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A NEW CHARACTERIZATION OF THE CLOSURE OF THE $(\mathscr{U}+\mathscr{K})$–ORBIT OF CERTAIN ESSENTIALLY NORMAL OPERATORS

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Abstract. The $(\mathscr{U}+\mathscr{K})$-orbit of a Hilbert space operator T is defined as $(\mathscr{U}+\mathscr{K})(T)=\{R^{-1}TR : R$ invertible of the form unitary plus compact$\}$. In this paper, we show that certain essentially normal operator with the same spectral picture as an essentially normal injective unilateral weighted operator generates the same closure of $(\mathscr{U}+\mathscr{K})$-orbit.

1. Introduction

Let H be a separable-dimensional Hilbert space over the complex field $\mathbb{C}$ and $\mathscr{L}(H)$ (resp. $\mathscr{K}(H)$) denote the algebra of all bounded linear operators on H (resp. the algebra of all compact operators on H). $T\in\mathscr{L}(H)$ is said to be essentially normal if $T^*T-TT^*\in\mathscr{K}(H)$.

For $T\in\mathscr{L}(H)$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, $\sigma_0(T)$, $\sigma_W(T)$, $\rho_{sF}(T)$, $\rho^r_{sF}(T)$ and $\rho_F(T)$ denote the spectrum, the point spectrum, the essential spectrum, the normal eigenvalues, the Weyl spectrum, the semi-Fredholm domain, the regular points of semi-Fredholm domain and the Fredholm domain of T, respectively. Let $H(\lambda;T)$ denote the Riesz eigenspace corresponding $\lambda\in\sigma_0(T)$. Let $\operatorname{nul}T=\dim\ker T$.

If $\lambda\in\rho_F(T)$, the index of $T-\lambda I$ is defined as

$$\operatorname{ind}(T-\lambda I)=\operatorname{nul}(T-\lambda I)-\operatorname{nul}(T-\lambda I)^*;$$

the minimum index of $T-\lambda I$ is

$$\min\operatorname{ind}(T-\lambda I)=\min\{\operatorname{nul}(T-\lambda I),\operatorname{nul}(T-\lambda I)^*\}.$$

Readers can refer to [8] for more information on the behaviour of these.

There are many ways to describe the equivalence relations of operators on H. Here, we are mostly interested in the $(\mathscr{U}+\mathscr{K})-$ equivalence of operators first introduced by D. A. Herrero ([9]). Let $(\mathscr{U}+\mathscr{K})(H)=\{X\in\mathscr{L}(H) : X$ is invertible with the form unitary plus compact$\}$.

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The $(\mathscr{U}+\mathscr{K})$-orbit of $T \in \mathscr{L}(H)$ is given by $(\mathscr{U}+\mathscr{K})(T) = \{XTX^{-1} : X \in (\mathscr{U}+\mathscr{K})(H)\}$. $\overline{(\mathscr{U}+\mathscr{K})(T)}$ denotes the norm closure of $(\mathscr{U}+\mathscr{K})(T)$. Let $A \in \mathscr{L}(H')$ for some Hilbert space H', We will write $A \cong_{\mathscr{U}+\mathscr{K}} T$ when $A \in (\mathscr{U}+\mathscr{K})(T)$ and $A \cong T$ if there is a unitary operator $U : H' \to H$ such that $A = U^*TU$.

The closures of $(\mathscr{U}+\mathscr{K})$-orbits of essentially normal operators on a Hilbert space have been studied by many authors (see, for example [1], [4], [5], [6], [11], [12], [14], [15], and [16]). For a survey on $(\mathscr{U}+\mathscr{K})$-orbit of operator, the reader is referred to [15]. In particular, L.W. Marcoux proved that an essentially normal operator T with the same spectral picture as unilateral shift operator S generates the same closure of $(\mathscr{U}+\mathscr{K})$-orbit as S in [14], and gave a conjecture $\overline{(\mathscr{U}+\mathscr{K})(M)} = \overline{(\mathscr{U}+\mathscr{K})(N)}$ whenever N and M are essentially normal and have same spectral picture in [15, Question 2].

