles* when I was taking my first course in quantum mechanics at Cornell University, taught by Richard Feynman. I found that it still overtaxed my powers of abstraction on the first two attempts, but it became highly readable and inspiring on the third try. Thus, I wonder whether it was the "humanizing" of the second edition, or merely the passage of time and increasing familiarity with Dirac's methods that made the second edition appear pedagogically superior to the first.

> equations between the symbols. A typical calculation in quantum mechanics will now be run as follows: One is given that a system is in a certain state in which certain dynamical variables have certain values. This information is expressed by equations involving the symbols that denote the state and the dynamical variables. From these equations other equations are then deduced in accordance with the axioms governing the symbols and from the new equations physical conclusions are drawn. One does not anywhere specify the exact nature of the symbols employed, nor is such specification at all necessary. They are used all the time in an abstract way, the algebraic axioms that they satisfy and the connexion between equations involving them and physical conditions being all that is required. The axioms, together with this connexion, contain a number of physical laws, which cannot conveniently be analyzed or even stated in any other way.[42]

By contrast, in the second ("more human") edition Dirac introduces the mathematics with a discussion of the principle of superposition:

> The superposition process is a kind of additive process and implies that states can in some way be added to give new states. Now any mathematical quantities which can be added to give new quantities of the same nature may be represented by vectors in a suitable vector space with a sufficiently large number of dimensions. We are thus led to represent the states of a system by vectors in a certain vector space. The vectors will be assumed all to radiate from a common origin.[43]

Dirac begins to present this mathematics more precisely in Chapter II:

> A convenient way of describing the geometrical nature of the vector space is by introducing a coordinate system of the simplest type possible and discussing the transformations of coordinates arising from the passage to other coordinate systems that are equally simple. Let the coordinates of a vector ψ_a be the set of numbers a_1, a_2, a_3,…. These numbers must in general be complex, since, as we saw … [earlier], we can multiply the vectors by complex numerical coefficients and then add them to other vectors. If we make a passage to a new coordinate system, in which the coordinates of the vector ψ_a are a_1*, a_2*, a_3*,…, then the new coordinates will be connected with the old ones by linear relations of the type
>
> $$a_r^* = \sum_s \gamma_{rs} a_s ,$$
>
> where the γ_{rs} are numbers which depend only on the two coordinate systems and not on the vector ψ_a.[44]

Thus, while Dirac utilized the algebraic approach to quantum mechanics in the first edition of *Principles*, he emphasized the geometry of the space of state vectors in the second edition. In a sense, this is more "visualizable" than the algebraic approach, although visualizing complex vectors in a multidimensional (or even infinite-dimensional) space does present a challenge! Dirac has remarked that his own thinking was more geometrical than algebraic, and that his lifelong interest in the beauty of mathematics was stimulated by projective geometry as a student.[45] He also explained why his preference for projective geometry was not more evident in his work:

> It was a most useful tool for research, but I did not mention it in my published work. I do not think I have ever mentioned projective geometry in my published work (but I am not sure about that) because I felt that most physicists were not familiar with it. When I had obtained a particular result, I translated it into an analytic form and put down the argument in terms of equations. That was an argument which any physicist would be able to understand without having had this special training.[46]

The Third and Fourth Editions of *Principles* and Dirac's Bracket Notation

The chapter headings for the third edition of *Principles* of 1947 are very similar to those of the second edition, except that Dirac (figure 3) now treats both discrete and continuous representations in a single chapter, and he called the final chapter Quantum Electrodynamics (which I will discuss below) instead of Field Theory. The main change, however, is that Dirac now uses the *bracket* notation that he had developed in 1939, and continues to use it in succeeding editions.

Dirac's new notation can be symbolized as <bra|c|ket>, where the ket or ket vector |ket> represents a quantum state that can be labeled by one or more parameters, for example, |a,b,…>. The bra or bra vector is the conjugate imaginary vector; for example, <a,b,…| is the conjugate imaginary of |a,b,…> in the sense that the product <a,b,…|a,b,…> is a real positive number (note that he shortened || to |). In general, <k|l> is a number whose complex conjugate is <l|k>. The c in the expression <bra|c|ket> is a linear operator, so that c|ket> is also a ket and <bra|c is also a bra.

Herman Feshbach pointed out in a review of the third edition that the chief effect of Dirac's new notation is "to render the relation between states and wave functions more transparent, many of the proofs become shorter and clearer."[47] Dyadics in state vector space also can be used to represent linear operators.

The fourth edition of *Principles* of 1958 is identical to the third except for its final chapter and one page that he added to emphasize the symmetry between occupied and unoccupied fermion states, so that: "The holes are just as much physical things as the original particles and are also fermions."[48] Dirac also made a few minor changes in the chapter on the relativistic electron and added a paragraph on electron-positron symmetry.

In his preface to the fourth edition, Dirac explained why he replaced the quantum electrodynamics (QED) in the third edition, in which the number of charged particles is conserved, with a newer version:

> In present-day high-energy physics the creation and annihilation of charged particles is a frequent occurrence. A quantum electrodynamics which demands conservation of the number of charged particles is therefore out of touch with physical reality. So I have replaced it by a quantum electrodynamics which includes creation and annihilation of electron-positron pairs. This involves abandoning any close analogy with classical electron theory, but provides a closer description of nature.[49]

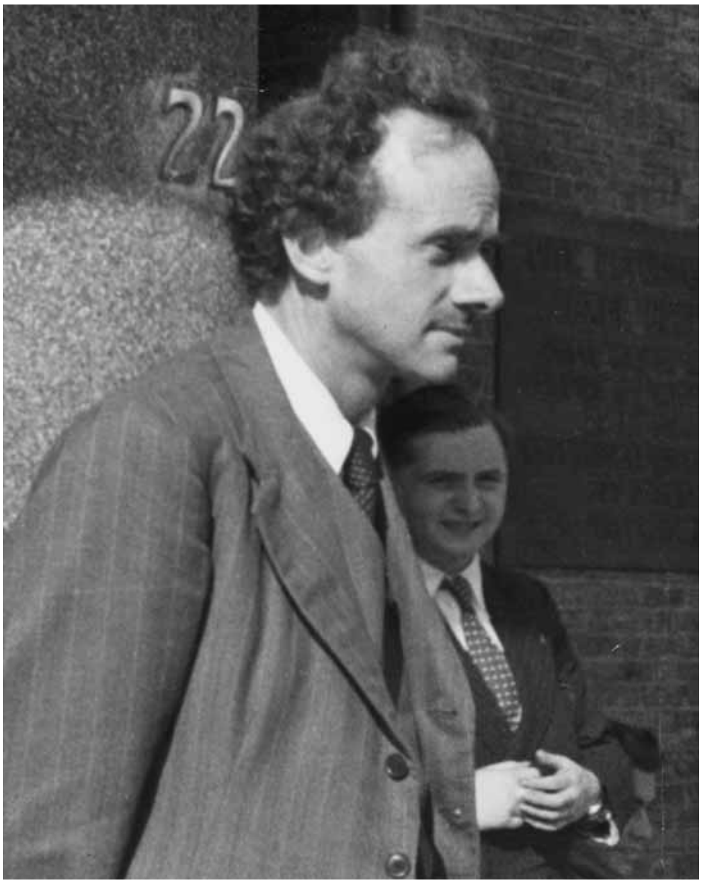

Fig. 3. Paul A.M. Dirac (1902–1984) in middle age. *Credit*: American Institute of Physics Emilio Segré Visual Archives.

Dirac sent a copy of the fourth edition to Heisenberg, who replied:

> I have in the past years repeatedly had the experience that when one has any sort of doubt about difficult fundamental mathematical problems and their formal representation, it is best to consult your book, because these questions are treated most carefully in your book.[50]

The fourth edition (revised) is the same as the fourth edition except for two new final sections, which I will discuss below.

Relativistic Quantum Mechanics and QED

In the first edition of *Principles*, Dirac begins Chapter XII, Theory of Radiation, by considering "an assembly of *n* similar [evidently identical] systems of any kind that satisfy the Einstein-Bose statistics and are all perturbed by some external field of force,"[51] which he specializes to a collection of photons interacting with an atomic system. His strategy is to obtain the interaction between a single photon and an atomic system by writing down the quantum-mechanical interaction for a large number of photons, which in that limit should agree with classical electrodynamics. (That is, Dirac uses Niels Bohr's correspondence principle, although he does not refer to it by that name.) He introduces the states of arbitrary numbers of identical Einstein-Bose systems and forms the number representation, in which the observable n_r (with integer eigenvalue) represents the number of systems in the state r, so that the state of the assembly is written as $(n_1, n_2, \ldots|)$. He summarizes the result of his derivation as follows:

> This shows that *an Einstein-Bose assembly is dynamically equivalent to a set of simple harmonic oscillators, there being one oscillator corresponding to each of a complete set of independent states of a system of the assembly, the quantum number of the oscillator corresponding to the number of systems in the state*.[52]

One can Fourier analyze a classical wave and regard each Fourier component as equivalent to a simple harmonic oscillator. If the Fourier components then are regarded as quantum oscillators, the wave can then be regarded as having been quantized.[53] Dirac, however, stands this procedure on its head, declaring that:

> We may replace the set of simple harmonic oscillators by a train of waves, each Fourier component of the waves being dynamically equivalent to a simple harmonic oscillator. Thus our Einstein-Bose assembly is dynamically equivalent to a system of waves. This provides us with a complete reconciliation between the corpuscular and wave theories of radiation. We may regard radiation either as an assembly of photons satisfying the Einstein-Bose statistics or as a system of waves, the two points of view being consistent and mathematically equivalent.[54]

In the former case, Dirac introduces the momentum and polarization of the photon as observables and writes the Hamiltonian for the proper energies of the photons and atomic system and their interaction energies. To include the possibility of photons being created or annihilated, he labels the state of a photon oscillator so that it has a "zero state" (occupation number zero) in which the photon has no energy or momentum. He then writes the interaction energy of a large number of photons with a potential V and compares it to the corresponding classical expression, saying: "For simplicity we shall consider the atom to consist of a single electron moving in an electrostatic field of force." That comparison yields expressions for matrix elements of the form $<\mathrm{p}|V|\mathrm{p}'>$, where the *p*s are momenta of photons of given polarization. He then uses these matrix elements to calculate the emission, absorption, and scattering of radiation. He concludes by discussing Einstein's Laws of Radiation, which he considered in an earlier chapter but now completes by including spontaneous radiation from an excited atom.

In his next chapter, Relativity Theory of the Electron, Dirac states that it is "fairly certain" that the general principles of quantum mechanics can be applied in the relativistic domain, but only a general field theory like Heisenberg and Pauli's would be satisfactory,[55] and this appears to be too difficult to use for practical applications. Dirac thus considers a single electron whose state is described by a function of space and time, that is, by a Schrödinger wave function, and derives his famous relativistic wave equation, which he solves for the case of the hydrogen atom.[56]

In the last section of this chapter, Dirac considers the Physical Meaning of the Negative Energy Solutions. He had already had the idea of the hole theory, that is, that the infinite sea of negative-energy states are almost all occupied by electrons and thus, because of the Pauli exclusion principle, are unavailable to transitions of electrons from positive-energy states.[57] The unoccupied states, the holes, would behave like positive charges of positive energy, which he thought could be protons, even conjecturing that a proton and an electron occasionally would combine to annihilate each other. He had not yet conceived the idea of the positron.

In the second edition of *Principles*, Dirac changed the heading of this final section to Theory of the Positron. Its language is identical to that of the first edition until the third paragraph, which in the first edition begins:

> In this way we are led to infer that the negative-energy solutions … refer to the motion of protons or hydrogen nuclei, although there remains the difficulty of the great difference in the masses.[58]

In the second edition, the corresponding sentence reads:

> In this way we are led to infer that the negative-energy solutions … refer to the motion of a new kind of particle having the mass of an electron and opposite charge. Such particles have been observed experimentally and are called *positrons*.[59]

In the rest of this paragraph, Dirac discusses occupied and unoccupied states in the negative energy region. His discussion is identical in both editions, except that "proton" in the first is replaced by "positron" in the second. The same is true for the rest of the chapter until the final paragraph, where Dirac emphasizes the new symmetry between electrons and positrons in the second edition.

The second edition also contains a new final Chapter XIII, Field Theory, in which Dirac explains how it differs from his earlier Chapter XI, Theory of Radiation:

> The present theory will go beyond that of Chapter XI in that the field quantities themselves will be used as dynamical variables, not merely the amplitudes and phases of their Fourier components, and the whole of the mutual interaction between electrified particles, including also the Coulomb interaction, will be shown to follow from the interaction between the particles and the field. The present theory will be relativistic throughout and we shall take the velocity of light c equal to unity.[60]

Dirac first applied the quantum conditions to the electric and magnetic field components. For simplicity of application, however, it also was necessary to explicitly quantize the electromagnetic potentials. He applied the necessary supplementary conditions on the potentials, in the form given by Enrico Fermi,[61] to separate the transverse vector photons from the longitudinal and scalar ones, which represent the Coulomb interaction. For relativistic covariance, Dirac used Heisenberg and Pauli's methods. Finally, following Pascual Jordan and Eugene Wigner,[62] he discussed the quantization of electron waves.

The third and fourth editions of *Principles*, like the earlier ones, contain a chapter on Theory of Radiation, now using Dirac's bracket notation. They are identical, except that Dirac added two paragraphs to it in the fourth edition. He obtained the same formulas as those in the second edition, but he replaced the section entitled Einstein's Laws of Radiation in the second edition with a new section entitled An Assembly of Fermions in the third and fourth editions, in which he stressed:

> The foregoing work shows that there is a deep-seated analogy between the theory of fermions and that of bosons, only slight changes having to be made in the general equations of the formalism when one passes from one to the other.[63]

That paragraph concluded his discussion in the third edition, but in the fourth edition he added two paragraphs in which he pointed out a place where this analogy fails to hold, namely, that fermion states can only be either singly occupied or unoccupied, which leads to a symmetry between them. For example, corresponding to the vacuum state $|0\rangle$, in which all states are empty (still in the nonrelativistic case), Dirac introduced the state $|0^*\rangle$, in which *all* states are occupied and destruction operators produce holes, writing:

> We may look upon the unoccupied fermion states as holes among the occupied ones and the η^* variables as the operators of creation of such holes. The holes are just as much physical things as the original particles and are also fermions.[64]

Dirac's chapter on Relativistic Theory of the Electron is almost the same in the third and fourth editions and contains only minor changes from the second edition. He again turns to the Schrödinger picture where, he says, space and time are on the same footing. He improves the relativistic notation somewhat, for instance by replacing the momentum W/c with p_o, where W is the energy, and introducing covariant and contravariant four-vectors at the outset in the fourth edition. In the section on the existence of electron spin, where he compares the squared Dirac equation with the classical relativistic Hamiltonian, he says that the Heisenberg picture is "always the more suitable one for comparisons between classical and quantum mechanics."[65] For a slowly moving electron, this allows him to display the term containing the electron's magnetic moment, without having to discuss its (unobservable) imaginary electric moment. His discussion of invariance under a Lorentz transformation is slightly simpler in the fourth edition, because he uses better relativistic notation.

Dirac introduced his chapter on Field Theory in the second edition by remarking that there exists a Hamiltonian theory of fields, and that there is a corresponding quantum mechanics of fields. However:

> It is of interest chiefly because of the mathematical beauty of its formal analogy with the classical theory when it is expressed in symbolic form. It has not so far led to any practical results which could not be obtained by more elementary methods.[66]

In the third edition, Dirac states that one "substantial alteration," in addition to the adoption of the bracket notation, has resulted from:

> A further development of quantum electrodynamics, including the theory of the Wentzel field. The theory of the electron in interaction with the electromagnetic

> field is carried as far as it can be at the present time without getting on to speculative ground.[67]

Dirac's Reservations About the New QED

The preface to the third edition of *Principles* is dated April 21, 1947. Thus, Dirac's Chapter XII, Quantum Electrodynamics, could not reflect the explosion in that field that began at the conference that took place on Shelter Island, New York, June 2–4, 1947,[68] where Willis Lamb reported the result of the Lamb-Retherford experiment, which showed that there is a splitting between the $2S_{1/2}$ and $2P_{1/2}$ states of the hydrogen atom, contrary to what Dirac's theory of the electron predicted. Following Lamb's talk, I.I. Rabi reported the results of an experiment that he, John E. Nafe, and Edward B. Nelson had performed at Columbia University on the hyperfine structure of hydrogen and deuterium.[69] In both cases, measurements showed a serious discrepancy with existing theory. These effects were immediately recognized to be "radiative corrections," that is, higher-order effects in QED, requiring new methods of approach. Later, Rabi, Polykarp Kusch, and Henry M. Foley showed that the electron's magnetic moment was "anomalous," that is, different from that predicted by Dirac's relativistic theory of the electron.

Dirac's Chapter XII, Quantum Electrodynamics, in the third edition of *Principles* thus can be regarded as the state of the art before the new QED. Dirac begins this chapter by introducing relativistic four-vectors $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ and discusses the Lorentz-invariant delta functions $\delta(\mathbf{x{\cdot}x})$ and $\delta(\mathbf{x{\cdot}x})x_0/|x_0|$, both of which vanish except on the past and future light cones, where they are singular.* He next considers the quantum conditions for the free fields in terms of the four-vector potential $A_\mu(x)$, writing the Hamiltonian for the field following Fermi's method.[70] To consider the interaction of the classical field with matter, he combines the advanced and retarded solutions of Maxwell's equations in such a way that there is an effective radiation damping at high frequencies. Finally, following Wentzel, he introduces new generalized potentials and proposes the so-called λ-limiting process, such that the Maxwell and Wentzel potentials become equal in the limit when λ goes to zero.[71]

In the quantum version of the theory, Dirac eliminates the longitudinal photons, obtaining the Coulomb interaction. To represent the interaction of a single charged particle with the transverse waves, he uses a power-series in its charge, assuming that the interaction is weak. Divergent integrals appear as the coefficients, however, so:

> We can conclude that *the wave equation … has no solution of the form of a power series in the charge e*. This conclusion must hold also for the wave equation for several particles – the transverse electromagnetic waves always lead to divergent integrals when one tries to get a solution of the form of a power series in the charges of the particles.[72]

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* The scalar product of two four-vectors ***a*** and ***b*** is defined as $\boldsymbol{a}{\cdot}\boldsymbol{b} = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$.

Dirac admits that the new QED "has many satisfying features in it" and that reasonable results can be obtained (for sufficiently long wavelengths) by applying cut-off procedures. Nevertheless:

> It is probable that some deep-lying changes will have to be made in the present formalism before it will provide a reliable theory for radiative processes involving short wavelengths. These changes may correspond to a departure from the point-charge model of elementary particles which provides the basis of the present theory. Already in the classical theory the point-charge model involves some difficulties … so it is not surprising that the passage to the quantum theory brings in further difficulties.[73]

In his chapter on Quantum Electrodynamics in the fourth edition of *Principles* of 1958, Dirac reformulates his discussion of QED in a somewhat clearer pedagogical form, introducing sections on the free electromagnetic field and on "the electron and the positron by themselves," in which the electron-positron field is subjected to "second quantization." Thus, he now can use QED to treat the creation and annihilation of electron pairs. In other sections, he discusses the relativistic form of the quantum conditions, the Schrödinger dynamical variables, the supplementary conditions, and the interactions. He introduces physical variables whose significance is as follows: If Ψ_{ax} creates a positron or annihilates an electron at the point **x**, then $\Psi^*_{ax} = e^{ieV(x)/h}\Psi_{ax}$ "creates a positron at the point **x** *together with its Coulomb field*, or else annihilates an electron at **x** *together with its Coulomb field*."[74] He replaces the Heisenberg-picture Hamiltonian, in which "the relativistic invariance of the theory is manifest," with a new Hamiltonian H^* that contains the physical variables and in which the unphysical photons are replaced by the Coulomb interactions of the particles.

This chapter is remarkable for ignoring the advances in QED that were made in the preceding decade, except for a negative comment at its end. Thus, in its final section, Dirac discusses Difficulties of the Theory, especially those concerning the vacuum fluctuations of the electromagnetic field, which give rise to the production of virtual electron-positron pairs. In his final two cautionary paragraphs, he tacitly refers to the current renormalization program:

> People have succeeded in setting up certain rules that enable one to discard the infinities produced by the fluctuations in a self-consistent way and have thus obtained a workable theory from which one can calculate results that can be compared with experiment. Good agreement with experiment has been found, showing that there is some validity in the rules. But the rules are applicable only to special problems, usually collision problems, and do not fit in with the logical foundations of quantum mechanics. They should therefore not be considered a satisfactory solution of the difficulties.
>
> It would seem that we have followed as far as possible the path of logical development of the ideas of quantum mechanics as they are at present understood. The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics.[75]

In the fourth edition (revised) of *Principles*, however, whose preface is dated in 1967, a decade after that of the fourth edition, Dirac replaced the paragraph labeled Difficulties in the fourth edition with two new sections, labeled Interpretation and Applications. He repeats some of the earlier material and again shows that the state without photons and electrons is not the true vacuum state and is not a stationary state because of the infinite vacuum fluctuations that take place. Therefore:

> A theory which gives rise to infinite transition probabilities of course cannot be correct. This result need not surprise us, because quantum electrodynamics does not provide a complete description of nature. We know from experiment that there exist other kinds of particles, which can get created when large amounts of energy are available. All that we can expect from a theory of quantum electrodynamics is that it shall be valid for processes in which there is not enough energy available for these other particles to be created to an appreciable extent, say for energies up to a few hundred MeV [million electron volts].[76]

In other words, QED is a "low-energy effective field theory," to use modern terminology, and the high-energy behavior of the theory must be modified by some form of cut-off, which spoils its relativistic invariance. Dirac then introduces the (unknown) true vacuum state, which contains many particles, and uses the Heisenberg picture, in which the state vector is constant.

