Definition

[edit]

A group can be defined in a number of equivalent ways; most commonly, it is defined as follows. A group is a set, G, together with an operation $\cdot$ (called the *group law* of G) that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the *group axioms*:[4]

Closure

For all a, b in G, the result of the operation, $a \cdot b$, is also in G.[b]

Associativity

For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity element

There exists an element e in G, such that for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds. Such an element is unique (see below), and thus one speaks of *the* identity element. The identity element of a group G is often written as 1 or 1_G,[5] a notation inherited from the multiplicative identity.

Inverse element

For each a in G, there exists an element b in G such that $a \cdot b = b \cdot a = e_G$.

The order in which the group operation is carried out can be significant. In other words, the result of combining element a with element b need not yield the same result as combining element b with element a; the equation

$$a \cdot b = b \cdot a$$

may not always be true. This equation always holds in the group of integers under addition, because $a + b = b + a$ for any two integers (commutativity of addition). However, it does not always hold in the symmetry group below. Groups for which the equation $a \cdot b = b \cdot a$ always holds are called *abelian* (in honor of Niels Abel). Thus, the integer addition group is abelian, but the symmetry group in the following section is not.

SU(2) and SU(3)

The special unitary group SU(n) can be represented by the set of n x n unitary matrices with determinant 1.

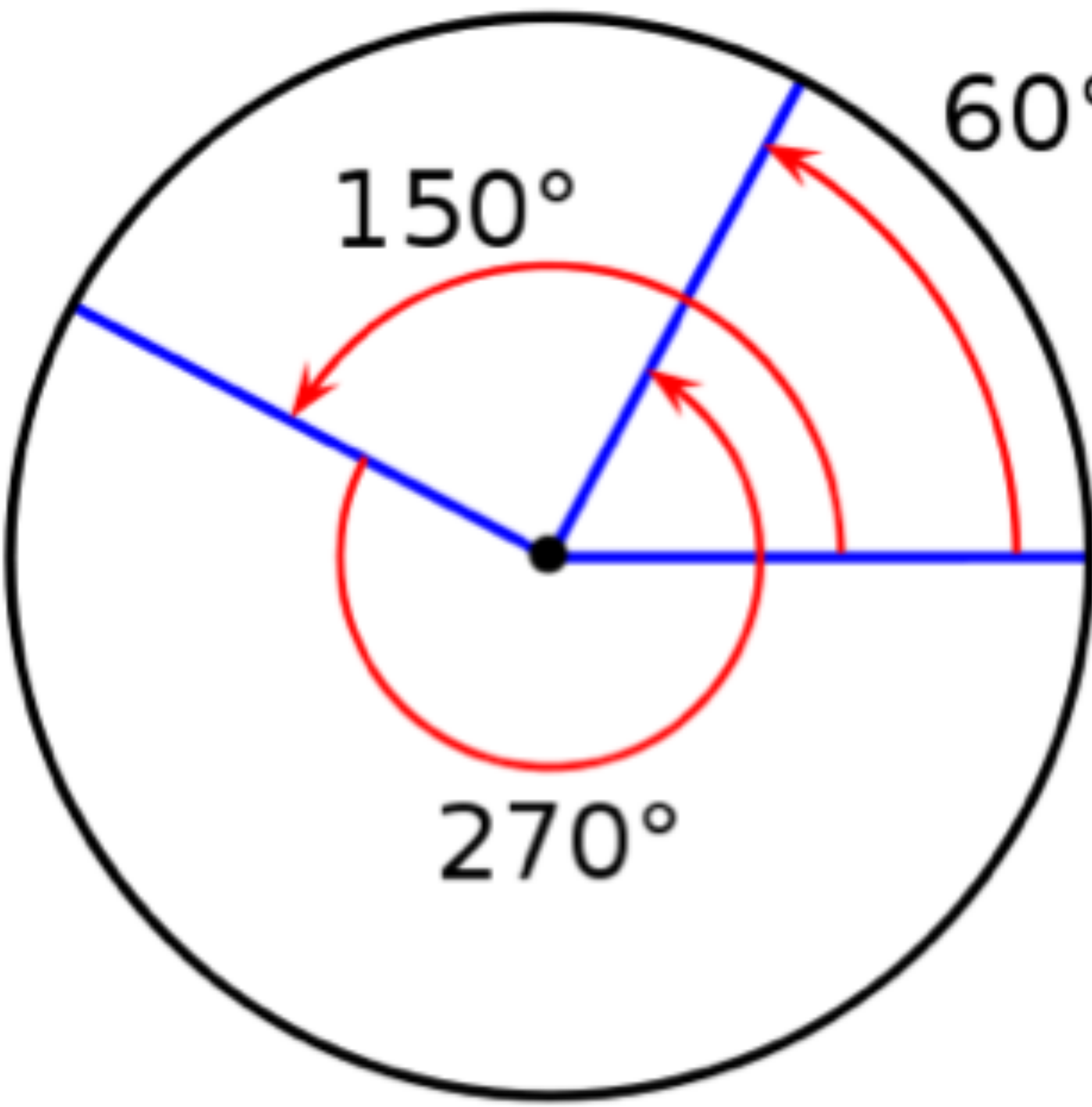
SU(2) is represented by the Pauli matrices

SU(3) is represented by the Gell-Mann matrices

Unitary means $UU^{+} = U^{+}U = 1$

Special means unimodular => determinant = 1

U(1) aka, the circle group



exp(i phi)

Appendix I

Table of Wigner Coefficients $(jj'mm'|JM)$

Note: Reading down one column in the table gives the coefficients in the expansion

$$W_M^{(J)} = \sum_{m,m'} (jj'mm'|JM) U_m^{(j)} V_{m'}^{(j')}, \tag{20.1}$$

and reading across one row gives the coefficients in

$$U_m^{(j)} V_{m'}^{(j')} = \sum_{J,M} (jj'mm'|JM) W_M^{(J)}. \tag{20.5}$$

The tables are taken from Cohen (1949)†, who also gives a table for $D^{(3/2)} \times D^{(5/2)}$ not included here. Algebraic tables have been given by the following:
$D^{(l)} \times D^{(1/2)}$, $D^{(l)} \times D^{(1)}$, $D^{(l)} \times D^{(2)}$, Condon and Shortley (1951);
$D^{(l)} \times D^{(3/2)}$, Cohen (1949);
$D^{(l)} \times D^{(5/2)}$, Melvin and Swamy (1957);
$D^{(l)} \times D^{(3)}$, Falkoff *et al.* (1952).

Extensive numerical tables in decimal form have been published by Rose (1957) and Simon (1954).

SU(2) x SU(2)

$D_{\frac{1}{2}} \times D_{\frac{1}{2}}$

	W_1^1	W_0^1	W_0^0	W_{-1}^1
$U_{1/2}^{1/2}\ V_{1/2}^{1/2}$	1			
$U_{1/2}^{1/2}\ V_{-1/2}^{1/2}$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$U_{-1/2}^{1/2}\ V_{1/2}^{1/2}$		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
$U_{-1/2}^{1/2}\ V_{-1/2}^{1/2}$				1

† The tables are reproduced here through the kind permission of Dr. Cohen.

The Clebsch-Gordon coefficients for converting from $|J, m_J>$ to $|j_1, m_1>$ and $|j_2, m_2>$.
The W labels are J superscript and m_J subscript.
The U labels are j_1 superscript and m_1 subscript.
The V labels are j_2 superscript and m_2 subscript.

$$\frac{1}{2} \times 1$$

SU(2) x SU(3)

	$W^{3/2}_{3/2}$	$W^{3/2}_{1/2}$	$W^{1/2}_{1/2}$	$W^{3/2}_{-1/2}$	$W^{1/2}_{-1/2}$	$W^{3/2}_{-3/2}$
$U^1_1 \quad V^{1/2}_{1/2}$	1					
$U^1_1 \quad V^{1/2}_{-1/2}$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
$U^1_0 \quad V^{1/2}_{1/2}$		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
$U^1_0 \quad V^{1/2}_{-1/2}$				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
$U^1_{-1} \quad V^{1/2}_{1/2}$				$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
$U^1_{-1} \quad V^{1/2}_{-1/2}$						1

$$\frac{1}{2} \times \frac{3}{2}$$

	W^2_2	W^2_1	W^1_1	W^2_0	W^1_0	W^2_{-1}	W^1_{-1}	W^2_{-2}
$U^{3/2}_{3/2} \quad V^{1/2}_{1/2}$	1							
$U^{3/2}_{3/2} \quad V^{1/2}_{-1/2}$		$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$					
$U^{3/2}_{1/2} \quad V^{1/2}_{1/2}$		$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$					
$U^{3/2}_{1/2} \quad V^{1/2}_{-1/2}$				$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$			
$U^{3/2}_{-1/2} \quad V^{1/2}_{1/2}$				$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$			
$U^{3/2}_{-1/2} \quad V^{1/2}_{-1/2}$						$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$	
$U^{3/2}_{-3/2} \quad V^{1/2}_{1/2}$						$\sqrt{\frac{1}{4}}$	$-\sqrt{\frac{3}{4}}$	
$U^{3/2}_{-3/2} \quad V^{1/2}_{-1/2}$								1

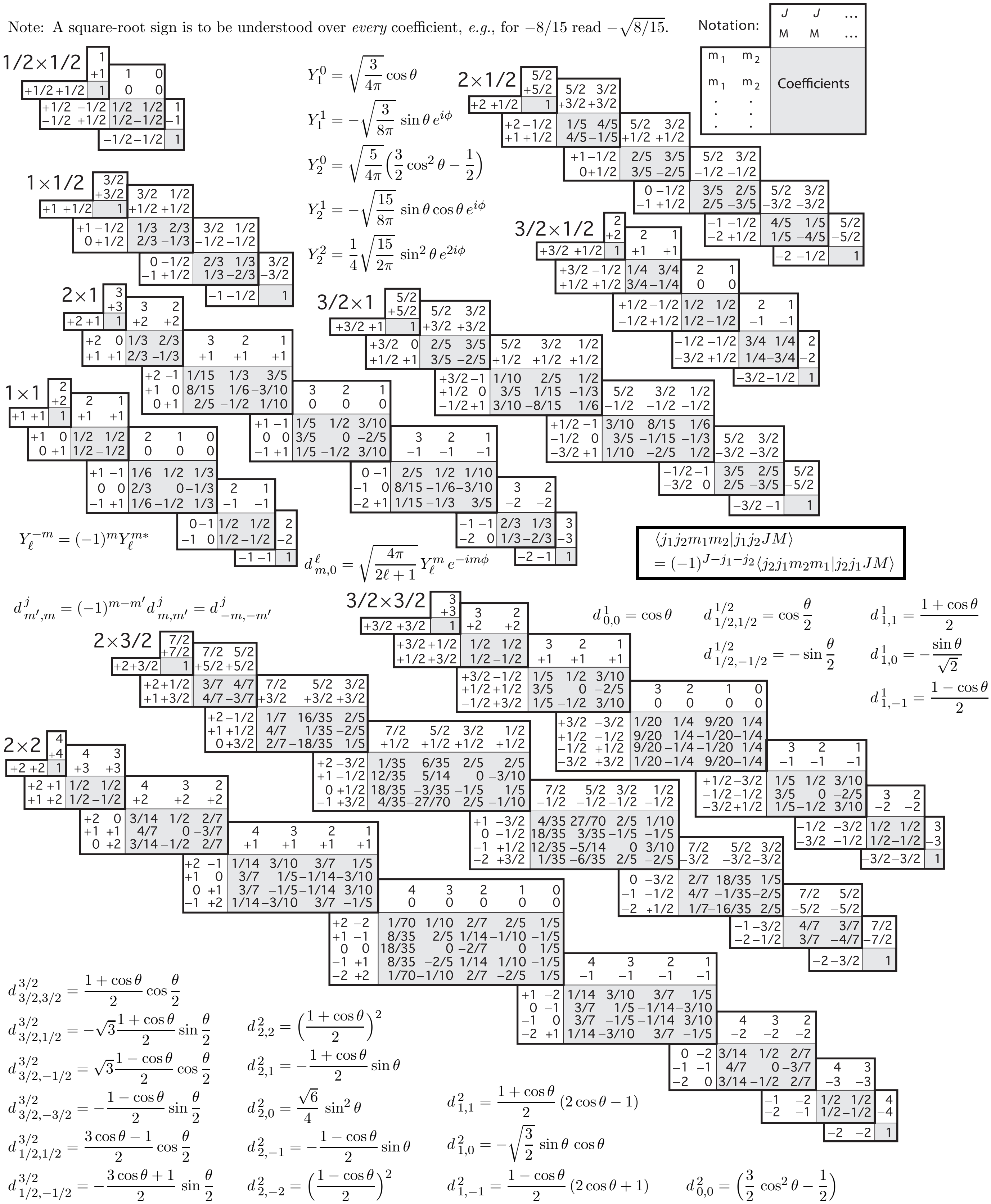
SU(3) x SU(3)

$D_1 \times D_1$

	W^2_2	W^2_1	W^1_1	W^2_0	W^1_0	W^0_0	W^2_{-1}	W^1_{-1}	W^2_{-2}
$U^1_1 \quad V^1_1$	1								
$U^1_1 \quad V^1_0$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$						
$U^1_0 \quad V^1_1$		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						
$U^1_1 \quad V^1_{-1}$				$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
$U^1_0 \quad V^1_0$				$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$			
$U^1_{-1} \quad V^1_1$				$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
$U^1_0 \quad V^1_{-1}$							$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$U^1_{-1} \quad V^1_0$							$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
$U^1_{-1} \quad V^1_{-1}$									1

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

$$D_1 \times D_2$$

	W^3_3	W^3_2	W^2_2	W^3_1	W^2_1	W^1_1	W^3_0	W^2_0	W^1_0	W^3_{-1}	W^2_{-1}	W^1_{-1}	W^3_{-2}	W^2_{-2}	W^3_{-3}
$U^2_2\ V^1_1$	1														
$U^2_2\ V^1_0$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$												
$U^2_1\ V^1_1$		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$												
$U^2_2\ V^1_{-1}$				$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$									
$U^2_1\ V^1_0$				$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$									
$U^2_0\ V^1_1$				$\sqrt{\frac{6}{15}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$									
$U^2_1\ V^1_{-1}$							$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{10}}$						
$U^2_0\ V^1_0$							$\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$						
$U^2_{-1}\ V^1_1$							$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{3}{10}}$						
$U^2_0\ V^1_{-1}$										$\sqrt{\frac{6}{15}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$			
$U^2_{-1}\ V^1_0$										$\sqrt{\frac{8}{15}}$	$-\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$			
$U^2_{-2}\ V^1_1$										$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$			
$U^2_{-1}\ V^1_{-1}$													$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
$U^2_{-2}\ V^1_0$													$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
$U^2_{-2}\ V^1_{-1}$															1

$D_{\frac{1}{2}} \times D_2$

		$W^{5/2}_{5/2}$	$W^{3/2}_{3/2}$	$W^{3/2}_{3/2}$	$W^{5/2}_{1/2}$	$W^{3/2}_{1/2}$	$W^{5/2}_{-1/2}$	$W^{3/2}_{-1/2}$	$W^{5/2}_{-3/2}$	$W^{3/2}_{-3/2}$	$W^{5/2}_{-5/2}$
U^2_2	$V^{1/2}_{1/2}$	1									
U^2_2	$V^{1/2}_{-1/2}$		$\sqrt{\frac{1}{5}}$	$\sqrt{\frac{4}{5}}$							
U^2_1	$V^{1/2}_{1/2}$		$\sqrt{\frac{4}{5}}$	$-\sqrt{\frac{1}{5}}$							
U^2_1	$V^{1/2}_{-1/2}$				$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$					
U^2_0	$V^{1/2}_{1/2}$				$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$					
U^2_0	$V^{1/2}_{-1/2}$						$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$			
U^2_{-1}	$V^{1/2}_{1/2}$						$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{5}}$			
U^2_{-1}	$V^{1/2}_{-1/2}$								$\sqrt{\frac{4}{5}}$	$\sqrt{\frac{1}{5}}$	
U^2_{-2}	$V^{1/2}_{1/2}$								$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{4}{5}}$	
U^2_{-2}	$V^{1/2}_{-1/2}$										1

$\frac{1}{2} \times 2$

$D_{\frac{1}{2}} \times D_3$

	$W^{7/2}_{7/2}$	$W^{7/2}_{5/2}$	$W^{5/2}_{5/2}$	$W^{7/2}_{3/2}$	$W^{5/2}_{3/2}$	$W^{7/2}_{1/2}$	$W^{5/2}_{1/2}$	$W^{7/2}_{-1/2}$	$W^{5/2}_{-1/2}$	$W^{7/2}_{-3/2}$	$W^{5/2}_{-3/2}$	$W^{7/2}_{-5/2}$	$W^{5/2}_{-5/2}$	$W^{7/2}_{-7/2}$
$U^3_3\ V^{1/2}_{1/2}$	1													
$U^3_3\ V^{1/2}_{-1/2}$		$\sqrt{\frac{1}{7}}$	$\sqrt{\frac{6}{7}}$											
$U^3_2\ V^{1/2}_{1/2}$		$\sqrt{\frac{6}{7}}$	$-\sqrt{\frac{1}{7}}$											
$U^3_2\ V^{1/2}_{-1/2}$				$\sqrt{\frac{2}{7}}$	$\sqrt{\frac{5}{7}}$									
$U^3_1\ V^{1/2}_{1/2}$				$\sqrt{\frac{5}{7}}$	$-\sqrt{\frac{2}{7}}$									
$U^3_1\ V^{1/2}_{-1/2}$						$\sqrt{\frac{3}{7}}$	$\sqrt{\frac{4}{7}}$							
$U^3_0\ V^{1/2}_{1/2}$						$\sqrt{\frac{4}{7}}$	$-\sqrt{\frac{3}{7}}$							
$U^3_0\ V^{1/2}_{-1/2}$								$\sqrt{\frac{4}{7}}$	$\sqrt{\frac{3}{7}}$					
$U^3_{-1}\ V^{1/2}_{1/2}$								$\sqrt{\frac{3}{7}}$	$-\sqrt{\frac{4}{7}}$					
$U^3_{-1}\ V^{1/2}_{-1/2}$										$\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$			
$U^3_{-2}\ V^{1/2}_{1/2}$										$\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{5}{7}}$			
$U^3_{-2}\ V^{1/2}_{-1/2}$												$\sqrt{\frac{6}{7}}$	$\sqrt{\frac{1}{7}}$	
$U^3_{-2}\ V^{1/2}_{1/2}$												$\sqrt{\frac{1}{7}}$	$-\sqrt{\frac{6}{7}}$	
$U^3_{-3}\ V^{1/2}_{-1/2}$														1

$\frac{1}{2} \times 3$

Pauli matrices

From Wikipedia, the free encyclopedia

The **Pauli matrices** are a set of three 2 × 2 complex matrices which are Hermitian and unitary.[1] Usually indicated by the Greek letter *sigma* (σ), they are occasionally denoted with a *tau* (τ) when used in connection with isospin symmetries. They are:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices were used by, then named after, the Austrian-born physicist Wolfgang Pauli (1900–1958), in his 1925 study of spin in quantum mechanics.

Each Pauli matrix is Hermitian, and together with the identity *I* (sometimes considered the zeroth Pauli matrix σ_0), the Pauli matrices span the full vector space of 2x2 Hermitian matrices.

Algebraic properties [edit]

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

where *I* is the identity matrix, i.e. the matrices are involutory.

- The determinants and traces of the Pauli matrices are:
  $$\det(\sigma_i) = -1,$$
  $$\operatorname{Tr}(\sigma_i) = 0.$$

From above we can deduce that the eigenvalues of each σ_i are ± 1.

- Together with the identity matrix **I** (which is sometimes written as σ_0), the Pauli matrices form an orthogonal basis, in the sense of Hilbert-Schmidt, for the real Hilbert space of 2×2 complex Hermitian matrices, or the complex Hilbert space of all 2×2 matrices.

The Group SU(2)

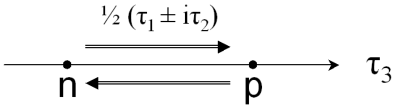
- **S**pecial **u**nitary group in **2** dimensions
- ($j = \frac{1}{2}$) is lowest-dimension nontrivial representation of SU(2) (isomporphic to rotation group SO(3))
- Generators $J_i = \frac{1}{2}\sigma_i$ with $i = 1, 2, 3$
  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  are traceless [S] and hermitian [U]
- Basis conventionally as eigenvectors of σ_3:
  $$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- describing a spin-$\frac{1}{2}$ particle, e.g. an electron

Application to Isospin

- SU(2) symmetry with (n,p) as fundamental representation
- SU(2) algebra, defining the group:

$$[I_j, I_k] = i\varepsilon_{jkl} I_l$$

- Generators $I_i = \frac{1}{2}\tau_i$ with τ_i equal to Pauli matrices



- Proton and neutron states represented by

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Gell-Mann Matrices [7.1]

SU(3) corresponds to special unitary transformation on complex 3D vectors.

The natural representation is that of 3×3 matrices acting on complex 3D vectors.

There are 3^2-1 parameters, hence 8 generators: $\{X_1, X_2, \ldots X_8\}$.
The generators are traceless and Hermitian.

The **generators** are derived from the Gell-Mann matrices: $X_i = \frac{1}{2}\lambda_i$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

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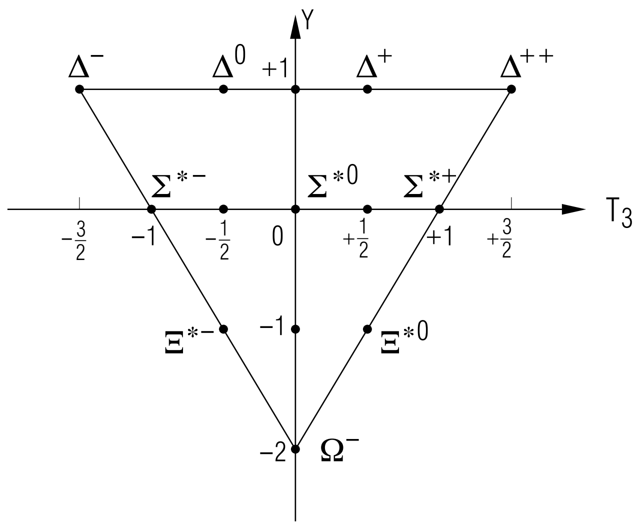
SU(3) AND THE QUARK MODEL



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SU(3) AND THE QUARK MODEL

by

J. R. Christman, U. S. Coast Guard Academy

SU(3) AND THE QUARK MODEL

by

J. R. Christman, U. S. Coast Guard Academy

1. Abstract

We here deal deal with a way of classifying the hadrons: according to SU(3) symmetry operations. This system is known as the quark model or the eight-fold way. The quark model of high energy physics is analogous to the periodic table of the elements in that it provides an ordering of the particles.

2. Readings

Longo, Chapter 8

G. F. Chew, M. Gell-Mann, and A. H. Rosenfeld, "Strongly Interacting Particles," *Scientific American* (Feb. 1964).

3. The Quark Concept

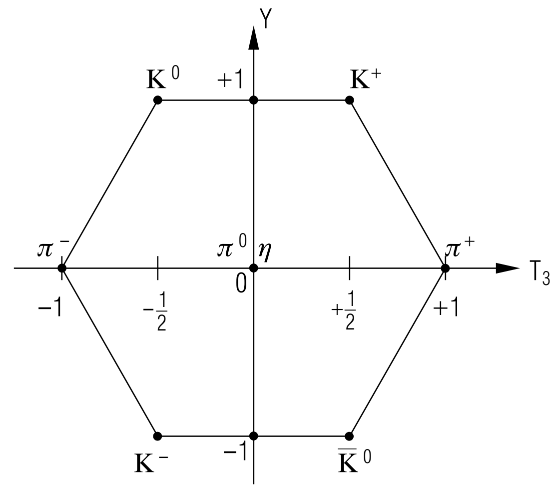
3a. Quarks as Elementary Particles. One of the goals of high energy physics is to explain the properties (mass, spin, charge, isotopic spin, strangeness) of the hadrons in terms of something more fundamental. The idea of the quark model is to invent a small set of particles, imbue them with appropriate properties, and use them to construct the hadrons much as neutrons and protons are used to construct the various atomic nuclei. The hypothetical fundamental building blocks are called quarks. They have never been isolated and have never been observed.

3b. Observed Symmetry Patterns in Hypercharge and Isospin. The quark model deals with the isotopic spin (both magnitude and 3-component), hypercharge, strangeness, and baryon number of the hadrons.

We begin with an observation that, if the particle properties are plotted a certain way, then there are only three forms to the plots. All of the plots have hypercharge on the vertical axis and 3-component of isotopic spin on the horizontal axis. We plot on the same graph only those particles which have the same baryon number, spin, and intrinsic

parity.[1] The particles on any one diagram have nearly the same mass.