Note that the essentially normal operator T with the same spectral picture as the essentially normal injective unilateral weighted shift operator W is different from the structure and essential spectrum of operators or operator models whose closure of $(\mathscr{U}+\mathscr{K})$-orbit were characterized in [1], [4], [5], [6], [11], [12], [14], [15], and [16], respectively. Thus, in this paper, we will consider Marcoux's conjecture for certain essentially normal operator T whose spectral picture is identical to that of an essentially normal injective unilateral weighted operator.

Throughout, for $T \in \mathscr{L}(H)$, let $H_r(T) = \overline{\text{span}}\{\ker(T-\lambda I) : \lambda \in \rho_{sF}^r(T)\}$, $H_l(T) = \overline{\text{span}}\{\ker(T-\lambda I)^* : \lambda \in \rho_{sF}^r(T)\}$, $H_0(T) = (H_r(T) \oplus H_l(T))^{\perp}$. $T_l = T|_{H_l(T)}$ and $T_0 = T|_{H_0(T)}$ denote the compression of T to $H_l(T)$ and $H_0(T)$, respectively. If $A, B \in \mathscr{L}(H)$, $\tau_{AB}(X) = AX - XB$ for $X \in \mathscr{L}(H)$ denotes Rosenblum operator. Let $\text{Rat}(\Omega)$ denote the set of rational functions of $\mathbb{C}$ whose poles lie outside of $\Omega \subset \mathbb{C}$, and $C(\Omega)$ denote the set of continuous functions on $\Omega \subset \mathbb{C}$.

Let W, $\alpha > 0$ and β be same as in [6, Theorem 3.4], that is, W is an essentially normal injective unilateral weighted operator with weight sequence $\{w_n\}_{n=1}^{\infty}$ satisfying $0 < \liminf_n \{w_n\} = \alpha$ and $\beta = \limsup_n \{w_n\}$. Let $\mathscr{A} = \{T : T \in \mathscr{L}(H)$ satisfies (i) T is essentially normal; (ii) $\sigma(T) = \{z \in \mathbb{C} : |z| \leqslant \beta\}$, $\sigma_e(T) = \{z \in \mathbb{C} : \alpha \leqslant |z| \leqslant \beta\}$; (iii) $\text{ind}(T-\lambda) = -1$ for all $z \in \mathbb{C}, |z| < \alpha$; (iv) $\text{nul}\,(T-z) = 0$ for all $z \in \mathbb{C}, |z| < \alpha$; (v) T_l is an essentially normal operator.$\}$, $\mathscr{B} = \{T : T \in \mathscr{L}(H)$ satisfies conditions (i), (ii), (iii) of $\mathscr{A}\}$.

We prove the following theorem.

THEOREM. *If $T \in \mathscr{A}$, $C(\sigma(T|_{H_l(T)^{\perp}})) = \overline{\text{Rat}(\sigma(T|_{H_l(T)^{\perp}}))}$. Then $\overline{(\mathscr{U}+\mathscr{K})(T)} = \mathscr{B}$.*

By [6, Theorem 3.4], then $\overline{(\mathscr{U}+\mathscr{K})(W)} = \overline{(\mathscr{U}+\mathscr{K})(T)}$ if T satisfies the conditions of Theorem. In addition, by the following example, we know that there exists an essentially normal operator satisfying the conditions of Theorem.

EXAMPLE. Let $\Gamma_1 \subset \{z \in \mathbb{C} : |z| = \alpha\}$ and $\Gamma_2 \subset \{z \in \mathbb{C} : |z| = \beta\}$ be compact sets, respectively. Let D_{Γ_1} and D_{Γ_2} be the diagonal operators satisfying $\sigma(D_{\Gamma_1}) = \sigma_e(D_{\Gamma_1}) = \Gamma_1$, $\sigma(D_{\Gamma_2}) = \sigma_e(D_{\Gamma_2}) = \Gamma_2$, respectively. Let $D = D_{\Gamma_1} \oplus D_{\Gamma_2}$,

K be a compact operator, $H_l=\overline{\text{span}}\{\ker(W-\lambda I)^*:\lambda\in\rho_{sF}(W)\}$, $H_0=H_l^{\perp}$,

$$T=\begin{bmatrix} D & K \\ 0 & W \end{bmatrix}\begin{matrix} H_0 \\ H_l \end{matrix}.$$

Then $T\in\mathscr{A}$. Moreover, $\ker(T-\lambda)^*=0\oplus\ker(W-\lambda)^*$ for $|\lambda|<\alpha$, $H_l(T)=H_l$. Note that $T|H_l^{\perp}=D$, $\sigma(D)$ is a prefect set with planar Lebesgue measure 0, by [10], then $C(\sigma(T|H_l^{\perp}))=\overline{\text{Rat}(\sigma(T|_{H_l(T)^{\perp}}))}$.