In his section on Applications in the fourth edition (revised), Dirac (figure 4) states that his new formulation of QED can be used to treat a single electron in a static electric or magnetic field, and he thus was able to calculate the Lamb shift and the anomalous magnetic moment of the electron.[77] One also can go to the Schrödinger picture and use the perturbation technique, but:

> One finds that the later terms are large and depend strongly on the cut-off, or are infinite if there is no cut-off. The perturbation procedure is not valid under these conditions. Nevertheless people have developed this method a long way and have devised working rules for discarding infinities (in a theory without cut-off) in a systematic manner, so that finite residual results remain.... The original calculations of the Lamb shift and the anomalous magnetic moment were carried out on these lines, long before the calculations in the Heisenberg picture. The results are the same by both methods.[78]

Dirac concludes with these paragraphs:

> Now there are other kinds of interactions, which are revealed in high-energy physics and are important for the description of atomic nuclei. The interactions are not at present sufficiently well understood to be incorporated into a system of equations of motion. Theories of them have been set up and much developed and useful results obtained from them. But in the absence of equations of motion these theories cannot be presented as a logical development of the principles set up in this book. We are effectively in the *pre-Bohr era* with regard to these other interactions.
>
> It is to be hoped that with increasing knowledge a way will eventually be found for adapting the high-energy theorems into a scheme based on equations of motion, and so unifying them with those of low-energy physics.[79]

Fig. 4. Paul A.M. Dirac (1902–1984) lecturing at Yeshiva University in New York in 1963–1964. *Credit*: American Institute of Physics Emilio Segré Visual Archives, Physics Today Collection.

Postscript

By surveying the five editions of Dirac's *Principles*, from the first of 1930 to the fourth (revised) of 1967, and by indicating what he included in them, we have gained some insight into Dirac's thinking about the fundamental aspects of quantum theory. We have seen that his views hardly changed regarding the nonrelativistic theory. However, in spite of the successes of his magnificent relativistic equation of the electron, he struggled to understand the relativistic treatment of several particles and quantum field theory – in which struggles, of course, he was not alone. Still, we can learn more about the difficulties he was facing by looking briefly at what he did not see fit to include in *Principles*.

Dirac made his first attempt to formulate a relativistic quantum theory in an article of 1926, in which he introduced a "quantum-time" variable to treat space and time in an even-handed way, as required by relativity theory.[80] As the generalized momentum conjugate to the q-number time he chose $-W$, where W is the energy, as in classical Hamilton-Jacobi theory. This led to the difficulty that the Hamiltonian equation, $H = W$, is "not consistent with the quantum conditions," because while an arbitrary function of the coordinates and momenta always commutes with W, it does not generally do so with the Hamiltonian H. Nevertheless, using Bohr's correspondence principle, Dirac applied his new method to solve the problem of Compton scattering for spinless electrons, as he and Walter Gordon showed later using the Klein-Gordon equation.[81]

Dirac did not mention "quantum time" in his subsequent papers, nor did he in *Principles*. But there a hint of his struggle in trying to formulate a relativistic quantum theory in the first chapter of the first edition of *Principles*, in which he gave the concept of "state" a four-dimensional significance:

> It is convenient … to modify slightly the meaning of the word "state" and to make it more precise. We must regard the state of a system as referring to its condition throughout an indefinite period of time and not to its condition at a particular time, which would make the state a function of the time. Thus a state refers to a region of 4-dimensional space-time and not to a region of 3-dimensional space.[82]

Dirac pointed out that there is an arbitrariness in this definition concerning what part of the system we decide to include in its "state," for example, whether the measuring apparatus is considered to be part of its state or as an "outside influence." He also remarked that one "needs a corresponding space-time meaning of an *observation*."

In the last chapter of the first edition of *Principles*, Relativity Theory of the Electron, Dirac argued that it is "fairly certain" that the transformation theory he developed earlier "will apply also to relativity treatments of dynamical systems."[83] However, one would have to relate dynamical variables at a given time to those at another time, and these relations "would in general be very complicated and artificial, as they would require us to connect distant parts of space-time." The most straightforward way to deal with this difficulty would be to formulate a "purely field theory," but this "involves complicated mathematics and appears to be too difficult for practical application." In the special case of a single particle, however, one can introduce a (Schrödinger) wave function whose domain "becomes identical with the ordinary space-time continuum, and this circumstance makes possible an elementary treatment of the problem which cannot be extended to more general dynamical systems."[84] He then derived the relativistic equation of the electron as he had done in his famous paper of 1928.[85]

Between the time that Dirac wrote the first and second editions of *Principles*, specifically in 1932 and 1933, he worked on the problem of relativistic quantum mechanics of several particles, publishing three papers[86] that strongly influenced those whom Silvan S. Schweber has called the "Men Who Made" QED, namely, Freeman Dyson, Richard Feynman, Julian Schwinger, and Sin-itiroTomonaga.[87] Three years later, however, Dirac did not mention these three papers in the context of relativistic quantum mechanics in the second edition of *Principles*.* In fact, he retreated from his earlier attempt to formulate a relativistic quantum mechanics. As he stated in its preface:

> The main change [from the first edition] has been brought about by the use of the word "state" in a three-dimensional non-relativistic sense. It would seem at first

* However, in the second edition of *Principles*, Dirac included a section on The Action Principle in his chapter The Equations of Motion, which he also included in all subsequent editions but with a footnote saying, "This section may be omitted by the student who is not specially concerned with higher dynamics." Richard Feynman quotes much of this section in his doctoral thesis; see Laurie M. Brown, ed., *Feynman's Thesis – A New Approach to Quantum Theory* (Singapore: World Scientific, 2005).

sight a pity to build up the theory largely on the basis of non-relativistic concepts. The use of the non-relativistic meaning of "state," however, contributes so essentially to the possibilities of clear exposition as to lead one to suspect that the fundamental ideas of the present quantum mechanics are in need of serious alteration at just this point, and that an improved theory would agree more closely with the development here given than with a development which aims at preserving the relativistic meaning of "state" throughout.[88]

Rejecting on philosophical and practical grounds the relativistic quantum field theory that Heisenberg and Pauli had published in 1929,[89] Dirac set out to construct an alternative QED in the first of the three papers noted above. He argued that the electromagnetic field should play a different role than that of the electrons, because:

The very nature of an observation requires an interplay between the field and the particles. We cannot therefore suppose the field to be a dynamical system on the same footing as the particles and thus something to be observed in the same way as the particles. The field should appear in the theory as something more elementary and fundamental.[90]

He proposed to generalize nonrelativistic quantum mechanics by borrowing the idea that "the probability of occurrence of any transition process is always given as the square of the modulus of a certain quantity … referring to the initial and final states." And, he continued:

These quantities, which we shall refer to as probability amplitudes, will then be the building stones analogous to Heisenberg's matrix elements. We should expect to be able to *set up an algebraic scheme involving only the probability amplitudes and to translate the equations of motion of relativistic classical theory directly into exact equations expressible entirely in terms of those quantities*.[91]

Dirac then showed how to set up the interaction between two electrons in terms of a wave function depending upon their space-time coordinates (thus with two times) and a single field. He showed that in the one-dimensional case this gives the expected relativistic interaction of the two electrons.

Shortly thereafter, Léon Rosenfeld showed that Dirac's "new quantum theory" was, in fact, equivalent to Heisenberg and Pauli's.[92] Nonetheless, there was a good deal of interest in Dirac's new theory in Russia, and later in 1933 Dirac, Vladimir Fock, and Boris Podolsky showed that equivalence in a different way and also developed Dirac's new theory further.[93]

Dirac's new quantum theory also attracted the attention of Sin-itiro Tomonaga and his mentor Yoshio Nishina in Japan. Tomonaga began applying it, now called the many-time theory, and he, Feynman, and Schwinger shared the Nobel Prize in Physics for 1965 for their work on renormalization. Tomonaga discussed his work in the first part of his Nobel Lecture.[94] His and Schwinger's formulations of renormalized relativistic QED are both based upon their generalizations of Dirac's new quantum mechanics to a super-many-time theory, involving the quantization of the electromagnetic field on an arbitrary space-like surface.[95]

In his paper of 1933, Dirac also assigned possible roles of the Lagrangian and the action principle in relativistic quantum mechanics. He specified the correspondence between the transformation function of quantum mechanics, connecting two times, finitely or infinitesimally separated, and an exponential function of the action, pointing out that the action is a relativistic invariant.[96] Feynman, in his Nobel Lecture,[97] described how he had used Dirac's action principle as his point of departure for the development of his path-integral method for quantum mechanics, for which he is justly famous.

Thus, although Dirac did not mention his work on the many-time theory and on the action principle in *Principles*, they played major roles in the later developments of renormalization theory. Dirac disregarded that connection and could never bring himself to accept renormalization theory, in spite of its precise agreement with experiment.

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Department of Physics and Astronomy
Northwestern University
Evanston, IL 60208 USA
e-mail: lbrown@northwestern.edu

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Dirac in 20th century physics: a centenary assessment

V I Sanyuk, A D Sukhanov

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Abstract. Current views on Dirac's creative heritage and on his role in the formation and development of quantum physics and in shaping the physical picture of the world are discussed. Dirac's fundamental ideas in later life (1948 – 1984) and their current development are given considerable attention.

1. Introduction

In August of 2002, the scientific community commemorated the birth centenary of P A M Dirac — one of the most original thinkers in 20th century physics. Our paper is concerned with the traits of his creative activity and his impact on the development of contemporary natural science.

Reminiscing about Dirac, A Salam, a Nobel Prize Laureate in Physics 1979, emphasized (see Ref. [1], p. 84) that "*Paul Adrien Maurice Dirac was undoubtedly one of the greatest physicists of this or any century. In three decisive years – 1925, 1926 and 1927 — with three papers, he laid the foundation, first of quantum physics, second of the quantum theory of fields, and third, of the theory of elementary particles ... No man except Einstein has had such a decisive influence in so short a time on the cause of physics in this century*".

Assessing the comparative role of personalities and separate accomplishments in the history of humankind's spiritual culture, the more so for a period of centuries and millennia (as done in Salam's statement), is an extremely difficult task. Even restricting oneself only to the scope of physics, one cannot but admit that it is conventional to equally admire the exquisite effects of experimenters — the authors of so-called *experimentum crucis* — and the brilliant insights of theorists — the authors of basic theories, which put the comprehension of a whole class of natural phenomena in order. What criteria should be applied to evaluate accomplishments so different in nature is a separate question.

Of course, it is possible to scrupulously count the number of references to the papers of one scientist or another in the publications of other authors and evaluate the so-called citation index. This relatively formal approach is the simplest to realize because there is no need to analyze the contents of the papers and their real value, and it is possible to resort to only the services of statisticians in lieu of the expensive services of analysts. Another way is to question specialists and experts engaged in a given or related field of knowledge, which is taken advantage of in one form or another when allotting grants, awarding prizes, etc. At best these methods allow us to determine the circle of best-known or most frequently cited scientists, but no more than that.

There exists a more objective criterion, namely, the assessment of accomplishments, which is only applicable, truth to tell, to acknowledged classics of science: to judge the contribution of a scientist by the number of 'nominal' results — principles, effects, phenomena, formulas, and equations bearing his name. Should this criterion be applied, Dirac would be among the indisputable leaders in 20th century physics: the Dirac equation, the Dirac transformation theory, the Dirac field, Dirac matrices, the Dirac delta function, Dirac brackets, the Dirac theory of holes, the Dirac interaction representation, the Dirac quantization rule, the Dirac monopole, Fermi – Dirac statistics, Dirac conjugation, the Dirac propagator, Dirac mechanics — this is by no means the complete list of appellations and terms that have firmly entered modern textbooks and monographs.

V I Sanyuk, A D Sukhanov Russian University of Peoples Friendship
ul. Miklukho-Maklaya 6, 117198 Moscow, Russian Federation
Tel. (7-095) 955 08 38, 952 35 83
E-mail: vsanyuk@sci.pfi.edu.ru; ogol@oldi.ru

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Getting somewhat ahead of our presentation, by and large it is valid to say that the words and notions introduced by Dirac form the basis of the language in which quantum physics expresses itself. Examples are an observable, a state, commutation relations, the Dirac unit of action (the Serbian letter $\hbar$), 'bra' and 'ket' vectors, c- and q-numbers to denote respectively classical and quantum quantities. The bracket notation for matrix elements, the particle creation and annihilation operators, and even the functional integral have also been inherited by modern physics from Dirac. (Some details of Dirac's 'quantum word creation' are given in Appendix 7.2.)

Reverting to the above criteria, we note that even the highest of levels specified is 'somewhat tight' for the evaluation of the creative work of scientists who are famous for more than just separate key ideas or accomplishments. The case in point is personalities who have actually changed our notions of the major aufbau principles of the surrounding world. It is generally recognized that among the creators of this rank are I Newton, who laid the foundations of the classical physical picture of the world (PPW), and A Einstein, who brilliantly completed its creation and paved the way for the nonclassical PPW. According to the aforementioned viewpoint of Salam, which is also shared by other famous scientists (see, for instance, Refs [1 – 6]), Dirac is among the personalities of precisely this scale.

Given below in support of this opinion, which is not universally accepted, are the arguments and facts which are testimony to Dirac's fundamental role in the formation of contemporary PPW. That is why we will not enumerate the well-known facts of Dirac's biography, which have repeatedly been published (in particular, by *Phys. – Usp.* [7]) and, if need be, can be found in the collected books [1 – 4, 8] as well as in autobiographic articles of Dirac himself [9]. To start with, we invoke an ancient truth that "everything is apprehended by comparison" and somewhat develop the above judgement of Salam, who ranked Dirac with Einstein. This comparison is instructive, the more so as the destinies and the features of creative activity of these two physicists have much in common.

2. Strokes on the canvas: Dirac and Einstein

Dirac and Einstein are similar primarily because of their profound and highly original thinking. In fact, neither one had tutors or numerous pupils and they seldom needed references to anybody else's works, while the papers each of them co-authored can be counted on one hand. Their deepest ideas were elaborated in practically complete solitude, either at the writing-desk of a patent clerk (Einstein) or during walks through the local environs (Dirac). Solitariness and isolation (in thought, creative activity, and everyday life) are the most distinctive features of both Dirac and Einstein, inherent in them until their very last years. [1]

Another distinguishing feature of Dirac's and Einstein's style of scientific creative work is simplicity, which is made a principle and yet has nothing in common with elementariness. Dirac remarked in the last page of the third edition (1947) of his *The Principles of Quantum Mechanics* [10] that "... *we should require of a satisfactory theory that its equations have a simple solution for any simple physical problem* ...". Einstein echoes him in his "Autobiographishes" [11]: "*The eminent heuristic significance of the general principles of relativity lies in the fact that it leads us to the search for those systems of equations which are in their general covariant formulation the simplest ones possible*...". Multipage computations and 'tedious' proofs are not found in their works, and the results and formulas they arrived at comply with the highest standards of 'mathematical beauty'. The amazing elegance and masterly ease with which logically harmonious theories emerged in the works published by Dirac and Einstein may be compared only with Mozart's style in music or with drawings made by Picasso and Dali.

It would be quite reasonable to suspect that there is some mystery behind all this... And it turned out that such was indeed the case! Each of them made use of his own 'know-how', which had long remained 'concealed from the uninitiated'. Einstein's magic wand of sorts was the preference he showed for 'the theory of principle', with thermodynamics being its embodiment for him. In his seventieth year, Einstein wrote in the above-mentioned "Autobiographishes": "*Reflections of this type made it clear to me as long ago as shortly after 1900, i.e., shortly after Planck's trailblazing work, that neither mechanics nor thermodynamics could* (*except in limiting cases*) *claim exact validity. By and by I despaired of the possibility of discovering the true laws by means of constructive efforts based on known facts. The longer and the more despairingly I tried, the more I came to the conviction that only the discovery of a universal formal principle could lead us to assured results. The example I saw before me was*

[1] We note that everything listed here pertains equally to Newton. However, a comparative analysis of the life and activities of the three greatest physicists would lead us far beyond the scope of this paper.

thermodynamics. The general principle was there given in the theorem: the laws of nature are such that it is impossible to construct a perpetuum mobile (*of the first and second kind*)...''. A detailed analysis of the thermodynamic origins in Einstein's thought can be found in Klein's article ``Thermodynamics in Einstein's thought'' [12] (see also Ref. [13]), where he showed that Einstein was directly or indirectly guided by the thermodynamic view of the world even when constructing theories outwardly remote from thermodynamics.

Dirac also had a 'secret' of his own, which emerged under the following circumstances. On receiving a bachelor's degree from Bristol University in 1921, youthful Paul made an attempt to continue education in Cambridge University. However, he, a recent immigrant, was refused a stipend and returned to Bristol where he was granted permission to attend lectures unofficially at the Mathematical Department, without payment for the education. But, as the saying goes, every dark cloud has a silver lining. The strongest impression of this period was produced by the lectures of the mathematician P Fraser, who managed to inculcate in his pupils the apprehension of the beauty of mathematical constructions and simultaneously a demand for the rigor of mathematical arguments. The mathematical beauty of physical laws, not without Frasers's influence, became for Dirac the intuitive criterion of correctness of physical theories.

It was Fraser who acquainted Dirac with projective geometry. ``*I was strongly impressed by its mathematical beauty*,'' — Dirac wrote later on. — ``*It seems to me that for the most part physicists know little of projective geometry, and I would say that this is a gap in their education. Projective geometry always operates on a plane space, but it is a powerful tool for its investigation, which equips us with methods, e.g., the method of unique correspondence, that yield results as if by magic... I have always invoked projective geometry considerations in my work... Projective geometry has been an extraordinarily useful research apparatus, but I have written nothing about it. It seems to me that I have not even mentioned it in my papers* (*though I am not quite sure of it*) [2], *for I realized that the majority of physicists are hardly familiar with it. On arriving at some result I would translate it into the analytical language and transform my arguments to equations. Any physicist could understand such an argument...*'' (see Ref. [9], p. 12). In this case, the story recurs (the other way round, though). In the 17th century, Newton, on obtaining the majority of his results with the aid of the methods of analysis he himself had elaborated, would convert them to geometrical language, in which his celebrated *Philosophiae Naturalis Principia Mathematica* were written, with the same purpose: to make the presentation clear to the majority of contemporary physicists.

While a post-graduate student at Cambridge, Dirac used to attend tea parties in Prof. G Baker's house (Fraser was also his student at his time), each of which ended with some communication on the results obtained employing projective geometry methods. After one of these tea sessions, a novice — Paul Dirac — had the courage to read a communication on a new method of solving projective geometry problems. That was the first lecture in his life [17]. The problems of the special relativity theory, which captivated Dirac early in his scientific career (the then obtained results were set forth in the second paper [18] of Dirac's publication list), were easily and simply solved in terms of projective geometry. [3]

From projective geometry Dirac derived not only the idea of spinors (homogeneous coordinates for isotropic lines), but he also transferred the Poncelet principle of duality to quantum mechanics, introducing not only the vectors of state, but dual vectors as well. It is noteworthy that, despite the fact that Dirac revealed his 'secret' which allowed him to arrive at outstanding results, the physical community practically refused to be interested in this information. In any case, projective geometry and Lobachevskian geometry have never been included in the list of obligatory courses of physics departments, and papers 'restoring' Dirac's original train of thought in terms of projective geometry are not found in the scientific literature. That is why Appendix 7.1 to our paper outlines briefly the simplest ideas of projective geometry.

It seems likely that the proximity of life and creative styles of Dirac and Einstein was by no means accidental. Worthy of note is their certain solitude in their families during childhood, as well as the oppressive feeling of being everlasting foreigners in society, which accompanied them throughout their lives, no matter where they were or in what capacity they worked. One cannot help noting a chain of astonishing coincidences: both received only basic technical education, both failed to get a job in their profession upon graduating with a higher education, and no one supported them during their first years of scientific research. They were compelled to live, devoting themselves to self-education, in small towns away from scientific centers. These circumstances undoubtedly slightly delayed the emergence of their first scientific papers. At the same time, they might have been the reason that the subsequent papers (only 1.5 years later!) of the young researchers fell right away into the category of unique works.

Indeed, Einstein and Dirac equally displayed an extraordinarily bright outburst of intellect, which embodied an original and many-sided blossoming at a relatively early age. [4] The ideas formulated in their early works exerted an immediate and sometimes decisive influence on their contemporaries and provided the basis for radically new physical theories. At a very early age they were elected respectively to the Prussian Academy of Sciences and the London Royal Society. At as early an age as thirty, they joined the world scientific elite as the main speakers at the First (Einstein, 1911) and Seventh (Dirac, 1933) Solvay Congresses. Naturally, both of them were Nobel Prize Laureates (Dirac, along with W Heisenberg, becoming a Laureate extremely early — at the age of 31). But Einstein and Dirac obviously stand out for the scale of their accomplishments even among Nobel Laureates.

These successes were great enough to go to their heads, but this did not happen. The great respect which Einstein and Dirac won from their contemporaries and progenies was based not only on the admiration for their scientific genius.

[2] Dirac's doubts are fully justified. Not only did he mention projective geometry methods, but he also made direct recourse to them, in particular, in papers which were mathematical in nature [14 – 16].

[3] Much has been written, in particular at a popular level in the book [19], about the relation between space geometry of the special relativity theory and Lobachevsky 'imaginary geometry' (which in turn is intimately related to projective geometry).

[4] Such examples are frequent in mathematics, for instance, E Galois, N H Abel, N N Bogolyubov, in music — W Mozart, in theoretical physics mention can be made of I Newton, J C Maxwell, L D Landau, Ya B Zel'dovich, R Feynman.

Both great physicists were distinguished by high human qualities, among which modesty is of special note. It manifested itself, of course, both in everyday life and in relations with other people. As regards science, they would never accentuate their role and, moreover, sometimes publicly underestimated their accomplishments [5]: no struggle for priority, all conceivable respect to the contributions to science made by their predecessors and contemporaries. Suffice it to refer to how Dirac throughout his life used to give Heisenberg his due for the initial idea.

Finally, we cannot help mentioning yet another feature which draws the scientific destinies of Dirac and Einstein closer. Having become classics of natural science relatively early in life, both of them experienced long periods of ideological solitude and even oblivion. The most active part of the physical community prematurely assumed that they no longer mattered, considering them has-beens. Many of their ideas advanced during the several last decades of their lives were underestimated by their contemporaries and have not been fully appreciated even to the present day. It is pertinent to note that the first comprehensive collections of the works of these outstanding physicists were issued not by the academic publishing houses in Germany, Great Britain, or the USA. The world's first four-volume collection of Einstein's works was published in the USSR in 1965 – 1967, while the publication of Dirac's collected scientific works is also now for the first time being undertaken in Russia [20].

It seems likely that the immanent properties of human consciousness require a significant historical distance to apprehend the true contribution of one personality or another. (Suffice it to remember what place Newton occupied in physics in the view of the scientific circles in the middle of the 18th century.) It therefore comes as no surprise that Dirac's true role in physics is gaining recognition in a gradual manner. We hope that our paper will convince the reader that Dirac was not only one of Newton's most deserving successors as a Lucasian Professor in Mathematics at Cambridge, but also continued the cause of constructing a proper physical picture of the world, pioneered by Newton.

