Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education

Prague, Czech Republic

July 16-21, 2006

Volume 1

Plenaries, Research Forums, Discussion Groups, Working Session, Short Oral Communications, Posters

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Charles University in Prague
Faculty of Education

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The Proceedings are also available on CD-ROM

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ISSN 0771-100X

Logo: Ivan Špirk
Cover Design: Jarmila Novotná & Ivan Špirk
Printing: Atelier Guimaec, s.r.o

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INTRODUCTION

WELCOME OF VÁCLAV HAVEL:

CONTRIBUTION TO CONFERENCE CATALOGUE PSYCHOLOGY OF MATHEMATICS EDUCATION

Prague, May 2006

Allow me to welcome you personally and your entire gathering to Prague Crossroads, located in the former, church of St Lawrence, long deconsecrated, which was once part of the Convent of St Anne in the Old Town. Several years ago our foundation, Vize 97, undertook to reconstruct it and make it available to the public. This was achieved in the autumn of 2004 after considerable efforts. Our intention is to draw on Prague's historical tradition as a crossroads of spiritual currents and revive it by means of cultural and social gatherings of various kinds, as well as inter-religious meditation, because here the locality radiates its remarkable and mysterious destiny and its long and turbulent history. It is said that St Wenceslas himself decided to build a rotunda here, and therefore it is assumed that it was a place that already had a special status in the pre-Slavonic period, being an intersection of various forces and influences with metaphysical protection. It may seem strange that I mention it in a somewhat unscientific fashion in connection with something so rational as a meeting on the psychological aspects of teaching mathematics. It is my belief, however, that it should be the ambition not only of this place, but of modem thinking in general, to emphasise the overlapping and interpenetration of scientific disciplines that were strictly separated until recently. An over-emphasis on rationality in modem civilisation stultified conscience – that ambassador of the moral and spiritual order within us – and with it the sense of responsibility for our actions that transcends by far the bounds of our physical existence. Only thinking that draws on the wealth of the manifold sources of human knowledge – without denying the uniqueness and strengths of any of them – can overcome our transitoriness in this world. I hope that you will feel at ease in these inspiring surroundings.

VÁCLAV HAVEL

Praha, květen 2006

Dovolte mi, abych vás osobně i celé vaše shromáždění přivítal v Pražské křižovatce, v bývalém a v minulosti již odsvěceném kostele sv. Vavřince při klášteře sv. Anny na Starém Městě. Naše nadace Vize 97 si jej před několika lety předsevzala zrekonstruovat a zpřístupnit veřejnosti, což se po nelehkém úsilí na podzim 2004 také stalo. Rádi bychom zde navázali na dějinnou tradici Prahy, jako křižovatky duchovních proudů, a obnovili ji rozmanitými kulturními a společenskými setkáními či mezináboženskými meditacemi, vyzařující zvláštní a tajemnou předurčenost místa a jeho prastarou a pohnutou historii. Vybudovat zde rotundu se prý rozhodl už svatý Václav, a právě proto se předpokládá, že to bylo místo se zvláštním postavením, průsečík rozmanitých sil a vlivů s metafyzickou ochranou už v době předslovanské. Může se zdát divné, že se o tom poněkud nevědecky zmiňuji v souvislosti s tématem tak racionálním, jakým je setkání o výuce matematiky a jejích psychologických aspektů. Domnívám se však, že právě důraz na přesahy a prolínání donedávna přísně oddělovaných vědních disciplin by měl být ambicí nejen tohoto místa, ale moderního myšlení vůbec. Přílišný důraz na racionalitu otupil v moderní civilizaci svědomí, onoho velvyslance mravního a duchovního řádu v nás a tím i pocit zodpovědnosti za činy, které daleko přesahují hranice naší fyzické existence. Pouze myšlení čerpající z bohatství rozmanitých proudů lidského poznání, aniž by kterémukoli z nich upřelo jeho jedinečnost a jeho přednosti, může překonávat naši dočasnost na tomto světě. Přeji vám, ať se v tomto inspirativním prostředí cítíte dobře.

Václav Havel

WELOME OF CHRIS BREEN, PME PRESIDENT

I would like to welcome the enormous number of participants who have chosen to join us in Prague this year in celebrating the 30^{th} birthday of PME. I believe that the overwhelming response that has led to our largest ever PME conference is indicative of both the attractiveness of the venue as well as the robustness and value of our organisation. The publication of this set of larger-than-usual proceedings is also a tribute to the large number of hours that our Conference Organiser, Jarmila Novotna, and her team have dedicated to PME. We are all extremely appreciative of her hours of dedication to PME30.

There are a number of additional reasons and events that will contribute to this 30^{th} PME conference.

- On Wednesday 19^{th} July, we will be celebrating the launch of the *Handbook of Research on the Psychology of Mathematics Education* and appreciating the hard work that has been put into it by the authors and editors.
- This conference is also the first to be held since the decision at last year’s AGM in Melbourne to broaden the aims of PME. PME30 participants will notice that there are several authors who have addressed these broader issues in their research reports. It seems appropriate that at the same time that we celebrate 30 years of existence, we also mark this new initiative that shows our organisation’s capacity and willingness to adapt to changing circumstances.
- This conference will also mark the first time that PME will be hosting our most recently elected Honorary Member, Joop van Dormolen, who has officially retired as our Executive Secretary but has still played a considerable role in assisting with the organisation of PME30.
- In many ways, Joop’s retirement has provoked a thorough review of a whole range of PME’s procedures and processes, and the Policy meeting on Monday 17^{th} July will provide members with an opportunity to air their views and make suggestions to the IC for further investigation. The involvement of PME members in this ongoing reflective process is particularly essential to enable the IC to continue to address the needs of the community.

I trust that participants at PME30 will also make the best possible use of the opportunity that will present itself for us to engage with and learn from the broad spectrum of participants from a wide range of geographical locations.

Chris Breen

Chris Breen, PME President.

WELCOME TO PME30:
MATHEMATICS IN THE CENTER

It is our pleasure to welcome you to the 30th Annual Conference of the International Group for the Psychology of Mathematics Education. At its 30th anniversary, the conference is held in Prague. The host institution is the Faculty of Education, one of the younger offspring of the renowned Charles University in Prague, which, at the same time as PME celebrates its 30th anniversary, celebrates 60 years from its foundation. The conference is held under the auspices of the Rector of Charles University and the Mayor of Prague.

History of Charles University is long and rich. It is the oldest university in the Czech Lands and in Central Europe. It was founded as early as 1348 by Charles IV, King of Bohemia and Emperor of the Holy Roman Empire. In the university's charter he wrote the following words, which are both beautiful and committed: "... In order that the inhabitants of the kingdom will not have to beg for alms abroad, but, on the contrary, that they will find a table rich with delicacies in the kingdom … that they will be proud to be able to invite inhabitants of other kingdoms to come and profit from the spiritual wealth …" Charles University was the first institution of higher education north of the Alps. At the time of its foundation the university consisted of four faculties – theology, law, medicine and the arts. Teachers from the faculty of arts also conducted courses in the fundamentals of algebra and geometry.

Charles University's Faculty of Education was officially opened on November 15, 1946. Currently it is one of the University's seventeen faculties. Its mission is to prepare teachers for all types and levels of schools and to prepare specialists and scientists in the area of pedagogy, educational psychology and didactics. Though the preparation of teachers is also provided at other five faculties, the Faculty of Education holds a unique position in that it fully focuses on the issues of education. Depending on the type of study, the Faculty of Education awards Bachelor, Master and Doctor diplomas and degrees. In the area of international co-operation, the Faculty of Education focuses its effort on the exchange of scientific knowledge and practical experience, on joint research projects and studies and on international meetings of teachers and students.

The tradition of didactics of mathematics and its co-operation not only with pedagogy but didactics of other subjects, psychology and other social sciences is long and rich. The number of international events in didactics of mathematics organized here testify the sole position of didactics of mathematics in the Czech Republic. From the many international events organized here let me list at least some: SEMT - Symposium on Elementary Math Teaching, a bi-annual conference, first held in 2001; CERME 2; events focusing on the beginning researchers (for example YERME 2004 Summer School); or conferences preceding PME 30 organized by other universities in the

Czech Republic (Fourth International Conference on "Creativity in Mathematics Education and the Education of Gifted Students, CIEAEM 58.

Prague, the capital of the Czech Republic, boasts with the reputation of one of the most beautiful cities in the world. Its history goes back to the 6th century AD. The legends say that the settlement was founded on the command of princess Libuše who prophesied its future glory that would touch the stars. The stars may not have been reached yet. However, the fame of the city is worldwide and the city centre is on the list of world heritage protected by UNESCO. Walking through the picturesque streets of the Old Town or among the imposing palaces of the Lesser Quarter, or past the monumental gothic or baroque cathedrals, temples and churches, one must not forget that Prague is not only an open-air museum or theme park. Its one million inhabitants live ordinary lives. Quality of education provided by local schools is one of their concerns. The feeling that education needs to undergo reformation and changes (after 40 years for totalitarian communist regime) is not rare. New trends in teaching are sought and welcome by many.

The theme of this year's PME conference is *Mathematics at the centre*. This theme was chosen with the intention of "going back to the roots". Without any doubt, psychology and pedagogy play crucial role in the teaching of mathematics. It is important to study the learning environments, learners' motivation and learning processes, teaching methods, classroom interaction and so on. However, at the heart of every effort to make mathematics teaching comprehensible, useful, interesting and thrilling must be mathematics itself and this is not to be neglected. Never before have so many mathematics educators and teachers come to the PME conference as this year. We believe this is a proof that many of us share this concern for mathematics. We hope that all that will be presented at the conference will make an interesting point and contribution to the main theme and that all the participants will be leaving the conference with the feeling that the time spent here in Prague was spent fruitfully. We invite everybody to active participation and sharing of their ideas and opinions. The more passionate and fierce our discussions will be, the greater progress will be made.

The Programme Committee and the Local Organizing Committee want to express our thanks to Chris Breen for kind, immeasurable help, encouragement and friendly stilling, Joop van Dormolen for technical support and work with the database, to Anne-Marie Breen, PME Project Manager for organizational support, Helen Chick for sharing her experience with organization of PME 29 and providing priceless advice. Without Helen's knowledge, experience and templates, the publication of these proceedings would have been far more difficult.

Finally, I would like to thank the many people who have done their best to secure smooth and successful course of this year's conference. The Progamme Committee, whose members are listed later, were not only responsible for careful consideration of all the proposals, but made other important decisions. I have appreciated the support

from the Board of the Faculty of Education and of colleagues from other departments and our administrative machinery. I would also like to thank the Department of Mathematics and Didactic of Mathematics, the Local Organizing Committee, and undergraduate and PhD students for their help during the course of the conference. The help of Guarant Int. (let me name at least the project manager Jitka Puldová) was also priceless. Their know-how in the area of organization of big events made the organization of this year's conference much easier. Finally, I would like to thank Hana Moraová whose assiduous work and unceasing support was crucial.

We hope you enjoy your stay here in Prague and find your participation at the conference fruitful and unforgettable.

Jarmila Novotná

Jarmila Novotná, Conference Chair

We want to thank the following sponsors:

The Conference organisers express gratitude to the following sponsors for their support of the PME30 Conference:

- Charles University in Prague, Faculty of Education
- Dagmar and Václav Havel Foundation VIZE 97
- Hewlett-Packard
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- ČEZ, a.s.

THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

History and Aims of PME

PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976. Its former presidents have been Efraim Fischbein (Israel), Richard R. Skemp (UK), Gerard Vergnaud (France), Kevin F. Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), Kathleen Hart (UK), Carolyn Kieran (Canada), Stephen Lerman (UK), Gilah Leder (Australia), and Rina Hershkowitz (Israel). The current president is Chris Breen (South Africa).

The major goals* of PME are:

- To promote international contacts and the exchange of scientific information in the field of mathematics education.
- To promote and stimulate interdisciplinary research in the aforesaid area.
- To further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

PME Membership and Other Information

Membership is open to people involved in active research consistent with the Group's goals, or professionally interested in the results of such research. Membership is on an annual basis and requires payment of the membership fees (40 €) for the year 2006 (January to December). For participants of the PME30 Conference, the membership fee is included in the Conference Deposit. Others are requested to contact their Regional Contact, or the PME Project Manager.

Website of PME

For more information about International Group for the Psychology of Mathematics Education (PME) as an association, history, rules and regulations and future conferences see its home page at http://igpme.org or contact the PME Project Manager.

Honorary Members of PME

Hans Freudenthal (The Netherlands, deceased)
Efraim Fischbein (Israel, deceased)
Joop van Dormolen (Israel)

* Item 2 of the Constitution of PME, http://igpme.org

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Treasurer: Markku S. Hannula (Finland)

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PME PROCEEDINGS OF PREVIOUS CONFERENCES

PME International

The tables indicate the ERIC numbers of PME conference proceedings.