We start with the 0^- mesons (spin 0, odd parity):

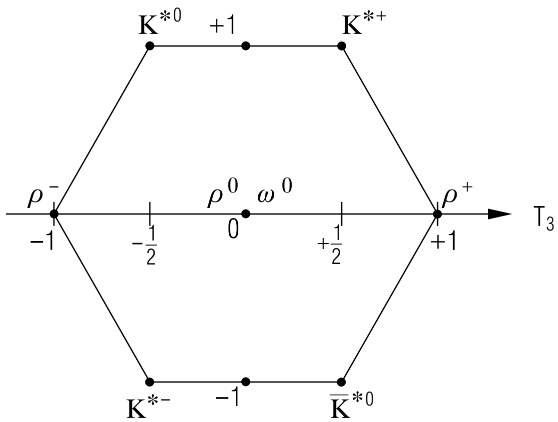


Both the π^0 and the η have $T_3 = 0$. However the π^0 belongs to an isospin triplet and has $T = 1$ while the η is an isospin singlet and has $T = 0$. This plot contains 8 particles and they are jointly called an octet.

There is an octet of 1^- mesons:

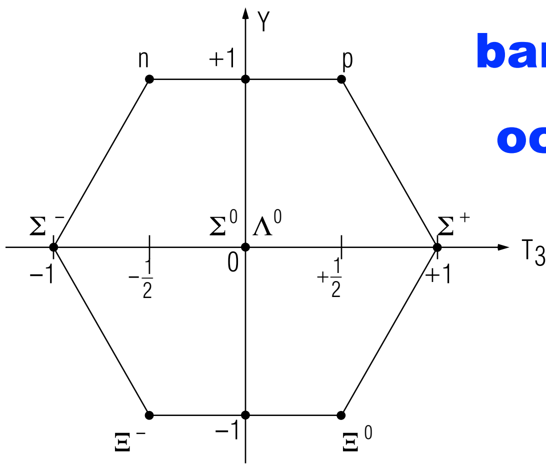
[1] A hadron can be assigned an intrinsic parity (+ or −) depending on whether or not the wave function of the particle, when the particle has zero orbital angular momentum, changes sign with operation by the parity operator. Intrinsic parity is denoted by a + or − superscript on the spin. This is a minor technical point for this discussion. You should realize that all particles in a given diagram have the same spin but there may be more than one diagram corresponding to a given spin.

spin 1 meson octet



There is an octet of $1/2^+$ baryons:

spin 1/2 baryon octet

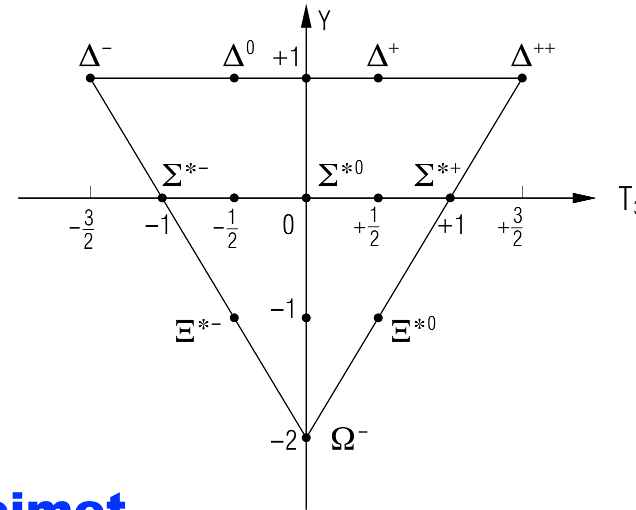


The octet always consists of an isospin doublet with $Y = +1$, an isospin triplet with $Y = 0$, an isospin singlet with $Y = 0$, and an isospin doublet with $Y = -1$.

The second type pattern to be considered is the decimet, composed of ten particles: an isospin quartet with $Y = +1$, an isospin triplet with $Y = 0$, an isospin doublet with $Y = -1$, and an isospin singlet with $Y = -2$. Only baryons have been found to form decimets and, as we

spin 3/2 baryon decuplet

shall see, the quark model provides a reason why. Here is a decimet of $3/2^+$ baryons:



decimet

This is the only complete decimet known but there are undoubtedly others at higher masses (yet to be discovered "resonances").

It is important to note that if a particle belongs to a given multiplet, all of its isospin partners belong to the same multiplet. The patterns shown here combine several sets of isospin partners to form a larger pattern than that provided by isospin alone.

We have now described two of the three forms, the octet and the decimet. The third form is the simplest. It consists of a single particle with $Y = 0$, $T = 0$, and $T_3 = 0$ and is called a singlet. It is easy to confuse one of these particles with the isotopic spin singlet which occurs in the octet of the same spin and parity. For example, the $\phi(1019)$ meson may be a 1^- singlet meson. If differs from the ω^0 only in mass. Which belongs to the octet and which to the singlet? We shall see that the quark model assigns different quark content to the $Y = 0$, $T = 0$ singlet and to the $Y = 0$, $T = 0$ particle in the octet. But quarks are not observable, so this distinction cannot be used. The point is that some assignments of particles to octets or singlets are arbitrary at present and, in fact, the physical particle may be some superposition of the two states.

5b. Hadron-Hadron Interaction: SU(3) from QQ Interaction. The abstract properties of the operators can be discussed and some general properties of the strong interaction derived. We shall not do this since the mathematics required (group theory) is generally not part of an undergraduate education and, although the ideas are not difficult, it would take too much time. Suffice it to say that the construction of hadrons from quarks, with properties as given, forces SU(3) symmetry on the strong interaction.

5c. Operations on Quarks: Quark Model Basic Postulates. Since we have already postulated the quarks, we can list the operators of SU(3) in terms of how they transform the quarks. The operators are denoted by λ_i:

	operator		causes:
$\lambda_1\mathrm{u} = 0$	$\lambda_1\mathrm{d} = \mathrm{u}$	$\lambda_1\mathrm{s} = 0$	$\mathrm{d} \to \mathrm{u}$
$\lambda_2\mathrm{u} = \mathrm{d}$	$\lambda_2\mathrm{d} = 0$	$\lambda_2\mathrm{s} = 0$	$\mathrm{u} \to \mathrm{d}$
$\lambda_3\mathrm{u} = \mathrm{u}$	$\lambda_3\mathrm{d} = \mathrm{d}$	$\lambda_3\mathrm{s} = 0$	
$\lambda_4\mathrm{u} = 0$	$\lambda_4\mathrm{d} = 0$	$\lambda_4\mathrm{s} = \mathrm{u}$	$\mathrm{s} \to \mathrm{u}$
$\lambda_5\mathrm{u} = \mathrm{s}$	$\lambda_5\mathrm{d} = 0$	$\lambda_5\mathrm{s} = 0$	$\mathrm{u} \to \mathrm{s}$
$\lambda_6\mathrm{u} = 0$	$\lambda_6\mathrm{d} = 0$	$\lambda_6\mathrm{s} = \mathrm{d}$	$\mathrm{s} \to \mathrm{d}$
$\lambda_7\mathrm{u} = 0$	$\lambda_7\mathrm{d} = \mathrm{s}$	$\lambda_7\mathrm{s} = 0$	$\mathrm{d} \to \mathrm{s}$
$\lambda_8\mathrm{u} = \mathrm{u}/3$	$\lambda_8\mathrm{d} = \mathrm{d}/3$	$\lambda_8\mathrm{s} = -\mathrm{s}/3$	

One of the operators turns a u into a d quark, another turns a u into an s, etc., so that each quark is turned into each of the others by one of the operators. In addition, λ_3 and λ_8 are special in that they do not change the character of the quark. λ_3 produces the same quark state but multiplied by twice its isotopic spin. λ_8 produces the same quark state but multiplied by its hypercharge.

In the language of quantum mechanics the quark states are chosen to be eigenstates of the operators corresponding to the 3-component of isotopic spin and hypercharge. Another way of saying the same thing is that each quark has a definite value for T_3 and Y, and these values are constants of its motion; a u quark, for example, always has $T_3 = 1/2$ and $Y = 1/2$. The basic postulates of the quark model are:

a. the fundamental interaction which produces the hadrons is invariant under the SU(3) operators,

b. a meson is composed of a quark and an antiquark,

c. a baryon is composed of three quarks.

These postulates give rise to the grouping of hadrons into singlets, octets, and decimets. Operating on one of the particles of an octet, for example, turns it into one or more of the other particles in the octet. SU(3) provides the rationale for grouping the hadrons in to supermultiplets, as the octets, decimets and singlets are collectively called.

5d. SU(3) Operators, States, Insufficiency. The eight operators are not arbitrarily chosen. Every operator which operates in a space which is specified by 3 basis states can be written as a linear combination of these 8, augmented by the identity operator ($\lambda_0\mathrm{u} = \mathrm{u}$, $\lambda_0\mathrm{d} = \mathrm{d}$, $\lambda_0\mathrm{s} = \mathrm{s}$). Several conclusions can be drawn from this statement. First, the 8 operators ($\lambda_1, \ldots, \lambda_8$) are not unique. One can form many other sets of 8 independent operators but these will always be linear combinations of those we have written down. Different authors, in fact, use different sets but all sets will lead to the same physical conclusions. Second, if strong interaction physics can be described in terms of what happens to 3 independent states (i.e. 3 quarks) then the theory can be written in terms of the 8 SU(3) operators and the identity operator. There is nothing that can be done to a quark which is not describable by some combination of these operators. There is however evidence that things do happen in nature which are not describable by the SU(3) operators and physicists no are forced to postulate a fourth quark and deal with the operators of su(4). You should also realize that the SU(3) operators deal with isotopic spin and hypercharge. There are two other important quantities, namely spin and baryon number, which are used to describe quarks and which are outside the domain of SU(3).

6. Su(3) and Interactions

6a. Interaction Invariance under Selected SU(3) Operations. The postulates and mathematical reasoning behind the quark model and SU(3) symmetry seem to be invalid physically. If the strong interaction is invariant under the SU(3) operators, then all the hadrons of a given octet or decimet should have the same mass. This follows because the operators change one quark into another, or what is the same thing, change one member particle of a supermultiplet (as the singlets, octets, and decimets are called) into another member of the same supermultiplet without changing any of the interactions. Since the interactions are presumably responsible for the masses, all the particles of a supermultiplet must have the same mass.

If the quark model is to be valid, it must be that the interaction responsible for the particles of the supermultiplet is not the strong interaction but some other interaction which is invariant under the operations of SU(3).

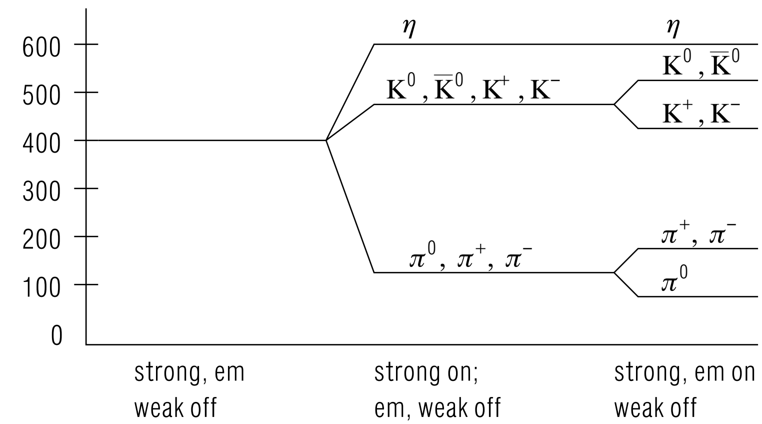
It is presumed that if all interactions would be turned off except this interaction, all particles in a supermultiplet would be identical. Particles in different supermultiplets would still be different (have different spins and different masses) because of internal quark dynamics. For example, the K^0, $\overline{\mathrm{K}}^0$, K^+, K^-, π^+, π^-, π^0 and mesons would be experimentally indistinguishable from each other as would the K^{*0}, K^{-*0}, K^{*+}, K^{*-}, ρ^+, ρ^-, ρ^0 and ω mesons but the particles of the second octet would have mass and spin which would be different from the mass and spin respectively of the first octet.

The complete strong interaction is not invariant under all the SU(3) operators. Since the complete strong interaction conserves isotopic spin and hypercharge, it must be invariant under λ_1, λ_2, λ_3, and λ_8, the operators associated with isotopic spin and hypercharge. It is not invariant under λ_4, λ_5, λ_6, or λ_7, may now have different mass.

Looking at the chart of the operators (Sect. 5c), we see that particles which have the same quark content, except for the interchange of a u and d quark, will have the same mass and still be indistinguishable when the strong interaction is turned on. Particles which differ by more than this interchange will generally have different masses and this mass difference is associated with the strong interaction. In more detail, the strong interaction is not invariant under the operators λ_4, λ_5, λ_6, λ_7 of SU(3). These operators interchange u and s quarks or d and s quarks. We conclude that particles which differ in quark content by the substitutions $\mathrm{u} \to \mathrm{s}$, $\mathrm{d} \to \mathrm{s}$, $\mathrm{s} \to \mathrm{u}$, or $\mathrm{s} \to \mathrm{d}$ differ in mass by virtue of the energy associated with the strong interaction. Particles which differ by the substitutions $\mathrm{u} \to \mathrm{d}$ or $\mathrm{d} \to \mathrm{u}$ do not differ in mass by virtue of the strong interaction. For example, K^0, $\overline{\mathrm{K}}^0$, K^+, K^- all have the same mass and π^+, π^-, π^0 all have the same mass but these two masses are different and differ from that of the η.

When the electromagnetic interaction is turned on, invariance with respect to isotopic spin is violated and a mass difference between the K^+ and K^0 is generated. Similarly, a mass difference between the π^+ and π^0 appears. The π^+ and π^- have the same mass because one is formed from the other by charge conjugation, and both the electromagnetic and strong interactions are invariant under this operation.

Presumably, the weak interaction produces a mass difference between particle and antiparticle, if charge conjugation invariance is violated. Mass effects of the weak interaction are too small to be observed at this time. The influence on mass of the various parts of the total interaction can be diagrammed as follows (for the lowest mass meson octet):

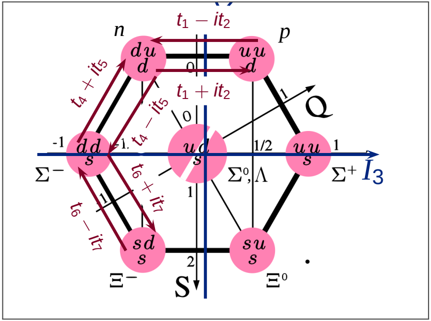


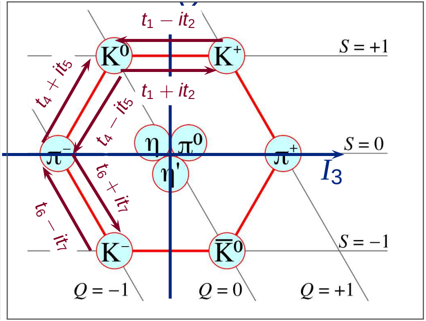
The mass of the original particle, plotted here at 400 MeV, is of course unknown since the interactions can not be turned off in practice.

6b. Comparison to Atomic Magnetic Splittings. The situation here is very similar to the magnetic states of an electron in an atom. For an electron with a specified principal quantum number corresponding to the values of m_ℓ are degenerate; they all have the same energy. When a magnetic field is turned on, the degeneracy is lifted and states with different m_ℓ have different energy. The splitting is given by $\Delta E = (e/mc)Bm_\ell$. For the spin 0 mesons, the original particle (with all interactions turned off) can be considered a quantum mechanical energy level that is 8-fold degenerate. The strong interaction splits the degeneracy to form 3 states, two of which are still degenerate (one 4-fold and one 3-fold) and the electromagnetic interaction further splits the degeneracy.

SU(3) multiplets

Hadrons can indeed be organized into representations of SU(3):





These are not the simplest representation

$\Longrightarrow$ but effect of ladder operators is the same.

But different hypercharge/strangeness states do not have the same masses.

We do not go into representations of SU(3) here, subject for a full group theory course.

The Group SU(3)

- **S**pecial **u**nitary group in **3** dimensions
- $3^2 - 1 = 8$ traceless and hermitian generators
- fundamental representation consisting of 3x3 matrices acting on triplet states
- standard choice for generators $F_i = \frac{1}{2}\lambda_i$ with Gell-Mann matrices λ_i:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \ldots \; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The Group SU(3)

- $\lambda_1, \lambda_2, \lambda_3$ correspond to the Pauli matrices $\Rightarrow$ SU(2) subgroup of SU(3)
- λ_3 and λ_8 are diagonal with simultaneous eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

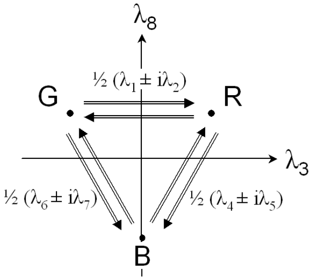
- Structure constants f_{ijk} define the SU(3) algebra:

$$[F_i, F_j] = if_{ijk}F_k$$

Color SU(3)

- Eigenvectors connected to 3 color charges of a quark:

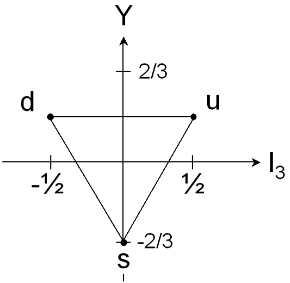
$$R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



- Quarks interact via octet of vector bosons: the gluons

Flavour SU(3)

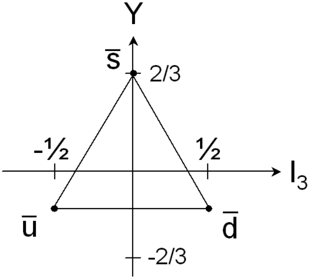
- in 1960's experimental evidence for a second additive quantum number called "strangeness"
- Isospin $I_3 \rightarrow$ SU(2); together with strangeness S $\rightarrow$ SU(3)
- Triplet $|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



- Hypercharge $Y \equiv B + S$ (baryon number + strangeness) centers multiplet at the origin
- Electric charge $Q = I_3 + \frac{Y}{2}$

Flavour SU(3)

- Analog antiquark multiplet:



- but Flavour SU(3) symmetry explicitly broken
  $\Rightarrow$ different masses of u, d, s quarks
- nevertheless very useful symmetry

Mass relations

Project in the course Group Theory
Henrik Jäderström

SU(3) flavour

Assumptions: 1) Strong force is flavour independent
2) u,d,s quarks have the same mass

Hamiltonian of the strong force is invariant under SU(3) transformations of the quarks

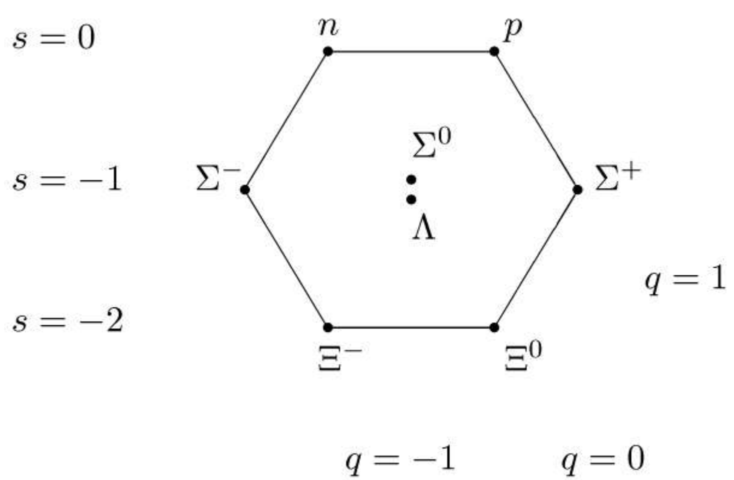
Isospin I and hypercharge are conserved and form a SU(2) $\times$ U(1) subgroup

$[I_a, I_b] = i\varepsilon_{abc} I_c$

$[I_a, Y] = 0$

3 x 3 x 3 = 10 + 8 + 8 +1

The baryon octet



Baryon	Y	I	Mass (MeV)	Mass formula
p	1	$\frac{1}{2}$	938	$m_0 + \delta m_1 + \frac{1}{2}\delta m_2$
n	1	$\frac{1}{2}$	939	$m_0 + \delta m_1 + \frac{1}{2}\delta m_2$
Λ	0	0	1116	m_0
Σ^+	0	1	1189	$m_0 + 2\delta m_2$
Σ^0	0	1	1193	$m_0 + 2\delta m_2$
Σ^-	0	1	1197	$m_0 + 2\delta m_2$
Ξ^0	-1	$\frac{1}{2}$	1315	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$
Ξ^-	-1	$\frac{1}{2}$	1321	$m_0 - \delta m_1 + \frac{1}{2}\delta m_2$

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda + m_\Sigma}{4}$$

1129 MeV 1135 MeV

Consider the DECUPLET representation. All of the particles had been previously observed except for the Ω^-. , which we will discuss further later. We end up with two OCTET representations. The first is antisymmetric in the second and third quarks and the second is symmetric in the second and third quarks. They are completely equivalent representations within SU(3) and represent the same physical particles.

The Mass Formula

We will simply state and discuss the mass formula here. The most general form for the mass formula is given by:

$$M = A + BY + C\left(I(I+1) - \frac{1}{4}Y^2\right) + DI_3$$

where the A, B, and C terms represent the average mass (within a row) and the D term represents electromagnetic splittings. Some OCTET experimental data implies that:

$$M_8 = 1109.80 - 189.83Y + 41.49\left(I(I+1) - \frac{1}{4}Y^2\right) - 2.45I_3$$

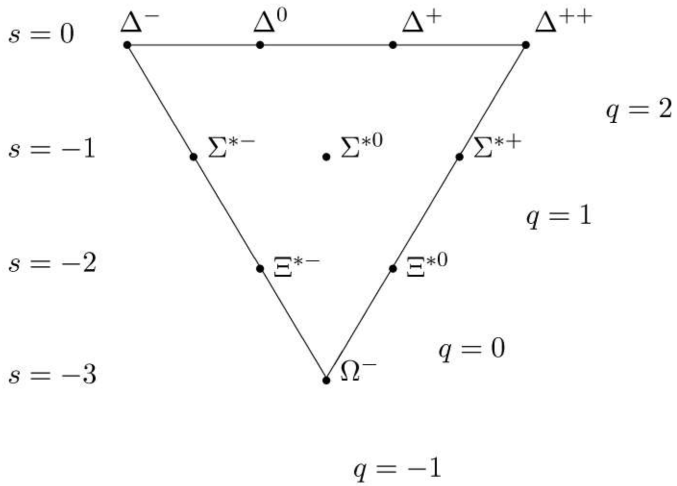
which gives the following tables :

particle	experiment	formula
n	939.5	943.2
p	938.2	938.3
Λ^0	1115.4	1109.8
Σ^-	1189.4	1190.3
Σ^0	1193.2	1192.8
Σ^+	1197.6	1195.2
Ξ^-	1316	1317.9
Ξ^0	1321	1322.8

This is an average error of only 0.19% The experimental data in the table below was also available. As we said above, the Ω^- particle had not yet been discovered.

spin 3/2

The Baryon decuplet



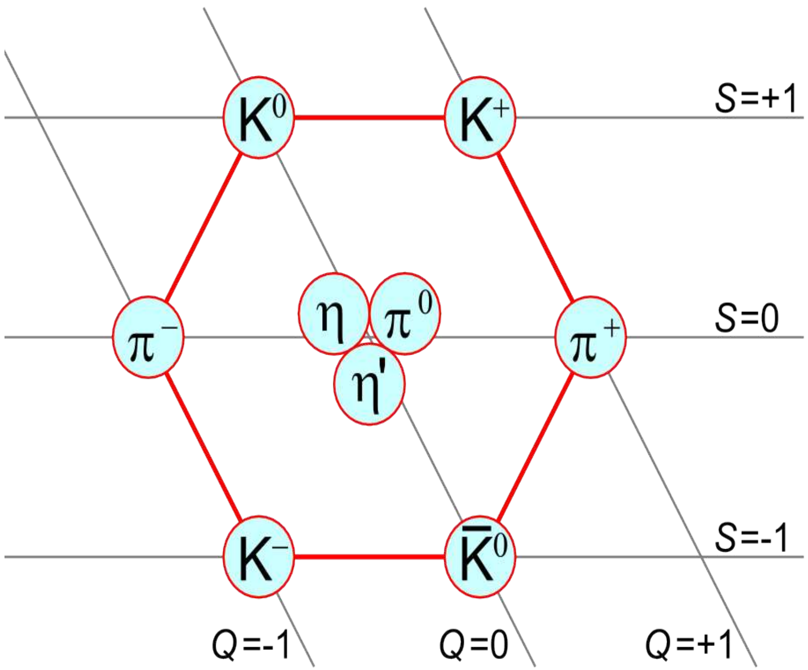
Baryon	Y	I	Mass (MeV)	Mass formula
Δ^{++}	1	$\frac{3}{2}$	1232	$m_0+\delta m_1+\frac{7}{2}\delta m_2$
Δ^{+}	1	$\frac{3}{2}$	1232	$m_0+\delta m_1+\frac{7}{2}\delta m_2$
Δ^{0}	1	$\frac{3}{2}$	1232	$m_0+\delta m_1+\frac{7}{2}\delta m_2$
Δ^{-}	1	$\frac{3}{2}$	1232	$m_0+\delta m_1+\frac{7}{2}\delta m_2$
Σ^{*+}	0	1	1383	$m_0+2\delta m_2$
Σ^{*0}	0	1	1384	$m_0+2\delta m_2$
Σ^{*-}	0	1	1387	$m_0+2\delta m_2$
Ξ^{*0}	-1	$\frac{1}{2}$	1532	$m_0-\delta m_1+\frac{1}{2}\delta m_2$
Ξ^{*-}	-1	$\frac{1}{2}$	1535	$m_0-\delta m_1+\frac{1}{2}\delta m_2$
Ω^{-}	-2	0	1672	$m_0-2\delta m_1-\delta m_2$