3. Founders of quantum mechanics: Heisenberg – Dirac – Schrödinger

The advent of quantum mechanics is one of the greatest events in the history of civilization. To reveal the true contribution to the common cause from each of the heroes of this epoch is therefore an important task not only for science historians. Of course, we are not dealing with priority matters, all the more since a man like Dirac attached no significance to them.

It is well known that quantum mechanics was for the most part the fruit of creative activity of very young physicists. In this connection it deserved the name 'Knabenphysik'(boys' physics) from W Pauli. (Indeed, Pauli himself was born in 1900, Heisenberg in 1901, and Dirac and P Jordan in 1902.) It therefore makes sense to compare the conditions in which these talented youths were educated and became scientists. It must be said that it is one thing to grow up in continuous communication with coryphaei — with A Sommerfeld (Münich), M Born (Göttingen), N Bohr (Copenhagen), and P Ehrenfest (Leiden) — which actually took place in the scientific lives of Pauli, Heisenberg, and Jordan. And it is quite another matter to be, like Dirac, a research student of the famous Cavendish Laboratory in Cambridge, in which, however, there were no prominent scientists engaged directly in the problems of atomic physics. [6]

The first idea which initiated the origin of 'new' quantum mechanics was undeniably stated by Heisenberg in the summer of 1925 [21]. However, the noncommutativity of dynamic variables, which came to light in Heisenberg's matrix mechanics, depressed primarily the author himself, who regarded it as a substantial fallacy of his theory. This judgement was initially shared by Born and Jordan, who became engaged in its development along with Heisenberg. On R Fowler's advice, Dirac took up the same work in September 1925. With the boldness of thought inherent in him and the knowledge of Hamiltonian dynamics, Dirac came to consider the noncommutativity of canonically conjugate variables as Heisenberg's main contribution to the construction of quantum dynamics. Upon familiarizing himself with the proof of Heisenberg's first paper, he prepared on his own a fundamental article "The fundamental equations of quantum mechanics" [22] by 7 November 1925, which saw light on 1 December 1925.

Interestingly, it is in this work that the modern form was imparted to the Heisenberg equations

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = [\hat{x}, \widehat{H}]\,,$$

this being done for an arbitrary observable x, an arbitrary Hamiltonian H, and any operator representation. This is precisely the equation form which has entered all textbooks and monographs on quantum mechanics. [7]

In point of fact, Dirac's paper turned out to be the second publication on quantum mechanics, for the well-known paper by Born and Jordan [23] (although it was submitted on 27 September 1925) was published somewhat later and had not been accessible to Dirac beforehand. Born, Heisenberg, and Jordan — the authors of the celebrated 'paper of three' [24], which proved to be the fourth paper on this topic submitted to publication — in its preparation had a copy of Dirac's paper [22] given by the author himself. Heisenberg's friendly letter of 20 November 1925 to Dirac runs as follows: "*I have read your excellent work with the keenest interest. All your results are undoubtedly correct, with the understanding, of course, that one has faith in the new theory... I hope you will not be grieved about the fact that a part of your results was obtained in our institute some time ago... In your results you have advanced much further, and this is especially true of the general definition of differentiation and the relation between quantum conditions and the Poisson brackets*". And that is indeed the

[5] When Salam asked Dirac what he regarded as his most significant contribution to physics, the answer astounded him — the Poisson bracket. "*But with characteristic modesty, he added after a pause that for a long time he felt ecstatic and pleased, till he found essentially the same remark made by Hamilton as a footnote in one of his papers written in the last century*" (see Ref. [1], p. 84).

[6] Dirac took his first journey to the 'continent' in September 1926, when his principal results in quantum mechanics had already been obtained and published in seven most important papers.

[7] Here, historical analogies suggest themselves again: it is well known that the modern form of Newtonian laws, in particular the second law, was first imparted by L Euler, while the Maxwell equations acquired their modern form of writing in H Hertz's works. It only remains to remark: while Euler did this within 70 years after Newton, and Hertz within 20 years, for Dirac it took only two (!!) months.

case, because Refs [23, 24] are only concerned with the equations for coordinate and momentum operators and only in the energy representation, though for a broader class of Hamiltonians in comparison with the pioneering paper by Heisenberg [21].

Furthermore, Dirac published in 1926 a series of papers on quantum mechanics [25], including "On quantum algebra" and "On the theory of quantum mechanics". Based on these papers, he prepared by May of 1926 a Ph.D. thesis "Quantum mechanics". In the history of physics this was the first purely 'quantum' thesis; four years later it formed the basis of his fundamental monograph *The Principles of Quantum Mechanics* (first edition — 1930) [26].

In his approach to the construction of quantum mechanics, Dirac proceeded from the Hamiltonian form of analytical dynamics. This enabled him not only to introduce, in the most natural way, the idea of noncommutativity of dynamic variables into the mathematical apparatus of the new science, but also to organically incorporate the qualitatively new concept of a *quantum state* — the basic concept of wave mechanics proposed by E Schrödinger at the end of January and published on 13 March 1926. The theory of transformations elaborated primarily by Dirac allowed its author to convincingly demonstrate the equivalence of the approaches of Heisenberg (matrices), Schrödinger (wave functions), and the most general one, belonging to Dirac himself (q numbers).

A remark must be made concerning the heroic period of quantum physics elaboration (1925–1934). Among theoretical physicists were supporters of either the Heisenberg–Born–Jordan matrix mechanics, or the de Broglie–Schrödinger wave mechanics. The standpoint of Dirac, whose works were oriented from the outset to the formation of quantum mechanics proper, clearly stood out against this background. In support of this statement we adduce the fact that in his 28 papers, written on the subject during that period, the term 'wave mechanics' is encountered in the title of only one paper, while 'matrix mechanics' is not used at all.

Therefore, we have every reason to believe that the 'new' quantum mechanics is the common creation of Heisenberg, Dirac, and Schrödinger, wherein the basic ideas which allowed for unifying different approaches and representing quantum mechanics as a qualitatively new science are due to Dirac. In this respect, the part played by Dirac is quite comparable to Einstein's role in the development of relativity theory, which also unified the contributions of three authors — H Lorentz, H Poincaré, and Einstein himself. In this case, the Nobel Committee made an adequate assessment of the contributions of each of the founders of quantum mechanics and awarded Nobel Prizes in Physics for its creation to Heisenberg (1932) and Dirac and Schrödinger (1933), with the Prizes presented (it so happened) simultaneously to all three of them in December 1933.

In this connection we allow ourselves only a few remarks. Firstly, the universally accepted statistical interpretation of quantum mechanics, which can be traced back to Einstein's ideas from his radiation theory, is commonly related only to Born's name. The latter did introduce it when discussing the interpretation of microparticle scattering in three-dimensional configuration space. Similar ideas were simultaneously and independently put forward by Dirac in his work "The physical interpretation of quantum dynamics" [27], with the only difference being they were formulated not for the wave function in ordinary space, but for the probability amplitude of any process (not only scattering) in an arbitrary Hilbert space of states.

Secondly, away back in autumn 1926 Dirac discussed the problem of simultaneous measurability of the coordinate and momentum of a microparticle, coming close to the formulation of the *uncertainty relation*. In his famous 1927 paper on uncertainty relations, Heisenberg directly pointed out that its source was the Dirac theory of transformations.

Thirdly, it is traditionally believed that the creation of quantum statistical mechanics is primarily related to the name of J von Neumann. Indeed, the original idea of the density matrix was advanced by L Landau and von Neumann in 1927. However, it is not generally known that this idea was realized in Dirac's works done during 1929–1931 [28] and in his monograph [26], wherein the principles of quantum statistical mechanics were developed even before von Neumann's well-known monograph saw light in 1932 [29].

Fourthly, it was Dirac [30] who first came to consider the scattering theory as a description of the transition between single-particle 'in' and 'out' states in the momentum representation with fixed values of momentum, spin, polarization type, etc. His approach, unlike the initial Born collision theory, has proved to be equally applicable in nonrelativistic and relativistic domains for any microparticles undergoing scattering, including photons, and for any targets. In fact, this work of Dirac contained the initial elements of S-matrix theory, whose development is associated with the names of Heisenberg, E Stückelberg, and Bogolyubov.

And finally, fifthly, Dirac made a substantial contribution to the progress of approximate techniques of quantum-mechanical calculations. Following Schrödinger, who worked out the perturbation theory for stationary states, he developed a version of this theory for unsteady states. Also, Dirac significantly improved the techniques for calculating multielectron systems. In particular, while the wave function of an electron system in the initial Hartree–Fock method is expressed as the product of two determinants, in Dirac's paper [31], where the spin variables are not separated out from the wave functions of individual electrons right from the start, it is expressed in terms of a single determinant, which significantly simplifies calculations. In Ref. [32], he introduced a correction to the theory of a Thomas–Fermi atom to allow for the electron exchange interaction, which significantly improved the accuracy of this computing method. Dirac expounded all the above-listed methods in a supplement to the first Russian edition (1932) of his monograph [33].

Dirac titled his main work on quantum mechanics — the monograph *The Principles of Quantum Mechanics* — in the spirit of Newton. This work, which ran into four revised editions during his lifetime, by its contents is the best exposition of the elements of quantum mechanics and has assumed its rightful place in the treasury of physical classics, along with Newton's *Philosophiae Naturalis Principia Mathematica*, Maxwell's *A Treatise on Electricity and Magnetism*, and J Gibbs's *Elementary Principles in Statistical Mechanics*. The book was written in a new quantum language elaborated by Dirac, which was initially disapproved of some physicists. Even Heisenberg wrote in his review of the German translation of the book that "... *Dirac supposedly conceives quantum mechanics, particularly its physical content, more 'symbolically' than is required*" [5].

Like Newton, Dirac began the exposition of quantum mechanics with basic definitions and axioms. [8] He considered in detail the distinctions between the classical and quantum approaches to the description of physical phenomena and the ensuing profound changes in the opinion of physicists on the mathematical foundations of their science. Dirac wrote, "*With the recognition that there is no logical reason why Newtonian and other classical principles should be valid outside the domains in which they have been experimentally verified has come the realization that departures from these principles are indeed necessary. Such departures find their expression through the introduction of new mathematical formalisms, new schemes of axioms and rules of manipulation, into the methods of mathematical physics*" (see Ref. [20], Vol. 1, p. 28).

The advantages of Dirac's approach to the exposition of the elements of quantum mechanics eventually received general acceptance. Interestingly, Einstein, who had never perceived the quantum theory as the unified scientific theory of the microscopic world and persistently sought contradictions in formulations and interpretations of quantum laws, would permanently carry precisely *The Principles of Quantum Mechanics* of Dirac, as attested to by witnesses. D D Ivanenko wrote in the foreword to the first Russian edition of *The Principles* ... (at that time translated as *The Elements* ...): "*Among all the books issued, Dirac's The Elements ... stands out primarily for its exceptional integrity and breadth of scope... Compared with other books on this subject in our field, one can say with some exaggeration that, alongside The Elements ..., Sommerfeld's supplementary volume Wellenmechanischer Ergangzungsband presents itself like collected solutions of a number of particular problems; de Broglie's Introduction a l'Etude de la Mecanique Ondulatoire is merely an introduction concerned primarily with the passage from classical to quantum mechanics; Elementare Quantenmechanik by Born and Jordan is an exposition of an intentionally limited part of the material...* (*the Schrödinger equation is absent in the book*)*, and, lastly, Frenkel's Einführung in die Wellenmechanik, the book most intelligible to the reader, is devoid, like all the above, of only one thing — the exposition of the system of quantum mechanics. It is precisely the exposition of the system that is afforded by Dirac's book, this being done in the most superior way, which is free from any provincialism, i.e., employing a restricted method, posing problems close to the author, etc.*" (see Ref. [20], Vol. I, p. 13).

4. Dirac's role in the elaboration of quantum field theory and the theory of elementary particles

That which was done by Dirac to lay the foundations of quantum mechanics alone would suffice to rank him among the 'immortals'. Meanwhile, at virtually the same period (1927 – 1934) Dirac was laying the foundations of two more exceptionally fruitful approaches to the study of the microworld — the quantum field theory and the theory of elementary particles. The former resulted from giving deep thought to Schrödinger's wave mechanics. According to his own reminiscences, Dirac asked himself the question: "*What if we take the Schrödinger wave equation and try to apply the quantization procedure to the wave function itself? It has always been assumed that the wave function is expressed in terms of ordinary numbers, i.e. c-numbers. The question now arises: what if they are transformed to q-numbers? ... Here is how the method known as the second quantization emerged*" [9]. [9]

4.1 Dirac as the founder of quantum field theory

It is generally recognized that the first work on quantum field theory was Dirac's paper "The quantum theory of emission and absorption of radiation" [34]. In this paper, for the first time the method of secondary quantization was proposed, the quantization of electromagnetic field was performed, and the coefficients entering Einstein's radiation theory were consistently calculated in the framework of the quantum theory. As a result of further development of the ideas outlined in this work, the arsenal of physicists was enriched with a qualitatively new object — *quantum field*, which allowed for the elimination of the contradictions between the corpuscular and wave interpretations of electromagnetic radiation.

Dirac's fundamental role in the elaboration of quantum field theory has been comprehensively investigated for a long time (see, for instance, articles by R Jost [6], V Weisskopf [35], and J Mehra [7]). For this reason we will not delve deeply into this topic, but will restrict ourselves to only a short summary of the most thorough, in our opinion, paper by B V Medvedev and D V Shirkov "P A M Dirac and formation of the basic notions of quantum field theory" [36]. The authors of the paper note that the theory of quantum fields has assumed different aspects more than once. In this case, "... *not only the details, but also, in a certain sense, the basic concepts*" of the theory experienced significant changes. This process is most naturally subdivided into the following three stages.

In the first stage (1927 – 1948), which may be referred to as the theory formation stage, the main effort was directed toward extending the methods of quantum mechanics to relativistic systems with an infinite number of degrees of freedom, i.e., to field systems. It was Dirac who contrived and proposed employing the majority of the technical means required for the solution of this problem. Apart from the general theory of transformations from one representation to another, which was proposed in Ref. [27], in the same paper Dirac introduced the first generalized function, the δ-function (present-day quantum field theory is unthinkable without employing generalized functions), as well as the rules for manipulating these functions. Subsequently proposed was the method of secondary quantization [34] and the so-called 'many-time formalism' [37] — the main working tool in relativistic quantum calculations right up to the emergence of the explicitly covariant formulation of quantum electrodynamics due to S Tomonaga, J Schwinger, R Feynman, and F Dyson.

"*However, the main obstacles to the transfer of the methods of quantum mechanics to field systems were not the technical problems,*" *the authors of the summarized paper* [36] *noted,* "*but supposedly the necessity to overcome the psychological*

[8] As stated by H Rechenberg [17], in doing this "... *he closely followed Baker's example, especially his book entitled The Principles of Geometry. From this book, Dirac practically copied the necessary statements in about the same order the mathematician had written them down. Also, as regards the geometric interpretation of the formalism, in two places he used Baker's scheme. On the one hand, he concluded from this book that it was possible to construct a mathematically consistent theory with noncommuting variables, and, on the other hand, he derived the geometric interpretation of what he named 'q-numbers'...*". Therefore, projective geometry has played its part in the creation of the masterpiece of the world's scientific literature.

[9] The term 'secondary quantization' itself was presumably proposed by V A Fock.

barrier of contraposing two forms of matter — particles and fields — which were perceived from the classical standpoint as absolutely different essences''. In fact, Dirac obviated the problem of wave–corpuscle dualism even in Ref. [34], wherein he established that ''... *the Hamiltonian which describes the interaction of the atom and the electromagnetic waves can be made identical with the Hamiltonian for the problem of the interaction of the atom with an assembly of particles moving with the velocity of light and satisfying the Einstein–Bose statistics...*''. The same paper first saw the emergence of a quantized electromagnetic field which satisfied the equations of classical electrodynamics but whose values were quantum-mechanical operators acting on the Schrödinger wave function; in this case, this wave function is often referred to as the state amplitude. The development of this central idea, in which the majority of the contrivers of quantum mechanics took an active part, was detailed in Ref. [36, Sections 2–5], which permits us to pass on to the results of the first stage at one.

Summarizing the activities of a large group of theorists (including Heisenberg, Pauli, Jordan, Fock, E Fermi, O Klein, E Wigner, and others), Medvedev and Shirkov concluded that ''... *These 15–20 years were actually a time of the agonizing development of a fundamental new paradigm (and of becoming accustomed to it) in which classical particles and fields come to have completely equal rights as two different manifestations of a single unitary object: a quantized field. The new understanding of a basic organizational mechanism of nature was developed by various people in small pieces, which only gradually combined to form a unified picture*'' [36].

It may be pertinent to note that this 'painful process' was brought to logical completion only 65 years later in Shirkov's work [38]. He noted, in particular, that the term 'quantized field', which was actively employed at the formation stage of quantum field theory, from the outset assumes the prime nature of the classical field and the secondary nature of the quantum one. But this reflects only the historical sequence of the origin of these terms since, as is well known, the quantum picture is more adequate to the physical reality and the classical picture is merely some approximation to it. It was therefore proposed to replace 'the historical ordering' of terms with the logical ordering and consider just the quantum fields as the prime essence. If this field is transformed according to Fermi–Dirac approach, in the classics it corresponds to the concept of a point particle. And if it is transformed according to Bose–Einstein approach, it corresponds to the concept of a classical relativistic field. In this case, once again there prevails a principle referred to as 'the Ockham razor': ''*essences should not be needlessly multiplied*''. To take the place of both the fields and particles of classical physics, a universal essence comes up — a quantum field, which boils down to primary matter constituents as well as quanta which transfer the interaction between the present-day prime elements.

In fact, quantum field theory almost entirely assumed its present-day aspect during the second stage, which can be dated to 1949–1964. The main problem of this stage was 'combatting divergences'; their inevitable emergence was first pointed out presumably by Ehrenfest almost immediately after the publication of Dirac's paper [34]. Ehrenfest noted that invoking the notion of a point electron would inevitably lead to its infinite intrinsic energy. Five years later, in Ref. [39] Dirac distinctly formulated the causes of this phenomenon, which was inherited from the classical problem of the electron interaction with the radiation field: ''*The classical equations which deal with this problem are of two kinds, (i) those that determine the field produced by the electron (which field is just the difference of the ingoing and outgoing fields) in terms of the variables describing the motion of the electron, and (ii) those that determine the motion of the electron. Equations (i) are quite definite and unambiguous, but not so equations (ii). The latter express the acceleration of the electron in terms of field quantities at the point where the electron is situated and these field quantities in the complete classical picture are infinite and undefined*''.

A year later, in his Solvay report [40], Dirac actually came up with the seed idea of charge renormalization. He stated that external charges should polarize the vacuum in his theory, with the effect that ''... *the electric charges which are normally observable for the electron, the proton, and other electrified particles are not the charges which are actually carried by these particles and which figure in the fundamental equations; they are instead smaller*''. He carried out calculations of this new physical effect, which reduced to a logarithmically diverging integral whose cut-off at momenta on the order of 100 ms (which corresponds to the classical electron radius) yielded a 'radiative correction' to the electron charge, which reduced it by about a factor of 1/137. Yet another year later, Weisskopf [41] also arrived at a similar result; he showed that the intrinsic electron energy with the inclusion of the Dirac vacuum diverges logarithmically, so that its addition to the 'mechanical' mass remains small even when the cut-off is effected at the Schwarzschild radius.

As a result, the development of these initial attempts 'to combat divergences' took two paths. On the one hand, Stückelberg [42] and H Kramers [43] formulated the central idea of the renormalization method: the final values for observables can be obtained, for instance, by appropriate subtraction of an infinite magnitude (of some characteristic) for a free electron from the similar infinite magnitude for a bound electron. This approach makes it possible to retain the deep-rooted notions of particles as points of geometrical space and of the local nature of quantum field theory. These ideas were brilliantly realized by Schwinger, Feynman, and Dyson in the late 1940s with a record accuracy of agreement between theoretical predictions and experiments. However, the unconventional technique of quantum–field calculations called for a sufficiently rigorous mathematical substantiation.

And such substantiation of renormalization technique did appear as a result of a thorough analysis of the mathematical nature of quantum–field infinities, which was reliant on the Sobolev–Schwartz theory of generalized functions. It transpired that the divergences (from the viewpoint of this theory) are a manifestation of the uncertainty in the operation of multiplication of the propagators of point particles (which are the generalized functions) in the event of coincidence of their spatio-temporal arguments. N N Bogolyubov and his pupils (O S Parasyuk, D V Shirkov, and others) [44–47] elaborated the R-operation technique: extension of the definition of the products of causal propagators in such a way as to ensure the finiteness of resultant expressions in all orders of the perturbation theory. In this way there came into existence the notion of renormalizable and nonrenormalizable models of quantum field theory, which became one more criterion for the selection of models rich in content. The modern treatment of the renormalizability concept was given by Shirkov in Ref. [48].

The ultimate embodiment of renormalization ideology and simultaneously the central result of the second stage of development of quantum field theory is the advent of renormalization group approach whose foundations were laid in Refs [49 – 51]. [10] The renormalization group method for the first time made it possible to go beyond the framework of weak coupling approximation and to obtain, on this basis, record-accurate data in the calculation of higher-order radiative corrections. However, the authors of Ref. [36] noted: "*As a result of all these studies, the outlook for the future prospects of renormalizable quantum field theories seemed a bit gloomy. It appeared that the qualitative diversity of renormalizable quantum field theories was negligible: for any renormalizable model, the only possible effects of interaction — for small coupling constants and moderate energies — were unobservable changes in the constants of free particles ... The existing theory — again, regardless of the specific model — was inapplicable to large coupling constants or asymptotically high energies. Quantum electrodynamics remained the only* (*although brilliant*) *application to the real world, which met these requirements*".

Now is as good a time as any to recall another line of 'combatting divergences', which Dirac chose for himself, working actually in complete 'solitude'. Having generated the initial idea of charge renormalization, he practically abandoned the further development of these ideas. In addition, more than once he argued against the development of QFT along these lines (see, for instance, Ref. [9]). Dirac would persistently seek the solution of the resultant problems by way of abandoning the notion of an electron as a point object. In particular, his quest resulted in the emergence of theories with indefinite metrics, one of the versions of which was first proposed in his Bakerian lecture "The physical interpretation of the quantum mechanics" [52]. Such theories later found numerous applications.

It is well known that Dirac did not achieve much success in quantum electrodynamics by following this path, but the original ideas and approaches suggested by Dirac became (in the majority of cases) the 'seeds' of the third stage of developing the quantum field theory, which will be discussed at length in Section 5.