No.	Year	Place	ERIC number
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943
7	1983	Shoresh, Israel	ED241295
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, Netherlands	ED411130 (vol.l), ED411131 (vol.2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris, France	ED411140 (vol.1), ED411141 (vol.2), ED411142 (vol.3)
14	1990	Oaxtepex, Mexico	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi, Italy	ED413162 (vol.1), ED413l63 (vol.2), ED41364 (vol.3)
16	1992	Durham,USA	ED383538
17	1993	Tsukuba, Japan	ED383 536
18	1994	Lisbon,Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia, Spain	ED453070 (vol. 1), ED45307 1 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch, South Africa	ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301 (vol. 1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Hawai‘i, USA	http://onlinedb.terc.edu
28	2004	Bergen, Norway	
29	2005	Melbourne, Australia	

Copies of some previous PME Conference Proceedings are still available for sale. See the PME web site at http://igpme.org/publications/procee.html or contact the Proceedings manager Dr. Peter Gates, PME Proceedings, University of Nottingham, School of Education, Jubilee Campus, Wollaton Road, Nottingham NG8 1 BB, UNITED KINGDOM, Telephone work: +44-115-951-4432; fax: +44-115-846-6600; e-mail: peter.gates@nottingham.ac.uk

PME North American Chapter

No.	Year	Place	ERIC number
2	1980	Berkeley, Califomia (with PME2)	ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montreal, Canada (with PME11)	ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132 (vol.1), ED411133 (vol.2)
12	1990	Oaxtepex, Mexico (with PME14)	ED411137 (vol.1), ED411138 (vol.2), ED411139 (vo1.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire (with PME16)	ED383538
15	1993	Califomia	ED372917
16	1994	Louisiana	ED383533 (vol.l), ED383534 (vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494 (vol.1), ED420495 (vol.2)
20	1998	Raleigh, North Carolina	ED430775 (vol.l), ED430776 (vol.2)
21	1999	Mexico	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	ED476613
24	2002	Athens, Georgia	ED471747
25	2003	Hawai‘i (together with PME27)	
26	2004	Toronto, Ontario	
27	2005	Roanoke, Virginia	

Abstracts from some articles can be inspected on the ERIC web site (http://www.eric.ed.gov/) and on the web site of ZDM/MATHDI (http://www.emis.de/MATH/DI.html). Many proceedings are included in ERIC: type the ERIC number in the search field without spaces or enter other information (author, title, keyword). Some of the contents of the proceedings can be downloaded from this site. MATHDI is the web version of the Zentralblatt fur Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing editor, fax: (+49) 7247 808 461, e-mail: Gerhard.Koenig@fiz-karlsruhe.de

THE REVIEW PROCESS OF PME30

Research Forums. The Programme Committee and the International Committee accepted the topics and co-ordinators of the Research Forum of PME30 on basis of the submitted proposals, of which all were accepted. For each Research Forum the proposed structure, the contents, the contributors and the role of the contributors were reviewed and agreed by the Programme Committee. Some of these proposals were particularly well-prepared and we thank their coordinators for their efforts. The papers from the Research Forums are presented on pages 1-95 to 1-184 of this volume.

Working Sessions and Discussion Groups. The aim of these group activities is to achieve greater exchange of information and ideas related to the Psychology of Mathematics Education. There are two types of activities: Discussion Groups (DG) and Working Sessions (WS). The abstracts were all read and commented on by the Programme Committee, and all were accepted. IPC recommended changing three Discussion Group proposals for Working Sessions which was accepted by the authors. Our thanks go to the coordinators for preparing such a good selection of topics. The group activities are listed on pages 1-187 to 1-208 of this volume.

Research Reports (RR). The Programme Committee received 412 RR papers for consideration. Each full paper was blind-reviewed by three peer reviewers, and then these reviews were considered by the Programme Committee, a committee composed of members of the international mathematics education community. This group read carefully the reviews and also in some cases the paper itself. The advice from the reviewers was taken into serious consideration and the reviews served as a basis for the decisions made by the Programme Committee. In general if there were three or two recommendations for accept the paper was accepted. Proposals that had just one recommendation for acceptance were looked into more closely before a final decision was made. Of the 412 proposals we received, 243 were accepted, 44 were recommended as Short Oral Communications (SO), and 49 as Poster Presentations (PP). The Research Reports appear in Volumes 2, 3, 4, and 5.

Short Oral Communications (SO) and Poster Presentations (PP). In the case of SO and PP, the Programme Committee reviewed each one-page proposal. A SO proposal, if not accepted, could be recommended for a PP and vice versa. We received 163 SO proposals initially, of which 129 were accepted and 9 were recommended as posters; later an additional 34 SO proposal were resubmitted from RR. We received 42 initial PP proposals, of which 37 were accepted; later an additional 33 PP proposals were resubmitted from RR or SO. The Short Oral Communications and Poster Presentations appear in this volume of the proceedings.

LIST OF PME30 REVIEWERS

The PME30 Program Committee thanks the following people for their help in the review process:

Abramovich, Sergei (USA)
Acuña-Soto, Claudia (Mexico)
Adler, Jill (South Africa)
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MATHEMATICS, DIDACTICAL ENGINEERING AND OBSERVATION

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Psychology is very important to mathematics education, but it is not the central domain for the study of the teaching of mathematics. Observation, didactical engineering and a renewed study of mathematics are the means essential to a scientific approach to phenomena that are specific to the act of teaching and learning mathematics.

INTRODUCTION

In 1976 at Karlsruhe, Efraïm Fishbein, Gérard Vergnaud and I discussed the various orientations that could be envisioned for scientific research on mathematics education in the framework of ICMI. At that time the mathematicians, who had just revitalized ICMI, had agreed to create a specific commission to develop scientific research. The only discipline with which they were prepared to share the responsibility for this research was cognitive psychology, which seemed to them to provide a guarantee of proven methods and concepts.

Research in sociology, pedagogy and methodology as well as large statistical studies were welcome in the general context of ICMI congresses (ICME), but there wasn't a place to accord them the particular support of a permanent commission. Direct experimental scientific study of the acts of teaching and learning seemed at that time unpromising and too complex, because the belief at the time was that it would necessitate first, and independently, better understanding the different components of the didactical act: material, students, teachers, institutions, etc. The rapid success of PME demonstrates how productive the route of psychology was.

Today, however, the position has been pushed yet further, to the point where a minister of education recently had no qualms in declaring that "cognitive neuroscience is *the* real science from which the study of teaching should take its inspiration." This reduction of psychology to the physiology of the brain points up the contradictions in the project of undertaking the study of teaching in the context of cognitive psychology that I noted at the time. A study of the materials to be taught and of the conditions for teaching them, inspired by the ingenious experiments used in genetic epistemology to detect the mathematical behaviors of children, convinced me of the interest of an alternative route, one not envisioned in traditional academic frameworks.

Only the community of mathematicians was in a position to conceive and support such a project. I therefore hoped that the commission would accept this idea and that its name would indicate an ambition of scientific research without giving it the name of any specific field. Efraïm thought it was impossible and Gérard convinced me that

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 3-18. Prague: PME.

mathematicians would not accept spiritual guidance from any field other than psychology. Today I understand how strange my proposal must have seemed to them and how little chance it had of being accepted.

But I wasn't alone. We had the good fortune to find mathematicians and psychologists who gave us the support and the means for this research, despite its originality. I also benefited greatly from the help of many teachers and researchers. As a result, today thirty years of work enable me to support and illustrate the propositions I put forth at that time.

LIMITATIONS OF PSYCHOLOGY FOR THE STUDY OF TEACHING

It is easier now to explain why from that period on I have refused to place my work under the control of any existing science, even psychology or the newly developing educational science. This attitude does not reflect the faintest negative opinion, or rejection or condescension. I have always behaved as a pedagogue, I positioned myself as a mathematician, and I have remained a student of psychology. But the direct study of teaching does not fit into any of the existing disciplines, just as, for example, economics does not.

Focused on experimental subjects and their behaviors, psychology can envision all of the experimental designs, but it cannot study them systematically. Likewise, it cannot study *a priori* the knowledge itself, nor the relationships among the elements of it, nor their relationship with the experimental situations. It follows that the conditions for actual use of knowledge appear in psychology as independent factors. The only usable models are terribly simplistic and general. For example, "the student" is not distinguished from "the psychological subject". It follows that actual knowledge of the subject, school-based knowledge, and reference knowledge are used interchangeably. This obscures the *didactical transposition*.

The knowledge to be taught to children is drawn from reference knowledge, but it must differ from it in form, context and use. The difference is produced by a *didactical transposition.* The way in which a transposition changes or conserves the functions of knowledge and the way in which the knowledge can arrive most swiftly at a final form and make a rich mathematical activity possible are essential objects of study for Didactics. These objects are not in the field of psychology, although it is easy to see their relationships to it. In order best to restore the functioning and meaning of knowledge in the transposition, we had to distinguish at least two functions of knowledge in the relations of an individual with his milieu. They do not correspond exactly to any of the forms of knowledge described in psychology, but they are differentiated in the Latin languages by the roots *cognoscere* and *sapere* (in French, *connaisance* and *savoir*. These we have translated by *c-knowledge* and *s-knowledge*.[1]

[1] C-knowledge develops spontaneously from an encounter or interaction of a subject with a milieu, in a situation. It enables her to manage the situation by all manner or means, empirical or

The student s-knows part of what she has been explicitly taught and c-knows what she has encountered and what she has gotten out of the encounter. Both can involve either concepts or procedures. The same person can respond in different ways to the same problem statement, depending on the circumstances. For example, in a very uncertain situation, a piece of acquired s-knowledge can function as a piece of c-knowledge. C-knowledge and s-knowledge are equally necessary for pieces of s-knowledge to function. The two functions contribute jointly to behavior and learning. Didactic transposition and teaching must take both into account.

Clearly one must not reproach a science for the absence of something that is not its objective. But the percolation of knowledge with psychological and scientific origins into teaching institutions denatures it, and has consequences that escape the vigilance of either one. Constructivism carried to the point of radical constructivism was first welcomed and then reviled, just as generalized structuralism had been earlier. The direct importation of experimental designs and exotic knowledge disarms teachers by opposing their practices with an authority that is in fact unjustified. Naïve adoptions and radical rejections succeed each other at the whim of scientific novelties, each time destroying more and more not only of the old practices but of the additions that could have been beneficial. Today the abusive use of evaluation and the rise of individualism are very visible. But their ideological, economic and political roots make them formidable scourges. Psychology cannot monitor and control the use of its results or its methods in the educational system. This fact alone is enough to demonstrate that it cannot be the science of teaching. Nor can the neurosciences to which it is now attached, for the same reason.

ALTERNATIVE PERSPECTIVES

Behavioral psychology adopted as logical evidence oversimplified conceptions of teaching and of the knowledge to be taught. For it, the black box on which it focused was the experimental subject, so that it systematically neglected the study of conditions and knowledge, both of which it reduced to the status of stimuli. Nothing that I know from my practice of mathematics or from its history corroborates this hypothesis. There is no automatic generator of mathematical knowledge. Every theorem can be seen as a logical deduction from others, but its presence and its use have a history that is different from its place in this deductive structure, and that history differs for each piece of knowledge. Although one theorem may facilitate the discovery and proof of another, the "discovery" of each, in particular, requires a process and conditions that are specific to it. The experiments that the Piagetians used to describe the cognitive development of children provided interesting examples of situations designed to reveal the acquisition of certain pieces of mathematical

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otherwise, and with a manner of understanding it that may be temporary and personal. A teacher "teaches" c-knowledge *implicitly*, with the situations he creates as intermediaries. S-knowledge comes out of the study of spontaneous c-knowledge and situations with the help of the culture. It makes it possible to identify parts of it, classify it and understand it by attaching it to other established s-knowledge and if necessary to prove it. The teacher teaches s-knowledge explicitly.

knowledge. But the study of the experimental designs themselves and their relationship with the mathematical knowledge were excluded. What epistemological study was made was rather theoretical and arose directly from ideas of mental development.

Studies of teaching cannot consider either the stimuli or the experimental designs to be transparent, as the psychological approach does. The conditions for the appearance and use of thought are our black box, and the behaviors of the students and the teacher are what reveal them. Consequently, a didactical project must begin with its objective. It is the nature and function of knowledge that are the mainspring of its comprehension, use and learning.

Our rejection of these premises constituted a considerable epistemological leap and an enormous scientific gamble, and the principal cause of its success was no doubt the modesty of our condition and the intelligence of the institutions we worked with.

Mathematics is produced by mathematical activity; why should it not be learned that way, by its necessity and its use? We were thus interested in the functioning of mathematics as a general human activity, as a scholarly activity and above all as an individual activity of children. We were therefore faced with redefining mathematics on the basis of the conditions of its appearance and use in human interactions, and not directly. Every theorem needed to be associated with at least one situation that determined it as a correct solution, but also offered incorrect options. These situations constituted models of the functioning of mathematics on which one could work *a priori* and predict their possibilities before realizing them, in order to compare the predictions with the actual occurrences. Inversely, one could model non-experimental scholastic episodes observed in the course of ordinary classes and try to explain their logic.

Transforming teachers and students into objects of observation, actors of an experiment or researchers presented real dangers. More or less classical experimental designs such as passive observation, comparative pedagogy, action research or systematic experimentation didn't hold up under deeper analysis. I had to imagine a design that would put the activity of a class into relationship with a system of research that would satisfy a large number of conditions of very different orders.

The three elements that seemed to me indispensable in order to make the act of teaching an object of scientific study are: observation, didactical engineering and the didactical study of mathematics itself.

First one must have a *means of observation* in order to submit contradictable statements to the test of reproducible events. It is essential that the observations be made on the conditions under which the actors operate, not on the actors themselves. The observations must be specific to the knowledge under consideration.

Next one must have a *didactical engineering* to conceive and realize appropriate phenomenotechnical designs, that is, precise conditions in which mathematical knowledge appears or is learned. Didactical engineering came into existence in our efforts to find credible justifications for the early teaching of some very general mathematical structures. It broke with the traditional structure of seeking ways to implement mathematical learning following a pre-existing sequence of topics. This it was able to do thanks to new means of observation and experimentation. Reflections on the organization of mathematical knowledge and the production of situations are consubstantial with mathematics. They are therefore just as old as the mathematics itself. Their study became scientific thanks to the systematic observation of the relationships between the practices and their effects. Our work thus places didactical engineering at the heart of a science whose aim is to take teaching as an object of study.