Difference in mass between the different hypercharges about 150 MeV

Gell-Mann used this to predict the mass, Isospin and Hypercharge of Ω

3 x 3 = 8 + 1

The meson octet and singlet



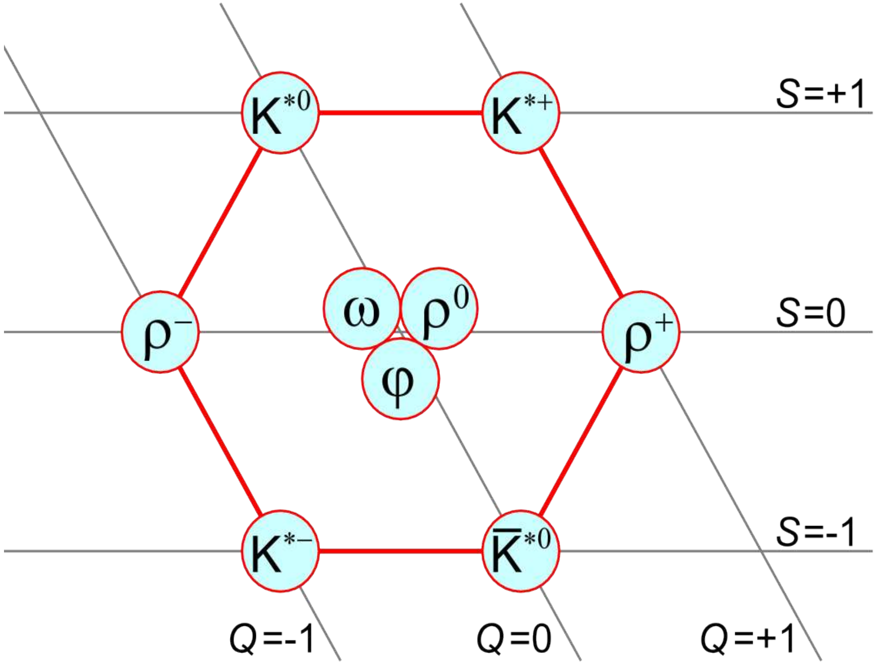
Meson	Y	I	Mass (MeV)	Mass formula
π^+	0	1	140	$m_8 + 2\delta m$
π^0	0	1	140	$m_8 + 2\delta m$
π^-	0	1	135	$m_8 + 2\delta m$
K^+	1	$\frac{1}{2}$	494	$m_8 + \frac{1}{2}\delta m$
K^0	1	$\frac{1}{2}$	498	$m_8 + \frac{1}{2}\delta m$
$\bar{K}^0$	-1	$\frac{1}{2}$	498	$m_8 + \frac{1}{2}\delta m$
K^-	-1	$\frac{1}{2}$	494	$m_8 + \frac{1}{2}\delta m$
η	0	0	548	–
η'	0	0	956	–

η and η' are a mix of the members of the octet and singlet

$$|\eta\rangle = \cos(\theta)|\eta_8\rangle + \sin(\theta)|\eta_1\rangle$$
$$|\eta'\rangle = -\sin(\theta)|\eta_8\rangle + \cos(\theta)|\eta_1\rangle$$

with mixing angle $\theta = 24^\circ$

Meson nonet with spin 1



Meson	Y	I	Mass (MeV)	Mass formula
ρ^+	0	1	776	$m_8 + 2\delta m$
ρ^0	0	1	776	$m_8 + 2\delta m$
ρ^-	0	1	776	$m_8 + 2\delta m$
K^{*+}	1	$\frac{1}{2}$	896	$m_8 + \frac{1}{2}\delta m$
K^{*0}	1	$\frac{1}{2}$	840	$m_8 + \frac{1}{2}\delta m$
$\bar{K}^{*0}$	-1	$\frac{1}{2}$	840	$m_8 + \frac{1}{2}\delta m$
K^{*-}	-1	$\frac{1}{2}$	896	$m_8 + \frac{1}{2}\delta m$
ω	0	0	783	$-$
ϕ'	0	0	1019	$-$

The mixing angle of the ω,ϕ is 53°

Conclusions

- Group theory can be used to derive the mass splititing within baryon octet and decuplet and meson nonet.
- The mixing angles of η,η' and ω,ϕ can be calculated
- Gell-Mann used this to predict the existence of the last member of the baryon decuplet

antiparallel spins and zero orbital angular momentum. This is the lowest mass set of particles. Spin 0 particles also result if the quark spins are parallel and the quarks have 1 unit of orbital angular momentum directed opposite to the spins. These particles evidently have masses on the order of 800 MeV larger than the first set of particles.

Spin 1 mesons result from any one of the following combinations:

a. Quark spins parallel and zero orbital angular momentum. This is evidently the lowest mass set of spin 1 mesons.

b. Quark spins antiparallel and 1 unit of orbital angular momentum.

c. Quark spins parallel and 2 units of orbital angular momentum directed opposite to the spins.

4e. Large Masses as Excited States. Quantum mechanically, one can consider the mesons of a large mass set (the spin 1 mesons, for example) to be excited states of mesons in the lowest mass set. All the K^-'s, for example, have the same properties except spin and mass and can be thought of as having the same quark content.

4f. Baryon Supermultiplets and Quarks. The baryons are constructed from 3 quarks, antibaryons from 3 antiquarks. Note that this prescription automatically satisfies the rules for assignment of baryon number and automatically makes the baryons fermions: they must have spin of half a positive odd integer. There are 10 ways to combine 3 quarks 3 at a time. They are:

	Y	T_3	T
uuu	1	3/2	3/2
uud	1	1/2	3/2 or 1/2
udd	1	−1/2	3/2 or 1/2
ddd	1	−3/2	3/2
uus	0	1	1
uds	0	0	1 or 0
dds	0	−1	1
uss	−1	1/2	1/2
dss	−1	−1/2	1/2
sss	−2	0	0

These form into groups of eight spin 1/2 baryons and ten spin 3/2 baryons.

The baryon octet:

quarks:	T	particle:
uud	1/2	p
udd	1/2	n
uds	0	Λ^0
uus	1	Σ^+
uds	1	Σ^0
dds	1	Σ^-
uss	1/2	Ξ^0
dss	1/2	Ξ^-

The baryon decimet:

quarks:	T	particle:
uuu	3/2	Δ^{++}
uud	3/2	Δ^+
udd	3/2	Δ^0
ddd	3/2	Δ^-
uus	1	Σ^{*+}
uds	1	Σ^{*0}
dds	1	Σ^{*-}
uss	1/2	Ξ^{*0}
dss	1/2	Ξ^{*-}
sss	0	Ω^-

Note that uud and udd can each form two different states, one with $T = 3/2$ and one with $T = 1/2$. The $T = 1/2$ states (uud with $T_3 = 1/2$ and udd with $T_3 = -1/2$) occur in the spin 1/2 octet while the $T = 3/2$ states (uud with $T_3 = 1/2$ and udd with $T_3 = -1/2$) are augmented with the other two states in the isospin multiplet (uuu with $T = 3/2$ and ddd with $T = 3/2$ and $T_3 = -3/2$) and occur in the spin 3/2 decimet.

Similarly the combination uds can form two different states, one with $T = 1$, $T_3 = 0$ and one with $T = 0$, $T_3 = 0$. The $T = 1$ state occurs in both groups while the $T = 0$ state occurs only in the octet.

5. Su(3) Operators

5a. The Quark-Quark Interaction and SU(3). There is more to the quark model than just the construction of particles from quarks. The chief idea behind the model is that the interaction which gives rise to the particles has a high degree of symmetry. The symmetry we are talking about is very much analogous to the ideas of invariance under parity or time reversal. That is, there is a group of operators (eight in number) which do not change the interaction when they operate on it (similar in nature to the fact that the electromagnetic interaction does not change when operated on by the parity operator i.e. when ($\vec{r}$) is replaced by (-$\vec{r}$) and ($\vec{p}$) by (-$\vec{p}$). The eight operators are collectively known as the SU(3) indicates that the basis of the group consists of 3 independent states (the 3 quarks).

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May 2, 2010, *5:00 pm*

Group Think

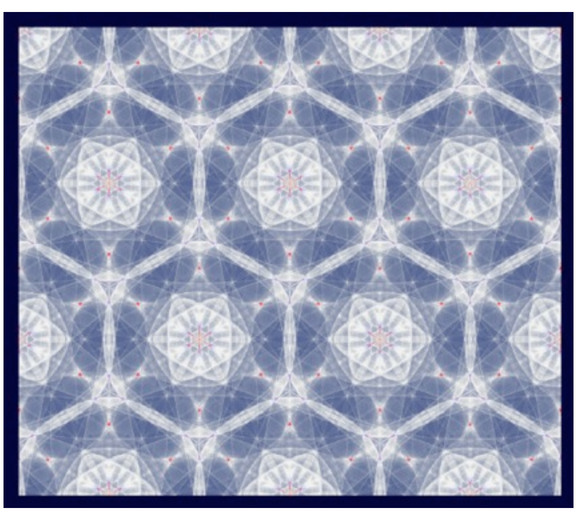
By STEVEN STROGATZ

My wife and I have different sleeping styles — and our mattress shows it. She hoards the pillows, thrashes around all night long, and barely dents the mattress, while I lie on my back, mummy-like, molding a cavernous depression into my side of the bed.

Bed manufacturers recommend flipping your mattress periodically, probably with people like me in mind. But what's the best system? How exactly are you supposed to flip it to get the most even wear out of it?

Brian Hayes explores this problem in the title essay of his recent book, "Group Theory in the Bedroom." Double entendres aside, the "group" in question here is a collection of mathematical actions — all the possible ways you could flip, rotate or overturn the mattress so that it still fits neatly on the bed frame.

By looking into mattress math in some detail, I hope to give you a feeling for group theory more generally. It's one of the most versatile parts of mathematics. It underlies everything from the choreography of contra dancing and the fundamental laws of particle physics, to the mosaics of the Alhambra and their chaotic counterparts like this image.



Michael Field

As these examples suggest, group theory bridges the arts and sciences. It addresses something the two cultures share — an abiding fascination with symmetry. Yet because it encompasses such a wide range of phenomena, group theory is necessarily abstract. It distills symmetry to its essence.

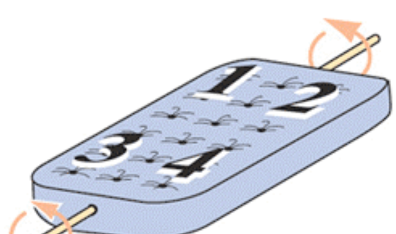
Normally we think of symmetry as a property of a shape. But group theorists focus more on what you can *do* to a shape — specifically, all the ways you can change it while keeping something else about it the same. More precisely, they look for all the transformations that leave a shape unchanged, given certain constraints. These transformations are called the "symmetries" of the shape. Taken together they form a "group," a collection of transformations whose relationships define the shape's most basic architecture.

In the case of a mattress, the transformations alter its orientation in space (that's what changes) while maintaining its rigidity (that's the constraint). And after the smoke clears, the mattress has to fit snugly on the rectangular bed frame (that's what stays the same). With these rules in place, let's see what transformations qualify for membership in this exclusive little group. It turns out there are only four of them.

The first is the "do-nothing" transformation, a lazy but popular choice that leaves the mattress untouched. It certainly satisfies all the rules, but it's not much help in prolonging the life of your mattress. Still, it's very important to include in the group. It plays the same role for group theory that zero does for addition of numbers, or that 1 does for multiplication. Mathematicians call it the "identity element," so I'll denote it by the symbol I.

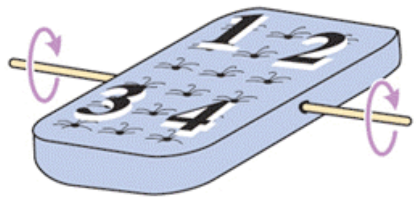
Next come the three genuine ways to flip a mattress. To distinguish among them, it helps to label the corners of the mattress by numbering them like so:

The first kind of flip is depicted near the beginning of this post. The handsome gentleman in striped pajamas is trying to turn the mattress from side to side by rotating it 180 degrees around its long axis, in a move I'll call H, for "horizontal flip."



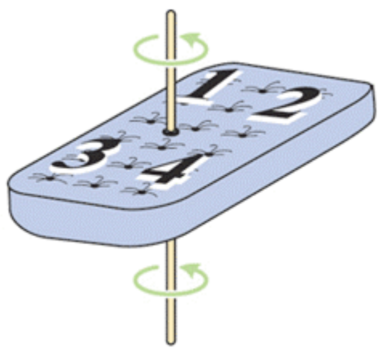
Horizontal Flip

A more reckless way of overturning the mattress is a "vertical flip" V. This maneuver swaps its head and foot. You stand the mattress upright, the long way, so that it almost reaches the ceiling, and then topple it end over end. The net effect, besides the enormous thud, is to rotate the mattress 180 degrees about the axis shown below.



Vertical Flip

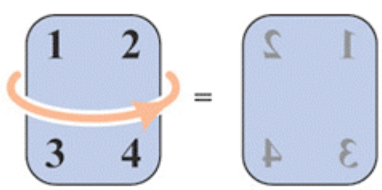
The final possibility is to spin the mattress half a turn while keeping it flat on the bed.



Rotation

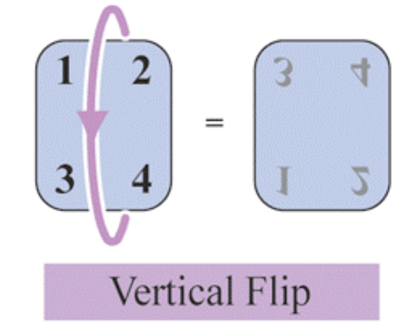
Unlike the *H* and *V* flips, this “rotation” *R* keeps the top surface on top. That difference shows up when we look at a top view of the mattress — now imagined to be translucent — and inspect the numbers at the corners after each of the possible transformations.

The horizontal flip turns the numerals into their mirror images. It also permutes them so that 1 and 2 trade places, as do 3 and 4.

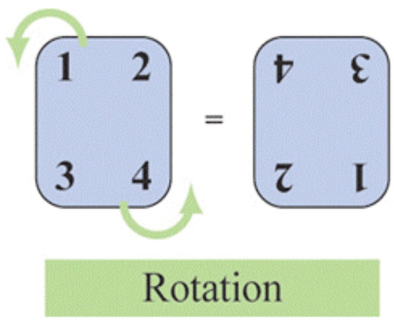


Horizontal Flip

The vertical flip permutes the numbers in a different way and stands them on their heads, besides mirroring them.

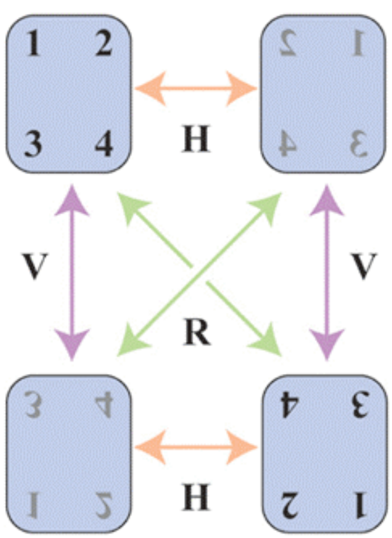


The rotation, however, doesn't generate any mirror images. It merely turns the numbers upside down, this time exchanging 1 for 4 and 2 for 3.



These details are not the main point. What matters is how the transformations relate to one another. Their patterns of interaction encode the symmetry of the mattress.

To reveal those patterns with a minimum of effort, it helps to draw the following diagram. (Images like this abound in a terrific new book called "Visual Group Theory," by Nathan Carter. It's one of the best introductions to group theory — or to any branch of higher math — I've ever read.)



The four possible "states" of the mattress are shown at the corners of the diagram. The upper left state is the starting point. The colored arrows indicate the moves that take the mattress from one state to another.

For example, the green arrow pointing from the upper left to the lower right depicts the action of the rotation R. The same green line also has an arrowhead on the other end, because if you do R twice, it's tantamount to doing nothing.

That shouldn't come as a surprise. It just means that turning the mattress head to foot and then doing that again returns the mattress to its original state. We can summarize this property with the equation $RR = I$, where RR means do R twice, and I is the do-nothing identity element. For that matter, the horizontal and vertical flip transformations also undo themselves: $HH = I$ and $VV = I$.

The diagram embodies a wealth of other information. For instance, it shows that the death-defying vertical flip V is equivalent to HR, a horizontal flip followed by a rotation — a much safer path to the same result. To check this, begin at the starting state in the upper left. Head due east along H to the next state, and from there go diagonally southwest along R. Because you arrive at the same state as if you'd simply followed V to begin with, the diagram demonstrates that $HR = V$.

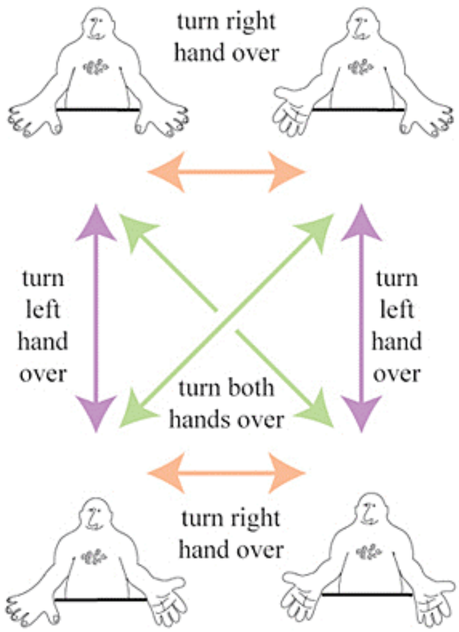
Notice, too, that the order of those actions is irrelevant: $HR = RH$, since both roads lead to V. This indifference to order is true for any other pair of actions. You should think of this as a generalization of the commutative law for addition of ordinary numbers, x and y, according to which $x + y = y + x$. But beware: the mattress group is special. Many other groups violate the commutative "law." Those fortunate enough to obey it are particularly clean and simple.

Now for the payoff. The diagram shows how to get the most even wear out of a mattress. Any strategy that samples all four states periodically will work. For example, alternating R and H is convenient — and since it bypasses V, it's not too strenuous. To help you remember it, some manufacturers suggest the mnemonic "spin in the spring, flip in the fall."

The mattress group also pops up in some unexpected places, from the symmetry of water molecules to the logic of a pair of electrical switches. That's one of the charms of group theory. It exposes the hidden unity of things that would otherwise seem unrelated … like this anecdote about how the physicist Richard Feynman got a draft deferment.

The army psychiatrist questioning him asked Feynman to put out his hands so he could examine them. Feynman stuck them out, one palm up, the other down. "No, the other way," said the psychiatrist. So Feynman reversed *both* hands, leaving one palm down and the other up.

Feynman wasn't merely playing mind games; he was indulging in a little group-theoretic humor. If we consider all the possible ways he could have held out his hands, along with the various transitions among them, the arrows form the same pattern as the mattress group!



But if all this makes mattresses seem way too complicated, maybe the real lesson here is one you already knew — if something's bothering you, just sleep on it.

NOTES

1. *Two recent books inspired this piece:*
   *N. Carter, **"Visual Group Theory"** (Mathematical Association of America, 2009).*
   *B. Hayes, "**Group Theory in the Bedroom, And Other Mathematical Diversions"** (Hill and Wang, 2008).*
   Carter introduces the basics of group theory gently and pictorially. He also touches on its connections to Rubik's cube, contra dancing and square dancing, crystals, chemistry, art and architecture.
   An earlier version of Hayes's mattress-flipping article appeared in American Scientist in the issue of September/October 2005.
2. *The mattress group is technically known as the "Klein four-group." It's one of the simplest in a gigantic zoo of possibilities. Mathematicians have been analyzing groups and classifying their structure for about 200 years. Among the earliest pioneers were two brilliant men who died tragically young: Évariste Galois, killed in a duel at age 20, and Niels Henrik Abel, dead from tuberculosis at age 26. The questions that concerned them were purely mathematical, having to do with the finding the roots of polynomials and proving the unsolvability of the quintic equation in terms of simple formulas involving radicals. For more about their stories, see:*
   *M. Livio, **"The Equation That Couldn't Be Solved"** (Simon and Schuster, 2005).*
   *A. Alexander, **"Duel at Dawn"** (Harvard University Press, 2010).*
   And for an engaging account of the quest to classify all "finite simple groups," see:
   *M. du Sautoy, **"Symmetry"** (Harper, 2008).*
3. *A word about some potentially confusing notation used above: in equations like $HR = V$, the H was written on the left to indicate that it's the transformation being performed first. Carter uses this notation for functional composition in his book, but the reader should be aware that many mathematicians use the opposite convention, placing the H on the right.*
4. *Readers interested in seeing a definition of what a "group" is should consult any of the authoritative online references or standard textbooks on the subject. The treatment I've given here emphasizes symmetry groups rather than groups in the most general sense.*
5. *Michael Field and Martin Golubitsky have studied the interplay between group theory and nonlinear dynamics. In the course of their investigations, they've generated stunning computer graphics of*

symmetric chaos. For the art, science and mathematics of this topic, see:
*M. Field and M. Golubitsky, **"Symmetry in Chaos,"** 2nd edition (Society for Industrial and Applied Mathematics, 2009).*

6. *For the anecdote about Feynman and the psychiatrist, see:*
   *R. P. Feynman, **" 'Surely You're Joking, Mr. Feynman!' "** (Norton, 1985), p. 158.*
   *J. Gleick, **"Genius"** (Random House, 1993), p. 223.*

Thanks to Mike Field and Marty Golubitsky for sharing their images of symmetric chaos; Margaret Nelson for preparing the illustrations; and Paul Ginsparg, Jon Kleinberg, Tim Novikoff, Diana Riesman and Carole Schiffman for their comments and suggestions.

Δ^{++} baryon, conflict with Pauli principle?

Consider the Δ^{++} baryon. We know experimentally that it is a *uuu* state has

- Spin $3/2$
- Isospin $|I, I_3\rangle = |3/2, 3/2\rangle$
- Electric charge $Q = +2$.
- Lightest baryon with these quantum numbers $\Longrightarrow L_{12} = L_3 = 0$

Only possible spin assignment $|u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle \Longrightarrow$ wavefunction completely symmetric in exchange of two identical $u\uparrow$-quarks.
Quarks are fermions, violate Pauli principle?
— No, $\exists$ component of wavefunction which is antisymmetric.
This is "color" of SU(3).

Two SU(3) groups

Mathematics the same, physics different.

- Flavor SU(3), $\sim$ symmetry between u, d, s, exact if $m_u = m_d = m_s$.
- Color SU(3), color states of quarks 3-component vectors; charge **operator** 3×3 matrix t_a.
  (Remember in QM states are vectors, observables, such as charges, operators i.e. matrices.)

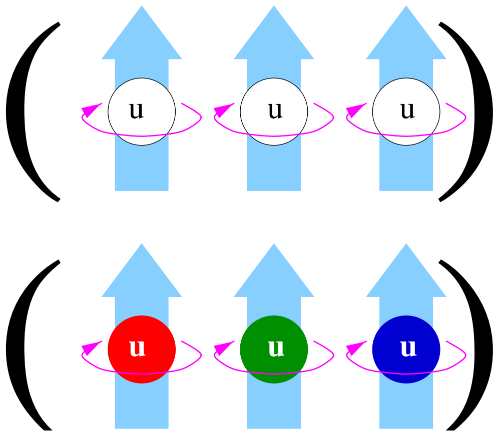


Figure 9.10: The Δ^{++} in the quark model.

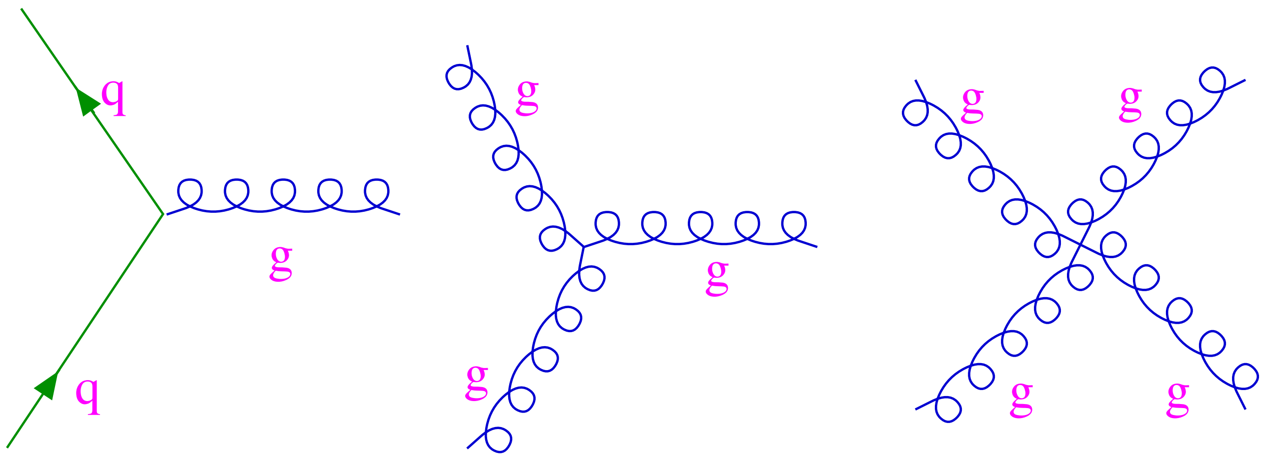


Figure 9.11: The basic building blocks for QCD feynman diagrams

quarks. These are all "hadrons", mesons and baryons, since they must couple through the strong interaction. By determining the energy in each if the two jets we can discover the energy of the initial quarks, and see whether QCD makes sense.