4.2 The Dirac equation and principles of elementary particle theory

Dirac's next basic result is his celebrated relativistic equation of an electron, which has not revealed all its properties to physicists nor to mathematicians. This is how Weisskopf, one of the first CERN directors, assessed this event in his semi-autobiographic article "Growing up with field theory" [35]: "*In 1928, Dirac published two papers dedicated to the new relativistic equation for the electron. This was his third outstanding contribution to the foundations of modern physics* (*the first contribution was the new formulation of quantum mechanics – 'The Transformation Theory...', and the second one was the theory of radiation ...*)". Apart from satisfying the principles of relativism and probabilistic interpretation of quantum mechanics, it contained information about the half-integer spin of an electron and its magnetic moment, and also provided a gauge invariant description of the electron interaction with electromagnetic field.

[10] An intelligible exposition of this approach is contained in Ref. [36, Section 8].

True, in this case an electron acquired a new degree of freedom — it could move into states with negative energy. This appeared to be so odd that one might as well abandon the results obtained. We are reminded that quantum mechanics in fact had inherited the problem of negative energies from the special relativity. According to the formula for relativistic energy $E = c\sqrt{m^2c^2 + p^2}$, which contains a square root, it can assume both positive and negative values. In other words, the particle energy can assume formally any value in the range between mc^2 and infinity, as well as from $-mc^2$ to minus infinity. In the classical theory, where particle trajectories are continuous, problems do not arise, for a particle cannot pass into a negative-energy state. In the quantum theory, the probability of such a transition is nonzero, so that the particle can change the sign of its energy in a stepwise manner, without going through the intermediate states.

The paradoxicality of the ensuing conclusions did not frighten Dirac. He chose another way — he believed in the reality of negative-energy states and, taking advantage of the Pauli exclusion principle, filled all unreal states with real electrons. Dirac termed the collection of these states a 'sea' or an 'ocean', which "*is occupied with electrons without the restriction for a negative energy and therefore there is nothing like a bottom in this electron ocean*" [9]. Dirac believed that electrons with a negative energy are not observed, because they make up a continuous invisible background against which all world events take place. However, when a high-energy photon finds itself in the 'Dirac electron sea', under certain conditions it can knock out one of the countless 'sea' electrons. The empty place, a 'hole', will behave like a quasi-particle with a positive charge. [11]

The situation changed when Dirac took the next step by assuming that the 'holes' in the electron sea should be treated not as quasi-particles, but as real positively charged particles which would be experimentally observable, in principle, as free objects. We are reminded that only electrons, protons, and photons were known from experiment late in the 1920s, so that even atomic nuclei were assumed to be collections of tightly coupled electrons and protons. It is proceeding from precisely the available opportunities that Dirac initially selected a proton as a candidate for a 'hole'. As a result, the 'elementary particle physics' known by that time would have actually been described with a single equation — everything would be simple and beautiful.

We emphasize that the proposed theory of 'holes' was not taken seriously by the majority of physicists and, whenever considered by individual theorists, the aim was primarily to disprove it. Dirac himself was not discouraged by these circumstances, and he continued to elaborate the theory under the title 'the theory of electrons and protons', assuming that the glaring difference in the masses of electrons and protons would later be possible to explain by the special features of interaction in the electron sea. In particular, as early as 1930 he calculated the annihilation cross section for electrons and 'holes', obtaining by so doing (as it turned out later) the correct cross section for the annihilation of electrons and ... the then unknown positrons.

[11] We emphasize that Dirac interpreted the vacancies among the occupied states of this type as 'holes' almost right away. He proceeded from the scheme of occupation of some atomic electron shells and their restructuring at molecular formation, which was employed in the theories of multielectron atoms and chemical valence, as well as in the description of the origin of X-ray atomic spectra.

In the May of 1931, in the paper "Quantized singularities in the electromagnetic field" [53] Dirac clearly pointed out for the first time that the combined employment of the principles of the quantum theory and the relativity theory requires that to each charged particle there corresponds its own oppositely charged antiparticle with the same mass. That is why the role of 'holes' with respect to electrons should be played by qualitatively new objects — antielectrons, which were termed positrons before long. Simultaneously, Dirac stated that there are also bound to exist the antipodes of protons — antiprotons. Slightly more than a year went by when an American physicist C Anderson announced on August 2, 1932 (not long before Dirac's birthday) the discovery of the positron in cosmic rays. (The antiproton was obtained at an accelerator in 1955, and the antineutron in 1956.)

The above events call for several comments, primarily concerning the role of R Oppenheimer's well-known letter [54] in the establishment of the positron concept. This letter contains a preliminary estimate of the cross section for electron – positron annihilation as a process which follows from previously advanced Dirac's theory. Since the resultant estimate did not correspond to the observed stability of these particles, Oppenheimer suggested that (i) the holes in the electron background should not be identified with protons; (ii) electrons and protons should be treated as absolutely independent particles; (iii) all negative-energy electron states should be completely filled with electrons to eliminate holes, and (iv) in order to compensate for the infinite negative charge of the electron background, a similar background with an infinite positive charge should be introduced, filling completely, i.e., without any holes, with protons the negative-energy levels of the similar proton background.

Therefore, according to the idea of Oppenheimer, both for electrons and protons, the holes in the corresponding backgrounds are lacking and cannot be produced in principle. That is why the processes of annihilation or production of massive particles should not take place at all. The only unconventional positively charged particles whose existence could be hypothesized on the basis of Oppenheimer's suggestions were protons in negative-energy states, but not positrons; far from it!

We next note that up to the present day it is possible to encounter the following assertion in the scientific literature: to discover positrons required cosmic photons with energies of more than 1 MeV. The collisions of the latter with nuclei made it possible to observe electron – positron pairs whose components were deflected differently by a magnetic field. In reality, this requirement was not necessary at all: even five years prior to Anderson's experiments, events were known which now are referred to as positive β decay of nuclei. [12] In these events, positrons emerged one at a time and with any arbitrarily low energy. However, observers interpreted their 'incorrect' deflection in the magnetic field as the backward (i.e., towards the source) motion of electrons.

We would also like to emphasize that the positrons themselves were not the point. The basic idea advanced by Dirac in these papers, which now is frequently overlooked, was the possibility of principle to produce and destruct particles of any mass on keeping the corresponding conservation laws. Of course, the theoretical possibility of the interconversion of kinetic energy and rest energy follows from the special relativity, and the majority of physicists agreed with it by the late 1920s. However, this did not in the least imply that the number and sort of particles could vary in elementary processes. The long-standing resistance to the recognition of a photon as one of elementary particles was supposedly due to this circumstance, for photons had the capacity to be radiated and absorbed. In the long run, an exception was made for massless photons. At the same time, the only corroboration of energy interconversion processes for nonzero-mass objects was the occurrence of radioactivity and the simplest nuclear reactions, which were commonly treated by analogy with molecular dissociation and chemical reactions. Even β decay was initially interpreted by analogy with the ionization of atoms. To put it another way, the number and sorts of nonzero-mass particles were always assumed to be the same at the onset and the end of any process, and only a relatively small energy redistribution was dealt with when the same particles moved from a bound state to the free state and back.

[12] The classical source on this problem and its history is monograph [55].

Having postulated the possibility of the production and annihilation of electron – positron pairs (and the production and annihilation operators themselves appeared even in Dirac's pioneering work on quantum theory in 1925 [22]), Dirac predicted for the first time the interconversion of elementary particles of any mass, including the processes wherein the rest energy of the initial particles was completely converted to the kinetic energy of the final particles. The success of this prediction subsequently had an enormous impact on changing world outlook (Weltanschauung) of the scientific community as a whole, for the implications of the special relativity enriched with the quantum theory were brought to their logical conclusion.

Finally, we are reminded that the existence of antiprotons predicted by Dirac, which now appears to be almost trivial, was disapproved by many physicists even after the discovery of positrons. The point is that anomalous magnetic moments were discovered in protons and neutrons by that time, and the question of whether the Dirac equation could be applied for their description proved to be an open question (with all the ensuing consequences) [13].

But Dirac was not confused by these doubts. His Nobel lecture [56] concluded with a new prevision: "*If we accept the view of complete symmetry between positive and negative electric charges so far as concerns the fundamental laws of Nature, we must regard it rather as an accident that the Earth* (*and presumably the whole solar system*) *contains a preponderance of negative electrons and positive protons. It is quite possible that for some of the stars it is the other way about, these stars being built up mainly of positrons and negative protons. In fact, there may be half the stars of each kind. The two kinds of stars would both show exactly the same spectra, and there would be no way of distinguishing them by present astronomical methods*". While the discovery of 'antistars' has never been reported, there is significant progress in the cause of experimental discovery of antimatter pioneered by Anderson 70 years ago. In August of 2002, in fact on the centenary of Dirac's birth, the international team of the 'ATHENA' project in CERN for the first time produced tens of thousands of antihydrogen atoms in one experiment, i.e., an almost macroscopic dose of antimatter. In principle, the door was thereby opened to the production of antimolecules and later 'antiliquids', 'anticrystals', etc.

[13] According to present-day notions, the Dirac equation underlies the description only of truly basic structureless fermions — leptons and quarks.

It is pertinent to add a few words about the fundamental significance of these Dirac's ideas. Of course, since the 1940s no one can be surprised by discoveries of theoretically predicted particles (from the Yukawa meson to the *t*-quark). However, Dirac's theory was more than the first successful prediction in this series. In his report presented at the XIVth International Conference on Cosmic Rays in 1975 Heisenberg emphasized: "*Of significance was by no means the discovery of yet another previously unknown particle; of significance was the discovery of a new symmetry, the particles-antiparticles conjugacy intimately related to the Lorentz group of special relativity theory as well as to the conversion of the kinetic energy of colliding particles to the rest energy of new particles and back*" [57]. Elaborating on this idea, I Yu Kobzarev noted that "... *the new symmetry of nature discovered by Dirac has proved to be significant not only for fermions. Its intimate connection to the relativistic invariance was subsequently embodied in the celebrated CPT theorem which presently underlies the theory of elementary particles. This symmetry was experimentally borne out by the discovery, for practically every particle, of its associated antiparticle different from it*" [58].

The subsequent destiny of Dirac's idea of the 'sea' of negative-energy electrons turned out to be quite extraordinary. It underwent a qualitative evolution in quantum electrodynamics itself and, more broadly, in quantum field theory. A radically new notion was eventually introduced — the physical vacuum, qualitatively different from the classical notion about 'void'. The vacuum is filled with virtual pairs of electrons and positrons, virtual photons, as well as virtual pairs and basic quanta of other types. The last exert an effect on the properties of real objects, which shows up in the renormalization of charge and mass as well as in polarization effects, which were also considered by Dirac for the first time [59], and so forth. But even today, despite all the modifications, the initial idea of the Dirac's negative-energy sea exhibits amazing vitality: it is employed to advantage, for instance, in the interpretation of 'anomalies' in the quantum field theory [60].

However, the theory of holes-quasiparticles advanced most significantly and found numerous applications outside quantum field theory proper. It underlies the band theory of electronic spectra in semiconductors and is employed in the theories of multielectron atoms and chemical valence, the nuclear shell model, the theory of supercharged nuclei [61], and, lastly, the theory of superconductivity. In fact, Dirac's notions of 'holes'-quasiparticles have proved to be extremely fruitful in all physical systems whose energy spectra possess a gap or a Fermi sphere.

It is pertinent to note that no one had anticipated so quick an experimental corroboration of the existence of antiparticles predicted by Dirac. For just on the eve of this event many famous theorists (L D Landau, V A Fock, N Bohr, and several others), to put it mildly, could hardly believe so crazy a hypothesis. Even Pauli, although a witty and slightly adventurous person who had just advanced a hypothesis (true, a speculative one) for the existence of the neutrino, cast strong doubt on Dirac's predictions in a famous review paper on quantum mechanics [62]: "*In this theory, the laws of nature are precisely symmetric about electrons and antielectrons, and it seems unsatisfactory for this reason alone... We do not think this way out should be considered in earnest*". However, Dirac's brilliant intuition and his conviction that pretty mathematical results are efficient in physics won out this time, too. This was a triumph. In this connection Weisskopf emphasized in the above-mentioned paper [35] that "*The theoretical predictions concerning new basic processes and the new properties of matter had been made before any experimental indications were made on that score. On the contrary, all previous experience contradicted the symmetry between positive and negative electric charge*".

The discovery of a positron as the confirmation of the existence of basic antimatter constituents produced an impression on the broad public, comparable only with the confirmation of the results on the general theory of relativity in the observations of light ray deflection in the solar gravitational field in 1919. Dirac, like Einstein in his time, instantly became a world celebrity, but this had no effect on his mode of life and style of scientific work. Meanwhile, everybody was expecting him to report equally quick and sensational results. However, such results were not to be. Regular routine scientific work was underway, which was oriented, as we now understand, to a distant perspective and therefore remained outside the scope of current attention (and sometimes of understanding) of colleagues. Furthermore, the Second World War broke out and after it the acute period of the cold war. The physics society's interest in Dirac's creative work began to gradually fade away.

5. Basic ideas of the 'later' Dirac

Since 1934, i.e., after laying the foundations for three basic theories, Dirac lived and went on working for 50 long years. It is inconceivable that a person of his intelligence and the power of engrossing in scientific work would rest on his laurels at the age of 32 and not make significant contributions to science any more. This viewpoint is nevertheless rather popular, largely due to some stereotypes created by famous Dirac biographers (see, for instance, Mehra's article [7] as well as Ref. [63]).

Thus, the article by R Dalitz [63], a famous theoretical physicist, which opened a collection of memories of Dirac published by his friends and colleagues, gave the list of 24 most significant (in the view of the author) Dirac's papers. The last paper in the list is dated 1948, and the 1934 – 1948 period is represented by only six papers. Therefore, strange as it may seem, the majority (150!!) of Dirac's papers which saw light after 1934 remained outside the field of view of Dalitz, who was seemingly treating Dirac's name and his heritage with benevolence and distinction. [14]

Meanwhile, these works, which have not engaged the attention of biographers, contain several basic ideas, each of which deserves at least a thorough paper, if not a separate monograph. In the subsequent discussion we therefore restrict ourselves to only a summary outline of consolidated series of his papers whose ideas (in our opinion) have either proved to be fruitful or contain incompletely revealed potential for the development of modern theoretical and mathematical physics.

[14] It is pertinent to establish the consistent conservatism of Dalitz's standpoint. Almost 10 years later, in 1995, as an editor he prepared the publication, which was unique in many respects, of Dirac's selected papers [64], again including only the papers published before 1949.

5.1 Classical Hamiltonian dynamics with constraints — Dirac mechanics — and quantization of gauge fields

It is likely that the series of papers on generalized Hamiltonian dynamics [65] (see also lectures [66]) constitutes Dirac's greatest contribution to theoretical physics in the 1950s–1980s. In fact, this is the next stage in the development of analytical dynamics after Hamilton himself, and for this reason the title 'Dirac mechanics' [67, 68] is increasingly often employed in the modern literature, side by side with Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics.

Dirac's works on the generalization of Hamiltonian dynamics made their appearance at the time when quantum field theory was going through its most difficult period. After the stunning successes of quantum electrodynamics related to the names of Tomonaga, Schwinger, Feynman, and Dyson, there set in the 'Time of Troubles' of heavily dispiriting failures at meson theories of nuclear forces, where the renormalization procedures, which had shown themselves to be advantageous in electrodynamics, would not do any good. The so-called 'zero charge problem' nonplussed the eminent theorists in all its magnitude. Their opinion was most clearly formulated by Landau: "... *the Hamiltonian method for strong interactions has become obsolete and should be buried, naturally, and rendered homage it has deserved*" [69]. On these grounds they attempted to 'discard' the whole quantum field theory as 'being out of date' and replace it with semiphenomenological approaches like the analytical theory of S-matrix, reggistics, current algebra, etc. [15] True, this viewpoint was by no means unanimously shared. An intensive search for new approaches and generalizations was underway, and an increasingly more powerful mathematical apparatus was invoked for the development of the quantum-field approach.

In his lectures [66] Dirac explained in detail why the development of the apparatus of relativistic quantum field theory called first and foremost for the extension of the capabilities of classical Hamiltonian dynamics and why on this path it is necessary to successively go through all the stages from the relativistic-invariant action principle to the Hamiltonian and only then to the quantum theory. As a preliminary, he elucidated the way in the situation when the conventional transfer from the Lagrangian $L(q, \dot{q})$ to the Hamiltonian $H(p, q)$ is impossible, i.e., when the conventional definition of the generalized momentum $p_i = \partial L/\partial\dot{q}_i$ is unsolvable for some set of generalized velocities $\dot{q}_i$. For systems with a finite number of degrees of freedom, this situation occurs when the rank of a Hessian $\partial^2 L/(\partial\dot{q}_i\partial\dot{q}_j)$ is smaller than the number of degrees of freedom. The corresponding Lagrangians are termed singular or special. In going over to systems with an infinite number of degrees of freedom (condensed media, field systems), the problem persists and is even aggravated. In real situations, the latter takes place for the majority of modern models in particle physics, such as the gauge Yang–Mills model, the supersymmetric generalizations of Yang–Mills fields, supergravity, superstring, membrane, and bag models, etc., in which the fields have one geometrical significance or another.

As is well known, the theories of non-Abelian gauge fields (or Yang–Mills fields) occupy a special place in the contemporary notions of the nature of fundamental interactions. First of all, based on the principle of gauge invariance, physicists had at their disposal a simple and efficient algorithm of constructing 'dynamics from symmetries'. Simple and elegant, yet amazingly informative, Yang–Mills Lagrangians came to replace the immense expressions for the Lagrangians of the meson theories of the late 1940s. In any case, the Standard Model, which represents our present-day understanding of the physics of elementary particles and fields, was constructed on the basis of such theories. However, we are reminded that the Yang–Mills fields were perceived by theorists, for more than ten years after their introduction, as an elegant but useless construction, which was, at most, of academic interest. The reason lay, in particular, with the massless gauge vector bosons predicted by the theory, which had never manifested themselves in experiments (for more details, see, for instance, Refs [71, 72]).

We note that Dirac 'betrayed', when solving this range of problems, his traditional 'emploi' of a researcher personally developing his ideas to all conceivable logical consequences and played the part of a 'playmaker' rather than the main 'goal-scorer'. The generalization of Hamiltonian formalism proposed by Dirac relies on reducing the initial phase space by imposing first- and second-class constraints corresponding to the system — the *Dirac reduction* — making it possible to find the modified Poisson bracket — the *Dirac bracket* — and construct the corresponding Hamiltonian formalism. Even in the first paper (1950) of the series [65] he also proposed the scheme of operator quantization of the systems with constraints (as a matter of fact, Dirac was developing his approach for precisely this purpose). However, in the application of this scheme to the gravitational field [73] the problems emerged with multiplier ordering and relativistic covariance, among others. [16] Feynman's attempt (1963) to carry out the quantization of Yang–Mills fields by employing the methods which had proven advantageous in quantum electrodynamics also encountered certain contradictions (the violation of unitarity condition was discovered).

The further narration of the creation of the quantum theory of gauge fields would be a digression from our main subject. Omitting the intermediate stages, we therefore point out straight away that the method of continual integration developed by Feynman (1948) has eventually proved to be the most adequate apparatus for the quantization of gauge fields. The starting point for Feynman was Dirac's idea, which was proposed in Ref. [74] as far back as 1933, that the temporal evolution of a quantum system on a finite time interval can be represented as a composition of a large number of evolutions over short time intervals. Relying on his theory of transformations developed earlier, Dirac showed that the final transform function appears in this case in the form of a multiple integral of the product of a large number of 'elementary' transform functions taken over the possible values of dynamic variables at intermediate points in time. Most significantly, Dirac suggested that the wave function transformations should be determined employing the exponent of the classical action of the system. This idea was further refined in an infrequently cited paper [75]. The development

[15] The author's preface to book [70] says: "...*The general level of the book assumes familiarity by the reader with the principles of nonrelativistic quantum mechanics* (*including scattering theory*) *as well as with the Lorentz group. No background in quantum field theory is required. Indeed, as pointed out in the preface to my 1961 lecture notes, lengthy experience with Lagrangian field theory appears to constitute a disadvantage when attempting to learn S-matrix theory*". No comment is necessary!

[16] It is well known that these problems were solved at a later time, but the construction problem of the quantum theory of gravitation still remains unsolved owing to definite nonrenormalizability.

and formalization of these ideas led to Feynman integrals, which are referred to as path integrals in the quantum mechanics of systems with a finite number of degrees of freedom, and as functional integrals in the quantum field theory (for more details, see Ref. [36]).

The first issue of the journal *Teoreticheskaya i Matematicheskaya Fizika* (Theoretical and Mathematical Physics) saw light in 1969; it opened with L D Faddeev's paper entitled "Integral Feĭnmana dlya singulyarnykh lagranzhianov" ("The Feynman integral for singular Lagrangians") [76]. The paper gave the general recipe for the quantization of systems with constraints within the formalism of a continual integral, which has gained general acceptance and is reproduced in practically all guides and textbooks on the quantum theory of gauge fields up to the present time. From the very title of the work it follows that doing this required accomplishing, at the very least, the synthesis of two of Dirac's ideas mentioned above: the generalized Hamiltonian formalism, and the continual integral.

However, the task was not limited to the synthesis alone. It took a certain development of the Dirac scheme to carry out gauge group reduction, since, owing to the gauge invariance of the theory, the principal objects in it are not the potentials A_μ but their equivalence classes (orbits). Next obtained was an explicit expression of the Feynman measure for Dirac's generalized Hamiltonian dynamics. It was found that the requisite reduction is most naturally realized employing precisely the generalized Feynman integral. As a result of this and other accomplishments, which we do not mention here and which the reader can familiarize himself utilizing monograph [72], the gauge field theories have occupied a fitting position in particle physics, and the dynamics of systems with constraints has become an actively advancing independent direction (see, for instance, Refs [67, 77]).

5.2 The Dirac monopole and topological ideas in physics

Another fruitful direction in modern theoretical physics, which is also closely related to Dirac's name, is the problem of a solitary magnetic charge (monopole). It reduces to the question: why are magnetic field sources similar to electric charges absent in nature? For otherwise, electric and magnetic fields enter the Maxwell equations quite symmetrically. This brings up the natural question: why did nature require so evident an asymmetry as regards the sources of electric and magnetic fields?