Finally, one must have a specific field of *mathematical studies* in order to guarantee the consistency of the designs and of their relationship with mathematical knowledge, as well as their adequacy for the intended didactical project and its realizations. Of the three components, certainly the most important is the mathematical one. The reasons for using and for learning a piece of mathematical knowledge are specific. Even though a theorem appears as a "necessary" piece of knowledge within a theory, its actual existence – its discovery, its formulation, its use – is the result of a "history" that has nothing obligatory about it. On the other hand, it is also not a chance outcome. Every theorem appears in particular conditions and for particular reasons, which ought to play their role in the learning and use of the theorem.

MATHEMATICS

Mathematics forms a world that is both organized and prolific. Whether one wishes to visit a few picturesque locales within that world as a tourist, or understand the geography of it, or navigate a long trip through it, or possess and dominate it as a conqueror, this world offers a universe of adventures. This metaphor cannot be taken very far, but it does point up some interesting distinctions between the areas constructed in the course of history by human activity, and the relationships to those areas (overview, visit, exploitation, habitat, etc.) produced by individual or collective trajectories, channeled by the demands of society, culture or Didactics. In any case, the metaphor lets us envisage a certain diversity in the sorts of "learnings" that we can conceive of.

In the sixties the teaching of basic school mathematics was a five thousand year old village. Successive generations had brought in their treasures without really rearranging them and one could still recognize the rather badly coordinated traces of their successive efforts. This village of elementary mathematics opened out into higher mathematics, a countryside won in the 16th century and cleared in the 17th and 18th. Beyond that begin the vast, wild solitudes of more modern mathematical theories that promised a real amplification of mathematical space. They removed the boundaries and blockages of the ancestral village and revealed far more convenient access routes. The first job for mathematicians concerned with teaching was to imagine how this new world could be invested by the young inhabitants.

Some of the new branches opened up alternatives to the old conceptions that were worth studying. It swiftly became apparent that classical Didactics, which I knew as a practitioner, did not allow the creation of the situations that the young students needed to "practice and acquire" mathematical knowledge as it was now defined. This Didactics led at best to commentaries on textbooks or to story problems as far removed from reality as the infamous bath tub problems. Reflections on the situations in which mathematics is used and learned revealed the flaws of the old system. Even for learning things as fundamental as counting and measuring the old routes were costly and awkward.

Areas like algebra, statistics and probability could usefully be introduced considerably earlier than had been thought. A mathematically correct construction of major structures like the natural, rational and decimal numbers was possible from the very beginning of school, and laid the foundations for a good study of functions. Provided it was not confused with spatial knowledge, deductive geometry could also be undertaken in an attractive way.

This is possible if at least two conditions are met. The first is the availability of a rich and precise didactical engineering, both at the level of short sequences and of curricula and sequences of courses. The second is that this engineering be supported by a culture that is not only sufficient but above all common to the community of teachers, so that they can take part in each other's work.

What can induce someone to "do" mathematics, to use it, to learn it, or to invent it? And why should one be interested in this or that particular statement?

The literature of mathematics is composite and complex, in its objects, its goals, its styles, its forms, its authors, and more. The classical image of teaching gives us a simplified idea of it: some definitions and their properties, or some axioms and some theorems, organized by a small number of rules of production that make it possible to verify the consistency of the whole set, plus some references, some languages and a metalanguage.

But this characterization leaves out a number of essential elements such as questions. They make a modest appearance in the teacher's folio, but they are nowhere to be found among the students' exercises. These distortions are undoubtedly due to the fact that teaching is guided by the literature, and not by the activity that produces the literature. It prepares readers and commentators rather than authors. Moreover, strangely, and unlike literature (the teaching of which tries to show it as an activity that is individual but also social, cultural and historical), mathematics is exhibited as the completed discourse of a sole epistemic and eternal subject. Consequently the real activity of mathematicians, knowledgeable or not, which is personal and above all social is completely obscured. This is not without consequences to the possibility of reproducing it. The motivation of the mathematical work of didacticians and teachers is to envisage and realize conditions in which students will develop the activities that are indispensable to an effective practice of the *mathematical culture.*

A concern not to repeat the letter of mathematics like a citation, but rather to have it be produced anew became visible very early in our work in the presentation of certain statements as problems. A theorem is a (true) statement that the students are allowed to cite without a new proof, a problem with its solutions is a (true) statement, but it cannot serve as a reference, even if it is well known. The question is generally posed by the teacher, and the proof or the calculation are to be given by the student. The organization and density of theorems in the field of statements of a mathematical theory are the affair of Didactics.

By nature, the field of mathematics is infinite. Each part, each element, can be combined with others to create new questions and objects, and the results are always different, in their objectives, in their presentation, in their function, etc. Not only that, but each part can become the metalanguage of another. Consequently, these parts lend themselves to an infinitude of interpretations. There does not exist an automatic generator of mathematical thought, neither within mathematics nor, *a fortiori*, outside of it. Moreover, if there did exist such a generator, mathematics would no longer really exist as a human activity. To construct a didactical model of a mathematical theory that isn't a simple "reproduction" requires mathematical work that can be very complex. In an original work of mathematics, the role of interpretation and of reconstruction of prior knowledge can be very important. This essentially mathematical work is also didactical to the extent that it integrates constraints related to the possibilities and habits of the people destined to receive the thought. But the authors of textbooks cannot break away from the forms that are familiar to the mathematicians, even though the rearrangements needed for diffusing information in the world of professional mathematicians are not the same as those required for students.

Exercises and problems appear as a motivation for calculations and for the theorems called for in their proof.[2] The *reason* to use these theorems is thus *mathematical and logical*. On the other hand, if the teacher has chosen this problem only to illustrate these theorems and have them used and learned, their *motivation* is purely didactic. Can we lessen the distance between the *reason* and the *motivation*, or perhaps even make them coincide?

This brings us back to the original question: do there exist non-didactical reasons to study and solve one problem rather than another? Why does one do mathematics? It is not just a matter of an anthropological study of a small world of professional mathematician and their "activities", it is a matter of extracting a representation, a model in the scientific sense, of that world that can be used by a society for the effective education of its future members!

[2] Teachers say that a problem is an "application" of the most important of these elements. The formulation is ambiguous because it leaves in the shadows the difficulty of establishing a relationship between the problem and its solution. Depending on the case, it may or may not be possible to recognize swiftly the s-knowledge of which it is an application, and its use calls at times for rather large rearrangements.

In the sixties this question produced a wealth of reflections on mathematics.[3] But Piaget appeared to be the only one to envisage putting these reflections to the test of scientific experiments and observations (still in the context of non-scholastic knowledge).

A small group of mathematics students and professors took up the challenge of extending by experiments the work carried out by the mathematical community for a century. Numerous historical circumstances and causes, including the movements in May '68, the emergence in France of an ambitious social and cultural project and the creation of the IREM's, made it possible for this group to put in place the necessary means for research.

In 1975 we called this domain "experimental epistemology". But this name does not evoke sufficiently either the study of teaching as a social project or the goal of our action. That goal was to aid the community of mathematicians in the didactical task incumbent on it by exploring the scientific means it had been looking for since Felix Klein.[4] "Why answer a question in mathematics?" "Because it's there!", said Hilbert. Dieudonné took it further: "For the honor of the human spirit!" "Because intellectual curiosity and the need to know the truth are universal human responses, however little the conditions justify and support them", we might reply thinking of school children.

What makes up the essence of mathematics can be found in life well beyond the borders that the culture assigns to it. Mathematics furnishes simplified models essential for most human activities. Separating it off as is the common practice in schools is an error. In a very simple environment, it shows how to act, make a decision, conceive and simplify a program of action, adapt it to the task at hand or to the person who is to accomplish it, formulate and communicate information, discuss it, and above all prove and explain it.

Let's start with the most obvious: how can a human being honestly induce someone else to share his opinion? By showing that the opposite opinion is incompatible with what he already knows (or hopes), because nobody wants to "lose face" by admitting a thing and its negation. Proving something is not dominating the other person, it is humbly and honestly serving him, renouncing all artifices, illegitimate means of influence, rhetoric, disconcerting procedures, authority and seduction. In mathematics, objects and properties depend on very little accessory and unexpected knowledge. Young children find it a propitious terrain for learning and exercising

[3] The small group of which I speak later was directly influenced by authors, mostly French-speaking mathematicians, who advanced their ideas under the aegis either of philosophy of science, among them Poincaré, Lebesgue, Bourbaki,…Wittgenstein, Cavaillès, F. Gonseth, Dieudonné…, or of philosophy (Brunschvicg, Desanti…), or of epistemology (C. Houzel, J. Piaget…), or of history of mathematics (J. Dhombres), of psychology (J. Hadamard) or of heuristics (Polya).

[4] Mathematicians thought that didacticians would be speaking in their name to teachers, whereas the need was also, and perhaps above all, to rearrange the knowledge of the mathematicians so that it could act on the teaching. A misunderstanding!

discussion and proof "among equals". Knowing how to argue is one thing, knowing how to debate is more than that, it's a culture, and we have been able to teach it, by practicing, to very young children (4 to 5 years old). Clearly, the arguments must rest on acknowledged and undoubted facts.

For example, a child makes a series of drawings to represent the contents of a box, which is then closed. Later, when she is supposed to remember and list the contents of the box, she is supposed use her drawings to name the objects. Her teacher sees what really is in the box and confirms or negates what she says, and at the end of the game shows the contents of the box and validates her success or failure. Later the author of the list gives it to another child who now must use it to list the contents of the box. If he doesn't succeed, both have lost. They discuss the representation so as to make it work better, and in doing so use properties, negation and other logical connectors.

One mathematical property can thus serve to reformulate another, to simplify it, etc. It can also describe an action, and above all it can serve to make a decision. It can intervene as knowledge without the subject's being able to formulate it. These considerations led us to enlarge the notion of "problem" into that of "situation", and to search out among the latter those that are the most characteristic of a given piece of mathematical knowledge. A mathematical object would no longer be characterized only by the properties it satisfies, but also by the class of situation to which it furnishes an optimal solution. The difficulty now consisted of conceiving and choosing such situations and of organizing them into a process that leads to the students' acquisition of stable, correct knowledge.

That's the business of didactical engineering.

In the process, the modeling – sometimes mathematical – of mathematical situations made it possible to describe and predict a certain number of curious phenomena such as epistemological obstacles in mathematics, and verifying their consistency required the development of a fair number of theoretical concepts and of methods.

DIDACTICAL ENGINEERING

From the beginning, all teachers of mathematics have used and devised problems, even at the most elementary levels. A good problem is one that is interesting and even exciting, adapted to the knowledge and objectives of the teacher and productive of knowledge and new questions. Be it the fruit of tradition or the imagination and art of the teacher, a problem is never good by chance. This "good" problem becomes an object of engineering once it is identified and reproducible, and when its characteristics compared to those of its alternatives are known, explained and proved.

In a simplified way, a problem includes: a "statement of the problem", composed of two collections of properties provided by the teacher: H (hypotheses) and C (conclusions), and a response provided by the student which should make explicit the manner in which H determines C. Otherwise stated, every problem expresses a theorem. The student is supposed to take H as "true", necessarily or conditionally, and on the other hand take C as open to doubt, until such time as he has established it, even if it seems perfectly obvious to him.

Inversely, any mathematical theorem H $\Rightarrow$ C can be formulated as a problem. Sometimes either H or C can be converted from a statement to a question. Apart from the formulation, the difference resides in the didactical status: theorems serve as official references, but not problems. This presentation of problems extends to the elementary level, al though there one is more likely to be treating relationships between constants than implications between statements.

For example a teacher asks her students "3 times 4?" and they answer "12". There we have an exercise in mental calculation, but not a problem. If she asks in an impromptu way "12?" and the students are supposed to answer "2 times 6" or "3 times 4", she is certainly not working on the automatic use of the multiplication table, as in the preceding exercise, and hence the exercise is a little more problematic. If she now announces "3 times 4 = 12" and requests the children to produce "proofs" along the lines of "2 times 4, 8, plus 4, 12" or "3 and 3, 6, and 6, 12", with the same mathematical relationship and almost the same answers, she is prefiguring a rather different problem. Moreover, "3 x 4" is equivalent to the descriptive statement "There are 3 x 4 objects" from which the student is supposed to deduce: "There are 12 objects".

Now let us imagine a student who is a bit less experienced. He only knows how to count. The teacher has put stacks of three counting chips in each of the four corners of the room. The child goes to find one stack and puts the three chips in box like a piggy bank, and then starts over. Only after he has put in all four stacks of counting chips does the teacher raise the question: "Now if you take the chips out of the box and count them, how many will you have?" The child must predict, and not just observe or recite the results of an operation that he knows. He can use a model: his fingers, or go find four stacks of counting chips and count them, or make a drawing, or if he knows a bit more use a known result: 4 and 4 are 8, and count the rest in his head. And when he gives his answer, he can verify it by opening the box. The decision does not rest on the knowledge of the teacher. That child is experiencing a situation.

Depending on the circumstances, the students, and the intentions of the teacher, the same mathematical statement can engender very different problems, situations or exercises. Inversely, situations and problems can profoundly modify the significance of mathematical statements and their role in learning.

The objective of didactical engineering is to produce, organize and test out the instruments of the teacher's didactical action, making explicit the possible options and justifying their choice by all of the theoretical and experimental means of Didactics. It consists of asking oneself at each step "Why?", and searching for convincing and/or verifiable answers.

Didactical engineering produces not just situations and problems or even curricula for entire sectors of mathematics, but also experimental designs involving the teachers as actors in a didactical play. For instance, as a defining situation in geometry a teacher might ask the class to find triangles such that the points of intersection of the medians are as far apart as possible (he has "shown" that there are three by means of a rapid sketch.) This request and the subsequent discussion constitute a lot more than just a problem and its solution.