2. Preliminaries

In order to simplify the proof of Theorem, we need the following lemmas. For convenience, we list [1, Theorem 4.15] ([14, P.1213]) and the claim in the proof of [11, Theorem 1] as our Lemma 2.1 and Lemma 2.2, respectively.

LEMMA 2.1. ([1, Theorem 4.15]). *Suppose* $T\in\mathscr{L}(H)$ *is essentially normal and that* $\sigma(T)=\sigma_e(T)\cup\sigma_0(T)$. *Assume, moreover, that* $C(\sigma(T))=\overline{\text{Rat}(\sigma(T))}$. *Then* $N\in\overline{(\mathscr{U}+\mathscr{K})(T)}$, *where* N *is a normal operator such that* $\sigma(N)=\sigma(T)$, $\sigma_0(N)=\sigma_0(T)$, *and* $\text{nul}(\lambda I-N)=\dim H(\lambda;T)$ *for all* $\lambda\in\sigma_0(N)$.

LEMMA 2.2. (The claim in the proof of [11, Theorem 1]). *If* $0\in\Omega$ *and* $T\in\mathscr{B}_n(\Omega)$ *is essentially normal,* $T_1=T|_{H\ominus\ker T^{k_0}}$. *Let* A *be an essentially operator whose spectral picture is identical to that of* T. $A_0=A|_{\ker(A^{k_0})}$ *is a* $k_0n\times k_0n$ *upper triangular matrix with the diagonal entries are zeros,* F *is a finite rank operator. Then for given* $\varepsilon>0$, *there exists a compact operator* K *with* $\|K\|<\varepsilon$ *such that*

$$\begin{bmatrix} A_0 & F \\ 0 & T_1 \end{bmatrix}+K\cong_{\mathscr{U}+\mathscr{K}}T.$$

where Ω *is a bounded connected open subset of* $\mathbb{C}$ *and* n *is a positive number.* $\mathscr{B}_n(\Omega)$ *denotes the set of operators* $R\in\mathscr{L}(H)$ *which satisfy* (a) $\Omega\subset\sigma(R)$; (b) *ran* $(T-\lambda)=H$ *for all* $\lambda\in\Omega$; (c) $\bigvee\{\ker(R-\lambda):\lambda\in\Omega\}=H$; (d) $\text{nul}(R-\lambda)=n$ *for all* $\lambda\in\Omega$.

LEMMA 2.3. *Let* $T\in\mathscr{A}$, N *be a diagonal operator of uniform infinite multiplicity satisfying* $\sigma(N)=\sigma_e(N)=\sigma(T)$. *Then* (i) T_0 *is an essentially normal operator and* $\sigma(T_0\oplus N)=\sigma_e(T_0\oplus N)=\sigma_e(T)$; (ii) T_l *is a lower triangular operator,* $\sigma(T_l\oplus N)=\sigma(T)$, $\sigma_e(T_l\oplus N)=\sigma_e(T)$, $\text{nul}(T_l-\lambda I)=0$ *and* $\text{nul}(T_l-\lambda I)^*=1$ *for all* $\lambda\in\sigma(T_l)\setminus\sigma_e(T_l)$, *and* $T_l^*\in\mathscr{B}_1(\Omega)$, *where* $\Omega=\{z\in\mathbb{C}:|z|<\alpha\}$; (iii) $\overline{(\mathscr{U}+\mathscr{K})(T)}\subset\overline{(\mathscr{U}+\mathscr{K})(T_l\oplus N)}$. *Moreover, if* $C(\sigma(T|_{H_l(T)^{\perp}}))=\overline{\text{Rat}(\sigma(T|_{H_l(T)^{\perp}}))}$, *then* $\overline{(\mathscr{U}+\mathscr{K})(T)}=\overline{(\mathscr{U}+\mathscr{K})(T_l\oplus N)}$.