Speaking at the symposium held at Loyola University (USA) and dedicated to his 80th birthday, Dirac explained his interest in the problem in the following way: "*Another example of pretty mathematics led to the idea of the magnetic monopole. When I did this work I was hoping to find some explanation of the fine-structure constant $\hbar c/e^2$. But this failed. The mathematics led inexorably to the monopole. From the theoretical point of view one would think that monopoles should exist, because of the prettiness of the mathematics*" [78].

After a thorough analysis of the known facts on the fundamental unobservability of the phases of the wave functions in quantum mechanics, which, in addition, are defined correct to 2π and become nonintegrable in the presence, for instance, of an electromagnetic field, Dirac showed in 1931 in Ref. [53] that the hypothesis of the existence of solitary magnetic monopoles with a charge μ is not at variance with the principles of quantum mechanics, provided that $e\mu = 2\pi\hbar cn$, where n is an integer. Therefore, if the monopoles were discovered, the above formula, termed the *Dirac quantization condition*, would be an explanation of the quantized nature of the electric charges of the known particles. "*Under these circumstances one would be surprised if Nature had made no use of it*", Dirac noted at the end of the paper [53].

In a series of papers [79], Schwinger generalized the Dirac quantization conditions to the interaction of two particles each of which possesses both electric and magnetic charges:

$$(e_1\mu_1 - e_2\mu_2) = 2\pi\hbar cn\,,$$

which he termed *dions*. In this case, when such a dion is produced from two bosons with nonzero total electric and magnetic charges, the resultant bound state should obey the Fermi – Dirac statistics, i.e., there occurs the so-called *Fermi – Bose transmutation*. Currently, such transmutations are actively being investigated in the framework of *supersymmetric theories*.

True, the Dirac monopole proved to be a highly exotic (according to the notions of those days) solution containing a chain of singularities — the *Dirac string* — which is unobservable with the fulfilment of quantization conditions. In the view of M Atiyah [80], Dirac's work was in point of fact the first application of topological ideas in quantum physics. In this connection he wrote that "*... topology around the monopole* (*a 3-dimensional version of the winding numbers in a plane*) *would affect the wave function of the particle, and this in turn would lead to the quantization of its electric charge. Thus, the discreteness of charge is directly related to the discreteness of topological 'winding numbers'...*" In a paper dated 1948 [81], Dirac developed the general theory of interaction between charges and magnetic poles (positive and negative) and, in particular, endeavored to explain the inseparability of magnetic poles by the fact that they are connected by the Dirac string (the so-called monopole confinement). This idea was subsequently harnessed many times in different versions of string models of baryon, in which quarks were placed in lieu of monopoles at the ends of strings (see, for instance, Ref. [82]).

The idea of the Dirac monopole received the most interesting development in the grand unified theory. In 1974, A M Polyakov and G 't Hooft found a soliton-type solution with a unit magnetic charge (topological in nature) in one of the versions of electroweak theory — the Georgi – Glashow model. Unlike the Dirac monopole, the 't Hooft – Polyakov monopole is finite in dimensions and possesses finite values of energy, momentum, etc. What is most important, the magnetic charge of these monopoles should be topologically nontrivial, and their mass should be 10^6 times the proton mass. The monopoles predicted by the grand unified theories should be still more massive. Their mass should be 10^{16} times the proton mass. It is evident that the energy of not only the most modern accelerators, but also of the highest-energy cosmic rays, would be too small to give birth to this 'mammoth of the microworld'. However, early in the universe's evolution, when energy was abundant, monopoles could well have been produced that survive to the present day. That is why the quest for the monopoles does not cease in circumterrestrial space and near space.

One of the possible ways of detecting monopoles was derived 'with a pen and a sheet of paper' by V A Rubakov in 1981 and somewhat later by C Callan (the Callan – Rubakov effect, or the monopole catalysis) [83]. They discovered that a proton in the presence of a monopole should instantly decay into a positron and mesons. The monopole itself remains safe and sound in the process (by the law of magnetic charge conservation) and further capable of destroying the ambient material. The monopole trace in the material would therefore be accompanied by an easily detectable chain of 'proton catastrophes'. Neither the idea of the Dirac monopole, nor the idea of the 't Hooft–Polyakov monopole has been directly borne out in experiment. Despite this fact, they have lent impetus to the development of new directions [17] not only in physics, but in mathematics as well, have impelled physicists to master the unconventional mathematical apparatus of algebraic topology, and have simultaneously generated considerable interest among pure mathematicians in physical problems (see, for instance, Ref. [85]).

It is noteworthy that Dirac introduced, in the style inherent only in him, a new mathematical object to describe dynamics in the monopole field — *a many-valued functional*. Investigating its properties called for a substantial development of variational methods carried out by S P Novikov [86]. Prior to Dirac, the employment of topology was at the periphery of physicists' attention. Having introduced the idea of a monopole and its attendant topological singularity, Dirac pioneered the penetration of the elements of topology and the corresponding language in physics. These have found numerous applications in the present-day versions of elementary particle physics, in the physics of condensed media, and in cosmology, particularly in the development scenarios of the early universe. Therefore, even though magnetic monopoles have not been discovered experimentally, their numerous 'twins' (skyrmions, thorons, holons, etc.) have occupied a fitting place in theoretical physics (see, for instance, Refs [87, 88]).

5.3 Dirac's ideas in the realm of gravitation and cosmology

Speaking on the occasion of the centennial anniversary of Einstein's birth in 1979, Dirac briefly outlined his hypothesis of large numbers advanced back in 1937–1938 [89]. Under this hypothesis, all very large numbers composed of various physical and astronomical constants are not in fact fixed but are related by simple laws to the epoch — the time elapsed from the instant of the universe's creation. [18] The stated hypothesis allows an unambiguous choice among three possible evolution scenarios of our universe. Should this hypothesis prove to be true, this would manifest itself in a reduction of the gravitational constant, in a variation of interplanetary distances, etc.

Dirac developed these ideas for almost half a century, although they found a relatively narrow response among the scientific community. In recent years, the situation has taken a turn for the better as regards these ideas. Firstly, Dirac's hypothesis for the existence of two time scales — gravitational and atomic (electromagnetic) — may be realized in modern supergravitation approaches, where the number of dimensions increases not only with reference to spatial variables, but with reference to temporal variables as well. It also correlates with the modern ideas [90] according to which the gravitational and electromagnetic interactions are realized in spaces of different dimensionality.

Secondly, the idea of the time decrease of the gravitational constant and its attendant weakening of the gravitational interaction between visible and 'dark' matter may prove to be verisimilar. The point is that the latest discoveries of observational astronomy are indicative of the significant part played in the universe by so-called 'vacuum matter', or 'quintessence', as a fundamentally new material object. In this connection, efforts could well be made to ascribe the effective, after Dirac, time decrease of the intensity of gravity to the time increase of the role of peculiar 'antigravity'. Dirac's idea itself, which consists in the possibility to relate the big numbers known in physics to the age of the universe, has never been disproved. However, all this is still beyond the range of the experimental capabilities of contemporary physics.

5.4 Dirac's work on mathematical physics

Apart from the above-listed ideas, the work carried out by Dirac during the last 50 years of his life contains a lot of other discoveries and findings. Of these we point out only the most striking ones (in light of modern views). Having actually pioneered the development of the theory of renormalizations, later Dirac would repeatedly characterize this approach merely as a temporarily inevitable approach, bearing in mind the necessity of eliminating divergences. He spent a lot of time and mounted a serious effort to construct a quantum field theory with renormalizations, but without divergences. On the one hand, it is conceivable that these efforts were spent in vain, for the modern renormalization procedure reliant on the Bogolyubov R-operation is mathematically irreproachable. However, the very idea of constructing a truly finite quantum field theory is nowadays being realized in the so-called supersymmetric models which exhibit the remarkable property of cancellation of ultraviolet divergences in all orders of the perturbation theory (for more details, see Ref. [91]). Singletons, which have recently come under intensive investigation in conformal field theories, also rely on the conformal group representation proposed by Dirac in 1936 [15].

By and large, Dirac's works concerned with the problems of group representation theory deserve special consideration. Investigating the Lorentz group representations in Ref. [92], Dirac observed: "*The finite representations of this group, i.e. those whose matrices have a finite number of rows and columns, are all well known, and are dealt with by the usual tensor analysis and its extension spinor analysis. None of them is unitary. The group has also some infinite representations which are unitary. These do not seem to have been studied much, in spite of their possible importance for physical applications*". In this paper he proposed a new method of studying such representations, which leads to a new variety of tensor quantities in spacetime with an infinite number of components and a positive definite square of their length. He termed them *expansors*. Not only did Dirac determine the properties of expansors, but he also applied them for the description of a 4-dimensional harmonic oscillator, as well as for a particle with a spin, deriving in doing so several amazing consequences. Nevertheless, this work has not come, according to D P Zhelobenko, to the attention of experts in this field.

In Ref. [16], Dirac took advantage of projective geometry techniques to construct the quaternion representation of the Lorentz group, making it possible not to restrict oneself (as is done in the majority of textbooks) to the Lorentz transforma-

[17] For instance, research into topological and geometrical phases in quantum theory and optics (the Berry, Vladimirskiĭ, Anandan, etc. phases). For more details, see Ref. [84].

[18] It is not difficult to trace the connection between this idea of Dirac and the ancient dream of the philosophers of the Pythagorean school: to relate the basic laws of nature to the properties of integer numbers.

tions along one axis, but to comprehensively study the relativistic particle kinematics in the case of arbitrary motion of the frame of reference. To the best of our knowledge, this work has also remained unnoticed.

In the physics of pre-Planck distances, rather many recent papers have been devoted to the study of the properties of membranes (two-dimensional generalizations of a string) and p-branes (its p-dimensional generalizations). Curiously, in Refs [93, 94] Dirac first introduced membrane-like objects and wrote for them the relativistic-invariant action (which is frequently referred to in the literature as the Nambu – Goto action) with the aim of explaining experimental data on muons. This is one more testimony in favor of the opinion that Dirac may also be regarded as one of the trailblazers of the rapidly advancing string theory and its various modifications.

In principle, practically all of Dirac's work can be regarded as particular realizations of a new powerful method which emerged in the course of the mutual progress of physics and mathematics toward unification. Dirac expounded this method in detail in Ref. [95]: "*The method of advance is to begin with the selection of a branch of mathematics which in your opinion can serve as a basis for the new theory. In doing this you should be guided in great part by the considerations of mathematical beauty. It is also likely that preference should be given to the branch of mathematics which relies on an interesting transformation group, since transformations play a great role in a modern physical theory; both the relativistic and quantum theories supposedly suggest that the significance of transformations is more fundamental than the significance of equations. On selecting the branch of mathematics, there is good reason to elaborate it in the corresponding directions, simultaneously bearing in mind how it can lend itself to a natural physical interpretation*". It may be said without gross exaggeration that Dirac's method today has been adopted by the majority of theoretical physicists. By the way, the issues of the interrelation between physics and mathematics were of concern to him throughout his life, he would readily discuss this subject, and he digressed to discuss it in his works dedicated to absolutely different problems (see, for instance, Ref. [96]).

Of course, this list of the fundamental ideas of the 'later' Dirac can be continued. However, based even on the foregoing one can arrive at a definite conclusion: the creative heritage of this genius of 20th century physics harbors a wealth of potential heretofore unknown and yet untapped.

6. Dirac and the present-day physical picture of the world

In summary, we would like to emphasize that Dirac's contribution to the progress of civilization is not limited to the above-listed fundamental theoretical discoveries. As evidenced by the course of time, his work has led to qualitative changes in our notions of nature as a whole, which is commonly referred to as the physical picture of the world. From the modern viewpoint, the main components of the PPW are, on the one hand, the abstract images of material objects and, on the other hand, the conceptual apparatus invoked to describe the most important properties of these objects. Dirac's ideas have led to significant additions and radical changes of both PPW components.

We are reminded that the main models of objects in physics for the first 150 years after Newton were massive material points (corpuscles) or their associations (solids, ideal liquids), with central forces acting instantly between them and all this taking place in an absolutely empty space for an absolutely continuous flow of time. In this case, the conceptual apparatus reduced only to the characteristics of material objects. In general terms, such was the first PPW. M Faraday and Maxwell supplemented this picture with fields and electromagnetic waves seemingly alien to it, and Lorentz was the first to guess that both the field and substance are the forms of matter, although qualitatively different. As is well known, the construction of the classical PPW version was completed by Einstein, whose relativity theory removed evident contradictions between the mechanical and field notions of the surrounding world; however, in this case our notions of the geometry of the universe changed significantly.

Proceeding from relativistic and quantum principles, Dirac in his turn showed that, along with conventional matter, there is also bound to exist its antipode — 'antimatter'. It may be said without exaggeration that Dirac actually discovered a 'second' nature for us by doubling the number of material objects amenable to observation and study. And Weisskopf's observation is absolutely correct [35] that "*... these predictions rank with the greatest achievements of natural science*".

From these predictions of Dirac there also followed the possibility of interconversion, including the creation and destruction, of nuclei and elementary particles, including those which are not observed under ordinary terrestrial conditions. Studying these processes in space and in terrestrial conditions has opened up the way to the cognition of the early stages of the evolution of the universe.

Furthermore, Dirac laid the foundations of quantum field theory which has elicited the qualitative unity of matter at the microlevel. According to modern views, the notion of the quantized field, which he introduced, is the most basic and universal form of describing matter, which underlies all its observable (both wave and corpuscular) manifestations. Finally, the qualitatively new conception of the physical vacuum, which is being actively developed in the modern models of quantum theory and cosmology scenarios, emerged under the impact of Dirac's work.

No less significant is Dirac's contribution to the second PPW component — the conceptual apparatus of physics. We dwell only on the most significant contribution, on the introduction of two fundamentally new notions in locution — observables and states, which pertain to two qualitatively different aspects of the physical reality — the object as such, and its macroenvironment. The natural development of this idea is the modern notion that all physical objects exist not by themselves, but as if in a 'fur coat', experiencing an uncontrollable quantum action (on a Planck constant scale) from macrosurroundings which may also include the means of observation. In this connection, the independent characteristics of both the object itself and its state, determined by the uncontrollable action of the environment, turn out to be equally the subject of the physical theory.

Dirac's viewpoint of principle concerning the role of macrosurroundings in the formation of the state of a microsystem was reflected in his discussion with Heisenberg at the Fifth Solvay Congress (1927) in connection with Bohr's report "Quantum postulate". Dirac spoke positively in the sense that the reduction of a wave packet takes place because "*... The Nature chooses and decides in favor of a specific state*

ψ_n *with a probability* $|C_n|^2$*. This choice cannot be rejected, and it determines the subsequent evolution of the state*'' (see Ref. [20], Vol. II, p. 206). At the same time, Heisenberg insisted that ''... *it is our observations that give us the reduction to the eigenfunction*'', obviously overestimating at that moment the part played by the subjective factor.

There is another question: to what extent should environmental action be taken into account in the description of macro- and microobjects? For the dynamics (but by no means for thermodynamics!) of macroobjects, the existence of a 'fur coat' does not ordinarily play a significant part, so that for them there is good reason to restrict ourselves to only one class of characteristics — the observables. However, we have a completely different situation with microobjects. The concept of a quantum state acquires an independent role, with the result that the number of characteristics describing the physical reality in the microscopic world is actually doubled. Moreover, underestimating the role of one or another characteristic leads to paradoxes of the Einstein – Podolsky – Rozen type. Furthermore, attempts to give an interpretation of quantum phenomena on the basis of our usual, 'obvious' notions are nothing more nor less than a veiled hope for the existence in nature of the so-called 'hidden parameters'... That is why the results of the well-known experiments on the verification of Bell inequalities can be regarded as the confirmation of the correctness of Dirac's approach to the description of quantum realities and, first and foremost, of the idea of the integrity of quantum states.

To appreciate the extraordinariness of Dirac's innovation specified above, we revert to the formation period of quantum mechanics. Prevailing at that time was a tradition which can be traced back to Newton: to reduce the description of the natural objects to the study of their physical characteristics by themselves. In this case, it went without saying that these characteristics were undoubtedly observable. In other words, unobservable quantities introduced into physics on the basis of some speculative considerations had, according to this tradition, to be eliminated in the construction of any theory.

Many physicists believed that Einstein, too, was among the adherents of this tradition. In any case, he was presumed to proceed from such considerations when constructing the relativity theory. In particular, Heisenberg also adhered to this tradition and initially considered the observability principle as the basis for the quantum theory he was constructing. That is why, according to his own recollections [97], he was hoping for mutual understanding and support of his views when he informed Einstein of his initial premise during their conversation in 1926. However, a kind of discomfiture was in store for him, for Einstein spoke on this subject quite definitely: ''*Theory alone decides on what precisely can be observed*''. It should be said straight away that this statement significantly extends the scope of notions on observability and is at variance with the usual principles of classical science.

It is likely that Heisenberg's excessive concern with the observability problem in its simplified interpretation was actually a manifestation of the rudiments of classical thinking, which were not so easy to abandon. In the years when the 'new' quantum mechanics was under construction, in fact, there existed no other way of thinking apart from the classical one and Heisenberg was by no means alone in this respect. For instance, Fock, following Heisenberg, at that time spoke of quantum mechanics as of ''*a relativity theory with respect to means of observation*'', which could be adopted merely as a useful metaphor. Bohr also paid certain tribute to classical views in his initial statements concerning the principle of complementarity.

Dirac's standpoint was radically different: even in his first paper on quantum mechanics he managed 'to hold himself aloof' from too straightforward a classical view of nature and began formulating the quantum language of its description. Eventually, he showed that, along with the characteristics of objects by themselves known from classical physics and being as if on the face of phenomena, there exists the second independent set of characteristics — the characteristics of object states theretofore concealed from the attention of researchers, much like the opposite side of the Moon. In fact, this has led to the doubling of the number of characteristics employed in the conceptual apparatus of physics, this being true, as it has turned out, of not only quantum physics.

As emphasized by Faddeev [98], in the modern view ''... *the main notions participating in the formulation of a physical theory are observables and states...*'' He next showed in what sense the existing physical theories — classical and quantum mechanics, nonrelativistic and relativistic dynamics — can be considered as different realizations of the corresponding algebraic structures, the quantum-to-classical mechanics transfer and the relativistic-to-nonrelativistic dynamics transfer being regarded in this case as the deformations of these structures in the parameters $\hbar$ and $1/c^2$, respectively. Based on this general scheme, Faddeev observed that ''*From the standpoint of modern mathematics, the two principal revolutions in physics and natural science in general are deformations of unstable structures into the stable ones. From this viewpoint fashionable talks about the change of paradigms are losing their luster, to say the least*''. In this case, a similar scheme could have been revealed even in the 19th century; quantum mechanics and the relativity theory could have been arrived at simply by searching for other realizations of these general schemes. But ''... *the scheme itself appeared only after the discovery of quantum mechanics in the description of its general structure. Here, the part of fundamental importance was played by P Dirac. Only then was it recognized that classical mechanics is another realization of the same scheme*''.

This implies that the conceptual apparatus elaborated by Dirac makes it possible to adequately formulate not only the nonclassical PPW version, but also the classical one, which traces its origin to Newton. More recently, it was found that the conceptual apparatus elaborated by Dirac is applicable not only to mechanics. Today it has proven to be efficient in classical and statistical thermodynamics, including the theories of fluctuations [99, 100] and Brownian motion [101, 102, 109].

Therefore, there are strong grounds to believe that Dirac's works have led to qualitative changes in the Weltanschauung of the scientific community, completing the epoch of transfer from the classical view to the quantum view and, what is more, to the nonclassical view of nature initiated by Planck [103, 104, 110]. To put it another way, the radical change of the contents of both PPW components is Dirac's contribution of paramount importance to the cognitive activity of humanity as a whole. Before our very eyes the PPW is progressively acquiring the form of an adequate basic model of nature, which embodies in indissoluble unity the ideas of Newton, Einstein, and Dirac.

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7. Appendices

7.1 Projective geometry: elementary concepts

Projective geometry (see, for instance, Ref. [105]) originated from the teaching of perspective of the Renaissance; well-known painters indulged in it prior to others — Albrecht Dürer and Leonardo da Vinci — and da Vinci's canvas 'the Last Supper' is considered to be the canon for that stage of development of the future branch of mathematics. As a mathematical discipline in its own right, this science took shape (by concurrence of circumstances) in Russia, in the town of Saratov, which was the residence of Jean-Victor Poncelet (a captive lieutenant of Napoleon's army) from March of 1813 through June of 1814. He took advantage of the 'spare time' to make notes of his future *Traité des Propriétés Projectives des Figures* (Treatise on the Projective Properties of Figures) later published in Paris in 1822. That year is considered to be the birthday of this mathematical discipline, although several of its assertions (theorems) were formulated and proved even in the 17th century by G Désargues and B Pascal.

If, in lieu of Cartesian coordinates (x, y) of some point in a plane, one introduces *homogeneous* coordinates $(x_1 : x_2 : x_3)$ related to the Cartesian ones as $x = x_1/x_3$; $y = x_2/x_3$, it is easily seen that the homogeneous coordinates of an arbitrary point in a plane cannot simultaneously all vanish and are defined correct to a constant factor, for the triplets (x_1, x_2, x_3) and $(\lambda x_1, \lambda x_2, \lambda x_3)$ define the Cartesian coordinates of the common point (hence there appears the designation adopted for them). The name of the coordinates is related to the fact that the equation of any straight line is written in these coordinates in a homogeneous form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \,. \tag{7.1}$$

Second-order curves are also represented in a similar homogeneous form

$$a_{ij} x_i x_j = 0; \quad i, j = 1, 2, 3 \,. \tag{7.2}$$

From Eqn (7.1) it follows that the equalities $x_1 = 0, x_2 = 0$ define, respectively, the Y- and X-axes in the plane, while the equality $x_3 = 0$ is the equation for an *ideal* (infinitely far) straight line, which is the locus of *ideal* points of the plane. In the ideal line there intersect any two parallel straight lines, for instance, the straight lines $x_2 = kx_1 + b_1$; $x_2 = kx_1 + b_2$ intersect at a point $(1 : k : 0)$, and so forth. A straight line supplemented with an ideal point is termed a *projective straight line* and is designated as RP^1, while a plane complemented with an ideal straight line is termed a *projective plane* RP^2. These are the simplest objects of projective geometry that allow a natural generalization to higher dimensionalities.