All authors and all teachers mobilize a large number of reasons and maneuvers to produce their texts, eliminate some possibilities and choose some others. But these scaffolds then disappear. They therefore escape analysis, reuse and improvement. By examining the designs used and submitting them to critique, engineering takes up the task of classical methodology on a solider and bolder theoretical basis.

A study of engineering has the objective of producing not only some elements of situations, but also a whole curriculum, usable for the teaching of a piece of mathematical knowledge. By inventorying and clarifying the diverse possibilities and justifying the choice of some, it makes itself open to objective criticism. It is thus the indispensable prerequisite, from the ethical and technical point of view, of all experimentation and of all experimental study of teaching. Studies of the engineering of the experimental designs should be an essential part of any experimental study of cognitive psychology.

For thirty years we have studied didactical designs for the teaching of all the important mathematical knowledge for grades K-12. Even so, we have never made a concrete proposition based on our work, nor published a single text book. We developed a great many techniques and concepts, but our didactical engineering has served above all to expose, produce and study didactical phenomena, or to test the validity of our theories and our methods of observation.

The result that we find the most interesting – although we have a lot of others – has been to demonstrate that "ordinary" children could learn mathematics and not just calculation, in an "ordinary school", and be fascinated by their construction, following routes that were deemed highly abstract. That contradicted the inferences that some arrived at from psychological considerations, and in an essential way the didactical presuppositions of those inferences. Correcting accepted ideas in this domain is our most difficult task.

Our concern was justified. Today we have observed the effect of presenting five of the rather original fundamental situations used in our experiment on rationales and decimals. We had made use of them to illustrate some characteristics of situations and some elements of the theory. But their success caused some people to forget the existence of the sixty other lessons in the curriculum, many with more classical characteristics that permitted the design to function. Our examples were taken as "innovations", whereas we are proceeding in the opposite direction. In principle, innovation is addressed to teachers who are masters of their art and capable of directly grasping the effective interest of a proposition and how to use it. Any detailed explanation even seems a sort of offence to their competence. As a result, a large part of the impact of innovation lies in the forgetting or ignorance of efficient practices and serious reasoning.

Didactical engineering rests on the putting under study, by any theoretical or experimental means, of every design, new or not. In truth, there is often a conflict between micro-didactical engineering, which attempts to produce "good" situations for teaching, and macro-didactical engineering, which permits the diffusion of good or bad didactical ideas. Didactics has not escaped this problem.

OBSERVATION

While reflections on mathematics and the arrangements most favorable to its teaching are as old as mathematics itself, systematic observation has practically never been attempted. No doubt this is because teaching combines complexity and extreme familiarity. Any didactical activity puts together some very complex components: the student, the teacher, the knowledge itself. It calls forth resources arising from all the disciplines, and consequently, a priori, so does its study.

Inversely, all human relations have a didactical component: any interaction is accompanied for everyone interacting by a certain intention of modifying the other person. To at least some extent, everyone wants to "teach" something that the other person doesn't know and isn't trying on his own to learn. As a result, without theoretical identification any teaching appears a marvel of complexity in an ocean of diversity.

The definitions given by Comenius in his classical study of Didactics leave no room for contingency or experimentation. The idea of submitting engineering choices or hypotheses relative to different possible presentations of mathematics to empirical observation – let alone experimentation – appeared completely illusory, given how much it seemed that the educational or psychological conditions could distort the results.

Starting in 1968, a project of the IREMs [Institutes for Research in Mathematical Education] was the creation of an arrangement for observation. The issue was to apply the principles if didactical engineering and scientific observation to research on teaching. The design was to create conditions that would lead the actors to devise and formulate didactical hypotheses, then make them notice the effects and permit them to attempt to correct them quickly.

Working from this idea and its consequences, we determined the conditions necessary to our project. Many of them were contrary to the ideas that were habitual and accepted. The conception and creation of the COREM (Center for Observation and Research on the Teaching ["Enseignement"] of Mathematics) were the first and most complex applications of the theory of situations and of didactical engineering. Conducting it was the principal source of our observations. We had the good fortune of finding the initial help and means, and then that of being able rather quickly to show some interesting results and some sub-products, which enabled us to survive.

I will not describe here the arrangements: teaching personnel, technicians, researchers, number of classes, material means, schedules, etc. In order to last, the school had to avoid competition with neighboring schools and disarm any ideas people might have for using it (or using its budget) for anything other than observation. Thus it had to appear to the parents and the authorities to be an ordinary school, with no other objective than teaching children. 80% of the supplementary money mobilized was used to guarantee that the observations could not possibly diminish the performance of the children. The lessons given and the children's test results were all collected for the use of the administration and the parents, who naturally retained their usual rights.

It did not take part in any other project, nor subscribe to any pedagogical school of thought. It was not an experimental school, nor a model school, nor a school requiring applications, nor a school for special children, gifted or in difficulties with mathematics or anything else, nor difference *a priori* in anything at all. It took a lot of work and luck for our project to survive while giving so few promises or bits of spectacular information to the politicians and the media. Some of our results were of interest to the people providing the structural support system for the teachers – people who taught teachers, curriculum decision-makers, school inspectors and others at that level. We avoided making any of our materials available in a form in which they could be directly transferred into the classroom, partly because they were not suitable, and partly to avoid the conflict with people making general decisions that would inevitably have occurred.

The teachers were volunteers. They were co-opted in a list provided by the administration and the school's only additional criterion was their capacity to cooperate with their group and to handle the observation sessions. They were hired for renewable periods of three years, and thus tacitly renounced their right to tenure. The fact that this system functioned for thirty years without a single conflict either with the parents, or with the teachers (more than a hundred, all told) nor with their unions, nor with the administration (which also had a lot of turnover) is a striking proof that the observation system was stabilized and that the whole educational system really wanted to function intelligently.

The density of observations was very small, with a mean of three hours per class per year, or around thirty hours for the whole collection. The more demanding of the projects extended over several years. Observations covered what we called a-didactical situations for more than a dozen years before we began to be able to analyze openly the didactical situations that involved live decisions by the teacher. That was the time required to sharpen the methods of observation and of statistical analyses, the theoretical instruments and above all, for the institution, to acquire the knowledge and practices necessary in order for each one to have confidence not only in the competence of the others, but also in the understanding of their work and of their decisions.

I believe that what made these observations fruitful was a very fine-grained hierarchy of our preoccupations, of which the first was the satisfaction that the children felt in their learning, followed by the comprehension of the necessity of decisions not already agreed to by the teachers at work, and the questioning, feverishly and without concession, of the facts.

I cannot enumerate here all the precautions we developed to make our observations possible. But I can perhaps bring up two examples that show how we made some elements observable and how we dealt with some others.

The first example is a type of arrangement arising from the model called that of formulation. A team prepares two successive lessons. The first is presented to the class by one teacher, and the second will be presented by another who was not present as the first was given. He can, however, question the first teacher, and benefit from her remarks and observations. Their conversation is recorded. The nature of the questions, the vocabulary and the concepts that they use are noted, and the essential points that they forgot to note will be made visible and discussable by the difficulties that turn up in the second class

The second is a principle of observation: what should be done about an error?

An error made by the teacher is only interesting to the researcher if she can see in it the prototype of a phenomenon that can be reproduced by others in circumstances that arise fairly frequently. If that is not the case, the observer should ignore it. It would have happened if she had not been there, and correcting it is not her business. On the other hand, there are objective reasons that cause an error to turn up frequently, and then the personal responsibility of the person who committed it is diminished. Though the immediate correction may depend on her, prevention depends on the arrangement of the conditions that made its production probable. The error can be combated by the study and modification of these conditions. Admonition has a very narrow domain of legitimacy in Didactics.

Analyzing this aspect of observation made us conscious of a strong parallel with teaching. An essential principle is that to learn, you have to take on situations where you risk making a mistake, and then to correct yourself. This doesn't happen without some difficulties and some risks. Teaching requires putting the errors on trial, not their authors. Analyzing the objective causes of errors is much the easier if one can forget the author after having listened to them as being victims. This opportunity means that – contrary to accepted ideas – collective teaching can be much more efficient and effective than individual teaching.

The possibility of giving counter-examples to declarations or hazardous theories by carrying out some "experiments" thanks to ingenious didactical engineering was precious to us. Demonstrating the possibility of realizing in a perfectly reproducible manner performances held to be impossible was an exciting challenge – but a dangerous one, because what you *can* do is not necessarily what you *should* do.

However, these demonstrations have been no more than the visible part of a much more complex research process. We spent a great deal of time studying more standard phenomena. In these areas whose appearance was well known, our research used non-standard approaches, new concepts and original methods. Our results were very difficult to publish, because we had to explain and illustrate too many subjects at a time.

CONCLUSIONS

The search for situations and for a process of teaching appropriate for old and new mathematics led to a deepening of the study of mathematics and of the forms in which it can be taught to children. Putting falsifiable statements resulting from these reflections to the test in reproducible experiments was made possible by a lot of work on the description and analysis of the arrangements for teaching – the didactical engineering – and by the creation of an original model for observation.

The wealth of unedited and useful results collected is encouraging, but the procedure that consists of separating off the inadequate solution seems very slow in the face of the impatience of diverse protagonists of education, all the more so in that doing so does not lead to the adoption of the extreme "opposite" solution.

The theoretical instruments developed in the course of this research form a coherent body of concepts. They are, however, quite different from the ones that tradition would offer to have us take as primary evidence, and the rather marked rupture has to do with a wide spectrum of beliefs. In particular, a re-examination of the commonly accepted psychological, pedagogical and methodological presuppositions, and of the way to import them into the domain of Didactics and of teaching presents difficulties that must not be underestimated. Knowledge that has no immediate application is less and less consideration.

I would have liked to be able to show how our work makes it possible to bring under discussion some heavy trends of our educational systems:

- a trend towards total individualization of teaching,

- an excessively "psychological" and neuroscienific conception of school knowledge confused with the knowledge students actually have,

- a senseless pretension of treating teaching as if it were a commercial distribution, with the resulting barbarous misuse of test scores,

- use without reflection of popular opinion and "extremist" reasoning,

but these discussions will have to happen elsewhere.

The most important conclusion that one can draw from our work and from the study of the evolution of teaching is without a doubt the following: it is not reasonable to want to produce a profound transformation in the teaching actually practiced on the basis of naïve inferences and superficial experiments. Practices should only be proposed to the extent that they are understood and managed by the system. This depends on the culture.

We must be willing to advance the science and culture of Didactics with teachers and with the public without requiring that there be immediate applications in practice, which are very difficult to obtain. In Didactics, it seems to me that the prime cause of difficulties is impatience.

IN LIEU OF A BIBLIOGRAPHY

Given the space restrictions, I have opted rather than crowding in a list of English-language publications about Didactics to give a single URL at which an on-going list can be found. Also at that address is "Invitation to *Didactique"*, a mini-book (or maxi-article) by V. Warfield geared to gently introducing neophytes to *Didactique* in general and the Theory of Situations in particular.

The URL is www.math.washington.edu/~warfield/Didactique.html For other publications see: http://perso.wanadoo.fr/daest/Pages%20perso/Brousseau.htm or

http://math.unipa.it/~grim/homebrousseau.htm

A SEMIOTIC VIEW OF THE ROLE OF IMAGERY AND INSCRIPTIONS IN MATHEMATICS TEACHING AND LEARNING

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Because the objects of mathematics cannot be apprehended directly by the senses, the role of mediating signs is crucial in all mathematical activity, including its teaching and learning. In particular, this semiotic account addresses visual sign vehicles in the form of mental images or externally presented inscriptions (written or on computer screens) and indicates by means of examples from research projects over the last 30 years and recently, how various interpretations of the relationships amongst signs may facilitate or hinder the construction of mathematical knowledge at all levels. Of particular significance is the investigation of ways that teaching may facilitate learners' building of connections amongst mathematical signs. By highlighting structures and patterns across domains, such connections may foster generalizations and combat the phenomenon of compartmentalization.

SOME RESEARCH RESULTS AND A VIGNETTE

In my original fieldwork (Presmeg, 1985), Mr. Blue (pseudonym) was a high school mathematics teacher who did not feel the need to use imagery or inscriptions in solving the problems on my test for preference for visuality in mathematics. In fact, his *mathematical visuality* score on this test was 3 out of a possible 36, the most nonvisual score of all of the 13 teachers in my study. Yet he used many of the classroom aspects that had been identified in the literature and tested in my research as being facilitative of visual thinking in mathematics classrooms (Presmeg, 1991). Of 12 such classroom aspects, he was observed to use 7 during the year in which I observed his lessons. This *teaching visuality* score placed him squarely in the middle group of teachers (as one of four teachers in this group), between those of the nonvisual group (four teachers) and the visual group (five teachers) according to the visuality of their teaching. It turned out that *mathematical visuality* and *teaching visuality* were only weakly correlated in this sample of teachers (Spearman's rho = 0.404, not significant)—a result that made sense because good teachers know when their students need more visual thinking than they do. In the classes of these 13 teachers, of the 54 senior students who preferred to think visually according to my test, those who performed well in their final school-leaving examinations in mathematics were in the classes of teachers in the middle group, contrary to the common sense notion that visualizers would do best with visual teachers.

Commenting in an interview on some of the rich characteristics that I had identified in his lessons—in which he stressed abstraction and generalization in addition to introducing and encouraging visual inscriptions—Mr. Blue spoke as follows:

Mr. Blue [with excitement]: You've got to be careful sometimes I think; you've got to be careful making, bringing things too, from the abstract, to too concrete. Then it's that way forever, then everything is like that. You've got to be careful with that because sometimes you must remember that our abstractness carries us to flights of imagination of where we can go with it. And that's what I would like them as often to see here, when we do something, this is another possible way of doing this problem; more algebraically, what you can do with it.