Antisymmetric color state

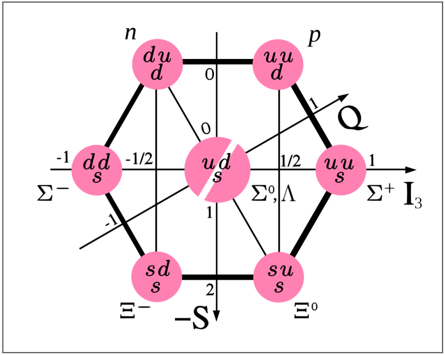
$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = r \text{ "red"} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = g \text{ "green"} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b \text{ "blue"}$$

Many quark states are tensor products, e.g. $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3 \equiv rgb$
$t_a^{\text{tot}} = t_a \otimes 1 \otimes 1 + 1 \otimes t_a \otimes +1 \otimes 1 \otimes t_a$ (I.e. total color = sum of quark colors)
rgb have eigenvalues $1/2, -1/2, 0$ and $1/(2\sqrt{3}), 1/(2\sqrt{3}), -1/\sqrt{3}$ for t_3, t_8.
Requirement of color neutrality: $t_a^{\text{tot}}|\psi\rangle = 0$ for all a ;
Total t_3, t_8 value 0 for 3 quarks $\Longrightarrow$ one quark in each color state:
$|\psi\rangle = \alpha_1 rgb + \alpha_2 gbr + \alpha_3 brg + \alpha_4 grb + \alpha_5 rbg + \alpha_6 bgr$
Also other t_a's vanish: $\Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = -\alpha_4 = -\alpha_5 = -\alpha_6$

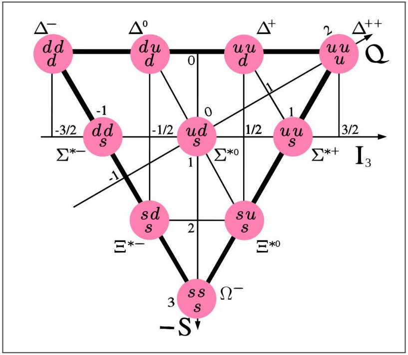
Color neutral 3-quark state is totally antisymmetric

- Thus combined orbital, spin part is **symmetric** under exchange of identical quarks
- Note: for N_c colors you need N_c quarks to form a color neutral state. Observation that baryons have 3 quarks $\Longrightarrow$ another indication that number of colors is 3.

Lightest baryon states



Spin 1/2 baryon octet



spin 3/2 baryon decuplet

Antiquarks in the antifundamental representation

For flavor SU(2) = isospin we had both u, d quarks and $\bar{u}, \bar{d}$ antiquarks in SU(2) doublets; 2 representation.
For SU(3) quarks are in fundamental representation 3, antiquarks in antifundamental $\bar{3}$ and **these are not the same**
$\Longrightarrow$ You can form color neutral state from 1 quark and 1 antiquark
$\Longrightarrow$ meson.

Allowed color neutral states

- Baryon q^3
- Meson $q\bar{q}$
- In general $q^{3n}(q\bar{q})^m \Longrightarrow$ possible for color neutrality; not observed.

Other combinations: qq, $qq\bar{q}$ ... not possible to form color neutral state.

Quarks

Strong, electromagnetic and weak interactions

Like leptons, quarks organized into 3 doublets and their antiparticles

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \qquad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \begin{pmatrix} \bar{c} \\ \bar{s} \end{pmatrix} \begin{pmatrix} \bar{t} \\ \bar{b} \end{pmatrix}$$

Electric charges

- up-type quarks u, c, t: $Q = +2/3$
- down-type quarks d, s, b: $Q = -1/3$

Type of quark (u, d, c, s, t, b) is called **flavor**

u up

d down

c charm, quantum number “charm”: $C_c = 1, \quad C_{\bar{c}} = -1$

s strange, quantum number “strangeness”: $S_s = -1, \quad S_{\bar{s}} = 1$ **sign!**

t top/truth, quantum number “truth” ? $T_t = 1 \quad T_{\bar{t}} = -1$

b bottom/beauty, quantum number “beauty” ? $\widetilde{B}_b = -1 \quad \widetilde{B}_{\bar{b}} = 1$ **sign!**

Conservation laws

Follow from symmetries

	Conserved quantity	Strong	Electromag.	Weak	
Spacetime	Energy/momentum		yes		more in part 5
	Parity P	yes	yes	no	more in part 5
	Charge conjugation* C	yes	yes	no	more in part 5
	$CP = T$	yes	yes	almost	
	CPT		always		
Internal	Electric charge		yes		
	Baryon number		yes		
	Lepton number		yes		
	Isospin	yes	no	no	more in part 6
	Strangeness, charm ...	yes	yes	no	
	e, μ, τ number ...	yes	yes	no†	

Weak interaction breaks many symmetries that are still useful for understanding strong interaction.

Process breaks strong interaction conservation law $\Longrightarrow$ happens through electroweak interaction $\Longrightarrow$ very slow.

(* Existence of antiparticles and thus C are classified here as a property of “spacetime”.)

(† Conserved in proper standard model, now understood to be broken.)

In the years 1950 - 1960 many elementary particles were discovered and one started to speak of the particle zoo. A quote: "The finder of a new particle used to be awarded the Nobel prize, but such a discovery now ought to be punished by a $10.000 fine."

1.3 The Eightfold Way

In the early 60's Murray Gell-Mann (at the same time also Yuvan Ne'eman) observed patterns of symmetry in the discovered mesons and baryons. He plotted the spin 1/2 baryons in a so-called octet (the "eightfold way" after the eighfold way to Nirvana in Buddhism). There is a similarity between Mendeleev's periodic table of elements and the supermultiplets of particles of Gell Mann. Both pointed out a deeper structure of matter. The eightfold way of the lightest baryons and mesons is displayed in Fig. 1.5 and Fig. 1.6. In these graphs the Strangeness quantum number is plotted vertically.

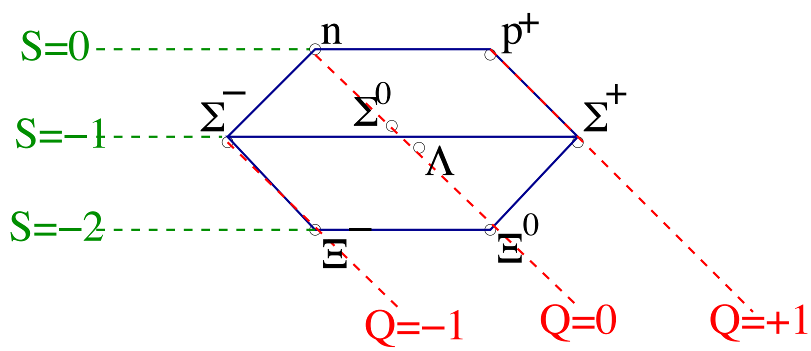


Figure 1.5: Octet of lightest baryons with spin=1/2.

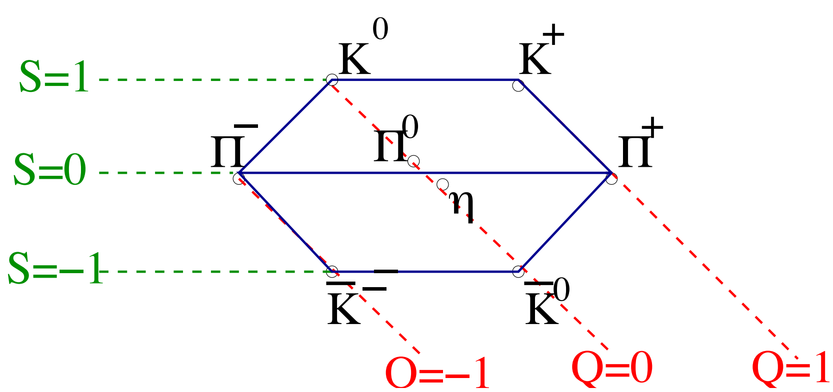


Figure 1.6: Octet with lightest mesons of spin=0

Also heavier hadrons could be given a place in *multiplets*. The baryons with spin=3/2 were seen to form a decuplet, see Fig. 1.7. The particle at the bottom (at S=-3) had not been observed. Not only was it found later on, but also its predicted mass was found to be correct! The discovery of the Ω^- particle is shown in Fig. 1.8.

Division	Eightfold Path factors
Wisdom (Sanskrit: *prajñā*, Pāli: *paññā*)	1. Right view
	2. Right intention
Ethical conduct (Sanskrit: *śīla*, Pāli: *sīla*)	3. Right speech
	4. Right action
	5. Right livelihood
Concentration (Sanskrit and Pāli: *samādhi*)	6. Right effort
	7. Right mindfulness
	8. Right concentration

The Blessed One said: "Now what, monks, is noble right concentration with its supports and requisite conditions? Any singleness of mind equipped with these seven factors—right view, right resolve, right speech, right action, right livelihood, right effort, and right mindfulness—is called noble right concentration with its supports and requisite conditions.

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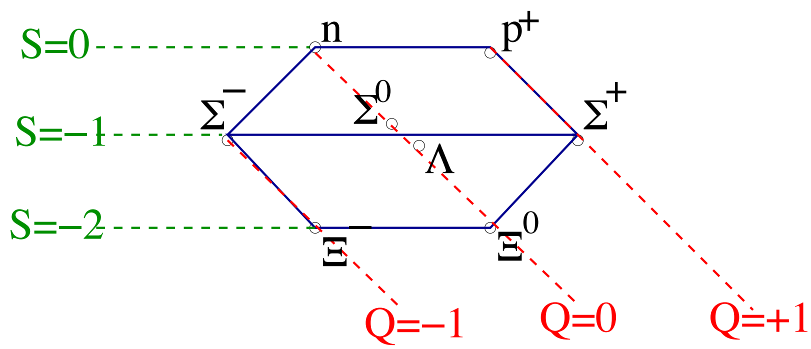

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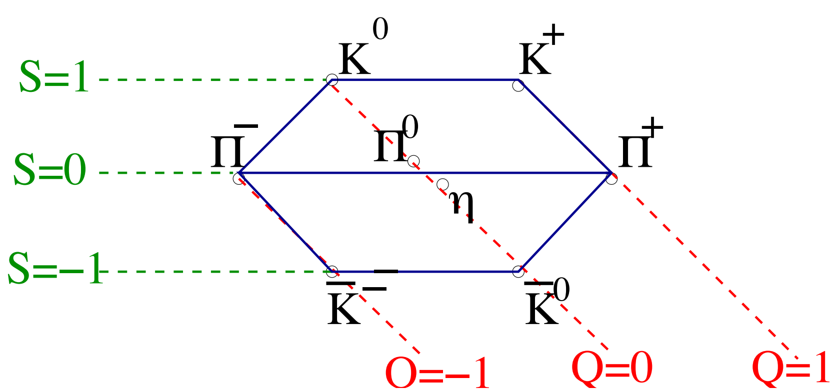

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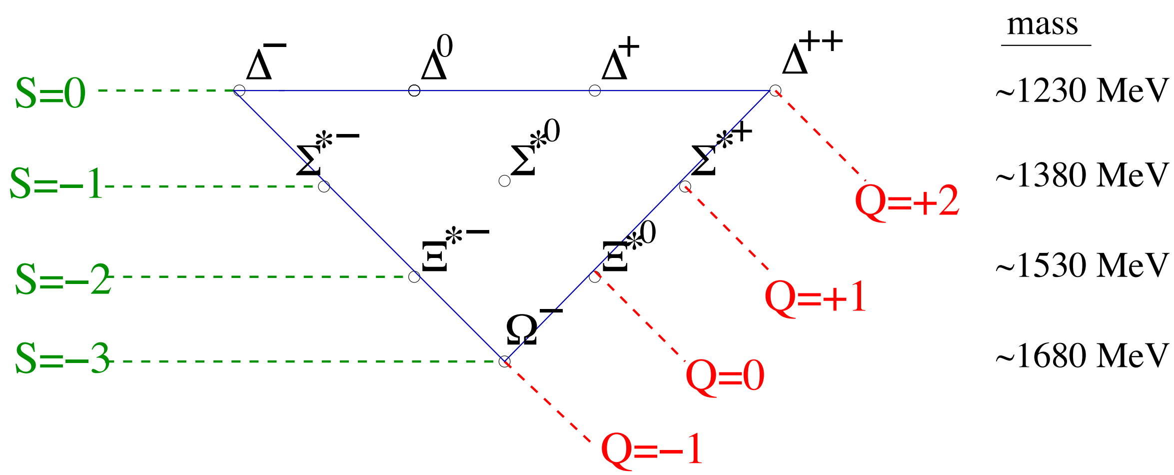

Figure 1.7: Decuplet of baryons with spin=3/2. The Ω^- was not yet observed when this model was introduced. It's mass was predicted.

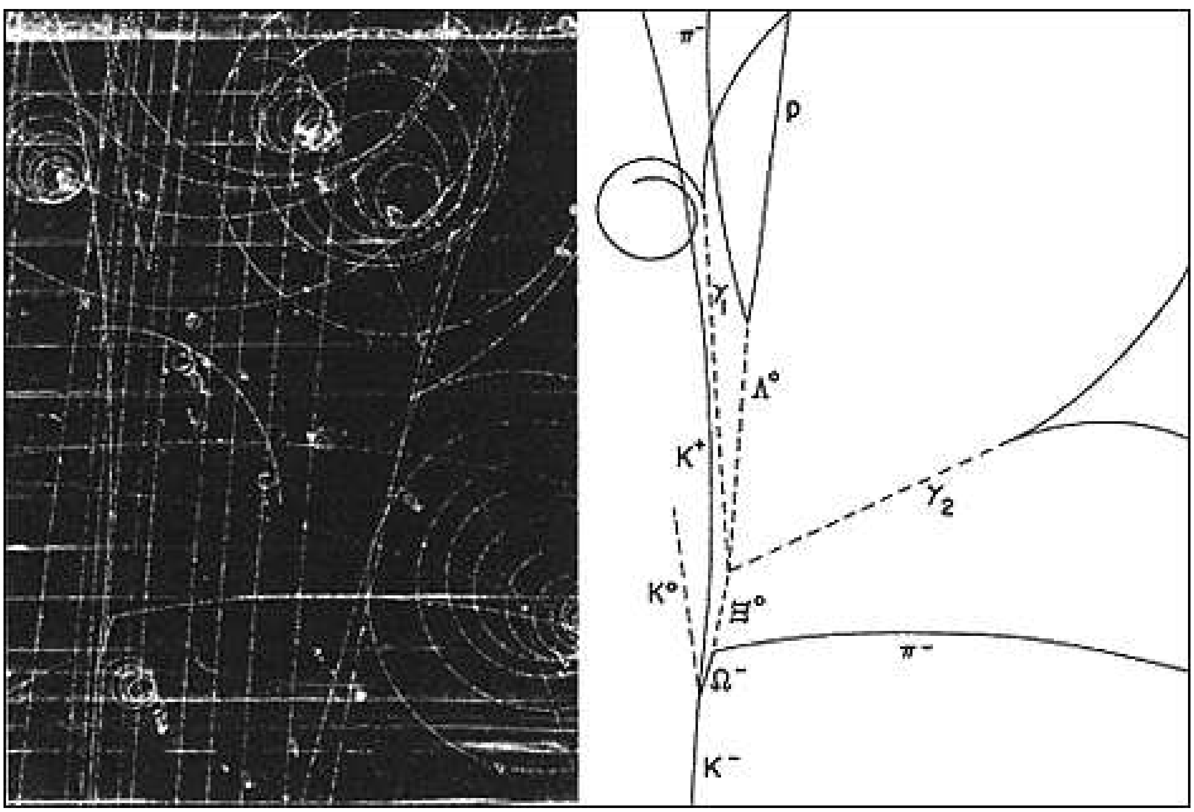

Figure 1.8: Discovery of the omega particle.

1.4 The Quark Model

The observed structure of hadrons in multiplets hinted at an underlying structure. Gell-Mann and Zweig postulated indeed that hadrons consist of more fundamental partons: the quarks. Initially three quarks and their anti-particle were assumed to exist (see Fig. 1.9). A baryon consists of 3 quarks: (q, q, q), while a meson consists of a quark and an antiquark: $(q, \overline{q})$. Mesons can be their own anti-particle, baryons cannot.

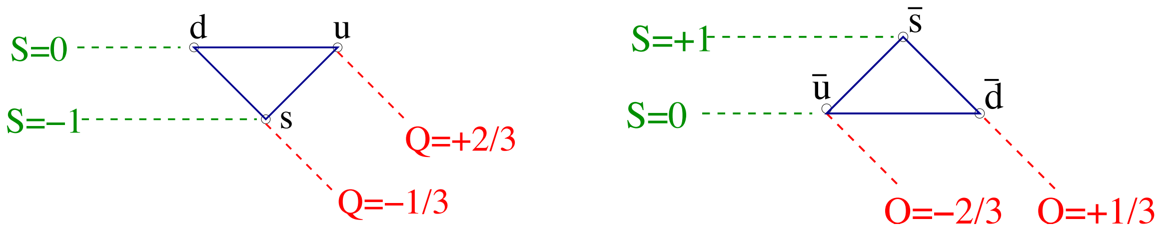

Figure 1.9: The fundamental quarks: u,d,s.

Exercise 2:
Assign the quark contents of the baryon decuplet and the meson octet.

How does this explain that baryons and mesons appear in the form of octets, decuplets, nonets etc.? For example a baryon, consisting of 3 quarks with 3 flavours (u,d,s) could in principle lead to 3x3x3=27 combinations. The answer lies in the fact that the wave function of fermions is subject to a symmetry under exchange of fermions. The total wave function must be anti-symmetric with respect to the interchange of two fermions.

$$\psi\,(baryon) = \psi\,(space) \cdot \phi\,(spin) \cdot \chi\,(flavour) \cdot \zeta\,(color)$$

These symmetry aspects are reflected in group theory where one encounters expressions as: $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ and $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

For more information on the static quark model read §2.10 and §2.11 in H&M, §5.5 and §5.6 in Griffiths, or chapter 5 in the book of Perkins.

1.4.1 Color

As indicated in the wave function above, a quark has another internal degree of freedom. In addition to electric charge a quark has a different charge, of which there are 3 types. This charge is referred to as the color quantum number, labelled as r, g, b. Evidence for the existence of color comes from the ratio of the cross section:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_C \sum_i Q_i^2$$

where the sum runs over the quark types that can be produced at the available energy. The plot in Fig. 1.10 shows this ratio, from which the result $N_C = 3$ is obtained.

Table 9.1: The properties of the three quarks.

Quark	label	spin	Q/e	I	I_3	S	B
Up	u	$\frac{1}{2}$	$+\frac{2}{3}$	$\frac{1}{2}$	$+\frac{1}{2}$	0	$\frac{1}{3}$
Down	d	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{3}$
Strange	s	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	-1	$\frac{1}{3}$

The masses are

$$
\begin{aligned}
M_\Delta &= 1232 \text{ MeV} \\
M_{\Sigma^*} &= 1385 \text{ MeV} \\
M_{\Xi^*} &= 1530 \text{ MeV} \\
M_\Omega &= 1672 \text{ MeV}
\end{aligned}
$$

(Notice almost that we can fit these masses as a linear function in Y, as can be seen in figure 9.6. This was of great help in finding the Ω.)

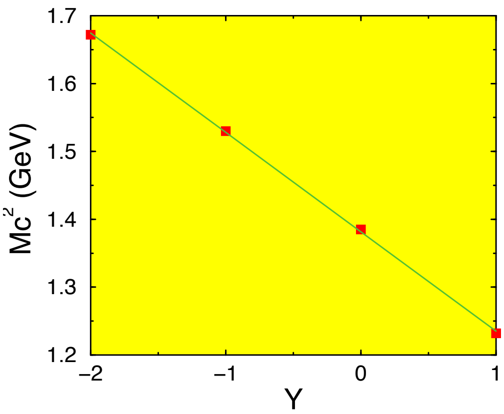

Figure 9.6: A linear fit to the mass of the decuplet

9.3 The quark model of strong interactions

Once the eightfold way (as the SU(3) symmetry was poetically referred to) was discovered, the race was on to explain it. As I have shown before the decuplet and two octets occur in the product

$$\boxed{\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}.} \quad \text{baryons} \tag{9.14}$$

A very natural assumption is to introduce a new particle that comes in three "flavours" called up, down and strange (u, d and s, respectively), and assume that the baryons are made from three of such particles, and the mesons from a quark and anti-quark (remember,

$$\boxed{\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}.\ )} \quad \text{mesons} \tag{9.15}$$

Each of these quarks carries one third a unit of baryon number. The properties can now be tabulated, see table 9.2.

In the multiplet language I used before, we find that the quarks form a triangle, as given in Fig. 9.7.

Once we have made this assigment, we can try to derive what combination corresponds to the assignments of the meson octet, figure 9.8. We just make all possible combinations of a quark and antiquark, apart from the scalar one $\eta' = u\bar{u} + d\bar{d} + c\bar{c}$ (why?).

A similar assignment can be made for the nucleon octet, and the nucleon decaplet, see e.g., see Fig. 9.9.

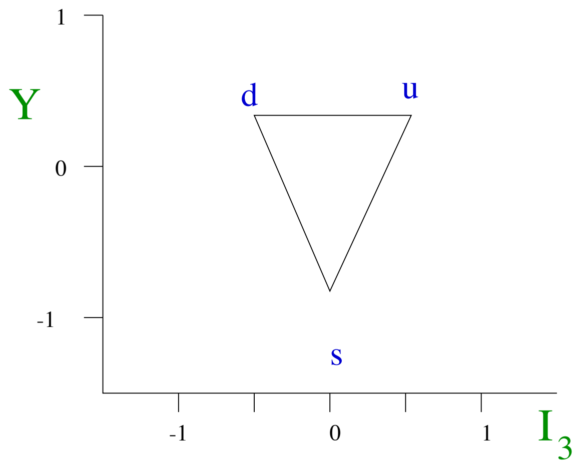

Figure 9.7: The multiplet structure of quarks and antiquarks

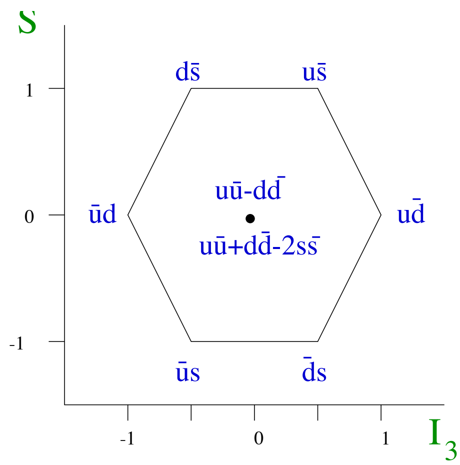

Figure 9.8: quark assignment of the meson octet

9.4 $SU(4)$, ...

Once we have three flavours of quarks, we can ask the question whether more flavours exists. At the moment we know of three generations of quarks, corresponding to three generations (pairs). These give rise to SU(4), SU(5), SU(6) flavour symmetries. Since the quarks get heavier and heavier, the symmetries get more-and-more broken as we add flavours.

9.5 Colour symmetry

So why don't we see fractional charges in nature? This is an important point! In so-called deep inelastic scattering we see pips inside the nucleon – these have been identified as the quarks. We do not see any direct signature of individual quarks. Furthermore, if quarks are fermions, as they are spin 1/2 particles, what about antisymmetry of their wavefunction? Let us investigate the Δ^{++}, see Fig. 9.10, which consists of three u quarks with identical spin and flavour (isospin) and *symmetric* spatial wavefunction,

symmetric
$$\boxed{\psi_{\text{total}} = \psi_{\text{space}} \times \psi_{\text{spin}} \times \psi_{\text{flavour}}.} \qquad (9.16)$$
NO!

This would be symmetric under interchange, which is unacceptable. Actually there is a simple solution. We "just" assume that there is an additional quantity called colour, and take the colour wave function to be antisymmetric:

antisymmetric
$$\boxed{\psi_{\text{total}} = \psi_{\text{space}} \times \psi_{\text{spin}} \times \psi_{\text{flavour}} \times \psi_{\text{colour}}} \qquad (9.17)$$
YES!

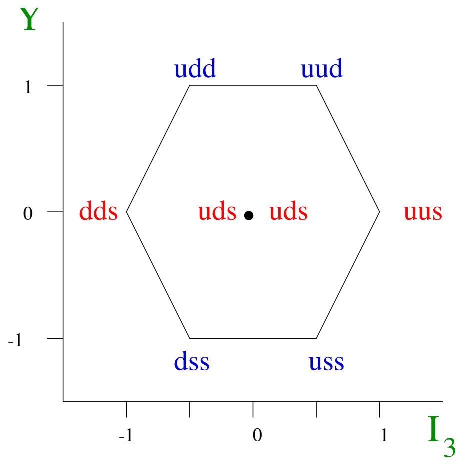

Figure 9.9: quark assignment of the nucleon octet

Table 9.2: The properties of the three quarks.