Proof. Since $T\in\mathscr{A}$, by [8, Theorem 3.38], we know that T is of the form

$$T=\begin{bmatrix} T_0 & B \\ 0 & T_l \end{bmatrix}\begin{matrix} H_0(T) \\ H_l(T) \end{matrix}$$

with respect to the decomposition $H = H_0(T) \oplus H_l(T)$, T_l is lower triangular, $\sigma(T_0) \subset \sigma_e(T)$, $\sigma_p(T_l) = \emptyset$.

By the assumptions on T and T_l, one has T and T_l are essentially normal operators, respectively; by simple computation, the BB^* is a compact operator. Thus B is a compact operator, T_0 is an essentially normal operator and $\sigma_e(T) = \sigma_e(T_0) \cup \sigma_e(T_l)$.

Since B is a compact operator, we have $\sigma(T_0 \oplus T_l) \subset \sigma(T) \cup \sigma_p(T_0 \oplus T_l) = \sigma(T) \cup \sigma_p(T_0) \cup \sigma_p(T_l)$, thus $\sigma(T_0 \oplus T_l) \subset \sigma(T)$. By [7, Problem 56], then $\sigma(T) = \sigma(T_0) \cup \sigma(T_l)$. Let $\Omega = \{z \in \mathbb{C} : |z| < \alpha\}$, then $\Omega \subset \sigma(T_l^*)$, $\operatorname{ran}(T_l - \lambda I)^* = H_l(T)$ and $\operatorname{nul}(T_l - \lambda I)^* = 1$ for all $\lambda \in \Omega$.

By the above discussion, the proof of (i) and (ii) are completed.

Since $H_l(T) = \overline{\operatorname{span}}\{\ker(T_l - \lambda I)^* : \lambda \in \Omega\}$, by [13, Proposition 1.41], then

$$\overline{\operatorname{span}}\{\ker[(T_l - \lambda_0 I)^*]^n : n \geqslant 1\} = H_l(T) \text{ for all } \lambda_0 \in \Omega.$$

By [8, Theorem 3.38], $\sigma(T_0) \subset \sigma_e(T)$, note that [13, Proposition 1.14], then $\ker \tau_{T_0^* T_l^*} = \{0\}$, $\ker \tau_{T_l T_0} = \{0\}$. Since B is compact, by [13, Lemma 1.10], there exist compact operators Z and K_1', $\|K_1'\| < \varepsilon/3$ such that $T_0 Z - Z T_l = B + K_1'$. Let

$$X_1 = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & K_1' \\ 0 & 0 \end{bmatrix},$$

then $X_1 \in (\mathscr{U} + \mathscr{K})(H_0(T) \oplus H_l(T))$, K_1 is a compact operator, $\|K_1\| < \varepsilon/3$,

$$X_1(T + K_1))X_1 = T_0 \oplus T_l.$$

By [2, Lemma 2.1], there exist a unitary operator U_1 and a compact operator K_2 with $\|K_2\| < \varepsilon/(3\|X_1\|\|X_1^{-1}\|)$ such that

$$U_1(T_0 \oplus T_l + K_2)U_1^* = (N \oplus T_0) \oplus T_l.$$

Let $T_0' = N \oplus T_0$, then $\sigma(T_0') = \sigma_e(T_0') = \sigma_e(T)$. Note that $\sigma_0(T_0') = \emptyset$, $\sigma(T_0')$ is prefect, by [8, Corollary 4.2], T_0' is of the form normal plus compact. By the proof of [5, Theorem 2.3], then there exist a compact operator K_3' with $\|K_3'\| < \varepsilon/(3\|X_1\|\|X_1^{-1}\|)$ and an invertible operator X_2' with form unitary plus compact such that $X_2'(T_0' + K_3')(X_2')^{-1} = N$, thus there exist a compact operator K_3 with $\|K_3\| < \varepsilon/(3\|X_1\|\|X_1^{-1}\|)$ and an invertible operator X_2 with form unitary plus compact such that $X_2(U_1(T_0 \oplus T_l + K_2)U_1^* + K_3))X_2^{-1} = N \oplus T_l$.

By the above proof, $\overline{(\mathscr{U} + \mathscr{K})(T)} \subset \overline{(\mathscr{U} + \mathscr{K})(T_l \oplus N)}$.