For Mr. Blue, algebra was often the vehicle of abstraction.

In his teaching, Mr. Blue frequently expressed his own pleasure in the beauty of mathematics. Indeed, he expressed his feelings often, not only towards mathematics, but also towards his boys (he taught in a boys' school). In a trigonometry lesson he spoke with his students about errors that some of them were making.

Mr. Blue: Don't just square things, and suddenly they disappear into space. … And then of course I was really saddened by this: now let me say this to you. <u>Don't</u> do this any more. Now you <u>know</u> better than that in this room. You cannot take the square root of individual what?

Boys: Terms.

Mr. Blue: Terms. … Don't force it! Maths just won't be forced. That's the beauty of it, that's its beauty: where it stands strong against this forcing things into it that don't have any place for it at all. It must go on the way it always has gone on.

The way that Mr. Blue encouraged metacognition was also apparent in an algebra lesson on change of base of logarithms, with the same class. The problem under discussion, which had been done by the boys in a test, involved a quadratic equation in logarithms:

$$(\log_3 x)^2 - 10\log_3 x + 9 = 0$$

Mr. Blue: So this would be the fastest way: factorise. You can do the change of base with tens, you can get it, it will be right, when you've finished [but it is slow]. … We could put a y in for log to the base 3 of x, couldn't we? Then factorise. … The whole thing in higher grade is to think in patterns, and relate the patterns of the former work received. And you get bigger and bigger problems. If you look at this one now, how many ideas were in this problem? This idea was a log idea, this turns into a quadratic idea, this turns into factorisation, this turns into exponentials to get the answer. All in one problem. That's what you must start getting used to.

The connections between domains that Mr. Blue was helping his boys to identify in this lesson are a central topic that I wish to highlight in this paper. The theoretical lens that I shall use is that of Peircean semiotics.

TERMINOLOGY

Semiotics is the study of activity with signs (Colapietro, 1993; Whitson, 1997). But what is a sign? Although in this paper I shall follow the triadic model of Charles

Sanders Peirce (1998) with basic components that he designated as *object*, *representamen* (standing for this object in some way) and *interpretant* (the result of interpreting this relationship), even in Peirce's own writings at various periods there is ambiguity in the sense in which he used the word *sign*. Thus it is necessary to specify how I am using the word. I shall take a sign to be the interpreted relationship between some representamen or signifier—called the *sign vehicle*—and an *object* that it represents or stands for in some way. In mathematics, the objects we talk about cannot be apprehended directly through the senses: for instance, "point", "line", and "plane" in Euclidean geometry refer to abstract entities that we can never see, strictly speaking, as in Sfard's (2000) *virtual reality*. We apprehend these objects, "see" them, and communicate with others about them, in a mediated way through their sign vehicles, which may be drawn by hand or through dynamic geometry software, labelled in conventional ways, moved and manipulated for multiple purposes. We work with these sign vehicles *as though* we were working with their objects: in Otte's (2006) terms, we become accustomed to *seeing an A as a B.* It is this interpreted relationship between a sign vehicle and its object that constitutes the sign.

At the same time, the sign vehicle and its object partake of different levels of generality: the sign vehicle is quite specific, e.g., *this particular scalene triangle that I drew*, whereas the object may be interpreted as *any triangle*. "Threeness", "the nature of being an exponential relationship", "the sinusoid", "a differentiable function"—all mathematical objects—are by their nature more general than their particular instantiations in sign vehicles, and more than one sign vehicle may refer to a particular mathematical object. Duval (1999) called such signs different *registers*, for instance as in the case of a drawn parabola and the corresponding algebraic notation for this particular quadratic function. The relationships may be depicted as in figure 1.

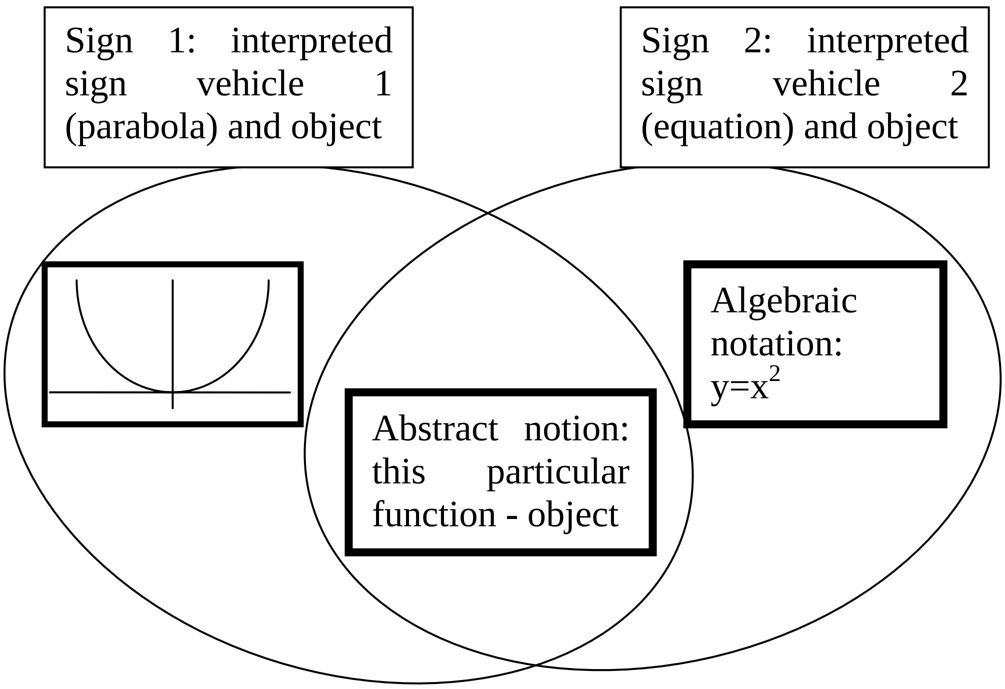


Figure 1. Two *registers* illustrated by two signs with the same object.

Thus issues of generality are implicated in a study of mathematical signs. I am particularly interested in ways that conversions amongst registers (signs) may facilitate what Radford (2002) called *objectification* in mathematics—the ability to treat a general structural relationship as an object in its own right, represent it with a sign vehicle, and interpret and work further with this sign. It is also a significant question how teaching may impact such objectification, enabling or retarding it. I shall illustrate some preliminary results from two current investigations of these issues later in this paper.

Peirce's well-known proclivity for thinking in threes resulted in ten trichotomies in his writings. One such trichotomy, according to Peirce (1998), is that signs may be *iconic*, *indexical*, or *symbolic*. These types are not inherent in the signs themselves, but depend on the interpretations of their constituent relationships between sign vehicles and objects. To illustrate by using some of Peirce's examples, in an *iconic* sign, the sign vehicle and the object share a physical resemblance, e.g., a photograph of a person representing the actual person. Signs are *indexical* if there is some physical connection between sign vehicle and object, e.g., smoke invoking the interpretation that there is fire, or a sign-post pointing to a road. The nature of *symbolic* signs is that there is an element of convention in relating a particular sign vehicle to its object (e.g., algebraic symbolism). These distinctions in mathematical signs are complicated by the fact that three different people may categorize the "same" relationship between a sign vehicle and its object in such a way that it is iconic, indexical, or symbolic respectively, according to their interpretations—thus effectively generating three different signs. Is the inscription "A", standing for this particular point called A, an icon, an index, or a symbol? It could be all three! The classification depends on the interpretation (personal communication with Colette Laborde, January 2006). However, these distinctions are included in my conceptual framework because they introduce a finer grain in analysing the ways that mathematical signs are interpreted and used, and the structural similarities that permit connections amongst signs.

Imagery and inscriptions

I am avoiding the use of the term "representations" because *imagery* and *inscriptions* capture concisely the visual aspects of what some other writers (e.g., Goldin, 1998) called internal and external representations respectively. I am aware that mental imagery may occur in various modalities (sight, hearing, smell, taste, touch). The modality that is most prevalent in mathematical imagery is the visual one (Presmeg, 1985). Both visual imagery and inscriptions are sign vehicles that are instantiations of visualization in mathematics, insofar as they depict the spatial structure of a mathematical object. Not only diagrams satisfy this purpose. Because numbers and algebraic expressions also have spatial structure, they may be depicted by various sign vehicles, including those of a visual nature (e.g., an algebraic formula).

THIRTY YEARS OF PME RESEARCH ON VISUALIZATION

Many of the issues that I am addressing in this paper are not new in mathematics education. Before illustrating a few semiotic aspects of more recent research, I shall summarize some pertinent results that have been reported in PME proceedings over the last three decades (Presmeg, 2006).

Affordances and constraints of use of imagery and inscriptions as sign vehicles

Already in the early 1980s, following on from the work of Krutetskii (1976), Clements (1981, 1982), and Suwarsono (1982), my research indicated that mental imagery and corresponding inscriptions constitute powerful sign vehicles that may be harnessed for mathematical generalization, in addition to their mnemonic advantages (Presmeg, 1985, 1997). At the same time, all of the difficulties experienced by the 54 high school students that I interviewed, in their use of visual sign vehicles, related in one way or another to problems of generalization. The affordances generated by being able to move flexibly amongst various mathematical signs were also evident in this study. This early research suggested that students can *learn* to use their mathematical imagery effectively, as was borne out by the fact that teachers such as Mr. Blue, with their emphasis on generalized patterns at the same time as they used and encouraged visual thinking, seemed to be helping these students to overcome some of the constraints of the *concreteness* of visual sign vehicles. Bishop (1988) reported many of these results at the PME-12 meeting. Other researchers in the decades since 1988 (e.g., Owens, 1999) have developed programs with the specific intent of encouraging students to use visual imagery and inscriptions.

Students' seeming reluctance to visualize

It was in 1991, at PME-15 in Assisi, Italy, that visualization in mathematics education came to fruition as a research field, with ten research reports listed in the initial category of *Imagery and Visualization* in the proceedings. In addition, two of the three plenary addresses that year directly concerned this topic (Dörfler, 1991; Dreyfus, 1991). While appreciating and illustrating the power of visual inscriptions in mathematical thinking, Dreyfus suggested that the basic reluctance of students to visualize in mathematics is the result of the low status accorded to visual aspects of mathematics in the classroom. However, the research of Presmeg and Bergsten (1995) on high school students' preference for visualization in three countries (South Africa, Sweden, and Florida in the USA) suggested that these issues are complex and that the claim that students are reluctant to visualize should not be interpreted simplistically to mean that students do not use this mode of mathematical thinking. On the contrary, preference for mathematical visualization follows a standard Gaussian distribution in most populations (Presmeg, 1997). Further, the research of Stylianou (2001) suggested that even in the learning of collegiate mathematics, the picture of "reluctance to visualize" had changed in the decade since Dreyfus's plenary address.

Objectification, compression, encapsulation, and reification

One aspect of Peircean semiotics that is salient in the well known reification of mathematical processes as objects (Sfard, 1991), is the proclivity of signs to form chains. Alternatively, signs may be regarded as nesting one within the other (Saénz-Ludlow & Presmeg, 2006). Peirce (1998) described a continuous process in which each interpretant in turn becomes objectified, represented by a new sign vehicle, and interpreted again, in an ongoing development of new signs. Implicit in this process is *compression*, which may be likened to a subroutine in a computer program. The relevance for mathematical generalization and abstraction is clear. However, there is one study (Herman et al., 2004) reported at PME-28, which suggested that in the learning of fractions at least, it is possible that "objects are not the encapsulation or reification of processes after all" (Vol. 4, p. 255). The results of this study were interpreted to suggest that the process-object duality of sign vehicles for fractions results in images for fraction as a product that are problematic in the sense that they cannot easily be converted into images of the process required in addition of fractions. It would seem that further research is required before conclusions can be drawn in this area. Many reported studies have indicated the usefulness of Peircean semiotic chaining in the objectification and compression of mathematical objects (see the papers in Saénz-Ludlow & Presmeg, 2006).

Theoretical developments: gesture and embodiment

In my original study, teachers' use of gesture was one of the surest indicators that they had a mental image that they were intentionally or inadvertently conveying to their students (Presmeg, 1985). In the last few years, interest in gesture as a sign vehicle indicating an object in someone's cognition has increased, bearing fruition in several new research reports in addition to a Research Forum at PME-29 (2005). These studies, which often draw on theories of embodiment for their conceptual frameworks, are too numerous to list here (see the summary in Presmeg, 2006) but present an interesting fine-grained line of research that is just emerging.

The need for an overarching theory of imagery and inscriptions in mathematics

Already in his plenary address at PME-16, Goldin (1992) outlined a unified model for the psychology of mathematics learning that incorporated cognitive and affective aspects of visualization. Also in a PME plenary address, Gutierrez (1996) posited a framework for visualization in the learning of three-dimensional geometry. Marcou and Gagatsis (2003, in Greek) developed a first approach to a taxonomy of mathematical inscriptions. However, none of these authors have availed themselves of the affordances of a Peircean semiotic model in their construction of theory, and the ongoing need for an overarching theory of imagery and inscriptions in mathematics learning and teaching is still present.

Big questions for research on imagery and inscriptions in mathematics

Didactics and curriculum development involving inscriptions, technological advances and their impact, and affective issues, are important areas that are also in need of further investigation. At the end of my Handbook chapter (Presmeg, 2006) I put forward a list of thirteen questions in this field that seem to be significant, as follows.