Quark	label	spin	Q/e	mass (GEV/c^2)
Down	d	$\frac{1}{2}$	$-\frac{1}{3}$	0.35
Up	u	$\frac{1}{2}$	$+\frac{2}{3}$	0.35
Strange	s	$\frac{1}{2}$	$-\frac{1}{3}$	0.5
Charm	c	$\frac{1}{2}$	$+\frac{2}{3}$	1.5
Bottom	b	$\frac{1}{2}$	$-\frac{1}{3}$	4.5
Top	t	$\frac{1}{2}$	$+\frac{2}{3}$	93

We assume that quarks come in three colours. This naturally leads to yet another $SU(3)$ symmetry, which is actually related to the gauge symmetry of strong interactions, QCD. So we have shifted the question to: why can't we see coloured particles?

This is a deep and very interesting problem. The only particles that have been seen are colour neutral ("white") ones. This leads to the assumption of confinement – We cannot liberate coloured particles at "low" energies and temperatures! The question whether they are free at higher energies is an interesting question, and is currently under experimental consideration.

9.6 The feynman diagrams of QCD

There are two key features that distinguish QCD from QED:

1. Quarks interact more strongly the further they are apart, and more weakly as they are close by – assymptotic freedom.

2. Gluons interact with themselves

The first point can only be found through detailed mathematical analysis. It means that free quarks can't be seen, but at high energies quarks look more and more like free particles. The second statement make QCD so hard to solve. The gluon comes in 8 colour combinations (since it carries a colour and anti-colour index, minus the scalar combination). The relevant diagrams are sketches in Figure 9.11. Try to work out yourself how we satisfy colour charge conservation!

9.7 Jets and QCD

One way to see quarks is to use the fact that we can liberate quarks for a short time, at high energy scales. One such process is $e^+e^- \to q\bar{q}$, which use the fact that a photon can couple directly to $q\bar{q}$. The quarks don't live very long and decay by producing a "jet" a shower of particles that results from the deacay of the

The associated observable is called **flavor.** Quarks now are assumed to come in three flavors, namely, red, blue and green.

The problem with the Ω^- state is fixed by assuming that it is completely antisymmetric in the flavor space or we write it as

$$\begin{aligned}\left|\Omega^-\right\rangle &= \left|s_r\right\rangle\left|s_b\right\rangle\left|s_g\right\rangle - \left|s_r\right\rangle\left|s_g\right\rangle\left|s_b\right\rangle - \left|s_b\right\rangle\left|s_r\right\rangle\left|s_g\right\rangle \\ &+ \left|s_b\right\rangle\left|s_g\right\rangle\left|s_r\right\rangle + \left|s_g\right\rangle\left|s_r\right\rangle\left|s_b\right\rangle - \left|s_g\right\rangle\left|s_b\right\rangle\left|s_r\right\rangle\end{aligned}$$

It is equal parts red, green and blue!

Another type of problem was that certain reactions were not occurring when SU(3) said they were allowed. This usually signals that an extra (new) quantum number is needed to prevent the reaction. The new quantum number would have a new conservation law that would be violated if the reaction occurred and hence it is disallowed.

The new quantum number introduced was **charm.** In the quark model this is treated as the appearance of a **new quark.** I will delineate the starting point of this discussion below. It leads to the formalism of SU(4) if we carried out the same steps as we did with SU(3).

Quark states:

$$\left|u\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad , \quad \left|d\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad , \quad \left|s\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad , \quad \left|c\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We now have

$$Q = I_3 + \frac{B+S+C}{2} \quad , \quad C = charm$$

The four quarks then have quantum numbers

Quark	Y	Q	B	S	C
u	1/3	2/3	1/3	0	0
d	1/3	-1/3	1/3	0	0
s	-2/3	-1/3	1/3	-1	0
c	1/3	2/3	1/3	0	1

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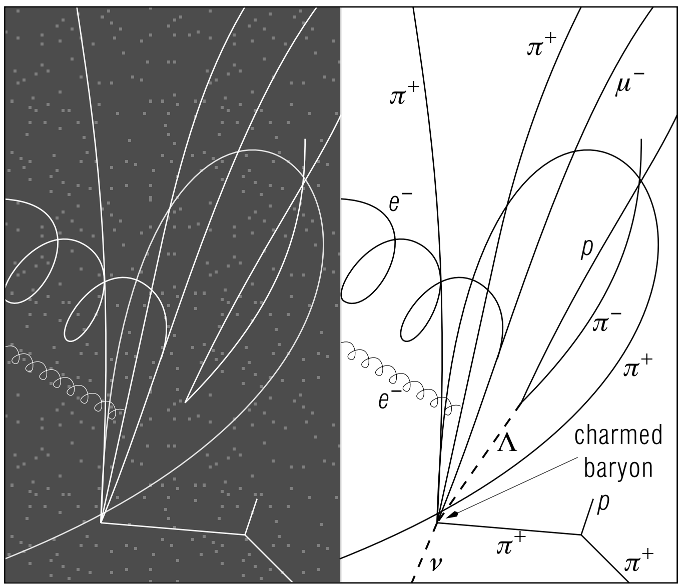
COLOR AND CHARM



Project PHYSNET•Physics Bldg.•Michigan State University•East Lansing, MI

COLOR AND CHARM

by

J. R. Christman, U. Coast Guard Academy

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1. Overview

This module covers material which has been extensively studied experimentally and theoretically over the past decade. The ideas of charm and color are frequently in the news and are often invoked to explain new discoveries. Because the words are used in press releases you should be aware of the concepts behind them. You should remember that charmed particles have been observed in experiments, but that the evidence for color is still indirect.

2. Assigned Reading

S. L. Glashow, "Quarks with Color and Flavor," *Scientific American*, (Oct. 1975).

3. Color

3a. Introduction. Color is a new quantum number assigned to the quarks. It is postulated that each quark can exist in one of three different states which are distinguished from each other by a quantity called "color." The three states are designated red, yellow, and blue by most authors although several different sets of names are in use. It should be emphasized that color, as used here, has absolutely nothing to do with hue or frequency of light. It is simply a convenient designation for states. With the addition of color, there are essentially nine kinds of quarks (red u, yellow u, blue u, red d, yellow d, etc). The postulate of color solves two problems which arise in connection with the quark model, although it raises another, perhaps equivalent question.

3b. Apparent Quark-Spin Violation of Pauli Principle. The first problem solved by the introduction of color arises from the quark model for baryons in the spin 3/2 decimet. The masses of these particles lead one to believe that the particle spin arises from aligned quark spins and not from quark orbital angular momentum. If this is true there are cases for which 2 or more quarks have exactly the same quantum

numbers — they are the same type of quark with the same spin and orbital angular momentum (namely zero). The most flagrant example is the Ω^- with quark content sss all in $\ell = 0$, $m_s = +1/2$ states. This quark content violates the Pauli exclusion principle which forbids more than one fermion from occupying the same state. This principle is derivable from long established principles of special relativity and to accept the violation would be tantamount to a rejection of relativity. The way out is to postulate that the three s quarks in the Ω^- are not really in identical states but rather differ in color; that is, one is red, one is yellow, and one is blue.

3c. Quark Triplets. The second problem that color helps to solve is the riddle of why our observable particles formed only from 3 quarks, 3 antiquarks or a quark-antiquark pair. Why not $(\mathrm{uuds\bar{d}})$ for example? To answer this it is assumed that:

1. there are 3 different colors;
2. the colors obey a color SU(3) symmetry, just like the SU(3) symmetry of the quarks themselves, but the color operators change the color instead of the quark type; and
3. all observable particles are color singlets (that is, they have zero net color).

There are exactly 3 ways to combine quarks and meet these conditions:

1. *A quark-antiquark pair* where both quark and antiquark are of the same color at any time and with the pair spending one-third of the time being each of the colors. The π^+ state, for example, is written (using the subscript "r" for "red" and similarly for yellow and blue):
$$\frac{1}{\sqrt{3}}\left[(\mathrm{u\bar{d}_r}) + (\mathrm{u\bar{d}_y}) + (\mathrm{u\bar{d}_b})\right].$$
2. *A quark triplet* in which each of the three quarks is a different color than the others and with each permutation of color occurring with equal probability. The p state, for example, can be written:
$$\frac{1}{\sqrt{6}}\left[\mathrm{u_r u_y d_b + u_y u_b d_r + u_b u_r d_y - u_y u_r d_b - u_b u_y d_r - u_r u_b d_y}\right]$$

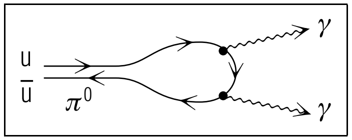

Figure 1. The decay of π^0 according to the quark model.

3. *An antiquark triplet* in which each of the three antiquarks is a different color than the others and with each permutation of color occurring with equal probability. If the color SU(3) symmetry is exact, all three uquarks have the same mass, spin, isospin, strangeness, hypercharge, baryon number, and charge. Similar statements can be made for the three dquarks and the three squarks.

These postulates lead to the new question: why are the particles color singlets? At present this question has not been answered.

3d. Quantitative Implications of Color. The existence of color has quantitative implications. According to the quark model the decay $\pi^0 \to \gamma + \gamma$ proceeds according to the diagram in Fig. 1, where q is either a u or a d quark. Without color considerations, this diagram leads to a prediction for the decay rate which is about one-third the observed rate. If color is included, there are 3 routes for the decay (via red quarks, via blue quarks, and via yellow quarks) and the predicted rate is close to the observed rate.

Production of hadrons from electron-positron annihilation proceeds according to the process illustrated in Fig. 2. Here each set of two adjacent parallel lines stands for a quark-antiquark pairs, representing a meson. Again, the predicted rate for hadron production, without color, is too low by a factor of 3. When color is introduced, the resulting agreement of predicted and observed rates can be taken as a strong argument for color.

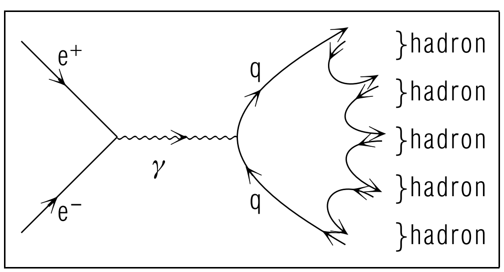

Figure 2. The annihilation of an electron-positron pair according to the quark model.

4. Charm

4a. Values, Conservation. Charm is a new particle quantum number that is somewhat similar to strangeness. Like strangeness it is embodied in a quark, denoted by c, that is called the charmed quark. The charmed quark has charm +1, the charmed antiquark has charm −1, and the other three quarks (u,d,s) have charm 0. Charm is evidently conserved in strong interactions.

4b. Relationships: C, Q, B, S, T_3 . The charmed quark has baryon number 1/3, spin 1/2, strangeness 0, isospin 0, and charge 2/3. With the advent of charm, the relationship between charge, baryon number, strangeness, and isotopic spin must be modified to:

$$q = \frac{1}{2}(B + S + C) + T_3.$$

Just as a hadron that exhibits net strangeness is called a strange particle, so a particle which exhibits net charm is called a charmed particle. Since a charmed particle and its anti-particle have opposite charm, the cbc meson is not charmed but ubc is a charmed meson and udc is a charmed baryon.

4c. Postulated Lepton-Quark Symmetry. The postulate of a charmed quark was made on several grounds. One is purely aesthetic. It is believed by many physicists that leptons and quarks are the truly fundamental particles and that nature must show a symmetry between these two set of particles. Since there are four leptons, it is not unreasonable to expect four quarks.

4d. Rarity of Strangeness - Conserving Neutral Interaction Decays. The other reason for introducing charm is more compelling. It has to do with a special class of weak interactions, called strangeness changing neutral interactions, in which the net strangeness of the hadrons involved changes but the net charge of the hadrons does not (hence the name "neutral"). Two examples are $K^0 \to \mu^+ + \mu^-$ and $K^+ \to \pi^+ + \nu_\mu + \bar{\nu}_\mu$. These interactions, described in terms of quarks, involve the transformation s → d and they are extremely rare; less than 0.7×10^{-3} per cent of hadron decays are via such strangeness-changing neutral interactions.

The theoretical problem which arises is this: why are these decays so rare? There is no conservation law which prevents $s \to d + W^0$, for example. The old answer to this question was that the W^0 does not exist. Its existence has now been implied by neutrino scattering experiments and a new reason must be sought.

The theory here is too complex to discuss in detail but the general idea is: the existence of a charmed quark provides another route by which these decays can take place. Classically, the existence of two routes to a particular final result increases the probability of reaching that result. However, quantum mechanically the probability amplitudes for the two routes can interfere destructively with each other and considerably reduce the probability of reaching the result. This mechanism gave rise to the name of this quark.[1] A charm wards off evil; the evil in this case is the direct, weak, neutral decay of the kaon.

5. Consequences of Color and Charm

5a. Multiplicities: Comparison of Charm and Color. Color, with the postulate that particles be color singlets, does not increase the number of particles predicted by the quark model. Charm, however, enormously increases the number of hadrons possible. The meson octets become 15 particle figures with the addition of $c = +1$ and $c = -1$ particles. The baryon octets become 20 particle figures with the addition of $c = +1$, $c = +2$, and c = +3 particles. There are also new $c = 0$ hadrons: particles with the combination $c\bar{c}$ and also perhaps in combination with other quarks.

5b. Color-Changing Gluon Exchange. Color enters into new theories in another, more fundamental, way. It is thought to be the characteristic that is responsible for the binding between quarks. Just as electric charge produces the electromagnetic field and photons, color ("charge") is responsible for the gluon field and gluons. The change from one color combination to another is accompanied by the emission or absorption of a gluon. These ideas are currently being studied.

Acknowledgments

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[1]Glashow, *Physics Letters*, (1964).

MODEL EXAM

1. See Output Skills K1-K6 in this module's *ID Sheet*. The actual exam may contain one or more of these skills.

2. Determine which of the following particles can exist, according to the 4-quark model. For each allowed particle, give a possible quark content. For each unallowed particle, tell why the 4-quark model prohibits it.

   a. a baryon with charm +1, strangeness +1, isospin 1.

   b. a baryon with charm -1, strangeness 0, charge 0.

   c. a meson with charm +2.

Brief Answers:

1. See this module's *text*.

2. a. No.

   This baryon would be [$c\bar{s}$(uor d)], but since c and $\bar{s}$ each have isospin of 0 and u and d have isospin 1/2, the baryon would have isospin 1/2 (not 1).

   b. Yes. ($\bar{c}\overline{dd}$).

   c. No. This meson would have to be cc, but a meson consists of a quark-antiquark pair.

SU(4) mesons

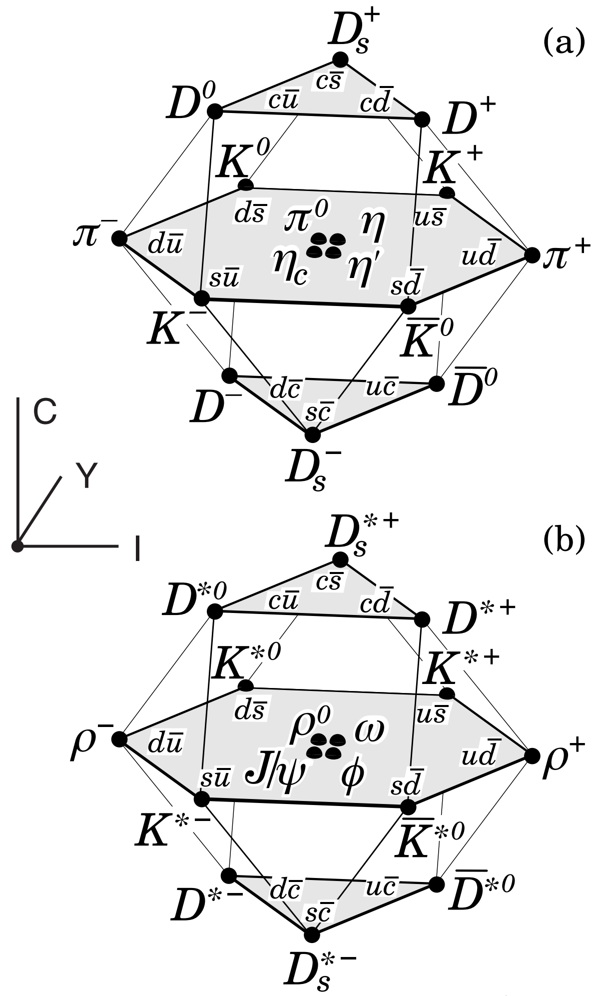

Figure 14.1: SU(4) weight diagram showing the 16-plets for the pseudoscalar (a) and vector mesons (b) made of the u, d, s, and c quarks as a function of isospin I, charm C, and hypercharge $\mathsf{Y} = \mathsf{S}+\mathcal{B} - \frac{\mathsf{C}}{3}$. The nonets of light mesons occupy the central planes to which the $c\bar{c}$ states have been added.

These mixing relations are often rewritten to exhibit the $u\bar{u} + d\bar{d}$ and $s\bar{s}$ components which decouple for the "ideal" mixing angle θ_i, such that $\tan\theta_i = 1/\sqrt{2}$ (or θ_i=35.3°). Defining $\alpha = \theta + 54.7°$, one obtains the physical isoscalar in the flavor basis

$$f' = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\cos\alpha - s\bar{s}\ \sin\alpha \ , \tag{14.8}$$

and its orthogonal partner f (replace α by α – 90°). Thus for ideal mixing (α_i = 90°), the f' becomes pure $s\bar{s}$ and the f pure $u\bar{u} + d\bar{d}$. The mixing angle θ can be derived from the mass relation

$$\tan\theta = \frac{4m_K - m_a - 3m_{f'}}{2\sqrt{2}(m_a - m_K)} \ , \tag{14.9}$$

which also determines its sign or, alternatively, from

$$\tan^2\theta = \frac{4m_K - m_a - 3m_{f'}}{-4m_K + m_a + 3m_f} \ . \tag{14.10}$$

SU(4) quarks

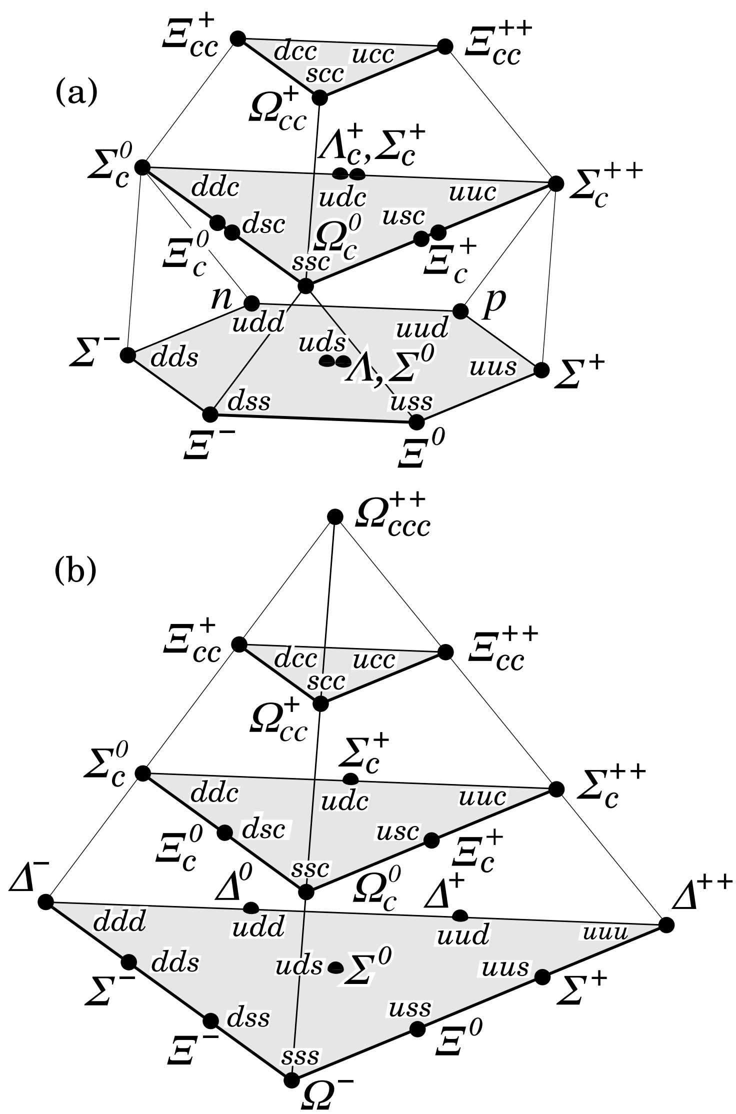

Figure 14.4: SU(4) multiplets of baryons made of u, d, s, and c quarks. (a) The 20-plet with an SU(3) octet. (b) The 20-plet with an SU(3) decuplet.

$$\mathbf{20} = {}^2\mathbf{8} \oplus {}^4\mathbf{1} \ , \tag{14.25c}$$

where the superscript $(2S+1)$ gives the net spin S of the quarks for each particle in the SU(3) multiplet. The $J^P = 1/2^+$ octet containing the nucleon and the $J^P = 3/2^+$ decuplet containing the $\Delta(1232)$ together make up the "ground-state" 56-plet, in which the orbital angular momenta between the quark pairs are zero (so that the spatial part of the state function is trivially symmetric). The **70** and **20** require some excitation of the spatial part of the state function in order to make the overall state function symmetric. States with nonzero orbital angular momenta are classified in SU(6)$\otimes$O(3) supermultiplets.

It is useful to classify the baryons into bands that have the same number N of quanta of excitation. Each band consists of a number of supermultiplets, specified by (D, L_N^P), where D is the dimensionality of the SU(6) representation, L is the total quark orbital angular http://pdg.lbl.gov

http://pdg.lbl.gov/2011/listings/contents_listings.html

14. QUARK MODEL

Revised August 2011 by C. Amsler (University of Zürich), T. DeGrand (University of Colorado, Boulder), and B. Krusche (University of Basel).

14.1. Quantum numbers of the quarks

Quantum chromodynamics (QCD) is the theory of the strong interactions. QCD is a quantum field theory and its constituents are a set of fermions, the quarks, and gauge bosons, the gluons. Strongly interacting particles, the hadrons, are bound states of quark and gluon fields. As gluons carry no intrinsic quantum numbers beyond color charge, and because color is believed to be permanently confined, most of the quantum numbers of strongly interacting particles are given by the quantum numbers of their constituent quarks and antiquarks. The description of hadronic properties which strongly emphasizes the role of the minimum-quark-content part of the wave function of a hadron is generically called the quark model. It exists on many levels: from the simple, almost dynamics-free picture of strongly interacting particles as bound states of quarks and antiquarks, to more detailed descriptions of dynamics, either through models or directly from QCD itself. The different sections of this review survey the many approaches to the spectroscopy of strongly interacting particles which fall under the umbrella of the quark model.

Quarks are strongly interacting fermions with spin 1/2 and, by convention, positive parity. Antiquarks have negative parity. Quarks have the additive baryon number 1/3, antiquarks -1/3. Table 14.1 gives the other additive quantum numbers (flavors) for the three generations of quarks. They are related to the charge Q (in units of the elementary charge e) through the generalized Gell-Mann-Nishijima formula

$$\mathsf{Q} = \mathsf{I}_z + \frac{\mathcal{B} + \mathsf{S} + \mathsf{C} + \mathsf{B} + \mathsf{T}}{2} , \tag{14.1}$$

where $\mathcal{B}$ is the baryon number. The convention is that the *flavor* of a quark (I_z, S, C, B, or T) has the same sign as its *charge* Q. With this convention, any flavor carried by a charged meson has the same sign as its charge, *e.g.*, the strangeness of the K^+ is +1, the bottomness of the B^+ is +1, and the charm and strangeness of the D_s^- are each −1. Antiquarks have the opposite flavor signs.

Table 14.1: Additive quantum numbers of the quarks.

	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	−1	0	0	0
C – charm	0	0	0	+1	0	0
B – bottomness	0	0	0	0	−1	0
T – topness	0	0	0	0	0	+1

14.2. Mesons

Mesons have baryon number $\mathcal{B} = 0$. In the quark model, they are $q\overline{q}\,'$ bound states of quarks q and antiquarks $\overline{q}\,'$ (the flavors of q and q' may be different). If the orbital angular momentum of the $q\overline{q}\,'$ state is ℓ, then the parity P is $(-1)^{\ell+1}$. The meson spin J is given by the usual relation $|\ell - s| \leq J \leq |\ell + s|$, where s is 0 (antiparallel quark spins) or 1 (parallel quark spins). The charge conjugation, or C-parity $C = (-1)^{\ell+s}$, is defined only for the $q\bar{q}$ states made of quarks and their own antiquarks. The C-parity can be generalized to the G-parity $G = (-1)^{I+\ell+s}$ for mesons made of quarks and their own antiquarks (isospin $\mathsf{I}_z = 0$), and for the charged $u\bar{d}$ and $d\bar{u}$ states (isospin $\mathsf{I} = 1$).