Projective geometry contains a wealth of amazing facts, which are quite unusual to a person with conventional (Euclidean-geometrical) thinking. In particular, from the equation for the best-known second-order curve [like Eqn (7.2)] — a circumference

$$x_1^2 + x_2^2 + a_0 x_3^2 + 2a_1 x_1 x_3 + 2a_2 x_2 x_3 = 0 \tag{7.3}$$

it follows that any circumference passes through two ideal imaginary points $(1 : i : 0)$ and $(1 : -i : 0)$, which are referred to as the *cyclic* points of the plane. The straight line which is defined by formula (7.1) and passes through any of the cyclic points is remarkable in that the length of any of its segments is equal to zero, while such straight lines themselves are termed *isotropic*. In this case, exactly two such isotropic straight lines pass through any point of the plane. This is but one step to spinors, which were discovered by the French geometrician E J Cartan in 1913, and introduced into physics by Dirac (see, for instance, Ref. [106]).

The second remarkable statement of projective geometry is the *principle of duality* (Poncelet): to any proposition with participation of the terms 'point' and 'straight line' there corresponds a dual proposition, which results from the first one by a simple permutation of these terms (in the case of projective space, 'plane' is added to these terms). For instance, the equation of a straight line (7.1), which is symmetric in form about a and x, for fixed x and variable a defines a set of straight lines passing through a point x, i.e., is the equation of a point.

In the general case, projective geometry studies the properties of figures which remain invariable under *projective transformations* of the form

$$x' = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3} \,, \qquad y' = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3} \,, \tag{7.4}$$

which define a one-to-one projective plane mapping onto itself. For spaces of higher dimensionality, projective transformations are obtained by a simple generalization of formulas (7.4) and in every case make up a *projective group*, which comprises as special cases the similarity group and the affine transformation group. On this basis, the mathematician Arthur Cayley in 1859 even enunciated a principle: *projective geometry is the entire geometry*!, which later proved to be only conventionally true. [19]

7.2 Dirac's 'quantum dictionary'

Dirac's ingenious way of thinking manifested itself even in his mode of inventing the terminology of quantum theory. He possessed an amazingly capacious spatial thinking, which enabled him to easily operate not only onto real bodies, but on abstract physical notions as well. That is why operating in the spirit of projective geometry it is as if he aspired to project many-dimensional physical abstractions onto the 'plane of thinking' of ordinary researchers. Having found that the conventional notions of vectors in finite-dimensional spaces are insufficient for describing the states of quantum-mechanical systems, he came up with the idea of generalizing these notions and going over to vectors in infinite-dimensional spaces (two years later, the mathematician J von Neumann

[19] On this occasion, see V G Boltyanskiĭ's notes to F Klein's lectures [107].

'recognized' these vectors as the elements of Hilbert spaces and gave a rigorous exposition of Dirac's apparatus in his monograph [29]). In his *The Principles of Quantum Mechanics* Dirac introduced this innovation as a quite natural one and continued: "*It is desirable to have a special name for describing the vectors which are connected with the states of a system in quantum mechanics, whether they are in a space of a finite or an infinite number of dimensions. We shall call them ket vectors, or simply kets, and denote a general one of them by a special symbol* $|\rangle$. *If we want to specify a particular one of them by a label, A say, we insert it in the middle, thus* $|A\rangle$. *The suitability of this notation will become clear as the scheme is developed*" (see Ref. [20], Vol. I, p. 29).

The term 'ket' is the second part of the word 'bracket'. For the vector conjugate to $|A\rangle$, Dirac introduced a 'bra' vector, which is the first part of the same word, and designated it by $\langle B|$. "*A scalar product* $\langle B|A\rangle$ *now appears as a complete bracket expression, and a bra vector* $\langle B|$ *or a ket vector* $|A\rangle$ *as an incomplete bracket expression. We have the rules that any complete bracket expression denotes a number and any incomplete bracket expression denotes a vector, of the bra or ket kind according to whether it contains the first or second part of the brackets* ..." (this is how simply and naturally introduced is the characteristic termed by modern physicists, after Feynman, as the $|A\rangle$-to-$|B\rangle$ transition probability amplitude).

In his *The Principles...*, Dirac made extensive use of the so-called δ-function, which he had introduced in Ref. [34] back in 1927, and which he needed "*to get a precise notation for dealing with ... infinities*". He considered the δ-function as "*a function of the real variable x which vanishes everywhere except inside a small domain, of length* ϵ *say, surrounding the origin* $x = 0$, *and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations...*".

Even someone who was not a mathematician understood that it was a 'trick' rather than a rigorous definition. But this did not confuse Dirac, who treated the δ-function without any respect — differentiated, integrated, multiplied by other functions, etc. The mathematicians of that time perceived Dirac's actions simply as a play on formulas. Those who harnessed the δ-function in their calculations preferred to conceal it in their publications and provided 'conventional' proofs of the theorems obtained with its aid. But then the mathematicians S L Sobolev and L Schwartz in their works elaborated the theory of generalized functions, the Dirac δ-function being their special case. All the results obtained by Dirac without substantiation thereby acquired 'legitimate status'.

"*I encountered the notation problem in connection with a Poisson bracket*", Dirac remembered. "*I borrowed all the information about it from Whittaker's book Analytical Dynamics, where parentheses were used for Poisson brackets, and square brackets were used for Lagrange brackets. The quantum theory does not employ the Lagrange brackets, it makes use of only the Poisson bracket. That it why Whittaker's designations seemed inconvenient to me. They suggest the idea of a scalar product known from the vector analysis. However, the scalar product is symmetric about permutation of the two terms involved, while the Poisson bracket is antisymmetric about their permutation. That is why I boldly took advantage of the other designation of the bracket... Since then, everybody does so. It turned out that the quantity antisymmetric about permutation of the two terms involved is quite convenient to designate by square brackets*" [9] [20].

When the equality $uv = vu$ is fulfilled, mathematicians-algebraists say that u is 'permutable' with v. The word 'permutability' seemed somewhat inappropriate to Dirac, since physicists, on the subject of permutations, commonly imply that rearranged are several quantities rather than two, as in our case. That is why Dirac introduced the word 'commute' (from the Latin *commutare* — 'change'). "*I do not think that mathematicians had used it before me*", he wrote. — "*I declared: when* $uv = vu$, *u and v commute with each other. Since then, this term has also come into use*".

Another typical example of Dirac's word creation is the introduction of c- and q-numbers. "*The situation was that I had to deal with new, quantum variables, which appeared quite mysterious to me, and therefore I invented a new word for them. I called them q-numbers to distinguish them from ordinary variables, which figured in mathematics and which I termed c-numbers... I next undertook to construct the theory of q-numbers; c-numbers can be treated simply as the special case of q-numbers which have the property that they commute with any quantities... I had no idea of the origin of q-numbers and believed that the Heisenberg matrices provided an example of q-numbers, but it might well turn out that q-numbers had a more general significance... I continued to elaborate the theory, and in doing this I was free to make any assumptions I needed, provided that they did not give rise to immediate contradictions. I was not going to find out the mathematical nature of q-numbers, nor did I intend to elucidate the accuracy of calculations with them*" [9].

Following Dirac's example, physicists would resort to such terms as 'fermions' (for particles with a half-integer spin) and 'bosons' (for particles with an integer-valued spin). He proposed their use in his lectures on elementary particles and their interactions, which were given in Princeton actually a year before the discovery of charged π mesons by S Powell, G Occhialini et al. in 1947. All massive particles known at that time possessed only half-integer spins, but Dirac had no doubt of the verity of Yukawa's hypothesis and believed that the discovery of mesons was only a matter of time.

Thus there gradually formed the vocabulary of terms that came to be 'spoken' by the new science – quantum physics. As justly observed by B V Medvedev in the introductory article to the collected works of Dirac [108]: "*Not only did Dirac turn quantum mechanics from a set of recipes for the solution of particular problems to a consistent and logically closed theory, but he also devised the language — of notions, terms, and symbols — in which we express ourselves in any division of the quantum theory. It can be said without gross exaggeration that in the event we are — like in a children's game — suddenly forbidden to use this language we would find ourselves in the situation of the builders of the tower of Babel*".

[20] We note that the Poisson brackets (the Poisson structures) play about the same part in classical Hamiltonian mechanics as the vector product in the vector algebra of Euclidean space, with the difference that the brackets should be nondegenerate. A more general notion of the Poisson structure that needs not necessarily obey the nondegeneracy requirement originated in the works of the Norwegian mathematician S Lie on the theory of continuous groups, which was elaborated for the integration of the systems of first-order partial linear differential equations. The interest in these works of Lie was rekindled due to Dirac and his work on the generalization of Hamiltonian mechanics (see Section 5.1)

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PETER GALISON

The Suppressed Drawing: Paul Dirac's Hidden Geometry

Purest Soul

For most of the twentieth century, Paul Dirac stood as the theorist's theorist. Though less known to the general public than Albert Einstein, Niels Bohr, or Werner Heisenberg, for physicists Dirac was revered as the "theorist with the purest soul," as Bohr described him. Perhaps Bohr called him that because of Dirac's taciturn and solitary demeanor, perhaps because he maintained practically no interests outside physics and never feigned engagement with art, literature, music, or politics. Known for the fundamental equation that now bears his name—describing the relativistic electron—Dirac put quantum mechanics into a clear conceptual structure, explored the possibility of magnetic monopoles, generalized the mathematical concept of function, launched the field of quantum electrodynamics, and predicted the existence of antimatter.

In this paper I will explore the meaning of drawing for Dirac in his work. In the thirteen hundred or so pages of his published work between 1924 and World War II, aside from a few graphs and a diagram in a paper that he coauthored with an experimentalist, Dirac had practically no use at all for diagrams. He never used them publicly for calculation, and I know of only two, almost trivial, cases in which he even exploited a figure for pedagogical purposes. His elegant book on general relativity contained not a single figure; his famous textbook on quantum mechanics never departed from words and equations.[1] If anything, diagrams appear to be antithetical to what Dirac wanted to be "visible" in his thinking. Dirac was known for the austerity of his prose, his rigorous and fundamentally algebraic solution to every physical problem he approached. (Even his fellow physicists found his ascetic style sometimes to be too terse—in response to questions, he would repeat himself *verbatim*; other physicists sometimes complained that his papers lacked words.) Now it is not the case that diagrams are simply absent from physics. To cite one famous example, there is the famous diagrammatic-visual reasoning of theorists like James Clerk Maxwell who insisted that full understanding would only come when joined to imagined, visualizable machines running with gears, straps, pulleys, and handles. Maxwell wanted objects described and drawn that could, in the mind's eye,

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be grasped with the hands and pulled with the muscles. Similarly visual were Einstein's thought experiments, his use of hurtling trains, spinning disks, and accelerating elevators. Dirac's papers contain none of this. Not even schematic diagrams appear in his writings, visualizations of the sort that Richard Feynman introduced to facilitate calculation and impart intuition about colliding, scattering, splitting, and recombining particles.[2]

It would seem, then, that the corpus of Dirac's work would be the last place to look for pictures. But in the Dirac archives something remarkable emerges. I was astonished, for example, to find these comments penned by Dirac as he prepared a lecture in 1972: "There are basically two kinds of math[ematical] thinking, algebraic and geometric." This sounds like the theoretical twin of a contrast I have long pursued between laboratory methods that yielded images (analogous here to Dirac's geometric thinking) and those methods predicated on the logical or statistical compilations of data points (analogous to Dirac's algebraic thinking).[3] So I was intrigued. Given Dirac's austere public predilection for sparse prose, crystalline equations, and the complete absence of diagrams of any sort, I assumed that in the next sentences he would go on to class himself among the algebraists. On the contrary, he wrote in longhand,

> A good mathematician needs to be a master of both.
> But still he will have a preference for one rather or the other.
> I prefer the geometric method. Not mentioned in published work because it is not easy to print diagrams.
> With the algebraic method one deals with equ[ations] between algebraic quantities.
> Even tho I see the consistency and logical connections of the eq[uations], they do not mean very much to me.
> I prefer the relationships which I can visualize in geometric terms.
> Of course with complicated equations one may not be able to visualize the relationships e.g. it may need too many dimensions.
> But with the simpler relationships one can often get help in understanding them by geometric pictures.[4]

These pictures were not for pedagogical purposes: Dirac kept them hidden. They were not for popularization—even when speaking to the wider public, Dirac never used the diagrams to explain anything. Astonishing: across the great divide of visualization and formalism that has, for generations, split both physics and mathematics, we read here that Dirac published on one side and worked on the other.

The poverty of print technologies in and of itself seems rather insufficient as an explanation for the privacy of Dirac's diagrams, but in another (undated) account his characterization may be more apt: "The most exciting thing I learned [in mathematics in secondary school at Bristol] was projective geometry. This had a strange beauty and power which fascinated me." Projective geometry provided this Bristolean student new insight into Euclidean space and into special relativity. Dirac added, "I frequently used ideas of projective geometry in my research work in later life, but did not refer to them in my published work because I was doubtful whether the average physicist would know enough about them to appreciate them."[5] Lecturing in Varenna, also in the early 1970s, he recalled the "profound influence" that the power and beauty of projective geometry had on him. It gave results "apparently by magic; theorems in Euclidean geometry which you have been worrying about for a long time drop out by the simplest possible means" under its sway. Relativistic transformations of mathematical quantities suddenly became easy using this geometrical reformulation. "My research work was based in pictures—I needed to visualise things—and projective geometry was often most useful—e.g. in figuring out how a particular quantity transforms under Lorentz transf[ormation]. When I came to publish the results I suppressed the projective geometry as the results could be expressed more concisely in analytic form."[6]

So Dirac had one way of producing his physics in his private sphere (using geometry) and another of presenting the results to the wider community of physicists (using algebra). Nor is this a purely retrospective account. For there remains among his papers a thick folder of geometrical constructions documenting Dirac's extensive exploration of the way objects transform relativistically. These drawings are not dated but on their reverse sides are writings dated from 1922 forward. None of these drawings were ever published or, as far as I can tell, even shown to anyone (figs. 1 and 2).

The question arises: how ought we to think about Dirac's "suppressed" geometrical work? Dirac himself saw projective geometry as key to his entrance into a new field: "One wants very much to visualize the things which we are dealing with."[7] Should one therefore split scientific reasoning, as Hans Reichenbach did, between a "logic of discovery" and a "logic of justification"? For Reichenbach there were some patterns of reasoning that were, in and of themselves, sufficient for public demonstration. Other procedures, more capricious and idiosyncratic, could not count as demonstrations though they might serve the acquisition of new ideas.[8] This distinction saturates the philosophy of science of the postwar era. In Karl Popper's hands it helped to ground his demarcation criterion between science and non-science: only scientific theories, in the context of justification, were falsifiable, only in the realm of the justifiable was there anything dignified of the word *logic*. "My view," Popper wrote, "may be expressed by saying that every discovery contains 'an irrational element', or 'a creative intuition', in Bergson's sense."[9] By contrast, Gerald Holton took the private-scientific domain to have a sharply articulable structure that can be characterized by commitments to particular thematic pairs (such as continuum/discretum or waves/particles). According to Holton, this rich, three-dimensional space of private thought is then "projected" onto the plane of public science (defined by the restricted axes of the empirical and the logical). In this empirical-analytic public plane, much of the private dynamic of science is necessarily lost.[10] Recent work in science studies has either denied the force of the Reichenbachian distinction, or maintained the public/private distinction in other terms. For example, Bruno Latour, in his early work with Steve Woolgar, characterized

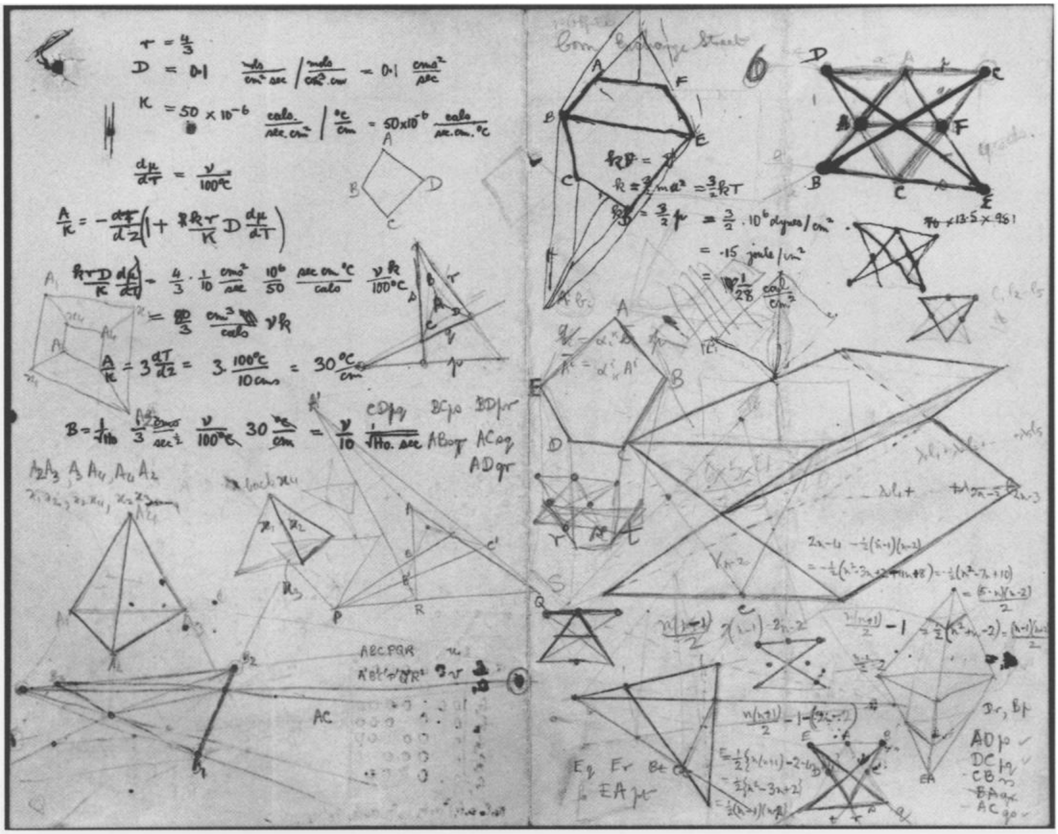

FIGURE 1. Paul Dirac, Geometrical Sketches, in the Paul A. M. Dirac Papers, Florida State University, Tallahassee, Florida; hereafter PDP. By permission of the Florida State University Libraries.

private science by a different grammar: the private is filled with modifiers, modal qualifications that slowly are filtered out until only a public, assertoric language remains.[11]

Certainly the common view of drawing as *preparation* would fit this sharp separation of public and private. Private sketches, in virtue of their schematic and exploratory form, would count as the precursors to the completed painting; private scientific visualization and sketches would, without requiring rigor, precede the public, published scientific paper. In such a picture the interior is psychological, aleatory, hermetic, and unrigorous while the exterior is fixed, formally constrained, communicable, and defensible. One thinks here of Sigmund Freud for whom the visual was primary, preceding and conditioning the development of language. To the extent that primitive reasoning is supplanted by language, the pictorial, uncon-

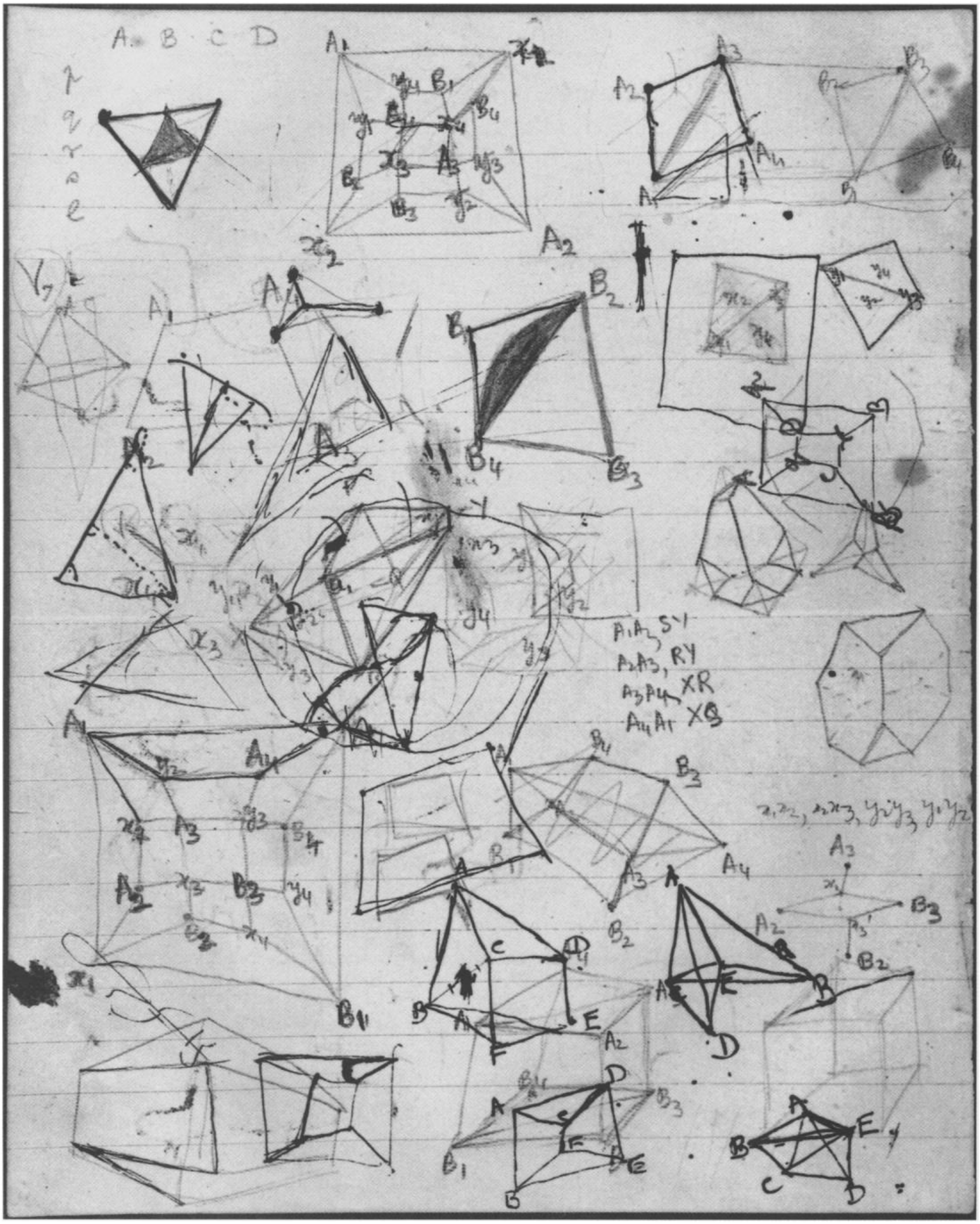

FIGURE 2. Dirac, Geometrical Sketches, in PDP. By permission of the Florida State University Libraries.

scious form of reason is of a different species from that of conscious, logical, language-based thought.