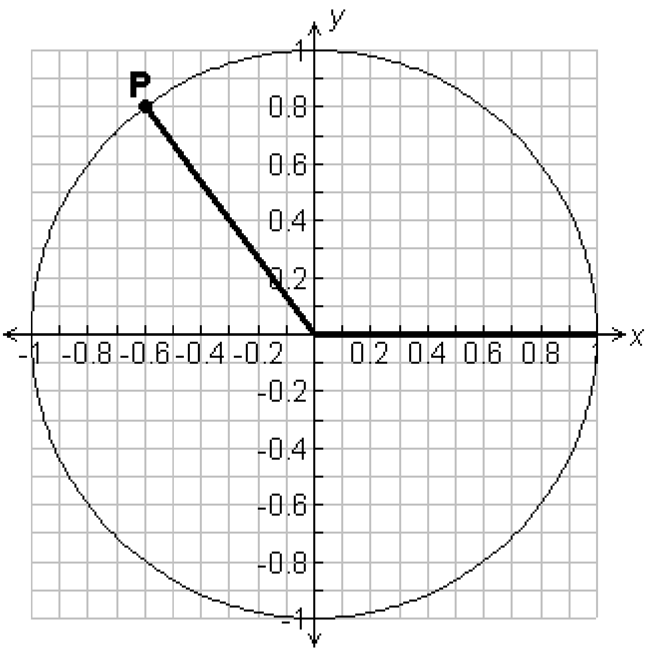
1. What aspects of pedagogy are significant in promoting the strengths and obviating the difficulties of use of visualization in learning mathematics?
2. What aspects of classroom cultures promote the active use of effective visual thinking in mathematics?
3. What aspects of the use of different types of imagery and visualization are effective in mathematical problem solving at various levels?
4. What are the roles of gestures in mathematical visualization?
5. What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of a visual nature, thus combating the phenomenon of compartmentalization?
6. What is the role of metaphors in connecting different registers of mathematical inscriptions, including those of a visual nature?
7. How can teachers help learners to make connections between visual and symbolic inscriptions of the same mathematical notions?
8. How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions, and conventional mathematical processes and notations?
9. How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?
10. How may visualization be harnessed to promote mathematical abstraction and generalization?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?
12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?
13. What is the structure and what are the components of an overarching theory of visualization for mathematics education?

RECENT RESEARCH

In the previous section I sketched some PME results that addressed various issues concerning imagery and inscriptions in the last three decades. In this section I shall provide a few illustrations of preliminary results from two very recent projects that begin to address the first and the fifth research questions in the foregoing list. The damaging effect of compartmentalization in mathematics education has been noted by several authors (Duval, 1999; Nardi, Jaworski, & Hegedus, 2005). Duval considered the ability to move freely and flexibly amongst mathematical registers (signs, in the terminology of this paper—figure 1) to be a *sine qua non* of effective performance in mathematics. In this section I shall illustrate both of these points using data from two ongoing research studies with colleagues (the first with Jeffrey Barrett and Sharon McCrone at Illinois State University, the second with Susan Brown in a Chicago high school—see her Short Oral presentation in these proceedings). In both studies my research question is the same, as follows. *How may* *teaching facilitate students' construction of connections amongst registers in learning the basic concepts of trigonometry?*

Compartmentalization in trigonometry

In the first case, illustrating the damaging effects of compartmentalization, preservice elementary teacher Sam (pseudonym) is trying to recall trigonometric principles that he learned several years earlier, and the "facilitative teaching" of trigonometry by my colleague Jeff Barrett in our multi-tiered teaching experiment has not yet taken place. Sam was chosen (as one of three students that I would interview, from a class of 27 taking a geometry content course taught by Jeff) because of his strong abductive thinking (Peirce, 1998) as observed in his contributions to class discussion. In our first interview, Sam was presented with three mathematical problems to solve while "thinking aloud" (Krutetskii, 1976). The third problem is as shown in figure 2 (Brown, 2005, p. 268).



This graph shows an angle. Give the approximate value of the sine of that angle.

Figure 2. Preliminary interview question 3.

Sam had figured out, using the x-axis, that the radius of the circle is of length one—a unit circle. He also knew from the right triangle definitions of the trigonometric ratios that the sign of an angle is the ratio of the lengths if the opposite leg and the hypotenuse of the triangle (using the mnemonic SOH CAH TOA, which he had written down). He called the rotation angle θ (theta) and marked its supplement, the reference angle in the second quadrant. From point P, he dropped perpendiculars to both the x and the y axes, and he also joined point P to the intersection of the y axis with the circle. Then he correctly identified the sine of the reference angle (in the right triangle he had drawn to the x axis) as having a value of 0.8.

At this point in the interview the following audio taped exchange took place.

(Note: … indicates a pause; […] indicates that transcript data have been omitted.)

Sam: That’s point 8, yeah. But, over here, let’s see, the whole sine of one … the sine of the whole angle is one.

Interviewer (I): The sine now of the obtuse angle?

Sam: So this would be point 2 [pointing to the arc between P and the y axis].

Sam appeared to have an image something like the inscription in figure 3.

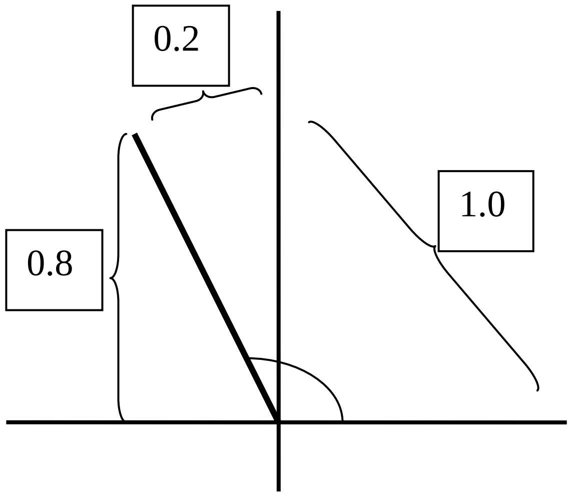


Figure 3. Sam’s imagined values of sine θ.

After some negotiation with the interviewer, he continued as follows.

Sam: So I’d say negative one point two. … I dunno.

I: How did you get one point two?

Sam: So, the sine of this angle is one.

I: The ninety degree angle?

Sam: Yes, so this is one. And then this is … let’s see, this point is … negative point 6, point 8.

I: Oh, I see. You’re figuring out the coordinates?

Sam: I was just thinking of a unit circle. And with coordinates … ’cause now like, the sine of this angle here [indicating point of intersection of circle and y axis] is, the cosine zero, the sine one. […] And then it goes, that’s 90, which, it still stays positive though, so … one point two, because this is point two.

The interviewer asked him to explain the positive and negative signs of the trigonometric ratios in the four quadrants, which he did correctly, knowing that the cosine and sign are coordinates of points on the unit circle. When she told him that the correct answer for the value of sine θ is point 8, he persisted, as follows.

Sam: But if this … if the sine of this angle here is one, how can a bigger angle be less?

I: Ah, that’s a good question! … Do you know what a sine graph looks like?

Sam: Yeah.

I: Can you draw me one? … Can you put values in there?

Sam drew the graph of y = sin θ for one revolution, and correctly inserted radian measures of π/2, π, 3π/2, and 2π in the appropriate places (except that π/2 was placed above the first hump instead of on the θ axis). He also marked one and negative one on the y axis.

I: There you go! Now you just said, how can it be less, if it's [the angle is] bigger than 90?

Sam: Yeah, if it's not … [following the curve with his finger].

I: So it goes down again.

Sam: So that spot is the same here. Yeah! [He marks symmetrical points on the sine curve on either side of π/2.].

Sam later indicated that he had always just used his calculator to find the values of trigonometric ratios. But the calculator values for reference angles would not have supplied direct answers for all quadrants, so he must at some point have been required to have connected knowledge of the sequence of trigonometric definitions that moves from the right triangle, through the coordinate plane to the unit circle, and finally to the graphs of the trigonometric functions. Although these connections were not robust enough to help him in the early part of this task, he seemed elated when he connected the sine graph with the problem he was trying to solve.

Teaching that encourages conversions amongst signs in trigonometry

The second case is drawn from a collaborative research project with Susan Brown, in her *Enriched Advanced Algebra/Trigonometry* class of 30 students in a Chicago high school. The aim of this research is similar to that in the project from which Sam's interview data were drawn, namely, to investigate ways in which teaching may facilitate students' building of connections between signs in moving from triangle trigonometric definitions, to the coordinate plane and the unit circle, and finally to the graphs of the trigonometric functions. Unlike Jeff's class of mostly preservice elementary school teachers, in which geometry was the focus although some trigonometry would be introduced, the main goal in Sue's trigonometry class is to foster skill in converting amongst signs as students build up comprehensive knowledge of trigonometry concepts. The methodology of this teaching experiment includes cycles of joint reflection based on interviews with students, followed by further teaching. Early in our collaboration, Sue listed ways in which she tries to facilitate connected knowledge in her class—actions that were confirmed in the researcher's observations of her lessons, and in documents such as tests and quizzes. In the analysis of data, her list will be compared with the connections constructed—or the lack of connections—by four students in a series of six interviews conducted at intervals during the semester. The four students were purposively chosen by the teacher in collaboration with the researcher to ensure a range of learning styles and proficiency.

All four of the students from Sue's class, in the second interview of the series, had no difficulty in completing the task given to Sam. Laura's solution is typical, although two of the four students felt no need to invoke the Pythagorean triples.

I: So first of all, where's the angle that you're looking at?

Laura: It's the … It goes through the first into the second quadrant.

I: Okay, the rotation angle?

Laura: Mm, the rotation angle. […] The x value over the radius, no, the y value over the radius. […] And I'd say the y value is approximately point 8.

I: Point 8. Where are you looking? On the y axis?

Laura: Yes. Hm … I'm not quite sure but I assume it [the radius] would be … about one. It has to be greater than …[5 seconds].

I: Is there a way that you can *see* what the radius is? … Have a look at other points on the circle.

Laura: Oh! Yeah. … So that would be point 6, minus point 6. … So, draw a triangle.

Laura drew triangles on her paper (not on the given diagram), as in figure 4.

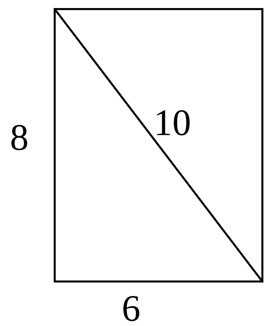


Figure 4. Laura's inscription of triangles in the second quadrant.

I: So you're thinking of a right triangle, 6, 8, 10? Is that …how did you know that? The 6, 8, 10?

Laura: It's just a, one of the triples that we learned.

I: The Pythagorean triples? Okay. So if that was 6, 8, that would be 10, and now you've got …

Laura: Point 6, point 8, and one. So the, it was one. Point 8 over one … point 8.

Although none of these four students had difficulty with *this* task, there was one interesting aspect of two of the students' thinking when I asked them to describe how they would work with angles in the *third* quadrant. Both Laura and Jim drew right triangles by dropping a perpendicular to the y axis rather than the x axis, and Jim in the discussion that ensued expressed resistance to working from the x axis.

Jim: Well I'd subtract the rotation angle from 270, to get the angle, and then I'd use, um, this up here. […] And you can do the same thing: multiply it by the radius. [In working with the second quadrant he had written "sinθ.r =opposite, and cosθ.r =adjacent" indicating multiplication by the length of the radius.]

I: Just be careful. Because if you now say the sine of that angle … which one is it going to give you? […] How can I put it? Your cosine gave you the x here [in the second quadrant], and the opposite gave you the y. Now is it going to be that the cosine gives you the x again?

Jim: No. It will still be the opposite and the adjacent legs, but it will switch from x to y.

I: I see. So you've got to be careful of the x and the y in this case.

In the fourth quadrant, Jim subtracted the rotation angle from 360 degrees, and drew a triangle by dropping a perpendicular to the x axis. It came out later in the conversation that he did not like drawing the triangle "backwards on itself" in the third quadrant, because it would be "blocking" the rotation angle. His image was apparently something like figure 5.

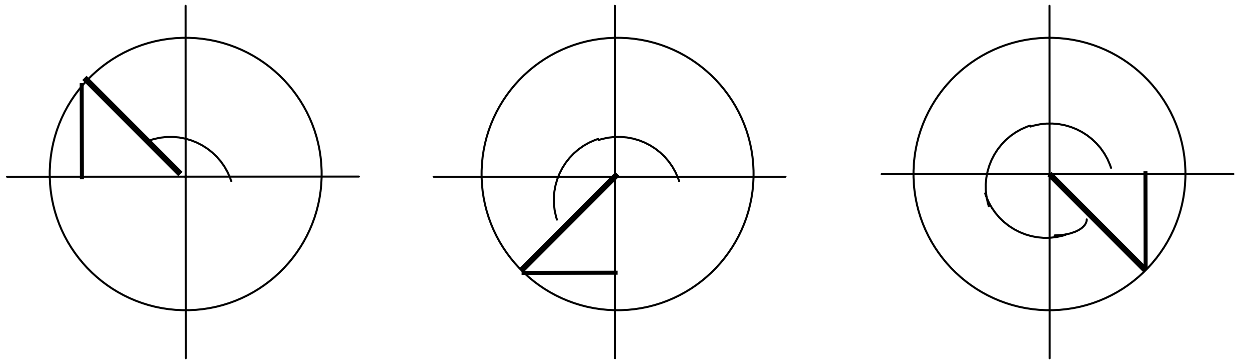


Figure 5. Triangles in the quadrants, as Jim apparently saw them.