The mesons are classified in J^{PC} multiplets. The $\ell = 0$ states are the pseudoscalars (0^{-+}) and the vectors (1^{--}). The orbital excitations $\ell = 1$ are the scalars (0^{++}), the axial vectors (1^{++}) and (1^{+-}), and the tensors (2^{++}). Assignments for many of the known mesons are given in Tables 14.2 and 14.3. Radial excitations are denoted by the principal quantum number n. The very short lifetime of the t quark makes it likely that bound-state hadrons containing t quarks and/or antiquarks do not exist.

States in the natural spin-parity series $P = (-1)^J$ must, according to the above, have $s = 1$ and hence, $CP = +1$. Thus, mesons with natural spin-parity and $CP = -1$ (0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, *etc.*) are forbidden in the $q\bar{q}\,'$ model. The $J^{PC} = 0^{--}$ state is forbidden as well. Mesons with such *exotic* quantum numbers may exist, but would lie outside the $q\bar{q}\,'$ model (see section below on exotic mesons).

Following SU(3), the nine possible $q\bar{q}\,'$ combinations containing the light u, d, and s quarks are grouped into an octet and a singlet of light quark mesons:

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \ . \tag{14.2}$$

A fourth quark such as charm c can be included by extending SU(3) to SU(4). However, SU(4) is badly broken owing to the much heavier c quark. Nevertheless, in an SU(4) classification, the sixteen mesons are grouped into a 15-plet and a singlet:

$$\mathbf{4} \otimes \overline{\mathbf{4}} = \mathbf{15} \oplus \mathbf{1} \ . \tag{14.3}$$

The *weight diagrams* for the ground-state pseudoscalar (0^{-+}) and vector (1^{--}) mesons are depicted in Fig. 14.1. The light quark mesons are members of nonets building the middle plane in Fig. 14.1(a) and (b).

Isoscalar states with the same J^{PC} will mix, but mixing between the two light quark isoscalar mesons, and the much heavier charmonium or bottomonium states, are generally assumed to be negligible. In the following, we shall use the generic names a for the $I = 1$, K for the $I = 1/2$, and f and f' for the $I = 0$ members of the light quark nonets. Thus, the physical isoscalars are mixtures of the SU(3) wave function ψ_8 and ψ_1:

$$f' = \psi_8 \cos\theta - \psi_1 \sin\theta \ , \tag{14.4}$$

$$f = \psi_8 \sin\theta + \psi_1 \cos\theta \ , \tag{14.5}$$

where θ is the nonet mixing angle and

$$\psi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \ , \tag{14.6}$$

$$\psi_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \ . \tag{14.7}$$

Table 14.2: Suggested $q\overline{q}$ quark-model assignments for some of the observed light mesons. Mesons in bold face are included in the Meson Summary Table. The wave functions f and f' are given in the text. The singlet-octet mixing angles from the quadratic and linear mass formulae are also given for the well established nonets. The classification of the 0^{++} mesons is tentative and the mixing angle uncertain due to large uncertainties in some of the masses. Also, the $f_0(1710)$ and $f_0(1370)$ are expected to mix with the $f_0(1500)$. The latter is not in this table as it is hard to accommodate in the scalar nonet. The light scalars $a_0(980)$, $f_0(980)$, and $f_0(600)$ are often considered as meson-meson resonances or four-quark states, and are therefore not included in the table. See the "Note on Scalar Mesons" in the Meson Listings for details and alternative schemes.

$n\ ^{2s+1}\ell_J$	J^{PC}	I = 1 $u\overline{d}$, $\overline{u}d$, $\frac{1}{\sqrt{2}}(d\overline{d}-u\overline{u})$	I = $\frac{1}{2}$ $u\overline{s}$, $d\overline{s}$; $\overline{d}s$, $-\overline{u}s$	I = 0 f'	I = 0 f	θ_{quad} [°]	θ_{lin} [°]
$1\ ^1S_0$	0^{-+}	$\boldsymbol{\pi}$	$\boldsymbol{K}$	$\boldsymbol{\eta}$	$\boldsymbol{\eta'(958)}$	−11.5	−24.6
$1\ ^3S_1$	1^{--}	$\boldsymbol{\rho(770)}$	$\boldsymbol{K^*(892)}$	$\boldsymbol{\phi(1020)}$	$\boldsymbol{\omega(782)}$	38.7	36.0
$1\ ^1P_1$	1^{+-}	$\boldsymbol{b_1(1235)}$	$\boldsymbol{K_{1B}}^\dagger$	$h_1(1380)$	$\boldsymbol{h_1(1170)}$		
$1\ ^3P_0$	0^{++}	$\boldsymbol{a_0(1450)}$	$\boldsymbol{K_0^*(1430)}$	$\boldsymbol{f_0(1710)}$	$\boldsymbol{f_0(1370)}$		
$1\ ^3P_1$	1^{++}	$\boldsymbol{a_1(1260)}$	$\boldsymbol{K_{1A}}^\dagger$	$\boldsymbol{f_1(1420)}$	$\boldsymbol{f_1(1285)}$		
$1\ ^3P_2$	2^{++}	$\boldsymbol{a_2(1320)}$	$\boldsymbol{K_2^*(1430)}$	$\boldsymbol{f_2'(1525)}$	$\boldsymbol{f_2(1270)}$	29.6	28.0
$1\ ^1D_2$	2^{-+}	$\boldsymbol{\pi_2(1670)}$	$\boldsymbol{K_2(1770)}^\dagger$	$\eta_2(1870)$	$\boldsymbol{\eta_2(1645)}$		
$1\ ^3D_1$	1^{--}	$\boldsymbol{\rho(1700)}$	$\boldsymbol{K^*(1680)}$		$\boldsymbol{\omega(1650)}$		
$1\ ^3D_2$	2^{--}		$\boldsymbol{K_2(1820)}$				
$1\ ^3D_3$	3^{--}	$\boldsymbol{\rho_3(1690)}$	$\boldsymbol{K_3^*(1780)}$	$\boldsymbol{\phi_3(1850)}$	$\boldsymbol{\omega_3(1670)}$	32.0	31.0
$1\ ^3F_4$	4^{++}	$\boldsymbol{a_4(2040)}$	$\boldsymbol{K_4^*(2045)}$		$\boldsymbol{f_4(2050)}$		
$1\ ^3G_5$	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
$1\ ^3H_6$	6^{++}	$a_6(2450)$			$f_6(2510)$		
$2\ ^1S_0$	0^{-+}	$\boldsymbol{\pi(1300)}$	$K(1460)$	$\boldsymbol{\eta(1475)}$	$\boldsymbol{\eta(1295)}$		
$2\ ^3S_1$	1^{--}	$\boldsymbol{\rho(1450)}$	$\boldsymbol{K^*(1410)}$	$\boldsymbol{\phi(1680)}$	$\boldsymbol{\omega(1420)}$		

† The $1^{+\pm}$ and $2^{-\pm}$ isospin $\frac{1}{2}$ states mix. In particular, the K_{1A} and K_{1B} are nearly equal (45°) mixtures of the $K_1(1270)$ and $K_1(1400)$. The physical vector mesons listed under 1^3D_1 and 2^3S_1 may be mixtures of 1^3D_1 and 2^3S_1, or even have hybrid components.

Table 14.3: $q\overline{q}$ quark-model assignments for the observed heavy mesons. Mesons in bold face are included in the Meson Summary Table.

$n\ {}^{2s+1}\ell_J$	J^{PC}	I = 0 $c\overline{c}$	I = 0 $b\overline{b}$	I = $\frac{1}{2}$ $c\overline{u}$, $c\overline{d}$; $\overline{c}u$, $\overline{c}d$	I = 0 $c\overline{s}$; $\overline{c}s$	I = $\frac{1}{2}$ $b\overline{u}$, $b\overline{d}$; $\overline{b}u$, $\overline{b}d$	I = 0 $b\overline{s}$; $\overline{b}s$	I = 0 $b\overline{c}$; $\overline{b}c$
$1\ {}^1S_0$	0^{-+}	$\boldsymbol{\eta_c(1S)}$	$\eta_b(1S)$	$\boldsymbol{D}$	$\boldsymbol{D_s^\pm}$	$\boldsymbol{B}$	$\boldsymbol{B_s^0}$	$\boldsymbol{B_c^\pm}$
$1\ {}^3S_1$	1^{--}	$\boldsymbol{J/\psi(1S)}$	$\boldsymbol{\Upsilon(1S)}$	$\boldsymbol{D^*}$	$\boldsymbol{D_s^{*\pm}}$	$\boldsymbol{B^*}$	B_s^*	
$1\ {}^1P_1$	1^{+-}	$\boldsymbol{h_c(1P)}$	$h_b(1P)$	$\boldsymbol{D_1(2420)}$	$\boldsymbol{D_{s1}(2536)^\pm}$	$\boldsymbol{B_1(5721)}$	$\boldsymbol{B_{s1}(5830)^0}$	
$1\ {}^3P_0$	0^{++}	$\boldsymbol{\chi_{c0}(1P)}$	$\boldsymbol{\chi_{b0}(1P)}$	$\boldsymbol{D_0^*(2400)}$	$\boldsymbol{D_{s0}^*(2317)^{\pm\dagger}}$			
$1\ {}^3P_1$	1^{++}	$\boldsymbol{\chi_{c1}(1P)}$	$\boldsymbol{\chi_{b1}(1P)}$	$D_1(2430)$	$\boldsymbol{D_{s1}(2460)^{\pm\dagger}}$			
$1\ {}^3P_2$	2^{++}	$\boldsymbol{\chi_{c2}(1P)}$	$\boldsymbol{\chi_{b2}(1P)}$	$\boldsymbol{D_2^*(2460)}$	$\boldsymbol{D_{s2}^*(2573)^\pm}$	$\boldsymbol{B_2^*(5747)}$	$\boldsymbol{B_{s2}^*(5840)^0}$	
$1\ {}^3D_1$	1^{--}	$\boldsymbol{\psi(3770)}$			$D_{s1}^*(2700)^\pm$			
$2\ {}^1S_0$	0^{-+}	$\boldsymbol{\eta_c(2S)}$						
$2\ {}^3S_1$	1^{--}	$\boldsymbol{\psi(2S)}$	$\boldsymbol{\Upsilon(2S)}$					
$2\ {}^3P_{0,1,2}$	$0^{++}, 1^{++}, 2^{++}$	$\chi_{c2}(2P)$	$\boldsymbol{\chi_{b\,0,1,2}(2P)}$					

† The masses of these states are considerably smaller than most theoretical predictions. They have also been considered as four-quark states (See the "Note on Non-$q\overline{q}$ Mesons" at the end of the Meson Listings). The open flavor states in the 1^{+-} and 1^{++} rows are mixtures of the $1^{+\pm}$ states.

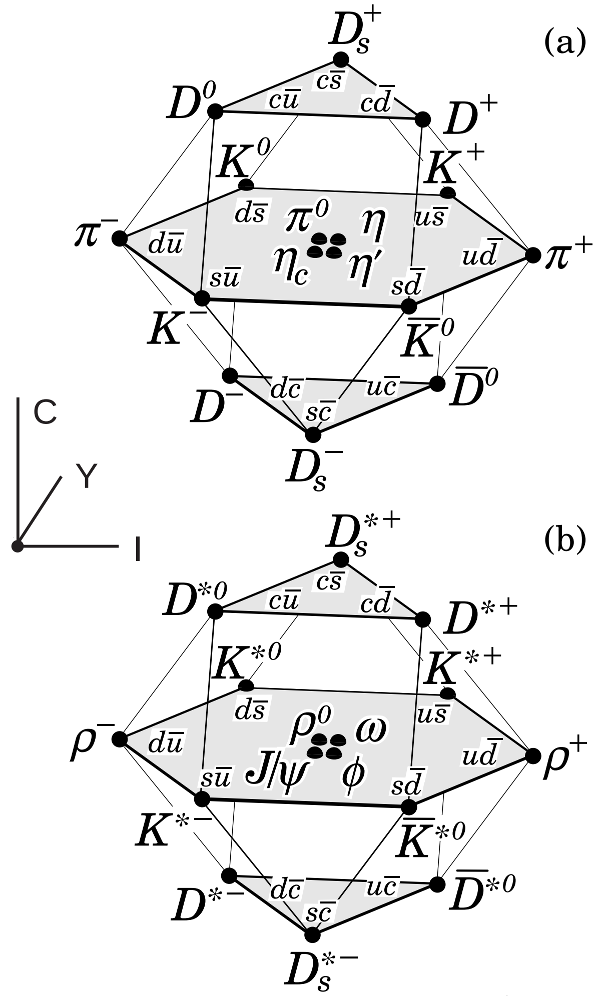

Figure 14.1: SU(4) weight diagram showing the 16-plets for the pseudoscalar (a) and vector mesons (b) made of the u, d, s, and c quarks as a function of isospin I, charm C, and hypercharge $\mathsf{Y} = \mathsf{S}+\mathcal{B} - \frac{\mathsf{C}}{3}$. The nonets of light mesons occupy the central planes to which the $c\bar{c}$ states have been added.

These mixing relations are often rewritten to exhibit the $u\bar{u} + d\bar{d}$ and $s\bar{s}$ components which decouple for the "ideal" mixing angle θ_i, such that $\tan\theta_i = 1/\sqrt{2}$ (or θ_i=35.3°). Defining $\alpha = \theta + 54.7°$, one obtains the physical isoscalar in the flavor basis

$$f' = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\cos\alpha - s\bar{s}\ \sin\alpha\ , \tag{14.8}$$

and its orthogonal partner f (replace α by α – 90°). Thus for ideal mixing (α_i = 90°), the f' becomes pure $s\bar{s}$ and the f pure $u\bar{u} + d\bar{d}$. The mixing angle θ can be derived from the mass relation

$$\tan\theta = \frac{4m_K - m_a - 3m_{f'}}{2\sqrt{2}(m_a - m_K)}\ , \tag{14.9}$$

which also determines its sign or, alternatively, from

$$\tan^2\theta = \frac{4m_K - m_a - 3m_{f'}}{-4m_K + m_a + 3m_f}\ . \tag{14.10}$$

Eliminating θ from these equations leads to the sum rule [1]

$$(m_f + m_{f'})(4m_K - m_a) - 3m_f m_{f'} = 8m_K^2 - 8m_K m_a + 3m_a^2. \tag{14.11}$$

This relation is verified for the ground-state vector mesons. We identify the $\phi(1020)$ with the f' and the $\omega(783)$ with the f. Thus

$$\phi(1020) = \psi_8 \cos\theta_V - \psi_1 \sin\theta_V \ , \tag{14.12}$$

$$\omega(782) = \psi_8 \sin\theta_V + \psi_1 \cos\theta_V \ , \tag{14.13}$$

with the vector mixing angle $\theta_V = 35^\circ$ from Eq. (14.9), very close to ideal mixing. Thus $\phi(1020)$ is nearly pure $s\bar{s}$. For ideal mixing, Eq. (14.9) and Eq. (14.10) lead to the relations

$$m_K = \frac{m_f + m_{f'}}{2} \ , \quad m_a = m_f \ , \tag{14.14}$$

which are satisfied for the vector mesons.

The situation for the pseudoscalar and scalar mesons is not so clear cut, either theoretically or experimentally. For the pseudoscalars, the mixing angle is small. This can be understood qualitatively via gluon-line counting of the mixing process. The size of the mixing process between the nonstrange and strange mass bases scales as α_s^2, not α_s^3, because of two rather than three gluon exchange as it does for the vector mesons. It may also be that the lightest isoscalar pseudoscalars mix more strongly with excited states or with states of substantial non-$\bar{q}q$ content, as will be discussed below.

A variety of analysis methods lead to similar results: First, for these states, Eq. (14.11) is satisfied only approximately. Then Eq. (14.9) and Eq. (14.10) lead to somewhat different values for the mixing angle. Identifying the η with the f' one gets

$$\eta = \psi_8 \cos\theta_P - \psi_1 \sin\theta_P \ , \tag{14.15}$$

$$\eta' = \psi_8 \sin\theta_P + \psi_1 \cos\theta_P \ . \tag{14.16}$$

Following chiral perturbation theory, the meson masses in the mass formulae (Eq. (14.9) and Eq. (14.10)) might be replaced by their squares. Table 14.2 lists the mixing angle $\theta_{\rm lin}$ from Eq. (14.10) and the corresponding $\theta_{\rm quad}$ obtained by replacing the meson masses by their squares throughout.

The pseudoscalar mixing angle θ_P can also be measured by comparing the partial widths for radiative J/ψ decay into a vector and a pseudoscalar [2], radiative $\phi(1020)$ decay into η and η' [3], or $\bar{p}p$ annihilation at rest into a pair of vector and pseudoscalar or into two pseudoscalars [4,5]. One obtains a mixing angle between –10° and –20°. More recently, a lattice QCD simulation, Ref. 6, has successfully reproduced the masses of the η and η', and as a byproduct find a mixing angle $\theta_{lin} = -14.1(2.8)^\circ$ We return to this point in Sec. 14.6.

The nonet mixing angles can be measured in $\gamma\gamma$ collisions, *e.g.*, for the 0^{-+}, 0^{++}, and 2^{++} nonets. In the quark model, the amplitude for the coupling of neutral mesons to two photons is proportional to $\sum_i Q_i^2$, where Q_i is the charge of the i-th quark. The 2γ partial width of an isoscalar meson with mass m is then given in terms of the mixing angle α by

$$\Gamma_{2\gamma} = C(5\cos\alpha - \sqrt{2}\sin\alpha)^2 m^3 \ , \tag{14.17}$$

for f' and f ($\alpha \to \alpha - 90°$). The coupling C may depend on the meson mass. It is often assumed to be a constant in the nonet. For the isovector a, one then finds $\Gamma_{2\gamma} = 9\ C\ m^3$. Thus the members of an ideally mixed nonet couple to 2γ with partial widths in the ratios $f : f' : a = 25 : 2 : 9$. For tensor mesons, one finds from the ratios of the measured 2γ partial widths for the $f_2(1270)$ and $f_2'(1525)$ mesons a mixing angle α_T of $(81 \pm 1)°$, or $\theta_T = (27 \pm 1)°$, in accord with the linear mass formula. For the pseudoscalars, one finds from the ratios of partial widths $\Gamma(\eta' \to 2\gamma)/\Gamma(\eta \to 2\gamma)$ a mixing angle $\theta_P = (-18 \pm 2)°$, while the ratio $\Gamma(\eta' \to 2\gamma)/\Gamma(\pi^0 \to 2\gamma)$ leads to $\sim -24\ °$. SU(3) breaking effects for pseudoscalars are discussed in Ref. 7.

The partial width for the decay of a scalar or a tensor meson into a pair of pseudoscalar mesons is model-dependent. Following Ref. 8,

$$\Gamma = C \times \gamma^2 \times |F(q)|^2 \times q\ . \tag{14.18}$$

C is a nonet constant, q the momentum of the decay products, $F(q)$ a form factor, and γ^2 the SU(3) coupling. The model-dependent form factor may be written as

$$|F(q)|^2 = q^{2\ell} \times \exp(-\frac{q^2}{8\beta^2}), \tag{14.19}$$

where ℓ is the relative angular momentum between the decay products. The decay of a $q\bar{q}$ meson into a pair of mesons involves the creation of a $q\bar{q}$ pair from the vacuum, and SU(3) symmetry assumes that the matrix elements for the creation of $s\bar{s}$, $u\bar{u}$, and $d\bar{d}$ pairs are equal. The couplings γ^2 are given in Table 14.4, and their dependence upon the mixing angle α is shown in Fig. 14.2 for isoscalar decays. The generalization to unequal $s\bar{s}$, $u\bar{u}$, and $d\bar{d}$ couplings is given in Ref. 8. An excellent fit to the tensor meson decay widths is obtained assuming SU(3) symmetry, with $\beta \simeq 0.5$ GeV/c, $\theta_V \simeq 26\ °$ and $\theta_P \simeq -17\ °$ [8].

Table 14.4: SU(3) couplings γ^2 for quarkonium decays as a function of nonet mixing angle α, up to a common multiplicative factor C ($\phi \equiv 54.7° + \theta_P$).

Isospin	Decay channel	γ^2
0	$\pi\pi$	$3\cos^2\alpha$
	$K\overline{K}$	$(\cos\alpha - \sqrt{2}\sin\alpha)^2$
	$\eta\eta$	$(\cos\alpha\cos^2\phi - \sqrt{2}\sin\alpha\sin^2\phi)^2$
	$\eta\eta'$	$\frac{1}{2}\sin^2 2\phi\ (\cos\alpha + \sqrt{2}\sin\alpha)^2$
1	$\eta\pi$	$2\ \cos^2\phi$
	$\eta'\pi$	$2\ \sin^2\phi$
	$K\overline{K}$	1
$\frac{1}{2}$	$K\pi$	$\frac{3}{2}$
	$K\eta$	$(\sin\phi - \frac{\cos\phi}{\sqrt{2}})^2$
	$K\eta'$	$(\cos\phi + \frac{\sin\phi}{\sqrt{2}})^2$

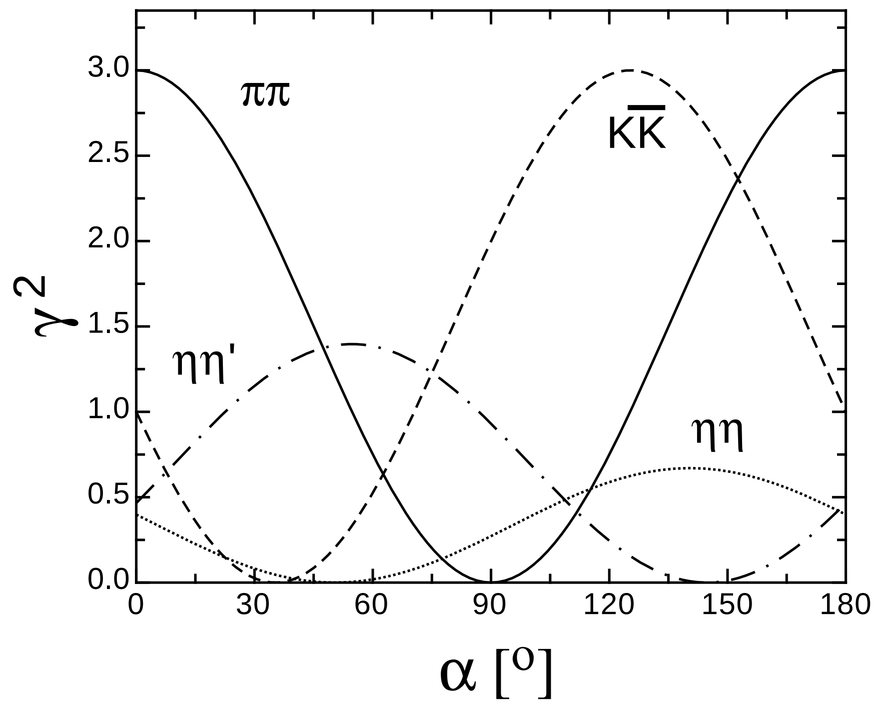

Figure 14.2: SU(3) couplings as a function of mixing angle α for isoscalar decays, up to a common multiplicative factor C and for $\theta_P = -17.3°$.

14.3. Exotic mesons

The existence of a light nonet composed of four quarks with masses below 1 GeV was suggested a long time ago [9]. Coupling two triplets of light quarks u, d, and s, one obtains nine states, of which the six symmetric (uu, dd, ss, $ud + du$, $us + su$, $ds + sd$) form the six dimensional representation **6**, while the three antisymmetric ($ud - du$, $us - su$, $ds - sd$) form the three dimensional representation $\overline{\mathbf{3}}$ of SU(3):

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \overline{\mathbf{3}} \ . \tag{14.20}$$

Combining with spin and color and requiring antisymmetry, one finds that the most deeply bound diquark (and hence the lightest) is the one in the $\overline{\mathbf{3}}$ and spin singlet state. The combination of the diquark with an antidiquark in the **3** representation then gives a light nonet of four-quark scalar states. Letting the number of strange quarks determine the mass splitting, one obtains a mass inverted spectrum with a light isosinglet ($ud\bar{u}\bar{d}$), a medium heavy isodoublet (*e.g.*, $ud\bar{s}\bar{d}$) and a heavy isotriplet (*e.g.*, $ds\bar{u}\bar{s}$) + isosinglet (*e.g.*, $us\bar{u}\bar{s}$). It is then tempting to identify the lightest state with the $f_0(600)$, and the heaviest states with the $a_0(980)$, and $f_0(980)$. Then the meson with strangeness $\kappa(800)$ would lie in between.

QCD predicts the existence of extra isoscalar mesons. In the pure gauge theory, they contain only gluons, and are called the glueballs. The ground state glueball is predicted by lattice gauge theories to be 0^{++}, the first excited state 2^{++}. Errors on the mass predictions are large. From Ref. 11 one obtains 1750 (50) (80) MeV for the mass of the lightest 0^{++} glueball from quenched QCD. As an example for the glueball mass spectrum, we show in Fig. 14.3 a recent calculation from Ref. 10. A mass of 1710 MeV is predicted for the ground state, also with an error of about 100 MeV. Earlier work by other groups produced masses at 1650 MeV [12] and 1550 MeV [13] (see also [14]). The first excited state has a mass of about 2.4 GeV, and the lightest glueball with exotic quantum numbers (2^{+-}) has a mass of about 4 GeV.