If $C(\sigma(T|_{H_l(T)^\perp})) = \overline{\operatorname{Rat}(\sigma(T|_{H_l(T)^\perp}))}$, let N_0 be a diagonal operator of uniform infinite multiplicity satisfying $\sigma(N_0) = \sigma_e(N_0) = \sigma(T_0)$. Then $N_0 \oplus T_0$ is essentially normal, $\sigma(N_0 \oplus T_0) = \sigma_e(N_0 \oplus T_0) = \sigma(T_0)$, $C(\sigma(N_0 \oplus T_0)) = \overline{\operatorname{Rat}(\sigma(N_0 \oplus T_0))}$. $\sigma(N_0) \subset \sigma_e(T_l)$, apply Lemma 2.1 to N_0 and $N_0 \oplus T_0$, note that [5, Proposition 2.7] and B is compact, we can imply

$$\overline{(\mathscr{U} + \mathscr{K})(N \oplus T_l)} \subset \overline{(\mathscr{U} + \mathscr{K})(N_0 \oplus N \oplus T_l)} \subset$$

$$\overline{(\mathscr{U} + \mathscr{K})(N_0 \oplus T_0 \oplus N \oplus T_l)} \subset \overline{(\mathscr{U} + \mathscr{K})(T_0 \oplus T_l)} \subset \overline{(\mathscr{U} + \mathscr{K})(T)}.$$

The proof of (iii) is completed. ∎

LEMMA 2.4. *Let $T \in \mathscr{A}$, $N \in \mathscr{L}(H)$ be the same as in Lemma 2.3, and S be a unilateral forward shift operator. Then for given $\varepsilon > 0$, there exist a natural number k and a compact operator K with $\|K\| < \varepsilon$ such that*

$$N + K \cong_{\mathscr{U}+\mathscr{K}} \begin{bmatrix} \beta S^* & 0 \\ B_{21} & T_l' \end{bmatrix} \oplus N,$$

where $T_l' = T_l|_{H_l(T)\ominus \operatorname{span}\{\ker((T_l-\lambda_0 I)^*)^k\}}$ for $\lambda_0 \in \{z \in \mathbb{C} : |z| < \alpha\}$, B_{21} is a finite rank operator.

Proof. By BDF Theorem ([8, Theorem 4.1]), there exist a unitary operator U_1 and a compact operator K_0 such that $U_1 N U_1^* = \beta S^* \oplus T_l \oplus N \oplus K_0$. By Lemma 2.3 (ii) and [13, Proposition 1.41], then

$$H_l(T) = \overline{\operatorname{span}}\{\ker((T_l - \lambda_0 I)^*)^n : n \geqslant 1\} \text{ for all } \lambda_0 \in \{z \in \mathbb{C} : |z| < \alpha\}.$$

Let $\{\lambda_n\}_{n=1}^\infty$ be a countable dense subset of $\sigma_e(T)$ such that $\sigma_e(T) = \sigma_e(N) = \sigma(N) = \overline{\{\lambda_n\}_{n=1}^\infty}$. Let $\{e_n\}_{n=1}^\infty$ (resp. $\{f_n\}_{n=1}^\infty$) be orthonormal basis of H such that $Ne_n = \lambda_n e_e$ (resp. $Sf_n = f_{n+1}$), and all the eigenvalues of N have infinity multiplicity. Let P_{2n} be an orthogonal projection of $H_l(T)$ onto span $\{(\ker(T_l - \lambda_0 I)^*)^n\}$, P_{1n} (resp. P_{3n}) be the orthogonal projections from H onto span $(\{f_i\}_{i=1}^n)$ (resp. span $(\{e_i\}_{i=1}^n)$), $P_n = P_{1n} \oplus P_{2n} \oplus P_{3n}$.

Note that K_0 is a compact operator. Thus there exists a natural number k such that $\|P_k K_0 P_k - K_0\| < \varepsilon/2$. Let $K_1 = P_n K_0 P_0 - K_0$, by the supper semicontinuity of spectrum, $\sigma(U_1 N U_1^* + K_1) \subset (\sigma(N))_{\varepsilon/2}$. Since $H \oplus H_l(T) \oplus H$ can be decomposed as span $(\{f_i\}_{i=1}^l) \oplus (H \ominus \operatorname{span}(\{f_i\}_{i=1}^l)) \oplus (\ker((T_l - \lambda_0 I)^*)^l) \oplus (H \ominus \ker((T_l - \lambda_0 I)^*)^l) \oplus \operatorname{span}(\{e_i\}_{i=1}^l) \oplus (H \ominus \operatorname{span}(\{e_i\}_{i=1}^l))$, consider the matrix representation of $U_1 N U_1^* + K_1$ with respect to this decomposition, and note that $\beta S^*|_{H\ominus \operatorname{span}(\{e_i\}_{i=1}^l)}$ (resp. $N|_{H\ominus \operatorname{span}(\{f_i\}_{i=1}^l)}$) is unitary equivalence to βS^* (resp. N), by simple computation, we can imply that there exists a unitary operator U_2 such that