For some analysts of science, the advantages of the radical public/private distinction is that it brought the private into a psychological domain that opened it up to studies of creativity. For others, the separation permitted a more formal analysis of the context of justification through schemes of confirmation, falsification, or verification. For those who saw published science as merely the last step of private science, the distinction helped shift the balance of interest toward "science-in-the-making" and away from the published end product.

I want here to pose the question differently and, specifically, to challenge the search for intrinsic markers of scientific drawing that would make it in some instances "private" and in others "public." As we learn from Jacques de Caso's essay on Théophile Bra, Bra's drawings surely cannot be understood as the expression of a purely interior or subjective sensibility. For example, at least one of Bra's cosmological sketches was clearly tied to his views of public discussion about changes in the structure of Saturn's rings; Bra even wrote to the French astronomer and optician, Dominique-Francois-Jean Arago, about the problem.[12] Nor does the geometry of Dirac issue from an isolated form of reasoning. Dirac's fascination with projective geometry is anything but a private language in Ludwig Wittgenstein's sense—as we will see momentarily (fig. 3).

In both instances (Bra's cosmologies, Dirac's geometry) the drawings neither issue entirely from the public domain nor are they sourceless fountains from a reservoir of pure subjectivity. Tracking Bra's worldly iconological sources or Dirac's public sources in geometry would surely prove both possible and profitable. And *yet* there is something important in the circumstance that both Dirac and Bra constructed a domain of interiority around these practices. It is not that Dirac's geometric drawing or Bra's cosmogenic images were *intrinsically* interior or psychological—there is no separate logic here that could provide a universal demarcation criterion splitting the public from the private. Rather, both Dirac and Bra drew a line (so to speak) around their drawings. Both assiduously hid their pictures from the public gaze, and refused (in the case of Dirac) even to admit them into his published arguments. One suggestive concept helpful in capturing this delineation of the private might be Gilles Deleuze's notion of the fold. For Deleuze the "content" of what is infolded is not intrinsically separate from the exterior; there is no metaphysical otherness dividing inside from outside. Instead, interiority is itself the product of an outside pulled in, a process that Michel Foucault called subjectivation because it makes contingent, not inevitable, the formation of what is understood as self.[13]

I want to push this notion of infolding or subjectivation in two directions. First, my concern here is with an aspect of the private that bears on the epistemic, rather than one that posits lines of individuation that separate a self from others and the world. That is, what interests me is the historical production of a kind of reason

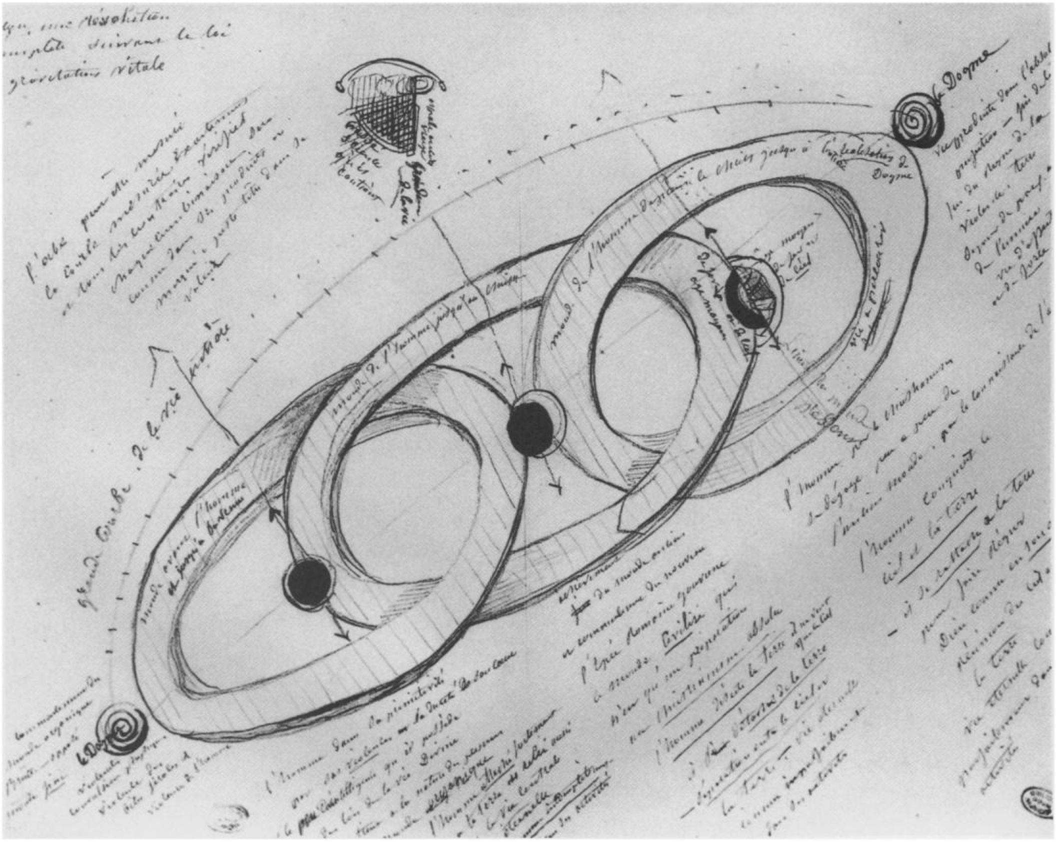

FIGURE 3. Théophile Bra, Untitled Drawing, in Jacques de Caso, *The Drawing Speaks: Théophile Bra, Works, 1826–1855* (Houston, 1997), plate 23.

that comes to count as private (rather than, for example, the production of the psychological sense of self more generally).[14] Second, building on this epistemic form of subjectivation, my concern is to explore the historical *process* by which this takes place. On such a view, the question shifts: How does a form of public inquiry and argument (geometry) come to count as private, cordoned-off reason?

Public Geometry, Private Geometry

The issue, therefore, is not what makes the interior or the private metaphysically distinct from the exterior and public, but rather how this inbound folding occurs over time. How, in our instance, did projective geometry pass from the status of a state religion at the time of the French Revolution to become, for Dirac, a repressed form of knowledge production that must remain consummately private—that is, how was geometry infolded to become, for Dirac, quintessentially an interior form of reasoning? What are the conditions of visibility that govern its place (or suppression) in demonstration?

So a new set of questions displaces those with which I began. Not the philosophical-psychological question: How do interior rules of combination differ from exterior rules of combination? But rather: What are the specific conditions that govern the separation of certain practices from the public domain? Not: How, linguistically or psychologically, does public science get created by successive transformations of the private domain? But rather the inverse: How do the "private" structures of visibility (specifically in drawing) get pulled in from the public arena to form a domain aimed, in the first instance, at the inward regulation of thought (rather than outward communication)? Consequently what we have is not quite the Deleuzian question either—not the transhistorical elucidation of what he calls the topology of the fold, but rather the historical process of the folding itself. What happens, over time and across places, such that features of public demonstration *become* private forms of reasoning?

During the late eighteenth century, descriptive geometry (later known as projective geometry) was first heralded by Gaspard Monge, as preeminent mathematician, as political revolutionary, and as director of the Parisian *Ecole polytechnique*. As Lorraine Daston and Ken Alder have shown, Monge's texts and the *Polytechnique* curriculum more generally were all oriented toward the school's mission to train engineers.[15] Descriptive geometry, the science of a mathematical characterization of three-dimensional objects in two-dimensional projections, was supposed to serve not only mathematicians and engineers but also the *Polytechniciens* who would become the nation's future high-level carpenters, stonecutters, architects, and military engineers.[16] For a generation of Monge's successors—Polytechnicien engineers including Charles Dupin, Michel Chasles, and Jean-Victor Poncelet—descriptive geometry became much more than a useful tool. Geometry, they contended, would hold together reason and the world.

For Monge and his school, physical processes including projection, section, duality, and deformation became means of discovery, proof, and generalization. This physicalized geometry defined a new role for the engineer as an intermediary lodged between the state and the artisan. Geometrical, technical drawing, "the geometry of the workshop" became at one and the same time a way of organizing the component parts of complex machines and a scheme for structuring a social and workplace order.[17] Geometry became a way of being as well as *the* proper way of founding a basis for mathematics. Indeed, at the *Ecole polytechnique*, geometry became an empirical science. Auguste Comte came to speak of an empirical mathematics, Lazare Carnot exploited physically motivated transformations in geometry and identified correlates between mathematical entities and their geometrical twins.

Geometry was practical and more than practical. Certainly for Dupin, Chasles, Poncelet, and their students, geometry towered above all other forms of knowledge as the paragon of well-grounded argumentation, *better* grounded, in particular, than algebra. Projective geometry came to stand at that particular place where engineering and reason crossed paths, and so provided a perfect site for pedagogy. As Monge insisted, projective geometry could play a central role in the "improvement" of the French working class—"Every Frenchman of sufficient intelligence" should learn it, and, more specifically, geometry would be of great value to "all workmen whose aim is to give bodies certain forms."[18] Enthusiastically Henri Saint-Simon and his followers adopted the cause in their utopian planning. Descriptive geometers established classes across Paris, joined the geometrical cause to republicanism, and launched a wider commitment to worker education. In 1825, Dupin proclaimed in his textbook that geometry "is to develop, in industrials of all classes, and even in simple workers, the most precious faculties of intelligence, comparison, memory, reflection, judgment, and imagination. . . . It is to render their conduct more moral while impressing upon their minds the habits of reason and order that are the surest foundations of public peace and general happiness."[19] Both before and after the French Revolution, geometry, as Alder notes, became the foundational skill in the training of workers—several thousand passed through the various popular art training programs. Geometry would teach both transferable skills crossing the trades and at the same time stabilize society by locking workers into the social roles previously occupied by fathers.[20]

Geometry did not, however, survive with the elevated status it had held in France at the highwater mark of the *Polytechniciens*' dominance. Analysts displaced the geometers. Among their successors was Pierre Laplace, for whom pictures were anathema and algebra was dogma. It was not in France, therefore, but rather in Britain and Germany that educators, scientists, and even politicians took up the cause of descriptive geometry with the conjoint promise of epistemic and pedagogical improvement. So although the French mathematical establishment had turned decisively to analysis in the last third of the nineteenth century, the British did not. Euclid had long reigned over British education as an exemplar of good sense and a pillar of mental training. By 1870, however, there was a widespread and disquieting sense that the British were losing to the Continent in the race for science-based industry. Geometry was no exception. In January 1871, leading mathematicians of the British Association for the Advancement of Science joined a committee known as The Association for the Improvement of Geometrical Teaching. Their goal was to produce a reform geometry better suited to technical and scientific education, in a form less rigid than that demanded by the purer mathematicians and enforced on schools. New methods of geometrical argument were introduced, and teachers began to step away from the definitions, forms of argument, and order of theorems dictated by the historical Euclidean texts. Such a loosening of Euclid's hold over the schoolchild's mind did not go undisputed. By 1901 the reformers (aiming to join geometry to the practical arts) and conservatives (hoping to preserve its purity) had settled into such powerfully opposed camps that separation seemed inevitable.[21]

These, then, were some of the nineteenth century's territories of geometry: Up until the 1860s or so, the French celebrated projective geometry as joining high reason with practical engagement of the working class; then this physicalized geometry faded from the scene. In Britain, accompanying the rapid expansion of industrial, technical education, Victorian descriptive geometry became the symbol and means of socio-educational uplift, improving the lot of young workers, including those of the working class. For the mathematician-logician Augustus De Morgan, for example, geometry was a route to knowledge in general—as he argued in 1868: "Geometry is intended, in education, . . . to [unmask] the tricks which reason plays on all but the cautious, *plus* the dangers arising out of caution itself."[22]

Over the last decades of the nineteenth century, the teaching of geometry in Britain gradually moved away from a rigid Euclid-based textual tradition toward a more expansive interpretation of geometry's basis. In part this shift issued from the marketplace. No longer would it be adequate for the teaching of geometry to exemplify sound reasoning as an end utterly unto itself. Instead, geometry came to have a practical significance as well—crucial for the upbringing of engineers, the upper tier of tradesmen, and scientists. One widely distributed encyclopedia of technical education put it bluntly: "It is impossible to overstate the importance of a knowledge of Geometry, forming as it does the basis of all mechanical and decorative arts, constituting, in fact, the grand highway from which the various branches of drawing diverge."[23] At the same time, part of the freeing of geometry from its purely descriptive roots was an increasing emphasis by reformers on "modern" methods including, prominently, non-Euclidean and projective geometry of higher dimensions. Pressured by both practical and research exigencies, geometry came to illustrate sound reasoning not by being purely descriptive of an ideal world, but rather by instantiating a reason best captured by a multiplicity of approaches.[24]

So much for the general historical condition of geometry as a very public epistemic ideal and educational method: as a defining feature first of republican and then working-class French pedagogy it continued into the 1870s and beyond in Germany, and re-emerged within the technical education movement of Victorian England. What, then, are the specific historical conditions under which drawing came to count for Dirac both as a reliable home of reason and as a "private" science, judged by him variously as too hard to print, too arcane for physicists to understand, insufficiently persuasive, or insufficiently concise to merit publication?

Dirac's trajectory in mathematical physics took him across several of geometry's territories, temporal-spatial regions where geometrical drawing was laid out differently from one to the next. The goal in following that arc is to see how it came to pass that what had been the most public of mathematical regimes could become, for Dirac as he moved across this shifting map of geometry's fortune, a most private refuge of thought. Here is an account that begins not with an assumed intrinsic dynamics of interior (psychological) style, but rather with the historical creation of a kind of science judged private: *the epistemic subjectivation of the geometrical.* This is, therefore, not so much an attempt to follow Dirac's biography, but rather to observe Dirac as a kind of movable marker in order to track the conditions under which reasoning through drawing came to be classed as something to be, in his word, "suppressed," interiorized, made to constitute the private scientific subject.

Zero in on Dirac as we turn from the generic Victorian British trade school to Dirac's secondary school, the Merchant Venturers' Technical College, in Bristol. This was where Dirac's father, Charles Dirac, taught, and where Dirac himself received his primary and secondary scientific-engineering education. Created out of various mergers of the Free Grammar and Writing School, the Merchant Venturers' Navigation School, and various forms of the Bristol Diocesan Trade and Mining School, Dirac's school had stabilized both its structure and name in 1894.[25] Charles Dirac took his degree at the University of Geneva and then, in 1896, came to Merchant Venturers' where he pursued a long career teaching French. A feared figure on the faculty ("a scourge and a terror" according to some of the students), Charles Dirac clearly reveled in the disciplined teaching of language—especially French, but others too, including Esperanto.[26] Dirac the younger often claimed that he simply stopped speaking to avoid having to perform at home in perfect, grammatically correct French. Dirac's wife put it this way: "His domineering father made it a rule to be spoken to only in French. Often he had to stay silent, because he was unable to express his needs in French. Having been forced to remain silent may have been the traumatic experience that made him a very silent man for life."[27]

Merchant Venturers', from its outset aimed, as such schools did across Britain, to provide a passage for students into specific trades including bricklaying, plasterwork, plumbing, metalwork, and shoemaking. Navigation had been central to its mission for decades, and continued to be of importance as did mathematics, chemistry, and physics.[28] In every way distant from British public education, this school was not, in mission, in curriculum, or in student body, designed to prepare the upper class for their stations in empire through a study of the classics. In the school archives of 1912, for example, there survives correspondence between Merchant Venturers' and the nascent University College, about the advisability of teaching firemen and preparing students for their Mine Manager's Certificates. "The more we do for the working classes," the then headmaster wrote, "the better for the university."[29] Like so many technical colleges around England, Merchant Venturers' held geometry front and center as a site for training in an appropriate, practical reason.

Paul Dirac entered Merchant Venturers' in 1914, at the age of 12, passing from it immediately into his study of electrical engineering at Bristol University, where the university's program was, in fact, run by Merchant Venturers' as an extension of their primary and secondary programs. Young Paul took up electrical engineering under the supervision of David Robertson; Dirac's notebooks show a diligent student, adept in the technical drawing that had accompanied geometry from France to Germany to England. Month after month, Dirac trained himself to confront the constant stream of practical problems: electrical motors, currents, shunts, circuits, generators. Graduating in June 1921, he had as his principal subjects electrical machinery, mathematics, strength of materials, and heat engines (fig. 4).[30]

While he was in the midst of this engineering program, Dirac watched Arthur Eddington's 1919 eclipse expedition, "hit the world with tremendous impact," and Dirac, along with his fellow engineering students, desperately immersed themselves in the new theory of relativity. They picked up what physics they could from Eddington; Dirac even took a relativity course with the philosopher Charlie D. Broad. The relativity Dirac seized upon was not that presented in Einstein's 1905 paper—it was not a relativity of neo-Machian arguments and *Gedankenexperimenten* about trains and clocks. No, what enthralled Dirac was Hermann Minkowski's space-time, relativity cast into the diagrams in which startling relativistic results issued from reasoning through well-defined, if not-quite Euclidean, geometry. The appeal of this geometrized relativity was no doubt doubled in virtue of the fact that Dirac himself had struggled, in vain, to formulate a consistent, physically meaningful four-dimensional space-time.[31]

While a student, Dirac did some practical engineering work with the British Thompson Houston Works in Rugby and on graduation applied there for a job for which he was rejected. But Robertson was impressed by young Dirac and, with his engineering colleagues at Merchant Venturers', tried to lure him further into their field. They were bested by the mathematicians, who offered to include Dirac, gratis, in their courses for two years.[32] Entranced by his Bristol mathematics instructor, Peter Fraser, Dirac seized on projective geometry as his favorite subject and immediately began applying it to relativity. More specifically, Dirac turned his attention to the geometrical version of relativity that Minkowski had developed and made so popular; with projective geometry Dirac could simplify the new space-time geometry even further.[33]

In 1923 Dirac moved out of Bristol and up to Cambridge, where as a physics research student at St. John's, he entered the research group of Ralph H. Fowler. Fowler immediately introduced Dirac to Bohr's theory of the atom. But it took no time at all for Dirac to gravitate, on the side, back to the geometry he had come to love at Bristol. At 4:15, once a week, aspiring geometers would join the afternoon geometry tea parties held by the acknowledged Cambridge master of the subject, Henry Frederick Baker. Baker himself had just authored the first volume of his multitome text on projective geometry where he announced that whatever algebra was included, the geometry was sufficient unto itself. It was a form of mathematics that, Baker judged, would naturally appeal to engineers and physicists.[34] Certainly this proved to be the case with Dirac; as Olivier Darrigol, Jagdish Mehra, and

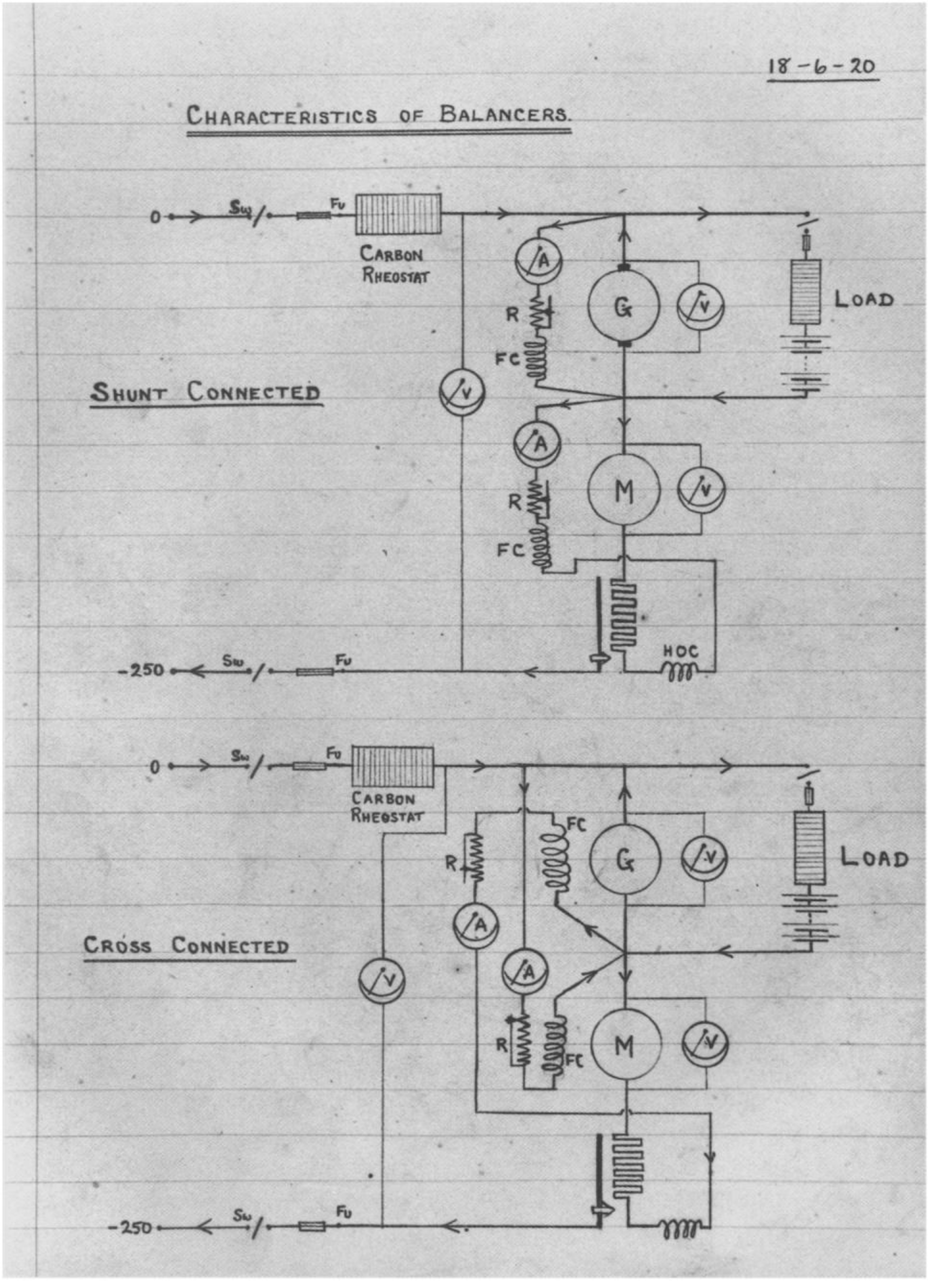

FIGURE 4. Paul Dirac, Electrical Engineering Diagrams. From Dirac's student notebooks, 18 June 1920, in PDP. By permission of the Florida State University Libraries.