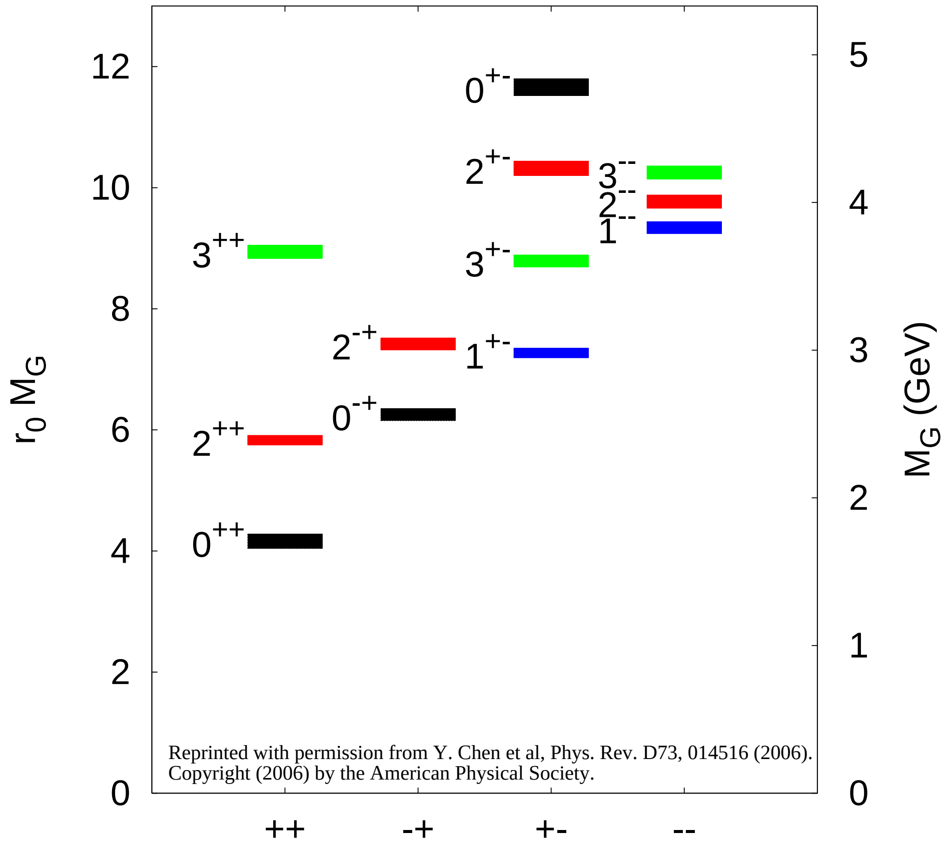

Figure 14.3: Predicted glueball mass spectrum from the lattice, in quenched approximation, (from Ref. 10).

These calculations are made in the so-called "quenched approximation" which neglects $q\bar{q}$ loops. However, both glue and $q\bar{q}$ states will couple to singlet scalar mesons. Therefore glueballs will mix with nearby $q\bar{q}$ states of the same quantum numbers. For example, the two isoscalar 0^{++} mesons around 1500 MeV will mix with the pure ground state glueball to generate the observed physical states $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ [8,15]. Lattice calculations are only beginning to include these effects. We return to a discussion of this point in Sec. 14.6.

The existence of three singlet scalar mesons around 1.5 GeV suggests additional degrees of freedom such as glue, since only two mesons are predicted in this mass range. The $f_0(1500)$ [8,15] or, alternatively, the $f_0(1710)$ [12], have been proposed as candidates for the scalar glueball, both states having considerable mixing also with the $f_0(1370)$. Other mixing schemes, in particular with the $f_0(600)$ and the $f_0(980)$, have also been proposed (more details can be found in the "Note on Scalar Mesons" in the Meson Listings and in Ref. 16).

Mesons made of $q\bar{q}$ pairs bound by excited gluons g, the hybrid states $q\bar{q}g$, are also predicted. They should lie in the 1.9 GeV mass region, according to gluon flux tube models [17]. Lattice QCD also predicts the lightest hybrid, an exotic 1^{-+}, at a mass of 1.8 to 1.9 GeV [18]. However, the bag model predicts four nonets, among them an exotic 1^{-+} around or above 1.4 GeV [19,20]. There are so far two candidates for exotic states with quantum numbers 1^{-+}, the $\pi_1(1400)$ and $\pi_1(1600)$, which could be hybrids or four-quark states (see the "Note on Non-$q\bar{q}$ Mesons" in the 2006 issue of this *Review* [21] and in Ref. 16).

14.4. Baryons: *qqq* states

Baryons are fermions with baryon number $\mathcal{B} = 1$, *i.e.*, in the most general case, they are composed of three quarks plus any number of quark - antiquark pairs. So far all established baryons are 3-quark (qqq) configurations. The color part of their state functions is an SU(3) singlet, a completely antisymmetric state of the three colors. Since the quarks are fermions, the state function must be antisymmetric under interchange of any two equal-mass quarks (up and down quarks in the limit of isospin symmetry). Thus it can be written as

$$|\, qqq \,\rangle_A = |\, \text{color} \,\rangle_A \times |\, \text{space, spin, flavor} \,\rangle_S \ , \tag{14.21}$$

where the subscripts S and A indicate symmetry or antisymmetry under interchange of any two equal-mass quarks. Note the contrast with the state function for the three nucleons in ^{3}H or ^{3}He:

$$|\, NNN \,\rangle_A = |\, \text{space, spin, isospin} \,\rangle_A \ . \tag{14.22}$$

This difference has major implications for internal structure, magnetic moments, *etc.* (For a nice discussion, see Ref. 22.)

The "ordinary" baryons are made up of u, d, and s quarks. The three flavors imply an approximate flavor SU(3), which requires that baryons made of these quarks belong to the multiplets on the right side of

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_M \oplus \mathbf{8}_M \oplus \mathbf{1}_A \tag{14.23}$$

(see Sec. 38, on "SU(n) Multiplets and Young Diagrams"). Here the subscripts indicate symmetric, mixed-symmetry, or antisymmetric states under interchange of any two quarks. The $\mathbf{1}$ is a uds state (Λ_1), and the octet contains a similar state (Λ_8). If these have the same spin and parity, they can mix. The mechanism is the same as for the mesons (see above). In the ground state multiplet, the SU(3) flavor singlet Λ_1 is forbidden by Fermi statistics. Section 37, on "SU(3) Isoscalar Factors and Representation Matrices," shows how relative decay rates in, say, $\mathbf{10} \to \mathbf{8} \otimes \mathbf{8}$ decays may be calculated.

The addition of the c quark to the light quarks extends the flavor symmetry to SU(4). However, due to the large mass of the c quark, this symmetry is much more strongly broken than the SU(3) of the three light quarks. Figures 14.4(a) and 14.4(b) show the SU(4) baryon multiplets that have as their bottom levels an SU(3) octet, such as the octet that includes the nucleon, or an SU(3) decuplet, such as the decuplet that includes the $\Delta(1232)$. All particles in a given SU(4) multiplet have the same spin and parity. The charmed baryons are discussed in more detail in the "Note on Charmed Baryons" in the Particle Listings. The addition of a b quark extends the flavor symmetry to SU(5); the existence of baryons with t-quarks is very unlikely due to the short lifetime of the top.

For the "ordinary" baryons (no c or b quark), flavor and spin may be combined in an approximate flavor-spin SU(6), in which the six basic states are $d\uparrow$, $d\downarrow$, $\cdots$, $s\downarrow$ ($\uparrow$, $\downarrow$ = spin up, down). Then the baryons belong to the multiplets on the right side of

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56}_S \oplus \mathbf{70}_M \oplus \mathbf{70}_M \oplus \mathbf{20}_A \ . \tag{14.24}$$

These SU(6) multiplets decompose into flavor SU(3) multiplets as follows:

$$\mathbf{56} = {}^4\mathbf{10} \oplus {}^2\mathbf{8} \tag{14.25a}$$

$$\mathbf{70} = {}^2\mathbf{10} \oplus {}^4\mathbf{8} \oplus {}^2\mathbf{8} \oplus {}^2\mathbf{1} \tag{14.25b}$$

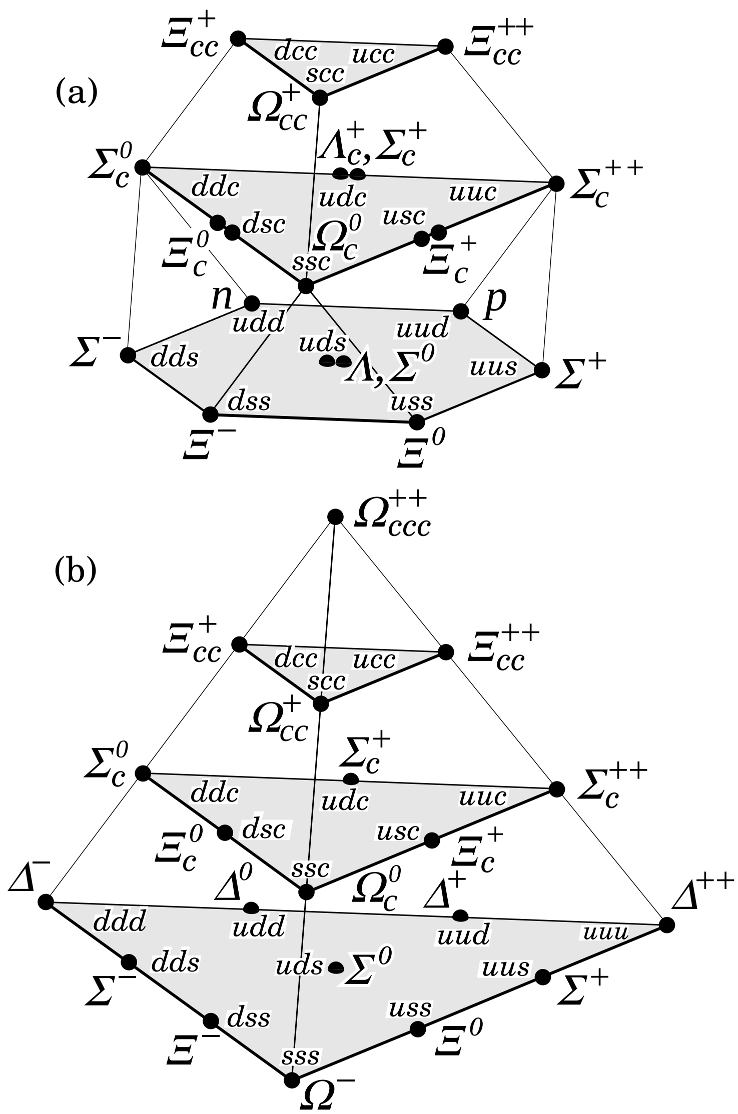

Figure 14.4: SU(4) multiplets of baryons made of u, d, s, and c quarks. (a) The 20-plet with an SU(3) octet. (b) The 20-plet with an SU(3) decuplet.

$$\mathbf{20} = {}^2\mathbf{8} \oplus {}^4\mathbf{1}\ , \tag{14.25c}$$

where the superscript $(2S+1)$ gives the net spin S of the quarks for each particle in the SU(3) multiplet. The $J^P = 1/2^+$ octet containing the nucleon and the $J^P = 3/2^+$ decuplet containing the $\Delta(1232)$ together make up the "ground-state" 56-plet, in which the orbital angular momenta between the quark pairs are zero (so that the spatial part of the state function is trivially symmetric). The **70** and **20** require some excitation of the spatial part of the state function in order to make the overall state function symmetric. States with nonzero orbital angular momenta are classified in SU(6)$\otimes$O(3) supermultiplets.

It is useful to classify the baryons into bands that have the same number N of quanta of excitation. Each band consists of a number of supermultiplets, specified by (D, L_N^P), where D is the dimensionality of the SU(6) representation, L is the total quark orbital angular momentum, and P is the total parity. Supermultiplets contained in bands up to N = 12 are given in Ref. 24. The N = 0 band, which contains the nucleon and $\Delta(1232)$, consists only of the $(56,0_0^+)$ supermultiplet. The N = 1 band consists only of the $(70,1_1^-)$ multiplet and contains the negative-parity baryons with masses below about 1.9 GeV. The N = 2 band contains five supermultiplets: $(56,0_2^+)$, $(70,0_2^+)$, $(56,2_2^+)$, $(70,2_2^+)$, and $(20,1_2^+)$.

Table 14.5: N and Δ states in the N=0,1,2 harmonic oscillator bands. L^P denotes angular momentum and parity, S the three-quark spin and 'sym'=A,S,M the symmetry of the spatial wave function. Only dominant components indicated. Assignments in the N=2 band are partly tentative.

N	sym	L^P	S	$N(I=1/2)$				$\Delta(I=3/2)$			
2	A	1^+	1/2	$1/2^+$	$3/2^+$						
2	M	2^+	3/2	$1/2^+$	$3/2^+$	$5/2^+$	$7/2^+$				
2	M	2^+	1/2		$3/2^+$	$5/2^+$			$3/2^+$	$5/2^+$	
2	M	0^+	3/2		$3/2^+$						
2	M	0^+	1/2	$1/2^+$				$1/2^+$			
				$P_{11}(1710)$				$P_{31}(1750)$			
2	S	2^+	3/2					$1/2^+$	$3/2^+$	$5/2^+$	$7/2^+$
								$P_{31}(1910)$	$P_{33}(1920)$	$F_{35}(1905)$	$F_{37}(1950)$
2	S	2^+	1/2		$3/2^+$	$5/2^+$					
					$P_{13}(1720)$	$F_{15}(1680)$					
2	S	0^+	3/2						$3/2^+$		
									$P_{33}(1600)$		
2	S	0^+	1/2	$1/2^+$							
				$P_{11}(1440)$							
1	M	1^-	3/2	$1/2^-$	$3/2^-$	$5/2^-$					
				$S_{11}(1650)$	$D_{13}(1700)$	$D_{15}(1675)$					
1	M	1^-	1/2	$1/2^-$	$3/2^-$			$1/2^-$	$3/2^-$		
				$S_{11}(1535)$	$D_{13}(1520)$			$S_{31}(1620)$	$D_{33}(1700)$		
0	S	0^+	3/2						$3/2^+$		
									$P_{33}(1232)$		
0	S	0^+	1/2	$1/2^+$							
				$P_{11}(938)$							

The wave functions of the non-strange baryons in the harmonic oscillator basis are often labeled by $|X^{2S+1}L_\pi J^P\rangle$, where S, L, J, P are as above, $X = N$ or Δ, and $\pi = S, M$ or A denotes the symmetry of the spatial wave function. The possible model states for the bands with N=0,1,2 are given in Table 14.5. The assignment of experimentally observed states is only complete and well established up to the N=1 band. Some more tentative assignments for higher multiplets are suggested in Ref. 25.

Table 14.6: Quark-model assignments for some of the known baryons in terms of a flavor-spin SU(6) basis. Only the dominant representation is listed. Assignments for several states, especially for the Λ(1810), Λ(2350), Ξ(1820), and Ξ(2030), are merely educated guesses. † recent suggestions for assignments and re-assignments from ref. [28]. For assignments of the charmed baryons, see the "Note on Charmed Baryons" in the Particle Listings.

J^P	$(D, L^P_N)\, S$		Octet members			Singlets
$1/2^+$	$(56,0^+_0)$ 1/2	$N(939)$	Λ(1116)	Σ(1193)	Ξ(1318)	
$1/2^+$	$(56,0^+_2)$ 1/2	$N(1440)$	Λ(1600)	Σ(1660)	Ξ(1690)†	
$1/2^-$	$(70,1^-_1)$ 1/2	$N(1535)$	Λ(1670)	Σ(1620)	Ξ(?)	Λ(1405)
				Σ(1560)†		
$3/2^-$	$(70,1^-_1)$ 1/2	$N(1520)$	Λ(1690)	Σ(1670)	Ξ(1820)	Λ(1520)
$1/2^-$	$(70,1^-_1)$ 3/2	$N(1650)$	Λ(1800)	Σ(1750)	Ξ(?)	
				Σ(1620)†		
$3/2^-$	$(70,1^-_1)$ 3/2	$N(1700)$	Λ(?)	Σ(1940)†	Ξ(?)	
$5/2^-$	$(70,1^-_1)$ 3/2	$N(1675)$	Λ(1830)	Σ(1775)	Ξ(1950)†	
$1/2^+$	$(70,0^+_2)$ 1/2	$N(1710)$	Λ(1810)	Σ(1880)	Ξ(?)	Λ(1810)†
$3/2^+$	$(56,2^+_2)$ 1/2	$N(1720)$	Λ(1890)	Σ(?)	Ξ(?)	
$5/2^+$	$(56,2^+_2)$ 1/2	$N(1680)$	Λ(1820)	Σ(1915)	Ξ(2030)	
$7/2^-$	$(70,3^-_3)$ 1/2	$N(2190)$	Λ(?)	Σ(?)	Ξ(?)	Λ(2100)
$9/2^-$	$(70,3^-_3)$ 3/2	$N(2250)$	Λ(?)	Σ(?)	Ξ(?)	
$9/2^+$	$(56,4^+_4)$ 1/2	$N(2220)$	Λ(2350)	Σ(?)	Ξ(?)	
			Decuplet members			
$3/2^+$	$(56,0^+_0)$ 3/2	Δ(1232)	Σ(1385)	Ξ(1530)	Ω(1672)	
$3/2^+$	$(56,0^+_2)$ 3/2	Δ(1600)	Σ(1690)†	Ξ(?)	Ω(?)	
$1/2^-$	$(70,1^-_1)$ 1/2	Δ(1620)	Σ(1750)†	Ξ(?)	Ω(?)	
$3/2^-$	$(70,1^-_1)$ 1/2	Δ(1700)	Σ(?)	Ξ(?)	Ω(?)	
$5/2^+$	$(56,2^+_2)$ 3/2	Δ(1905)	Σ(?)	Ξ(?)	Ω(?)	
$7/2^+$	$(56,2^+_2)$ 3/2	Δ(1950)	Σ(2030)	Ξ(?)	Ω(?)	
$11/2^+$	$(56,4^+_4)$ 3/2	Δ(2420)	Σ(?)	Ξ(?)	Ω(?)	

In Table 14.6, quark-model assignments are given for many of the established baryons whose SU(6)⊗O(3) compositions are relatively unmixed. One must, however, keep in mind that apart from the mixing of the Λ singlet and octet states, states with same J^P but different L, S combinations can also mix. In the quark model with one-gluon exchange motivated interactions, the size of the mixing is determined by the relative strength of the tensor term with respect to the contact term (see below). The mixing is more important for the decay patterns of the states than for their positions. An example are the lowest lying $(70, 1_1^-)$ states with J^P=1/2$^-$ and 3/2$^-$. The physical states are:

$$|\, S_{11}(1535)\rangle = \cos(\Theta_S)|N^2P_M 1/2^-\rangle - \sin(\Theta_S)|N^4P_M 1/2^-\rangle \tag{14.26}$$

$$|\, D_{13}(1520)\rangle = \cos(\Theta_D)|N^2P_M 3/2^-\rangle - \sin(\Theta)_D|N^4P_M 3/2^-\rangle \tag{14.27}$$

and the orthogonal combinations for $S_{11}(1650)$ and $D_{13}(1700)$. The mixing is large for the J^P=1/2$^-$ states ($\Theta_S \approx$ -32^o), but small for the J^P=3/2$^-$ states ($\Theta_D \approx +6^o$) [26,31].

All baryons of the ground state multiplets are known. Many of their properties, in particular their masses, are in good agreement even with the most basic versions of the quark model, including harmonic (or linear) confinement and a spin-spin interaction, which is responsible for the octet - decuplet mass shifts. A consistent description of the ground-state electroweak properties, however, requires refined relativistic constituent quark models.

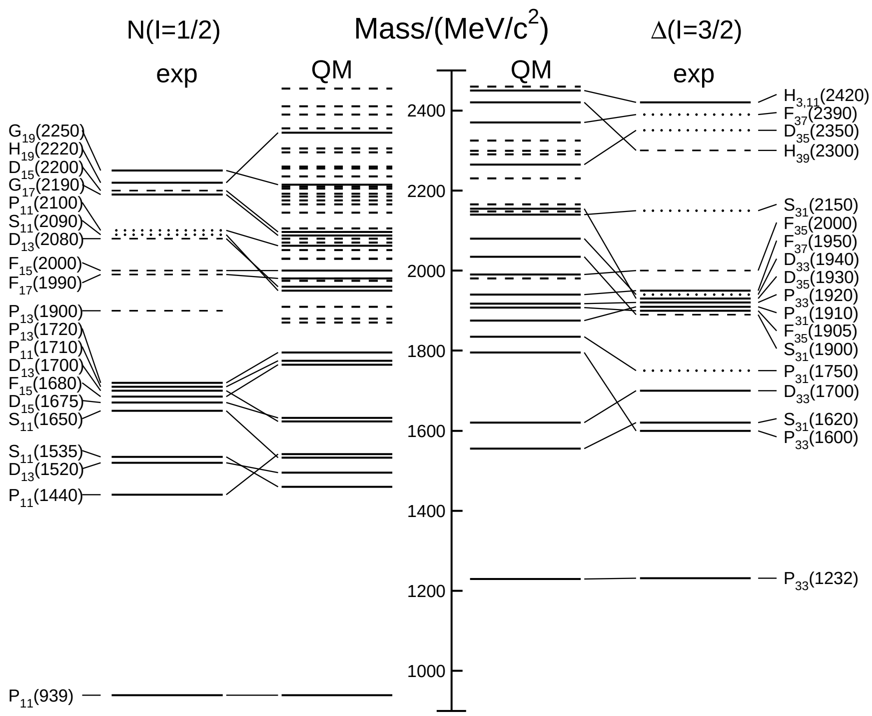

Figure 14.5: Excitation spectrum of the nucleon. Compared are the positions of the excited states identified in experiment, to those predicted by a relativized quark model calculation. Left hand side: isospin $I = 1/2$ N-states, right hand side: isospin $I = 3/2$ Δ-states. Experimental: (columns labeled 'exp'), three- and four-star states are indicated by full lines (two-star dashed lines, one-star dotted lines). At the very left and right of the figure, the spectroscopic notation of these states is given. Quark model [27]: (columns labeled 'QM'), all states for the N=1,2 bands, low-lying states for the N=3,4,5 bands. Full lines: at least tentative assignment to observed states, dashed lines: so far no observed counterparts. Many of the assignments between predicted and observed states are highly tentative.

The situation for the excited states is much less clear. The assignment of some experimentally observed states with strange quarks to model configurations is only tentative and in many cases candidates are completely missing. Recently, Melde, Plessas and Sengl [28] have calculated baryon properties in relativistic constituent quark models, using one-gluon exchange and Goldstone-boson exchange for the modeling of the hyperfine interactions (see Sec. 14.5 on Dynamics). Both types of models give qualitatively comparable results, and underestimate in general experimentally observed decay widths. Nevertheless, in particular on the basis of the observed decay patterns, the authors have assigned some additional states with strangeness to the SU(3) multiplets and suggest re-assignments for a few others. Among the new assignments are states with weak experimental evidence (two or three star ratings) and partly without firm spin/parity assignments, so that further experimental efforts are necessary before final conclusions can be drawn. We have added their suggestions in Table 14.6.

In the non-strange sector there are two main problems which are illustrated in Fig. 14.5, where the experimentally observed excitation spectrum of the nucleon (N and Δ resonances) is compared to the results of a typical quark model calculation [27]. The lowest states from the N=2 band, the $P_{11}(1440)$, and the $P_{33}(1600)$, appear lower than the negative parity states from the N=1 band (see Table 14.5) and much lower than predicted by most models. Also negative parity Δ states from the N=3 band ($S_{31}(1900)$, $D_{33}(1940)$, and $D_{35}(1930)$) are too low in energy. Part of the problem could be experimental. Among the negative parity Δ states, only the D_{35} has three stars and the uncertainty in the position of the $P_{33}(1600)$ is large (1550 - 1700 MeV).

Furthermore, many more states are predicted than observed. This has been known for a long time as the 'missing resonance' problem [26]. Up to an excitation energy of 2.4 GeV, about 45 N states are predicted, but only 12 are established (four- or three-star; see Note on N and Δ Resonances for the rating of the status of resonances) and 7 are tentative (two- or one-star). Even for the N=2 band, up to now only half of the predicted states have been observed. The most recent partial wave analysis of elastic pion scattering and charge exchange data by Arndt and collaborators [29] has made the situation even worse. They found no evidence for almost half of the states listed in this review (and included in Fig. 14.5). Such analyses are of course biased against resonances which couple only weakly to the $N\pi$ channel. Quark model predictions for the couplings to other hadronic channels and to photons are given in Ref. 27. A large experimental effort is ongoing at several electron accelerators to study the baryon resonance spectrum with real and virtual photon-induced meson production reactions. This includes the search for as-yet-unobserved states, as well as detailed studies of the properties of the low lying states (decay patterns, electromagnetic couplings, magnetic moments, *etc.*) (see Ref. 30 for recent reviews). This experimental effort has currently entered its final phase with the measurement of single and double polarization observables for many different meson production channels, so that a much better understanding of the experimental spectrum can be expected for the near future.