One of the strategies that Sue used in her teaching was to introduce the metaphor of a "bow tie" in talking about the triangles in the quadrants. At the time of the interviews quoted earlier, the bow tie had been an implicit feature in a computer program the students had constructed previously, but some students (such as Jim) had not yet connected this metaphor with the unit circle. After Sue used this metaphor explicitly, none of the four students dropped perpendiculars to the y axis in further interviews. In my original research, Alison constructed her own metaphor of a "water level, with a ship sailing on it" to help her remember the same principle (Presmeg, 1985).

Some of Sue's facilitative principles that have the intent of helping students to move freely and flexibly amongst trigonometric registers are summarized as follows:

connecting old knowledge with new, starting with the "big ideas", providing contexts that demand the use of trigonometry, allowing ample time, and moving into complexity slowly;

connecting visual and nonvisual registers, e.g., numerical, algebraic, and graphical signs, and requiring or encouraging students to make these connections in their classwork, homework, tests and quizzes;

supplementing problems with templates that make it easy for students to draw and use a sketch, or asking students to interpret diagrams that are given;

providing contextual ("real world") signs that have an iconic relationship with trigonometric principles, e.g., a model of a boom crane that rotates through an angle θ, $0^o<\theta<180^o$ on a half plane;

providing memorable summaries in diagram form, which have the potential of becoming for the students prototypical images of trigonometric objects, because these inscriptions are sign vehicles for these objects;

providing or requiring students to construct static or dynamic computer simulations of trigonometric principles and their connections, in many cases giving a sense of physical motion;

using metaphors that are sometimes based on the students' contextual experiences, e.g., the bow tie and the boom crane.

Analysis of the complete corpus of data in terms of Sue's full list (abridged here) will assess the effectiveness of these principles in accomplishing their goal, at least for the four students interviewed.

DISCUSSION: THE POWER OF SEMIOTICS

What stands out in the brief excerpts from interviews with Sam, Laura, and Jim is that all the mathematical thinking portrayed in these episodes involves *activity with signs*, i.e., semiosis. Whether their interpretations of the relationships connecting sign vehicles with their objects are mathematically correct or not, there is an internal logic in these interpretations, as revealed in the imagery associated with the students' interpretations of the relationships. For instance, Sam's image that results in his claim that the sine of the rotation angle is 1.2 is a sign vehicle that is connected *iconically* with the way he is seeing the relationships in the mathematical object (figure 2), which is the sine ratio defined in the unit circle on the coordinate plane. When he constructs a different sign, based on his inscription of the sinusoid graph—which was requested by the interviewer—then his previous icon is no longer viable: the value of the sine of an angle cannot exceed 1.

I am calling the relations between these images or inscriptions *iconic*, because there is a spatial or perceptual likeness between the sign vehicles and the inferred trigonometric object. However, there is also a sense in which they are *indexical*, because they *point* to what Sam sees as the structure of the relationships involved (as smoke points to fire). But in standard mathematics there is an element of *convention* associated with the principles governing the trigonometric ratios defined in the coordinate plane (e.g., that the radius vector rotates counter clockwise from the positive x-axis), and with the way the sinusoid is organized. (With the genesis of non-Euclidean geometry in mind, one might even argue that a different trigonometry could be constructed based on Sam's "incorrect" definition of the sine of the rotation angle and other trigonometric ratios in the coordinate plane.) Thus the correct interpretation of the relations between sign vehicles and mathematical objects is also *symbolic*, while it may partake at the same time of iconic or indexical features.

In a similar vein, when Laura and Jim want to subtract the rotation angle from 270^o to find the reference angle in the third quadrant (figure 5), the *iconic* sign they have constructed gives way to a conventional *symbolic* sign under the influence of the bow tie metaphor. This change is not arbitrary; it partakes of necessity according to the consistency of mathematical principles. As Mr. Blue claimed in the opening vignette, that is the beauty of mathematics—that it "just won't be forced."

And what of objectification and compression? Because the nature of metaphor is to compare two disparate domains and identify common structures (the *ground* of the metaphor: Presmeg, 1992, 1998), these signs often provide memorable prototypes or vehicles for mathematical objectification and concomitant compression. Metaphors are particularly memorable if they are iconic, as in the case of Sue's powerful bow tie metaphor for reference triangles in the four quadrants of the unit circle. However, a cautionary note is in order. Not all images or inscriptions that provide powerful prototypes are metaphoric. Even when they are, metaphors always have a *tension* (dissimilar elements in the two domains compared) as well as a ground. The constraints of mathematical prototypes, when students use these inflexibly or in a rote manner, have been documented in the literature (Presmeg, 1992). It is noteworthy in the previous section that when Laura constructed a right triangle in the second quadrant, with lengths of legs that reminded her of values she had encountered previously (6, 8, and hence 10), a Pythagorean prototype appeared to be invoked, from which she reasoned that the radius of the circle had to be one (because the actual values of the legs were 0.6 and 0.8). This prototype prevented her from seeing that she could have read off the value of the radius as one directly from the points of intersection of the circle with the x axis. Objectification as a semiosic process is powerful in the learning and doing of mathematics (Radford, 2002). But the flexibility of having the ability to convert freely back and forth amongst different signs for the same mathematical objects (Duval, 1999) is paramount.

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SCHOOL MATHEMATICS AS A DEVELOPMENTAL ACTIVITY

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INTRODUCTION

Many imagine mathematics to be an almost emblematic example of school education detached from life. It consists in a highly abstract exercise of the mind that serves to classify children as "talented" or not. And which does not prepare children for anything useful which may serve them in their later life – perhaps with the exception of simple calculations close to the everyday protoarithmetic. The Czech Republic has recently approved a new A-level exam (Baccalaureate), in which mathematics has been dropped as a compulsory subject. This was the result of emotive resistance against a compulsory exam by the public and by politicians. The discussions, amongst psychologists among others, have proved that **the relation between mathematics at school and its influence on the mental development** of the individual (child) is far from understood. There are various implicit epistemologies of mathematics shared by didacticians and teachers which are transmitted to pupils and, indirectly, to their parents. These epistemologies have for their part different consequences for the conception of mathematics in school.

What I want to deal with first are the basic epistemological approaches inherent in educational work in school. Those different approaches have naturally been applied at various times in the history of mathematics. What I want to emphasize is that those approaches reveal different answers to essential questions: What is mathematics? Or: What does it mean to be "doing maths"?

This type of implicit questioning gave rise to an often shared answer, namely: to be an efficient mathematics teacher/learner presupposes an active method, constructivism, situated learning. Only then do mathematics and the knowledge it communicates make sense to the child. Activating the child, for instance in solving problems or in mathematical games has undoubtedly contributed to the history of teaching the discipline.

Nevertheless, I am going to attempt to show the limits of this approach. I will point at certain weaknesses of situated learning based on the everyday context or based on the utility imperative. Activity theory of A.N. Leontiev and others (Engestrom, Clot, Rabardel) reveals the structure of the cognitive activity in which mathematical concepts represent the tools to resolve specific tasks. It also makes it possible to distinguish a merely instrumental "managing of the situation" of the mathematical school assignment from the state in which the apprehension of mathematical terms has contributed to the development of mental functions and structures and of the whole personality. Teaching/learning of mathematical concepts at school has an exceptional developmental potential which is not always made use of.

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IMPLICIT EPISTEMOLOGY: WHAT IS THE SUBJECT-MATTER OF MATHEMATICS AND WHAT DOES IT MEAN "TO BE DOING MATHS"?

It is well known that the most ancient epistemological conception of mathematics is the Platonic version of a certain ***"celestial mathematics"*** (Desanti 1968). It is widespread not only in general but also among teachers. This conception is based on the idea that mathematical forms pre-exist the grasp by a mathematician, as if "in themselves". They are pure and evident ideas and the mathematician (and the mathematics teacher) only discovers them (their relations, structure, etc.). This world of mathematical ideas is basically independent of his activities; it is transcendent, and it is accessible by perception and contemplation. The French epistemologist René Thom says that according to this conception, mathematical structures are not only independent of man, but man also has only an incomplete and fragmentary notion of them (1974). The task of school education consists in that the teacher presents the world of mathematical ideas with maximum clarity (we are here using the metaphor of light and perception/view, where the pupil's soul stands for the "eye of the soul") and assists the pupil in mastering the principles of abstract thought. This implicit epistemological conception is the foundation of the so-called traditional education which focuses mainly on exposition followed by exercises.

Another influential conception of mathematics may be described as ***"terrestrial".*** It does not presuppose the existence of autonomous mathematical entities. Mathematical knowledge only reflects the structure of the natural and perhaps even social world. The mathematician does not contemplate independent abstract entities; on the contrary, he abstracts the ideal – mathematical - structure of the world from the world itself. Again, mathematics exists outside of the individual, yet as a structure that he has to extract, not in the form of independent ideas. It is not transcendent but immanent. This implicit epistemological conception is the foundation of reformist education, i.e. new pedagogy, which endeavours to make the child discover mathematics above all (or only) by manipulation with particular mathematical "objects". Great emphasis is therefore put on the "use" of mathematics in various practical situations. The child is thus shown that (a) maths is useful, i.e. can serve a purpose in practical life and that (b) mathematical concepts, laws and structures exist, have a rationality of their own and that it is important to learn to operate with this rationality as the authorities can do. In any case, doing maths means to rediscover that which is already given. Yet, this time, analytic manipulation is the method and not perception (which requires above all memorization and automatization of paradigmatic procedures).

The third conception of mathematics can be described as ***"instrumental"*** – mathematical knowledge represents tools which serve the solution of *problem situations.* Mathematics does not pre-exist – either in the skies or hidden in the world around us. To do maths is not to discover but to create. The main conclusion is that mathematics is a historical creation by particular people under certain conditions, by people who themselves sought answers to particular problems.

The mathematical activity consists in the generation of particular instrumental operations and, at the same time, in the establishment of a certain field of operations, of their interconnected network.

This epistemological conception is the basis of education which relies methodologically on the belief that learning is the result of a successful demonstration that mathematical knowledge serves as a tool in the solution of initial problem situations; of situations which need not always be concrete and based on everyday experience. And it is assumed that learning is the result of the pupil's invention of a concept or of a rule which makes it possible to find a solution for such a situation. At the same time, the child cannot come up with anything, for the situations in question need to have a potential for the creation of mathematical instruments and need to display inner normativity, or requirements on the activity that the child may perform in the situation. He cannot therefore simply play or disrespect the limits of the situation.

Furthermore, the metaphor of light and vision related to perception ("to see a solution", to clarify the assignment") leads us to fairly unproductive schemes of interpretation (gifts and talents – be it in biological terms, "he's got a genius for it" or "he is a maths prodigy"; or socio-cultural: "he lacks the cultural capital of the abstract code"). I consider it very positive that, unlike the perception metaphor of light and vision, the activating and above all, the instrumental conception of mathematics sets the interpretation of learning by means of mental work or activity against the mechanical interpretation as due to "talents" and "capital". This activity includes both the activity of mathematicians in history, in particular situations which they had to resolve and the activity of the child during the learning process.

In the last decades, the changing views of mathematics, and subsequently, of the teaching/learning of the discipline lead to the dominant Platonic epistemology being increasingly complemented by play-oriented activating methods and, occasionally, by the instrumental or constructivist conception. The idea that to learn mathematics means "to be doing it", i.e. to create, produce, make mathematical concepts and procedures as tools for the resolution of tasks (problem situations) is now generally recognized. However, it is accepted mainly in the discourse of didacticians and mathematicians. In schools, the application of this notion is rather hesitant and often fails.

This should lead us to consider whether learning mathematical terms in problem situations that are modelled after everyday experience is the most efficient procedure. Should it not rather be the task of teaching/learning to underline the specificity of formalized mathematics at school as opposed to everyday mathematics? After all, the goal of school socialization in the cognitive domain in general is to contribute to the development of mental functions and of the child's personality.

Furthermore, the number of negative results lead us to consider whether "activity" or rather, cognitive activity should not deserve a more differentiated analysis. Does it not suggest that learning maths is a complexly compounded activity which may encompass the memorizing of definitions, routine exercise as well as difficult formulations of hypotheses in a problematic situation?

In my search for answers, I rely above all on the cultural-psychological tradition of L.S. Vygotsky, and the activity theory of A.N. Leontiev.

THE BENEFITS AND LIMITATIONS OF LEARNING CLOSE TO PRACTICAL CONTEXTS AND SITUATIONS

After fifty years of rule of the individual-psychological approach to cognition and learning, the last two decades have seen renewed interest in the socio-cultural character of human cognition and of mental development in general.

This emphasis has been remarkably rising in prominence since the 1980’s. It could draw on earlier inspirations: the unachieved work of Vygotsky from 1925-1934 followed by the works by Luria and Leontiev. Unfortunately, these were usually published relatively late and translated into foreign languages only from the 1980’s onwards – remaining virtually unknown till then. Further, in the 1970’s and 1980’s there was the cultural anthropological research and theoretical work in the field of intercultural psychology by Cole, Gay et al. (1971), Scribner and Cole (1981), Lave (1977; 1988) and above all M. Cole (1996), admirer and indirectly the pupil of A.R.Luria, concerned with the influence of formal scholarization on the mental development and ways of thinking of natives in Africa.

What I have in mind is the boom of literature on the so-called **situated or distributed learning**. Significantly, this translates into French as “learning in context” (apprentissage en contexte), a somehow inaccurate expression, but one which makes explicit reference to an important dimension of situated learning, to **context –** which in turn reminds us of the other necessary term of the relation, “text”.

The turn towards situated learning, towards forms of cognition and learning in practical situations in the life of natives (of J. Lave’s Liberian tailors; of seafarers in the Pacific), towards learning in practice (e.g. everyday arithmetic in the research concerning milkmen conducted by B. Rogoff, in 1990) led to a **full appreciation of cognition as a set of cultural practices.** At the same time, it led to the **overestimation** of this form of learning at the expense of the importance and function of the school form of cognition and learning. In concurrence with the reviving educational reformism and the come-back of pedocentrism, this led to the overall negation of the developmental significance of the school form of cognition. Situated learning in contexts of practical life of the individual was placed on a pedestal, almost as a model for learning at school. The activating and reformist (pedocentric) conceptions of teaching/learning are strongly nurtured by this conception.