$$U_2(U_1 N U_1^* + K_1)U_2^* = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & \beta S^* & 0 & 0 \\ A_{31} & 0 & T_l' & 0 \\ 0 & 0 & 0 & N \end{bmatrix},$$

where $T_l' = T_l|_{H_l(T)\ominus \operatorname{span}(\ker((T_l-\lambda_0 I)^*)^k)}$, A_{11}, A_{12} and A_{31} are finite rank operators, respectively.

Since $\sigma(A_{11}) \subset \sigma(U_1 N U_1^* + K_1) \cup \sigma(\beta S)$, and A_{11} acts on a finite dimensional space, apply Schur lemma to A_{11}, again perturb, we can choose compact operator K_2', X_1' such that $\|K_2'\| < \varepsilon/2$ and $X_1'(A_{11} + K_2')(X_1')^{-1} = F_d$ is a diagonal matrix with distinct diagonal entries,

$$\sigma(F_d) \subset \sigma(\beta S^*), \quad \sigma(F_d) \cap \{z \in \mathbb{C} : |z| = \beta\} = \emptyset.$$

By the above argument, apply functional calculus and [5, Corollary 4.5] to $\begin{bmatrix} F_d & X_1'A_{12} \\ 0 & \beta S^* \end{bmatrix}$ $\oplus N$, then there exist a compact operator K_2 with $\|K_2\| = \|K_2'\|$ and an invertible operator X_1 with form unitary plus compact such that

$$X_1(U_2(U_1NN^* + K_1)U_2^* + K_2)X_1^{-1} = \begin{bmatrix} \beta S^* & 0 \\ B_{21} & T_l' \end{bmatrix} \oplus N,$$

where B_{21} is a finite rank operator. The proof is complete. □

LEMMA 2.5. *Let* $R, T \in \mathscr{A}$. $N \in \mathscr{L}(H)$ *is the same as in Lemma 2.3. Then for given* $\varepsilon > 0$, *there exists a compact* K *with* $\|K\| < \varepsilon$ *such that*

$$R_l \oplus N + K \cong_{u+k} T_l \oplus N.$$

Proof. Let $R' = R_l \oplus N$, $T' = T_l \oplus N$, $N_0 = N$. Apply [2, Lemma 2.1] to R', there exist a unitary operator and a compact operator K_1' with $\|K_1'\| < \varepsilon/25$ such that $U(R' + K_1)U^* = N \oplus R_l \oplus N_0$. Note that the matrix representation of R_l with respect to $H_l(T) = \ker(R_l^*)^n \oplus (\ker(R_l^*)^n)^\perp$, and apply Lemma 2.4 to N_0. Then there exist a natural number k, a compact operator K_1 with $\|K_1\| < \varepsilon/5$ and an invertible operator X_1 with form unitary plus compact such that

$$X_1(R' + K_1)X_1^{-1} = N \oplus \begin{bmatrix} R_{l1} & 0 \\ R_{l2} & R_l' \end{bmatrix} \begin{matrix} \ker(R_l^*)^k \\ (\ker(R_l^*)^k)^\perp \end{matrix} \oplus \begin{bmatrix} \beta S^* & 0 \\ B_{21} & T_l' \end{bmatrix} \begin{matrix} \ker(T_l^*)^k \\ (\ker(T_l^*)^k)^\perp \end{matrix} \oplus N,$$

where $R_l' = R_l|_{(\ker R_l^{*k})^\perp}$, $T_l' = T_l|_{(\ker T_l^{*k})^\perp}$. R_{l2} and B_{21} are finite rank operators, respectively.