In quark models, the number of excited states is determined by the effective degrees of freedom, while their ordering and decay properties are related to the residual quark - quark interaction. An overview of quark models for baryons is given in Ref. 31, a recent discussion of baryon spectroscopy is given in Ref. 25. The effective degrees of freedom in the standard nonrelativistic quark model are three equivalent valence quarks with one-gluon exchange-motivated, flavor-independent color-magnetic interactions. A different class of models uses interactions which give rise to a quark - diquark clustering of the baryons (for a review see Ref. 32). If there is a tightly bound diquark, only two degrees of freedom are available at low energies, and thus *fewer* states are predicted. Furthermore, selection rules in the decay pattern may arise from the quantum numbers of the diquark. *More* states are predicted by collective models of the baryon like the algebraic approach in Ref. 33. In this approach, the quantum numbers of the valence quarks are distributed over a Y-shaped string-like configuration, and additional states arise *e.g.*, from vibrations of the strings. *More* states are also predicted in the framework of flux-tube models (see Ref. 34), which are motivated by lattice QCD. In addition to the quark degrees of freedom, flux-tubes responsible for the confinement of the quarks are considered as degrees of freedom. These models include hybrid baryons containing explicit excitations of the gluon fields. However, since all half integral J^P quantum numbers are possible for ordinary baryons, such 'exotics' will be very hard to identify, and probably always mix with ordinary states. So far, the experimentally observed number of states is still far lower even than predicted by the quark–diquark models.

Recently, the influence of chiral symmetry on the excitation spectrum of the nucleon has been hotly debated from a somewhat new perspective. Chiral symmetry, the fundamental symmetry of QCD, is strongly broken for the low lying states, resulting in large mass differences of parity partners like the J^P=1/2$^+$ $P_{11}(938)$ ground state and the J^P=1/2$^-$ $S_{11}(1535)$ excitation. However, at higher excitation energies there is some evidence for parity doublets and even some very tentative suggestions for full chiral multiplets of N^* and Δ resonances. An effective restoration of chiral symmetry at high excitation energies due to a decoupling from the quark condensate of the vacuum has been discussed (see Ref. 35 for recent reviews) as a possible cause. In this case, the mass generating mechanisms for low and high lying states would be essentially different. As a further consequence, the parity doublets would decouple from pions, so that experimental bias would be worse. However, parity doublets might also arise from the spin-orbital dynamics of the 3-quark system. Presently, the status of data does not allow final conclusions.

The most recent developments on the theory side are the first unquenched lattice calculations for the excitation spectrum discussed in Sec. 14.6. The results are basically consistent with the level counting of SU(6)⊗O(3) in the standard non-relativistic quark model and show no indication for quark-diquark structures or parity doubling. Consequently, there is as yet no indication from lattice that the mis-match between the excitation spectrum predicted by the standard quark model and experimental observations is due to inappropriate degrees of freedom in the quark model.

14.5. Dynamics

Quantum chromodynamics (QCD) is well-established as the theory for the strong interactions. As such, one of the goals of QCD is to predict the spectrum of strongly-interacting particles. To date, the only first-principles calculations of spectroscopy from QCD use lattice methods. These are the subject of Sec. 14.6. These calculations are difficult and unwieldy, and many interesting questions do not have a good lattice-based method of solution. Therefore, it is natural to build models, whose ingredients are abstracted from QCD, or from the low-energy limit of QCD (such as chiral Lagrangians) or from the data itself. The words "quark model" are a shorthand for such phenomenological models. Many specific quark models exist, but most contain a similar basic set of dynamical ingredients. These include:

i) A confining interaction, which is generally spin-independent (*e.g.*, harmonic oscillator or linear confinement);

ii) Different types of spin-dependent interactions:

a) commonly used is a color-magnetic flavor-independent interaction modeled after the effects of gluon exchange in QCD (see *e.g.*, Ref. 36). For example, in the S-wave states, there is a spin-spin hyperfine interaction of the form

$$H_{HF} = -\alpha_S M \sum_{i>j} (\vec{\sigma}\lambda_a)_i (\vec{\sigma}\lambda_a)_j \ , \tag{14.28}$$

where M is a constant with units of energy, λ_a $(a = 1, \cdots, 8,)$ is the set of SU(3) unitary spin matrices, defined in Sec. 37, on "SU(3) Isoscalar Factors and Representation Matrices," and the sum runs over constituent quarks or antiquarks. Spin-orbit interactions, although allowed, seem to be small in general, but a tensor term is responsible for the mixing of states with the same J^P but different L, S combinations.

b) other approaches include flavor-dependent short-range quark forces from instanton effects (see *e.g.*, Ref. 37). This interaction acts only on scalar, isoscalar pairs of quarks in a relative S-wave state:

$$\langle q^2; S, L, T|W|q^2; S, L, T\rangle = -4g\delta_{S,0}\delta_{L,0}\delta_{I,0}\mathcal{W} \tag{14.29}$$

where $\mathcal{W}$ is the radial matrix element of the contact interaction.

c) a rather different and controversially discussed approach is based on flavor-dependent spin-spin forces arising from one-boson exchange. The interaction term is of the form:

$$H_{HF} \propto \sum_{i<j} V(\overrightarrow{r}_{ij})\lambda_i^F \cdot \lambda_j^F \overrightarrow{\sigma}_i \cdot \overrightarrow{\sigma}_j \tag{14.30}$$

where the λ_i^F are in flavor space (see *e.g.*, Ref. 38).

iii) A strange quark mass somewhat larger than the up and down quark masses, in order to split the SU(3) multiplets;

iv) In the case of spin-spin interactions (iia,c), a flavor-symmetric interaction for mixing $q\overline{q}$ configurations of different flavors (*e.g.*, $u\overline{u} \leftrightarrow d\overline{d} \leftrightarrow s\overline{s}$), in isoscalar channels, so as to reproduce *e.g.*, the η - η' and ω - ϕ mesons.

These ingredients provide the basic mechanisms that determine the hadron spectrum in the standard quark model.

14.6. Lattice Calculations of Hadronic Spectroscopy

Lattice calculations are a major source of information about QCD masses and matrix elements. The necessary theoretical background is given in Sec. 17 of this *Review*. Here we confine ourselves to some general comments and illustrations of lattice calculations for spectroscopy.

In general, the cleanest lattice results come from computations of processes in which there is only one particle in the simulation volume. These quantities include masses of hadrons, simple decay constants, like pseudoscalar meson decay constants, and semileptonic form factors (such as the ones appropriate to $B \to Dl\nu$, $Kl\nu$, $\pi l\nu$). The cleanest predictions for masses are for states which have narrow decay widths and are far below any thresholds to open channels, since the effects of final state interactions are not yet under complete control on the lattice. As a simple corollary, the lightest state in a channel is easier to study than the heavier ones. "Difficult" states for the quark model (such as exotics) are also difficult for the lattice because of the lack of simple operators which couple well to them.

Good-quality modern lattice calculations will present multi-part error budgets with their predictions. A small part of the uncertainty is statistical, from sample size. Typically, the quoted statistical uncertainty includes uncertainty from a fit: it is rare that a simulation computes one global quantity which is the desired observable. Simulations which include virtual quark-antiquark pairs (also known as "dynamical quarks" or "sea quarks") are often done at up and down quark mass values heavier than the experimental ones, and it is then necessary to extrapolate in these quark masses. Simulations can work at the physical values of the heavier quarks' masses. They are always done at nonzero lattice spacing, and so it is necessary to extrapolate to zero lattice spacing. Some theoretical input is needed to do this. Much of the uncertainty in these extrapolations is systematic, from the choice of fitting function. Other systematics include the effect of finite simulation volume, the number of flavors of dynamical quarks actually simulated, and technical issues with how these dynamical quarks are included. The particular choice of a fiducial mass (to normalize other predictions) is not standardized; there are many possible choices, each with its own set of strengths and weaknesses, and determining it usually requires a second lattice simulation from that used to calculate the quantity under consideration.

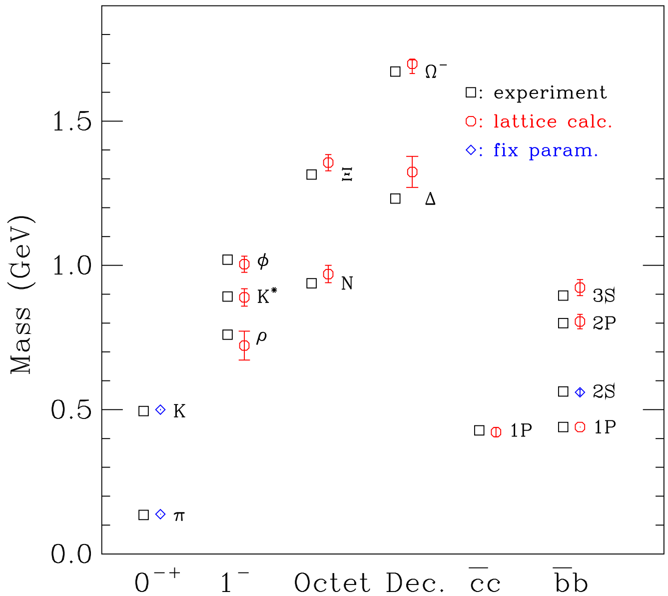

Figure 14.6: A recent calculation of spectroscopy with dynamical u, d, and s quarks. The pion and kaon fix the light quark masses. Only the mass splittings relative to the $1S$ states in the heavy quark sectors are shown. The Υ $2S-1S$ splitting sets the overall energy scale.

A systematic error of major historical interest is the "quenched approximation," in which dynamical quarks are simply left out of the simulation. This was done because the addition of these virtual pairs presented an expensive computational problem. No generally-accepted methodology has ever allowed one to correct for quenching effects, short of redoing all calculations with dynamical quarks. Recent advances in algorithms and computer hardware have rendered it obsolete.

With these brief remarks, we turn to examples. The field of lattice QCD simulations is vast, and so it is not possible to give a comprehensive review of them in a small space. The history of lattice QCD simulations is a story of thirty years of incremental improvements in physical understanding, algorithm development, and ever faster computers, which have combined to bring the field to a present state where it is possible to carry out very high quality calculations. We present a few representative illustrations, to show the current state of the art.

By far, the major part of all lattice spectroscopy is concerned with that of the light hadrons, and so we illustrate results from two groups. First, a recent calculation of spectroscopy with dynamical u, d, and s quarks is shown in Fig. 14.6. The pion and kaon masses are used to set the light quark masses. The Υ $2S-1S$ splitting is used to set the lattice spacing or equivalently, the overall energy scale in the lattice calculation. This is an updated figure from Ref. 39, using results from Ref. 41 and Ref. 42 (D. Toussaint, private communication).

These results come from simulations using dynamical up and down quarks which are heavier than their physical values. As a result, the error bars on all the particles which decay strongly and are above their decay thresholds (the vector mesons and the Δ, for example) do not include the effect of coupling to the decay channels.

A more recent result by Ref. 40 goes farther, in that its simulations include the coupling of resonances to open channels in their analysis. Their plot of light hadron spectroscopy is shown in Fig. 14.7.

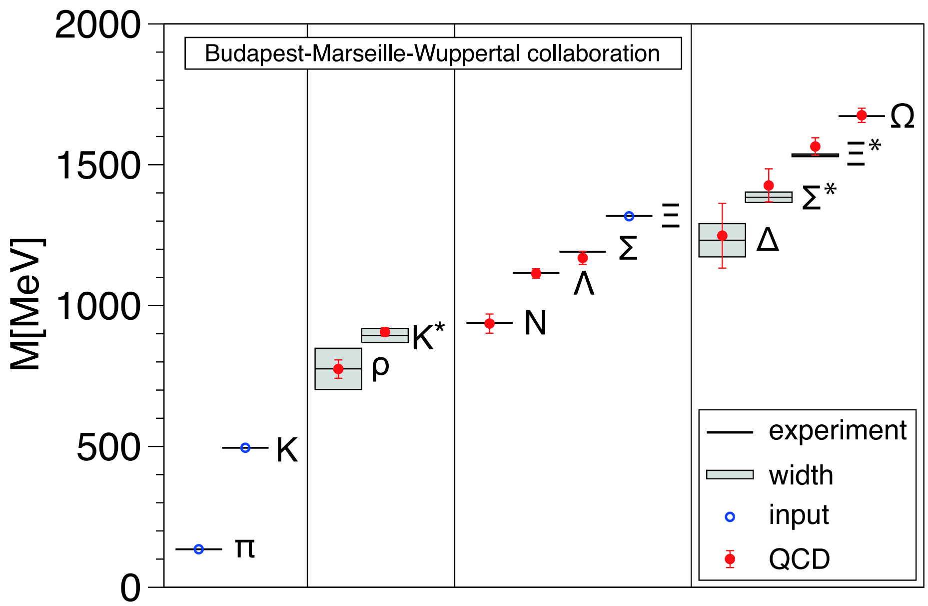

Figure 14.7: Light hadron spectroscopy from Ref. 40.

Flavor singlet mesons are at the frontier of lattice QCD calculations, because one must include the effects of "annihilation graphs," for the valence q and $\bar{q}$. Recently, the RBC and UKQCD collaborations, Ref. 6, have reported a calculation of the η and η' mesons, finding masses of 573(6) and 947(142) MeV, respectively. The singlet-octet mixing angle (in the conventions of Table 14.2) is $\theta_{lin} = -14.1(2.8)^\circ$.

The spectroscopy of mesons containing heavy quarks has become a truly high-precision endeavor. These simulations use Non-Relativistic QCD (NRQCD) or Heavy Quark Effective Theory (HQET), systematic expansions of the QCD Lagrangian in powers of the heavy quark velocity, or the heavy quark mass. Terms in the Lagrangian have obvious quark model analogs, but are derived directly from QCD. For example, the heavy quark potential is a derived quantity, extracted from simulations. Fig. 14.8 shows the mass spectrum for mesons containing at least one heavy (b or c) quark from Ref. 42 and Ref. 43. The calculations uses a discretization of nonrelativistic QCD for bottom quarks with charm and lighter quarks being handled with an improved relativistic action. Three flavors of light dynamical quarks are included.

Finally, Fig. 14.9 shows recent lattice calculations of singly and double charmed baryons. Here we are at the forefront of theory and experiment.

Recall that lattice calculations take operators which are interpolating fields with quantum numbers appropriate to the desired states, compute correlation functions of these operators, and fit the correlation functions to functional forms parameterized by a set of masses and matrix elements. As we move away from hadrons which can be created by the simplest quark model operators (appropriate to the lightest meson and baryon multiplets) we encounter a host of new problems: either no good interpolating fields, or too many possible interpolating fields, and many

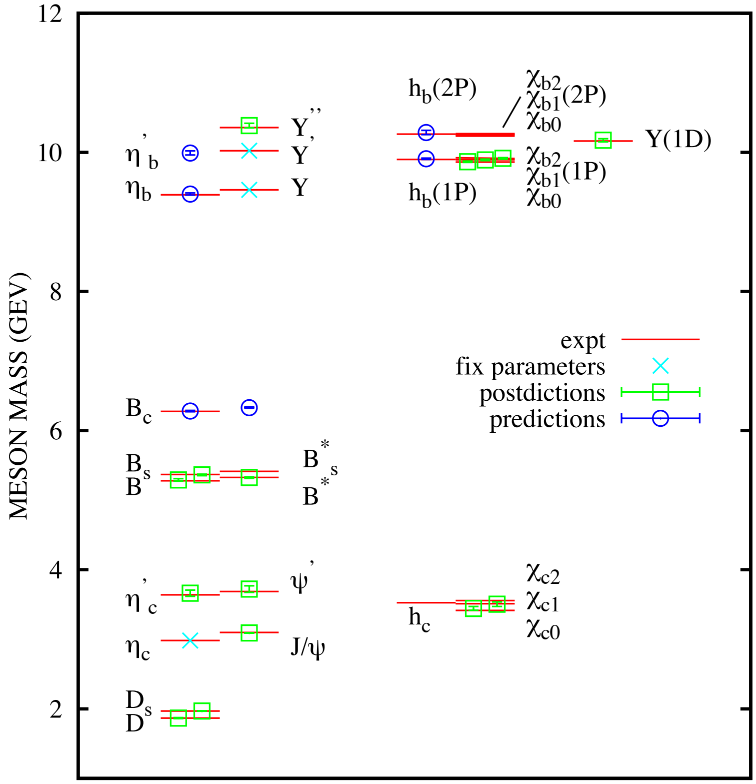

Figure 14.8: Spectroscopy for mesonic systems containing one or more heavy quarks (adapted from Ref. 42 and Ref. 43). Particles whose masses are used to fix lattice parameters are shown with crosses; the authors distinguish between "predictions" and "postdictions" of their calculation. Lines represent experiment.

states with the same quantum numbers. Techniques for dealing with these interrelated problems vary from collaboration to collaboration, but all share common features: typically, correlation functions from many different interpolating fields are used, and the signal is extracted in what amounts to a variational calculation using the chosen operator basis. In addition to mass spectra, wave function information can be garnered from the form of the best variational wave function. Of course, the same problems which are present in the spectroscopy of the lightest hadrons (the need to extrapolate to infinite volume, physical values of the light quark masses, and zero lattice spacing) are also present. We briefly touch on three different kinds of hadrons: excited states of baryons, glueballs, and hybrid mesons. The quality of the data is not as good as for the ground states, and so the results continue to evolve.

Ref. 49 is a good recent review of excited baryon spectroscopy. The interesting physics questions to be addressed are precisely those enumerated in the last section. An example of a recent calculation, due to Ref. 50 is shown in Fig. 14.10. Notice that the pion is not yet at its physical value. The lightest positive parity state is the nucleon, and the Roper resonance has not yet appeared as a light state.

Exotic mesons share the difficulties of ordinary excited states, and some recent calculations actually include both kinds of states in their combined fits. Ref. 51 provides a good summary of the theoretical and experimental situation regarding mesons with exotic quantum numbers, including a compilation of lattice data. The lightest exotics, the h_0, η_1, and h_2, have long been

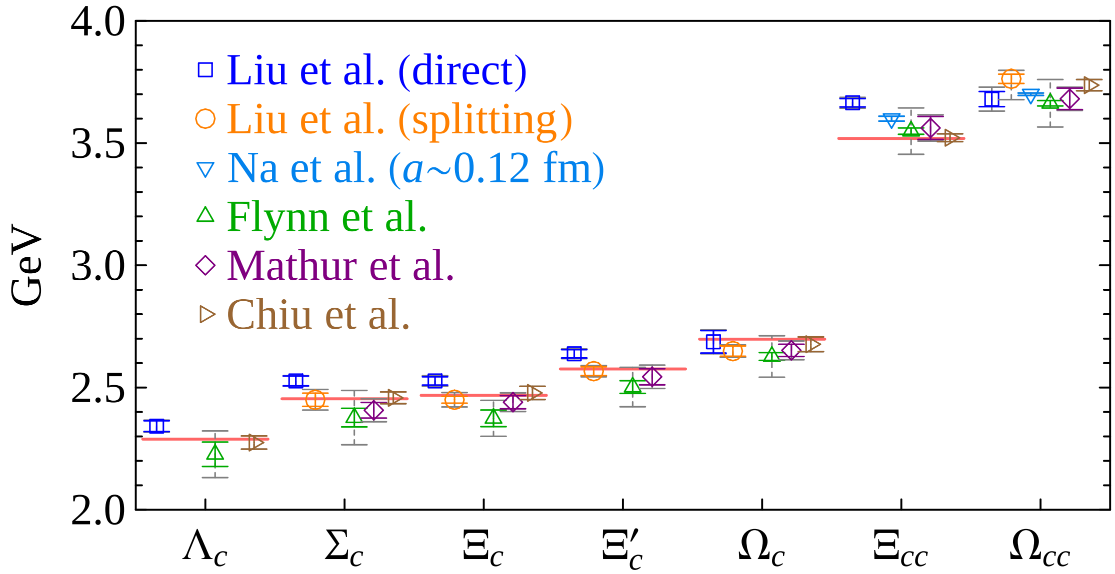

Figure 14.9: Lattice predictions for masses of charmed baryons. Data are Liu, *et al.*, Ref. 44; Na *et al.*, Ref. 45; Flynn *et al.*, Ref. 46; Mathur *et al.*, Ref. 47; and Chiu *et al.*, Ref. 48. The first two references use full QCD; the latter three are quenched. Two mass extractions are taken from Ref. 44; the lighter (orange) circular points come from a calculation of mass splittings while the darker (blue) square points are from a direct mass extrapolation. Lines are from experiment.

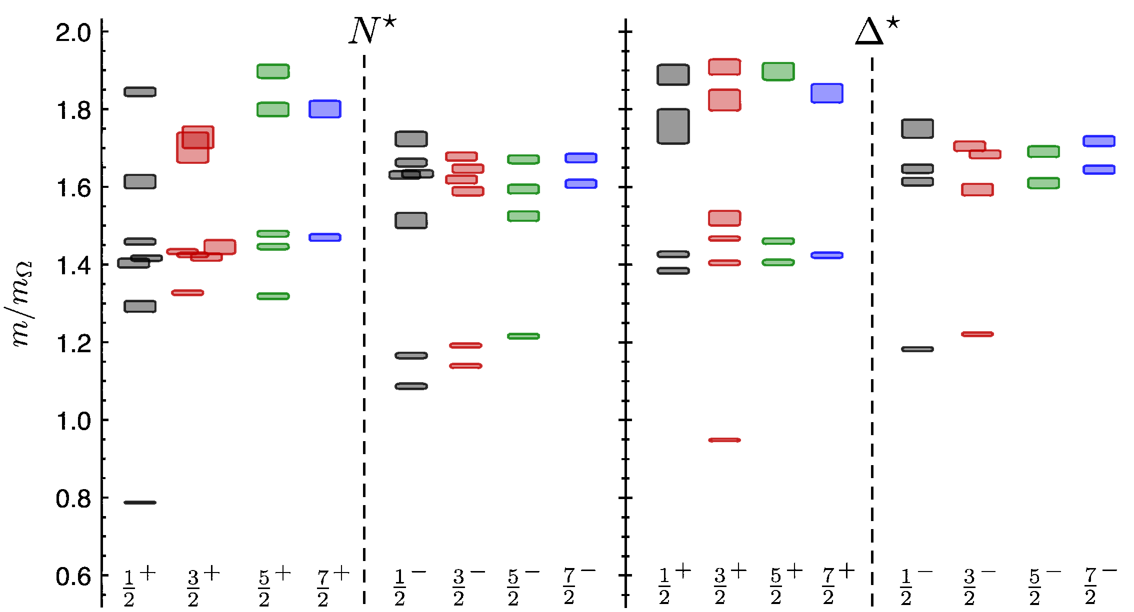

Figure 14.10: Spin-identified spectrum of nucleons and deltas, from lattices where $m_\pi = 396$ MeV, in units of the calculated Ω mass, from Ref. 50. The colors just correspond to the different J assignments: grey for $J = 1/2$, red for $J = 3/2$, green for $5/2$, blue for $J = 7/2$.

targets of lattice studies. Recently, the authors of Ref. 52 have presented new results for isoscalar and isovector meson spectroscopy, which observe the three states around 2 GeV. Again, the light quark masses in the simulations are higher than in nature; the pion is at 396 MeV.

Finally, glueballs. In Fig. 14.3 we showed a figure from Ref. 10 showing a lattice prediction for the glueball mass spectrum in quenched approximation. A true QCD prediction of the glueball spectrum requires dynamical light quarks and (because glueball operators are intrinsically noisy) high statistics. Only recently have the first useful such calculations appeared. Fig. 14.11 shows results from Ref. 53, done with dynamical u, d and s quarks at two lattice spacings, 0.123 and 0.092 fm, along with comparisons to the quenched lattice calculation of Ref. 11 and to experimental isosinglet mesons. The dynamical simulation is, of course, not the last word on this subject, but it shows that the effects of quenching seem to be small.

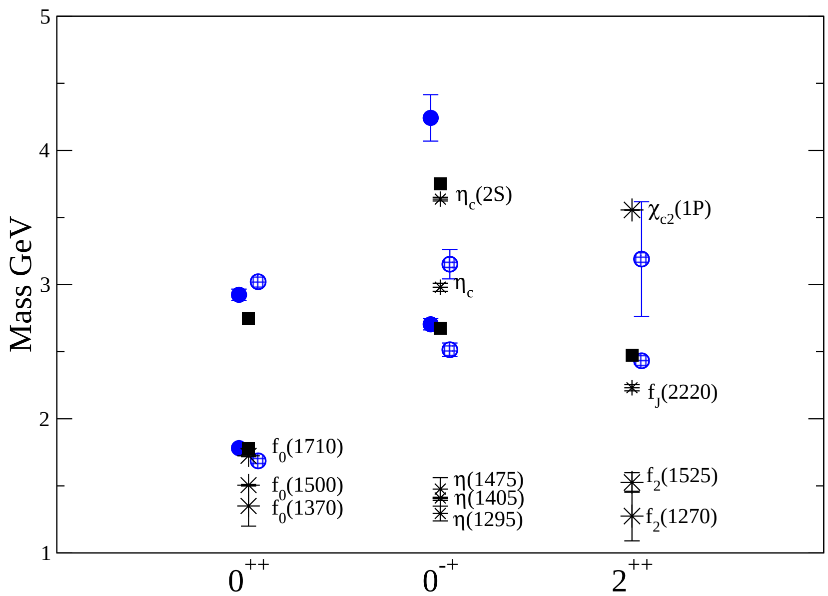

Figure 14.11: Lattice QCD predictions for glueball masses. The open and closed circles are the larger and smaller lattice spacing data of the full QCD calculation of glueball masses of Ref. 53. Squares are the quenched data for glueball masses of Ref. 11. The bursts labeled by particle names are experimental states with the appropriate quantum numbers.

References:

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