Helmut Rechenberg have shown, even Dirac's notation seems to follow in some detail the choices made by Baker in his 1922 text.[35]

Sometime in 1924—the date cannot be deduced exactly from the handwritten fragment—Dirac delivered a talk to Baker's tea party. This was a tough audience to please. All of Baker's students and associates understood that silences would promptly be filled by grilling, and no quarter would be given in discussion.[36] Dirac immediately turned to the intersection of relativity with geometry and expressed his heartfelt sense that pure mathematics had nothing over the applied. On the contrary, so Dirac contended, there was a deep mathematical beauty in the specificity of the "actual world" that was obscure to the pure mathematician.[37] "I think," Dirac penciled onto his handwritten notes,

> the general opinion among pure mathematicians is that applied mathematics consists of finding solutions of certain differential equations which are the mathematical expression of the laws of nature. To the pure mathematician these equations appear arbitrary. He can write down many other equations which are equally interesting to him, but which do not happen to be laws of nature. The modern physicist does not regard the equations he has to deal with as being arbitrarily chosen by nature. There is a reason, {which he has to find} why the equations are what they are, of such a nature that, when it is found, the study of these equations will be more interesting than that of any of the others.

Old Newtonian gravity had a force that varied as the distance squared—but from the pure mathematician's view, there was nothing special about the square—it could have been cube or the fourth power. But the new theory of gravity, built out of Riemannian geometry, was (from the physicist's perspective) anything but arbitrary.[38]

"Again," Dirac added, "the geometrician at present is no more interested in a space of 4 dim[ensions] than space of any other number of dimensions. There must, however, be some fundamental reason why the actual universe is 4 dim[ensional], and I feel sure that when the reason is discovered 4 dimensional space will be of more interest to the geometrician than any other." Questions of applied mathematics, questions from the physical world, would, he believed, become of central concern to the mathematician. That which is arbitrary in pure terms became fixed, definite, and unique when put into the frame of a real-world geometry.[39] To draw diagrams, to picture relationships—these were the starting points for grasping why the universe was as it was.

These words would have been music to Baker's ears, for he had little truck with the new, vastly more abstract, rigorous, and algebraic mathematics that was coming into prominence. For example, when the Indian abstract number theorist Ramanujan wrote to the leading mathematicians at Cambridge, Baker had evinced no particular interest in him or his work. G. F. Hardy and J. E. Littlewood welcomed the unknown Indian number theorist as something of a mathematical prophet.[40] Hardy, who helped shape a generation of British mathematics, emphasized rigor, axiomatic presentations, and perfect clarity in definitions. By stark contrast, Baker began volume 5 of his famous series of works on geometry with the words, "The study of the fundamental notions of geometry is not itself geometry; this is more an Art than a Science, and requires the constant play of an agile imagination, and a delight in exploring the relations of geometrical figures; only so do the exact ideas find their value."[41]

Dirac's fascination with the confluence of physical reasoning, geometrical pictures, and mathematical aesthetics became a theme to which he returned throughout his life. In a fragment called "The Physicist and the Engineer," Dirac contended that mathematical beauty existed in the approximate reality of the engineer, not in the realm of pure and exact proof. Mathematical beauty was *the* guide but it was a guide through the approximate reality of the engineer's world, the one actual world in which we live. Many times Dirac insisted that *all* physical laws—Isaac Newton's, Einstein's, his own, were but approximations. "I think I owe a lot to my engineering training because it did teach me to tolerate approximations," Dirac recalled. "Previously to that I thought any kind of an approximation was really intolerable. . . . Then I got the idea that in the actual world all our equations are only approximate. . . . In spite of the equations' being approximate they can be beautiful."[42]

In a sense, Dirac's trajectory can be seen as a series of flights from world to world, flights away from home, no doubt from his dominating father specifically. Margit Dirac, his wife, recalled after Paul's death, that "The first letter he wrote to me [in 1935] after his father's death was to say, 'I feel much freer now.' "[43] But my interest is not in reducing Dirac's views to his familial relations, but rather in following Dirac's path as it traversed a series of worlds of learning, a path that left mechanisms for circuits, circuits for geometry, projective geometry for physics, and eventually projective geometry and engineering for an algebra-inflected physics. It was a path at once ever further from trade work and from home. Schematically, one might summarize Dirac's trajectory as taking him across a surface that folded the geometrical, drawn world of pictures into a private space beneath the algebraic structures of the new quantum physics:

Merchant Venturers' (technical drawing) →

Bristol Electrical Engineering (mechanical and circuit diagrams) →

Bristol Mathematics (projective geometry) →

Cambridge (relativity/projective geometry) →

Cambridge (algebraic structures of quantum mechanics).

It was in the final transition beginning in 1925, just a few months after his tea party talk, that Dirac interiorized and privatized geometry, making public presentation purely in the mode of algebra. From this moment on, Dirac spoke the public ascetic language in which he couched all of his great contributions to quantum mechanics. But he had no affective relation to algebra—it was, in his words, an equation language that for him "meant nothing." Reflecting back on the years since his Bristol days in projective geometry, Dirac told an interviewer: "All my work since then has been very much of a geometrical nature, rather than of an algebraic nature."[44] These are statements characterizing Dirac as a subject in mathematical physics, carving out what is simultaneously a language, an affective structure, a form of argumentation, and a means of exploring the unknown.

The final step toward abstraction and toward the algebraic world for which he came to be considered a heroic figure in physics began in 1925 when his thesis advisor, Fowler, received the proof sheets for a new article from young Werner Heisenberg. The crux was this: he had dispensed with the Bohr orbits, he had developed a consistent calculus of the spectra emitted by various atomic transitions, and he had extended Bohr's "old" quantum theory of 1913 to cover a vastly more general domain. For Dirac there was something else that had fascinated him in Heisenberg's paper—the mathematics. In the course of his calculations Heisenberg had noted that there were certain quantities for which *A* times *B* was not equal to *B* times *A*. Heisenberg was rather concerned by this peculiarity. Dirac seized on it as the key to the departure of quantum physics from the classical world. He believed that it was precisely in the modification of this mathematical feature that Heisenberg's achievement lay. It may well be, as Darrigol, Mehra, and Rechenberg have argued, that the very idea of a multiplication that depends on order came from Dirac's prior explorations in projective geometry.[45] Perhaps it was here that Dirac began to feel that he could recreate the public algebraic world in an interior geometrical one. In any case, from there Dirac was off and running with a new mathematics, accurate predictions, no (public) visualization at any level. On the side, geometry ruled.

Dirac's steps into the unvisualizable domain of quantum mechanics were taken with a certain ambivalence. As he generalized the basic equation of quantum mechanics to include relativity, as he accrued a sense of departing from safe land, the cost to him was movingly captured in an essay he wrote repeatedly over several years titled "Hopes and Fears in Theoretical Physics." In an early fragment Dirac scribbled:

> The effect of fears are perhaps not so obvious.
> The fears are of two kinds.
> The first one is the fear of putting forward a new idea which may turn out to be quite wrong.
> The fear of sticking one's neck out.
> perhaps having to retract and being exposed to humiliation.
> It may be that such a fear acts largely subconsciously
> and inhibits one from making a bold step forward.
> A man may get close to a great discovery and fail to make the last vital step.
> Possibly it is such a fear that blocks this step.[46]

In these highly inflected lines, Dirac explicitly touched on his own terror of the humiliating failure that abutted any chance of success, a terror expressed in an ambivalence at once drawn toward risk and success (in the form of the quantum theory he helped create) and yet recoiling with fear from possible failure and "sticking his neck" out from his own place of security. There is here a psychological story of the ambivalence of leaving home, a "home" that is conjointly familial, social, and epistemic—Merchant Venturers' was the workplace of his father, his training ground in engineering, and the place of his first encounter with the projective geometry to which Fraser (and later Baker) had introduced him.

But there is a further story that is only incompletely lodged in this geography of the psychological. This other narrative entails an account of how the logic of drawing was "suppressed"; how thinking through drawing diagrams went from being celebrated across Europe in the mid–nineteenth century to being marginalized at the beginning of the twentieth. To complete this broader narrative properly would take us into the shifting fortunes of geometry in France and Germany, and into fundamental changes in pedagogy at Cambridge.[47] I have only begun to sketch here the shifting role of persuasive visibilities in physics and their function in shaping an epistemological interior life for Dirac.

The Suppression of Geometry

To the mathematical generation that came of age after 1900 in England, geometry was no longer a science with claims to being descriptive of the world. Instead geometry, once the sun in the scientific sky, was being eclipsed by the formalized, devisualized system of logical relations exemplified on the Continent by mathematicians associated with David Hilbert and by physicists linked to Heisenberg. In Cambridge, it was Hardy who epitomized this new world of rigor—expressing the new mathematics in the formal relations of number theory, not in a descriptive, physicalized, and *drawn* geometry. By the early 1920s, drawn diagrams felt ever more like a disappearing trace, a vestige of a system of inquiry, pedagogy, and values that was fast fading from the Cambridge scene. For the historian of mathematics Herbert Mehrtens, the geometrical-intuitive mathematicians in many ways stood for a *Gegen-Moderne*, an antimodernism fighting to bind mathematics to the physical world and beyond—to psychology, pedagogy, and progressive technology. The moderns, he argues, wanted to bound and restrict mathematics, guarding their authority through a professional autonomy; mathematics, they argued, was not "about" anything exterior to its own formal structure.[48]

Dirac stood with one foot in the Cambridge of the older sort (through his association with Baker) and the other in the "new" Continent-leaning Cambridge (through his alliance with Heisenberg, Hilbert, and Hardy). It was a choice between Victorian geometrical tea parties and a post-Victorian modernism. Even as Dirac gave his own tea party talk in 1924, Baker's projective geometry was on the wane. Dirac had moved into the wing of Cambridge mathematics that had already lost the war to set the exam standards for the next generation of students and the mathematical standards for the next generation of researchers. Drawing diagrams gave Dirac an older safe point from which to venture into the new and, as he repeatedly emphasized, more fearsome unknown.

Heisenberg's paper of 1925 was antivisual without being, for that, formally and rigorously mathematical. It was physical and yet completely unvisual. Here was a final step away from the legacy of the *Ecole polytechnique*'s physicalized geometry, away from Felix Klein's tactile mathematical models that formed part of his Erlanger program, away from the British Victorian effort to make descriptive geometry into the centerpiece of skilled reason binding head and hand. And yet, as Dirac launched a long and extraordinarily successful career expressed entirely in the language of algebra, there was another Dirac, privately sketching, figuring, reasoning with diagrams, translating the results back into algebra, and all but burying the scaffolding around an interior furnished with formerly public effects.

My inclination, then, is to use the biographical-psychological story *not* as an end in itself, but rather as a registration of Dirac's arc from Bristol to Cambridge, to an identification with Bohr's and Heisenberg's Continental physics. In that trajectory, Dirac was sequentially immersed in a series of territories in which particular strategies of demonstration were valued. Bristol University was a step away from the technical drawing of Merchant Venturers', the whole electrical engineering curriculum with its codified, abstracted, applied physics removed drawing to a form of depiction less tied to quasi-mimetic technical renderings and linked instead to more functional, topological circuit diagrams. Bristol's applied mathematics again took Dirac further away from engineering, as did Heisenberg's matrix mechanics.

Technical drawings idealize by removing nonfunctional textures; circuit drawings drop any pretense of mimetic depiction—they are topological insofar as they represent relationships and use icons to refer to component parts. Actual spatial positions and distances do not matter. Projective geometry is also topological in this sense—the distances are eliminated from consideration and only intersections and their relative locations count. Projective geometry began in the domain of the physical, crept somewhat away in higher dimensions and its representation of non-Euclidean geometries. But Dirac kept bringing projective geometry back to the world, using it to track each new topic in mathematical physics across a long career.

When Dirac moved to Cambridge to begin studying physics, he took with him this projective geometry and used it to think. But that thinking had now to be conducted only on the inside of a subject newly self-conscious of its separation from the scientific world. Dirac's maturity was characterized again by flight, this time to Heisenberg's algebra, an antivisual calculus that at once broke with the visual tradition in physics and with the legacy of an older school of visualizable, intuition-grounded descriptive geometry. With an austere algebra and Heisenberg's quantum physics, Dirac stabilized his thought through *instability*: working through a now infolded projective geometry joined by carefully hidden passageways to the public sphere of symbols without pictures.

Freud often argued that what cannot be expressed in private is manifested in public. In a sense I am suggesting the contrary here: at the turn of the century in Britain, projective geometry was shifting away from the status of a state-endorsed liberal epistemology that joined university to factory and toward a form of knowledge that was distinctly second class. Physicalized geometry—geometry grounded in spatial intuitions, visualizations, diagrammatics—collapsed under the language of an autonomous science. In a sense Dirac's suppressed drawings were the hidden remnants of an infolded Victorian world. Public geometry became private reason.

Notes

I would like to thank Sharon Schwerzel, Beverly McNeil, and especially Joseph R. McElrath for their assistance with the Dirac Papers.

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2. On James Clerk Maxwell and mechanical analogies, see the following, along with references therein: P. M. Harman, *Energy, Force, and Matter: The Conceptual Development of Nineteenth-Century Physics* (Cambridge, 1982); C. W. F. Everitt, *James Clerk Maxwell: Physicist and Natural Philosopher* (New York, 1975); Crosbie Smith and M. Norton Wise, *Energy and Empire* (Princeton, 1989); Jordi Cat, "On Understanding: Maxwell on the Methods of Illustration and Scientific Method" (forthcoming). On Albert Einstein and visualization, see Gerald Holton, *Thematic Origins of Scientific Thought* (Cambridge, Mass., 1973); and on Richard Feynman, see S. S. Schweber, *QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomanaga* (Princeton, 1994); P. Galison, "Feynman's War: Modelling Weapons, Modelling Nature," *Studies in the History and Philosophy of Modern Physics* 29B (1998): 391–434; and David Kaiser, Stick-Figure Realism: Conventions, Reification, and the Persistence of Feynman Diagrams, 1948–1964," *Representations* 70 (Spring 2000): 49–86. Schweber, *QED and the Men Who Made It*, chap. 8.
3. Peter Galison, *Image and Logic: A Material Culture of Microphysics* (Chicago, 1997).
4. Paul Dirac, "Use of Projective Geometry in Physical Theory," Boston, 30 October 1972, in Paul A. M. Dirac, Scientific Papers, Florida State University, Talahassee, Florida, file: Lectures B4 F26. Quoted with the permission of the Florida State University Libraries. Hereafter, the Dirac papers will be referred to as PDP.

5. Paul Dirac, Draft version "My Life as a Physicist," PDP, file: Lectures B3 F10.
6. Paul Dirac, "Recollections of an Exciting Era," three lectures given at Varenna, 5 August 1972. Quotations until "My research work" from typescript (10); and from that point forward from handwritten draft (3B). PDP, file: Lectures B4 F3.
7. Ibid.
8. H. Reichenbach, *Experience and Prediction* (Chicago, 1938).
9. Karl Popper, *Logic of Scientific Discovery* (New York, 1959), 32.
10. Holton, *Thematic Origins*, esp. chaps. 1, 3.
11. Bruno Latour and Steve Woolgar, *Laboratory Life: The Construction of Scientific Facts* (Cambridge, Mass., 1987). For challenges to the Reichenbachian distinction, see, e.g., Peter Galison, *How Experiments End* (Chicago, 1987), 3, 277; Karin Knorr-Cetina, *The Manufacture of Knowledge: An Essay in the Constructivist and Contextual Nature of Science* (Oxford, 1981), 84; Andrew Pickering, *Constructing Quarks: A Sociological History of Particle Physics* (Chicago, 1984), 414.
12. Jacques de Caso, "Process as Signification: The Drawings of Théophile Bra," in *The Drawing Speaks: Théophile Bra, Works, 1826–1855* (Houston, 1997), 26.
13. Gilles Deleuze, *Foucault* (Minneapolis, 1988), esp., "Foldings, or the Inside of Thought (Subjectivation)," 94–123.
14. For an excellent study of the historical attempt to foster individualism in nineteenth-century France, see the work of Jan Goldstein on Victor Cousin: Jan Goldstein, "Foucault and the Post-Revolutionary Self: The Uses of Cousinian Pedagogy in Nineteenth-Century France," in Jan Goldstein, ed., *Foucault and the Writing of History* (Cambridge, Mass., 1994); and Jan Goldstein, "Eclectic Subjectivity and the Impossibility of Female Beauty," in Caroline A. Jones and Peter Galison, eds., *Picturing Science, Producing Art* (New York, 1998).
15. Lorraine Daston, "Physicalist Tradition in Early Nineteenth-Century French Geometry," in *Studies in the History and Philosophy of Science* 17 (1986): 269–95; Ken Alder, *Engineering the Revolution: Arms and Enlightenment in France, 1763–1815* (Princeton, 1997), esp. 136–40; Bruno Belhoste, "Un modèle à l'épreuve, L'école polytechnique de 1794 au Second Empire," in Bruno Belhoste, Amy Dahan Dalmedico, and Antoine Picon, eds., *La formation polytechnicienne, 1794–1994* (Paris, 1994), esp. 12 ff.
16. Daston, "Physicalist Tradition," 279.
17. The "geometry of the workshop" is from Alder, *Engineering the Revolution*, 138.
18. Daston, "Physicalist Tradition," 291.
19. Ibid., 292.
20. Alder, *Engineering the Revolution*, 143–53.
21. John Perry, speaking to the British Association for the Advancement of Science in Glasgow in 1901, declaimed of the two sides: "What we want is a great Toleration Act which will allow us all to pursue our own ideals, taking each from the other what he can in the way of mental help. We do not want to interfere with the students of pure mathematics. . . . The more they hold themselves in their studies as a race of demigods apart the better it may be for the world. . . . I belong to a great body of men who apply the principles of mathematics in physical science and engineering; I belong to the very much greater body of men who may be called persons of average intelligence. In each of these capacities I need mental training and also mathematical knowledge"; cited in Joan Richards, *Mathematical Visions: The Pursuit of Geometry in Victorian England* (Boston, 1988), 197.
22. Cited in Richards, *Mathematical Visions*, 167.
23. *The Technical Educator: An Encyclopedia of Technical Education* (London, n.d.), 1:63.

24. Ibid., 175.
25. Information on flow charts, contained in letter from Mrs. P. M. Denney, Secretary to the Treasurer, The Society of Merchant Venturers, Merchant's Hall, to Mrs. R. Good, Librarian, Cotham Grammar School, 14 May 1999, and to author, 21 May 1999.
26. D. G. Pratten, *Tradition and Change: The Story of Cotham School* (Bristol, Eng., 1991), 24.
27. Margit Dirac, "Thinking of My Darling Paul," in Kursunoglu and Wigner, *Reminiscences*, 5.
28. Pratten, *Tradition and Change*, 11. Also: D.J. Eames, "The Contribution Made by the Society of Merchant Venturers to the Development of Education in Bristol," unpublished thesis, 1966, a copy of which is in papers of The Society of Merchant Venturers. My thanks to Mrs. P. M. Denney for making this available to me.
29. Julius Wertheimer to Sir Isambard Owen, 11 Sept. 1912, University of Bristol Special Collections. Happily we have a new interest in Victorian technical pedagogy—see, e.g., two excellent theses linking industrial arts with scientific-technical pedagogy: J. Graeme Gooday, "Precision Measurement and the Genesis of Physics Teaching Laboratories in Victorian Britain" (Ph.D. diss., University of Kent, 1989); and Nani Clow, "The Laboratory of Victorian Culture; Experimental Physics, Industry, and Pedagogy in the Liverpool Laboratory of Oliver Lodge, 1881–1900" (Ph.D. diss., Harvard University, 1999).
30. Faculty of Engineering, List of Candidates, 1910–1930, University of Bristol, Special Collections, DM 275.
31. Relativity, Dirac reported, felt like an escape from thoughts of World War I. It was in the immediate postwar period that he began to think that an extra (fourth) dimension might provide the link between space and time—though he had no sense that the metric was pseudo-Euclidean. Dirac in C. Wiener, ed., *History of Twentieth Century Physics* (New York, 1977), 109, cited in Abraham Pais, "Paul Dirac: Aspects of His Life and Work," in Goddard, *Paul Dirac*, 4.
32. R. H. Dalitz, "Another Side to Paul Dirac," in Kursunoglu and Wigner, *Reminiscences*, 71.
33. Peter Galison, "Minkowski's Space-Time: From Visual Thinking to the Absolute World," *Historical Studies in the Physical Sciences* 10 (1979): 85–121.
34. H. F. Baker, *The Principles of Geometry* (Cambridge, 1922).
35. See Mehra and Rechenberg, *Historical Development*, 161–68; and Darrigol, *c-Numbers*, 291 ff.
36. W. V. D. Hodge, "Henry Frederick Baker, 1866–1956," *Biographical Memoirs of Fellows of the Royal Society* 2 (1956): 49–68, on 53.
37. Paul Dirac, "Manuscript Notes for a Talk on Relativity to Prof. Baker's Mathematical Tea Party," draft lecture, c. 1924, Prof. Baker's Tea Party, PDP, file: Lectures B3 F25.
38. Ibid. Braces indicate cross-out in original.
39. Ibid.
40. Robert Kanigel, *The Man Who Knew Infinity: A Life of the Genius Ramanujan* (New York, 1991), 170.
41. Cited in Hodge, "Henry Frederick Baker," 56.
42. Paul Dirac, interview with Thomas S. Kuhn and Eugene Wigner, 6 May 1963, 5, in Archive for the History of Quantum Physics, Oral History Interviews, Niels Bohr Library, American Institute of Physics. Hereafter AHQP.
43. Dirac, "Thinking of My Darling Paul," 5.
44. P. A. M. Dirac, interview with Thomas Kuhn and Eugene Wigner, 1 April 1962, 3, in AHQP.

45. Darrigol, *c-Numbers*, 291–95.
46. Paul Dirac, draft lecture, 2 November 1971, "Hopes and Fears," PDP, file: Lectures B2 F19.
47. Andrew Warwick, "Cambridge Mathematics and Cavendish Physics: Cunningham, Campbell and Einstein's Relativity, 1905–1911," part 1: "The Uses of Theory," *Studies in History and Philosophy of Science* 23 (1992): 625–56, and part 2: "Comparing Traditions in Cambridge Physics," *Studies in History and Philosophy of Science* 24 (1993): 1–25.
48. Herbert Mehrtens, *Moderne Sprache Mathematik: Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme* (Frankfurt am Main, 1990).