Since $\sigma(R_l' \oplus \beta S^* \oplus N) = \sigma(T)$, $\sigma_e(R_l' \oplus \beta S^* \oplus N) = \sigma_e(T)$, $\sigma_0(R_l' \oplus \beta S^* \oplus N) = \emptyset$ and $\operatorname{ind}(R_l' \oplus \beta S^* \oplus N - \lambda I) = 0$ for $\lambda \in \sigma(T) \setminus \sigma_e(T)$. By [5, Theorem 2.3] and its proof, there exist a compact operator K_2' with $\|K_2'\| < \varepsilon/(5\|X_1\|\|X_1^{-1}\|)$ and an invertible operator X_2' with form unitary plus compact such that

$$X_2'(R_l' \oplus \beta S^* \oplus N + K_2')(X_2')^{-1} = N.$$

Thus there exist a compact operator K_2 with $\|K_2'\| = \|K_2\|$ and an invertible operator X_2 with form unitary plus compact such that

$$X_2(X_1(R_1' + K_1)X_1^{-1} + K_2)X_2^{-1} = E = \begin{bmatrix} R_{l1} & 0 & 0 \\ A_{21} & N & 0 \\ 0 & A_{32} & T_l' \end{bmatrix} \oplus N,$$

where A_{21} and A_{32} are finite rank operators, respectively.

Simultaneously apply [3, Corollary 4.5, P. 42] to A_{21} and A_{32}, there exist a compact operator K_3 with $\|K_3\| < \varepsilon/(5\|X_1\|\|X_1^{-1}\|\|X_2\|\|X_2^{-1})$ and a natural number n such that

$$E + K_3 = \begin{bmatrix} R_{l1} & 0 & 0 & 0 \\ B_{21}' & N_1 & 0 & 0 \\ 0 & 0 & N' & 0 \\ 0 & B_{42} & 0 & T_l' \end{bmatrix} \oplus N,$$

where B'_{21} and B_{42} are finite rank operators, respectively. $N = N_1 \oplus N'$, $N' \cong N$, N_1 acts on a Hilbert space whose dimension is n.

Claim. There exists a finite rank operator Z such that

$$N_1^* Z^* - Z^*(T'_l)^* = B^*_{42}\,.$$

Note that [11, Lemma 3.1] and [13, Lemma 3.10], we can assume that $(T'_l)^*$ is of upper triangular matrix representation

$$(T'_l)^* = \begin{bmatrix} 0 & a_{12} & \cdots & \cdots \\ & 0 & a_{23} & \cdots \\ & & \ddots & \ddots \end{bmatrix}$$

where $a_{ii+1} \neq 0$ for $i = 1, 2, \cdots$.

Let $N^* = \operatorname{diag}\{\lambda_i\}_{i=1}^n$, $Z^* = (z_{ij})_{n\times\infty}$ be $n \times \infty$ matrix whose elements are z_{ij}, $i = 1, 2 \cdots, n$; $j = 1, 2 \cdots$. Since B^*_{42} is known, $\lambda_i \neq 0$ for $i = 1, 2, \cdots, n$. $a_{ii+1} \neq 0$ for $i = 1, 2, \cdots$, by solving equation $N_1^* Z^* - Z^*(T'_l)^* = B^*_{42}$, we can get z_{ij} step and step. Thus such Z^* exists, by [3, Proposition 3.4, P.70], Z is bounded. The proof of the Claim is completed.

Let $X_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & Z & 0 & 1 \end{bmatrix}$, then

$$X_3(E+K_3)X_3^{-1} = \begin{bmatrix} R_{l1} & 0 & 0 & 0 \\ B'_{21} & N_1 & 0 & 0 \\ 0 & 0 & N' & 0 \\ ZB'_{21} & 0 & 0 & T'_l \end{bmatrix} \oplus N = \begin{bmatrix} R_{l1} & 0 & 0 \\ B''_{21} & N & 0 \\ B'_{31} & 0 & T'_l \end{bmatrix} \oplus N\,,$$

where B''_{21} and B'_{31} are finite rank operators, respectively.

Let $\delta = \|X_1\|\|X_2\|\|X_3\|\|X_1^{-1}\|\|X_2^{-1}\|\|X_3^{-1}\|$, apply [2, Lemma 2.1] to $N \oplus T'_l$, then there exist a compact operator K_4 with $\|K_4\| < \varepsilon/(5\delta)$ and a unitary operator U_2 such that

$$U_2(X_3(E+K_3)X_3^{-1} + K_4)U_2^* = F = \begin{bmatrix} R_{l1} & 0 \\ G & T'_l \end{bmatrix} \oplus N\,.$$

where G is a finite rank operator.

Apply Lemma 2.2 to $\begin{bmatrix} R_{l1} & 0 \\ G & T'_l \end{bmatrix}$, then there exist a compact K_5 with $\|K_5\| < \varepsilon/(5\delta)$ and an invertible operator X_4 with the form unitary plus compact such that

$$X_4(F+K_5)(X_4)^{-1} = T_l \oplus N.$$

By the above proof, the conclusion follows. □

LEMMA 2.6. *Let* $K \in \mathscr{L}(H)$ *be a compact operator and* $R = W + K \in \mathscr{L}(H)$ *satisfy the conditions* $(i), (ii), (iii), (iv)$ *in* $\mathscr{A}$. *Then* R_0 *and* R_l *are essentially normal operators, respectively.*

Proof. By [8, Theorem 3.38], R with respect to the decomposition $H = H_0(R) \oplus H_l(R)$ is of the form

$$R = \begin{bmatrix} R_0 & B \\ 0 & R_l \end{bmatrix} \begin{matrix} H_0(R) \\ H_l(R) \end{matrix},$$

and $\sigma(R_0) \subset \sigma_e(R)$. Let π denote the canonical map from $\mathscr{L}(H)$ to $\mathscr{L}(H)/\mathscr{K}(H)$, we imply that $\pi(R^*R) = \pi(W^*W)$,

$$\begin{bmatrix} \pi(R_0^*R_0) - \pi(D_1) & \pi(R_0^*B) \\ \pi(B^*R_0) & \pi(B^*B + R_l^*R_l) - \pi(D_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where D_1 and D_2 are diagonal operators, respectively.

Note that $\pi(B^*R_0) = 0$, $\sigma(R_0) \subset \sigma_e(R)$, R_0 is an invertible operator. Thus B^* is compact, so is B. Since R is an essentially normal operator, by simple computation, we can imply that R_0 and R_l are essentially normal operators, respectively. □

3. The Proof of Theorem and Remark

The Proof of Theorem. Let N be the same as in Lemma 2.3, $R \in \mathscr{B}$. Note that $\sigma_0(R) = \emptyset$, apply [8, Theorem 3.48] to R, we can imply that there exists a compact operator K_1 with $\|K_1\| < \varepsilon/2$ such that $\sigma(R+K_1) = \sigma_W(R)$ and $\operatorname{minind}(R+K_1-\lambda) = 0$ for $\lambda \in \rho_F(R)$. Thus $R + K_1$ satisfies the conditions (i), (ii), (iii), (iv) in $\mathscr{A}$.

Apply BDF theorem ([8, Theorem 4.1]) to $R + K_1$ and W, then there exist a unitary operator U and a compact operator C such that $U(R + K_1)U^* = W + C$. By Lemma 2.6, $U(R + K_1)U^* \in \mathscr{A}$. By Lemma 2.5 and the proof of Lemma 2.3, there exist a compact operator K_2 with $\|K_2\| < \varepsilon/2$ and an invertible operator X with form unitary plus compact such that $X(U(R + K_1)U^* + K_2)X^{-1} = T_l \oplus N$. Note the the assumptions of Theorem, by Lemma 2.3 (iii), $R \in \overline{(\mathscr{U} + \mathscr{K})(T)}$.

Conversely, by [8, Theorem 1.2] and [4, Proposition 0.6], the proof of Theorem is completed. □

REMARK. Note that the proof of Theorem is independent of [6, Theorem 3.4]. In fact, by [6, Theorem 3.4], we can also give a short proof about that $R \in \mathscr{B}$ implies $R \in \overline{(\mathscr{U} + \mathscr{K})(T)}$. Since, by [6, Theorem 3.4], $\mathscr{B} = \overline{(\mathscr{U} + \mathscr{K})(W)}$, in order to prove that $R \in \overline{(\mathscr{U} + \mathscr{K})(T)}$ when $R \in \mathscr{B}$, it is sufficient to show that $W \in \overline{(\mathscr{U} + \mathscr{K})(T)}$. Note that $W \in \mathscr{A}$, by Lemma 2.5 and Lemma 2.3, then $W \in \overline{(\mathscr{U} + \mathscr{K})(T)}$.

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