I’ll attempt to show that there is a substantial difference between situated learning, i.e. learning in the extra-curricular, everyday (e.g. family) context, and school learning which was described by Vygotsky as learning of “scientific” concepts in *Thought and Language* (1976).

It is certain that analyses of situated learning have re-oriented educational conceptions which had still been under a strong influence of the individual-cognitivistic tradition. What do they stand for, though? The pivotal idea is the following: learning, apprehending an item of knowledge can only be construed in a "situation", i.e. is dependent on the pupil's participation in social and material contexts, the person and his/her world being mutually constitutive. This idea underlies according to Moro (2002) the following theories: learning as ***apprenticeship*** associated with the works of Lave (1977, 1988) and Lave and Wenger (1991); learning as ***guided participation*** associated with the theoretical work by Barbara Rogoff (1990) and learning in the man – tool(s) system usually described as **distributed learning,** associated with the names of E.Hutchins studying pilots in a cockpit, or subway dispatchers in work (1995; 1990) and L.B.Resnick (1987).

All theories of situated learning redirect our attention towards the analysis of the situations in which learning takes place, and each in its own way puts the emphasis on one of the elements of Vygotsky's cultural-historical approach towards psychological functions (the prime importance of social activities, i.e. of the inter-psychological nature of psychological functions; the key importance of mediation and the role of the adult-expert; the formative effect of the artifact-tool). Thanks to these theories, Leontiev's concept of activity and the question of the unit of analysis in examining psychological phenomena rise in prominence. What, then, is the problem? Why not rank these theories within the stream of socially mediated approach to learning and make use of them at school?

First of all, it is necessary to note that these theories (1) localize the dynamic of learning almost exclusively into the world of everyday experience and neglect the importance of activities provided and made necessary by the school, i.e. of activities directed at reflection and abstraction. Thus, they hinder investigations into the differences and tension between an item of knowledge in its everyday form and one which is formalized - and therefore bypass the decisive moment of the cognitive and personal development of the individual. (2) In effect, these theories overestimate the formative influence of artifacts and situational configurations on mental functions - as if these were embodied in tools. This is because they fail to distinguish between the capacity to operate in context on the basis of the tool and the mental work of an individual transforming particular psychological functions. (3) They fail to dispel the impression that in their psychology of situations "the psyche in fact belongs to situations", thus only mechanically transposing mental *gestalts* originally localized in the minds of individuals into situations.

LEARNING IN THE SCHOOL CONTEXT

The theory of learning in the everyday practical context differs significantly from the approach of Vygotsky's school in its conception of the unit of analysis and in its conception of mediation. Along with Leontiev, in using the term *unit of analysis* I refer to the isolation of units of enquiry which enable the objectivation of psychological facts in their inter- and intra-psychological dimensions. "Participation in apprenticeship" can help grasp activities in the socio-cultural framework and can substitute for the mechanical understanding of internalization; however, the nature of the intra-individual activity itself largely escapes it. On the other hand, mediation is considered by Lave and Rogoff above all as communication between individuals and the prospective zone of proximal development as a communicative-relational network. The cognitive activity itself, i.e. the apprehension of the item of knowledge qua apprehension of norms of activities with the given item of knowledge, is left aside. Similarly, Hutchins' tool in the pilot's cockpit is admittedly instrumental and mediating; however, it is not Vygotsky's psychological tool, since it cannot demonstrate how permanent transformation of psychological functions and the development of the individual come about. Finally, what is striking is the insensitivity to the fact that learning at school is also learning in a context with its own specificities, a context which represents a community of practices derived from science. A comparison with extra-curricular contexts makes it evident that its objective is epistemic. It aims at the transformation of modes of thinking, of experiencing, of the self. This requires a clear conception of the relations between spontaneous learning, education, formal learning and development. What are, in Vygotsky's terms, the main differences between apprehending spontaneous concepts and those which are scientific (acquired mainly at school)?

The practical, utilitarian vs. epistemic attitude to the world and to language

Let us recall one of the classical comparisons – the apprehension of spoken language vs. language learning at school with the support of writing.

Formalized learning can start where spontaneous learning in contexts of everyday life comes to an end (i.e. where it reaches its limit). The latter stands on instrumental usage (knowing how to say something; say how the notion "brother" works, or who is a particular brother, to make oneself understood). P.Bourdieu (1996) says that in practical action the word used fits the situation. The former paves the way for reflection and builds on it (knowing why something can/cannot be said in this particular way; what is essential about the structures of "kinship" and why a "sister" is the same as a brother according to these laws, even if this is sheer nonsense in the context of everyday usage).

Although formalized learning of de-contextualized "scientific" knowledge makes use of spontaneous learning (is based on it), the important thing is that it transforms substantially the knowledge thus acquired. Due to formal learning and its tendency to de-contextualize, the child is brought to reflect upon and realize the specificities of the mother tongue, and to the necessary generalization of linguistic phenomena. By means of the new attitude towards language, its attitude towards the world changes into one which is epistemic and not practical. This in turn opens new horizons in other domains of knowledge.

Olson and Torrance (1983) introduce another striking criterion. On their view, both the context and the text are available to man in his practical attitude to the world. But the situation of spontaneous learning forces him to give priority to information from the context that is to say to rely on what is most probable in the given context. Olson and Torrance cite the following example.

> They observe that according to classical Piagetian tests children up to 8 years of age understand instructions contextually (and proceed in their thoughts on the basis of such understanding). The critique of these tests features the classical example of a logical sub-class – class relation (there are 9 flowers in the picture, 6 of them tulips and 3 roses). The question is: "Are there more tulips or more flowers in the picture?" Children answer on the basis of comparing the sub-class "tulips" with the sub-class "roses" and conclude that there are more tulips than flowers. Olson points out that children answer not on the basis of text but depending on the context, i.e. on their everyday experience and act as is common in such contexts. For we usually compare sets of the same kind or level (e.g. girls and boys; in everyday life, we rarely ask if there are more girls than pupils in a class). The child is thus guided by the context and not by the linguistic contents of the question and its logical structure, i.e. the "text". To follow the text, the child must undergo another type of learning than the more or less "spontaneous" reaction recorded by Piaget.

At school, meanings and interpretations are not only practiced; writers and readers are forced to engage in reflection on meanings themselves. The processes of acquisition of written knowledge are thus the decisive factor in the change of ways of thinking. Olson cites a Vygotskian distinction to that effect: *"thanks to writing, we have moved* himself says *from thinking about things to thinking about the representations of things"* (Vygotsky that spontaneous notions are generalizations about things, while scientific concepts are generalizations of these generalizations).

This is what Vygotsky describes as the key effect of school teaching/learning in *Thought and language* (1976). School education brings about (a) a rupture and (b) the intellectualization of mental functions. What does this rupture consist in?

The aim of spontaneous everyday learning is to deal with a practical situation in life. The child that enters school has thus already mastered some knowledge, say in arithmetic. This is proto-arithmetic knowledge: he/she can divide marbles into two even parts, knows how many people there are in the family, can compare his/her own age to that of a sibling, can add and subtract from the number of objects etc.

At school, nevertheless, this spontaneous knowledge serves as a basis for the child to develop real operations of addition and subtraction with the help of a teacher; the child constructs (abstracts – not extracts) numerical properties of empirical objects. Whether we deal with marbles, apples or books is of no importance – in any case, it is true that 2 + 1 = 3 and 3 – 2 = 1. The child performs a de-contextualization based on generalization as an empirical abstraction of the concept of quantity. According to Vygotsky, this is the above mentioned generalization of a lower order, a "generalization about things". Yet, arithmetic operations do not lie in (are not immanent to) the empirical situation, they are not additional properties of objects (besides colour or size, say). They are necessary non-empirical operations that the child must perform.

However, the important breaking point occurs when, by virtue of these operations, the child discovers the properties of the decimal system. At a certain stage of development, the child begins to understand how the decimal system fulfils its purpose and how it works, and that it is also possible to count using other numerical systems (binary, ternary or other systems). From then on, the child understands the decimal system as a *particular instance* of other possible numerical systems and founds a generalization of a higher order. This is the generalization of generalizations, a generalization based only on the relations between numerical entities.

Unreflected, or not consciously developed vs. planned and conscious procedure

The mastery of systems of higher generalizations makes possible the *distance from particular tasks and situations,* relatively permanent from the point of view of development, and, at the same time, the *realization* not only of a particular set of knowledge (the results and operations of addition, division etc.), but above all the realization of *one's own mental processes* and of oneself. This is exactly what Vygotsky calls the intellectualization of mental functions: it sets in when the mental function becomes dependent on the idea (concept) or is subordinate to it.

The example of intellectualization of memory and of the relation between thought and memory is well known. A small child thinks by remembering. His/her representations of things and of ways of handling them are not conscious and organized systematically around a certain idea or concept. An older child or a teenager already remembers and recollects by (and thanks to) thought. The intellectualization of memory consists in the organization of knowledge for the purpose of remembrance. The child thus increasingly works consciously and deliberately on his/her own memory processes. From a certain point on, the relationship between memory and intellect gets reversed. The introduction of conscious and planned (volitional) relations of the child towards his/her own mental processes is what cultural psychologists perceive as the criterion of a higher level of development.

It is valid universally that the emergence or discovery of the *relations of a higher generality* between concepts is the critical point (motor) of mental development.

The remarkable geographical metaphor of Vygotsky's makes it possible describe the concept as a geographical point at the longitude and latitude intersection. The "longitude" of the concept determines its place on the meridian leading from the most concrete to the most general meaning. The "latitude" of the concept then represents the point which it takes in relation to other concepts of equal "longitude" (of equal generality) but relating to other points of reality. The combination of both key characteristics of the concept determines the extent of its generality. It is given not only by the concrete/abstract scale, but also by the richness of connections to other concepts of the given conceptual network which form the domain in question.

Let us demonstrate "intellectualization" on the relation between the apprehension of arithmetic and algebra. The result of the operational development so far – e.g. the operations of addition and subtraction – becomes the "source" of new processes, algebraic operations with variables and unknown quantities. The performance of a thought operation (to define, compare, factor out, divide etc.) presupposes the establishment of relations between various concepts within the corresponding conceptual system. A six year old child cannot "define" an operator or a straight line, for instance, because the terms that he/she masters are not in relations of sufficient generality to other concepts. If, however, the child masters operations of the decimal system, an infinite number of means to express the concept, for instance of the number "four" are available to him/her (2+2; 8 – 4; 16 : 4 etc.) The concept of a higher order of generalization thus represents a point which makes possible several proceedings within the entire system.

School education plays a decisive part in this process of transformation of mental functioning. In learning "close to everyday life", the child observes, discovers, considers, argues, etc. (M. Brossard says, that he/she "coincides with the significations he/she practices", 2004). It is due to school education that, along with all of this, the child also focuses his/her attention on mental processes, which he/she performs when observing, discovering, considering etc. The child works on "pure meanings", which are the main object of his/her reflection. Thus, the ability to define the number "four" in several different ways involving various operators and their combinations necessarily places comparative reflection, the analysis of one's own attention and memory, knowledge about one's efficiency etc. at the forefront. That is, such processes, which would never come about, if the child were struggling with the ignorance and absence of automatic mastery of the elementary operator.

However, the child can never reach this reflective activity "spontaneously". It requires a teacher, a plan, a logic of the curriculum and of the teaching process, a programme which is at first only external to the child. Especially mathematical concepts of a higher of lever of generality are distinguished by the necessity to "introduce them from the outside"; these are "top-down" conceptualizations. Intellectualization stands on an increasing subordination of individual operations to the higher organizational principle (with the two characteristics expounded in the above mentioned geographical metaphor). From this point of view, mathematics represents activities in which - with a growing generality of a concept - the motive of the introduction of the concept is always "external" in respect to the child and his/her "spontaneous interest". The "new", conscious learning at school is guided by the requirements of the contents, or by the object of the cognitive activity. The pupil studies the "programme" proper to a given type of thinking whose observance is guaranteed by the institution of the school and the teacher. If we put this in Olson's terms, the "textual" approach is exercised at school – sometimes with success, sometimes less so; an approach which is supervised, systematic and planned. School mathematics which is supposed to fulfil its evolutional psychological function must provoke that which is of greatest value: tension between various levels of conceptualization (the development level achieved by the pupil to date vs. the elaborate form of conceptualization constructed in a didactic school situation in cooperation with a teacher). Brossard (2004) describes this as the internal motor of development alongside the external (socially motivational) motor.

I have repeatedly been using the terms operation, task, activity etc. Learning in a school context is however characterized by certain specificities which can be better understood with the activity theory model of A.N. Leontiev (1978).

THE MEANING OF LEARNING AS A RELATION OF OPERATIONS, TASKS AND THE OBJECT OF COGNITIVE ACTIVITY

Leontiev points at the hierarchical and internally differentiated structure of every activity, including the cognitive activity. He understands activity as a fairly molar unit consisting in partial levels represented by tasks or actions. Every task is formed by operations at a subordinate level (1978).

For Leontiev, it is above all the contents of the given activity, i.e. its object, that is crucial. What is also important is whether the cognitive activity makes sense to the pupil (and what sense it makes). There is therefore not so much question of who is setting the assignment and the problem to the child or whether the form of the activity is playful or utility-centred enough.

Interesting about this conception of activity are the relations between different levels of the activity and their functions. These above all point to the necessity to distinguish between the relations of ***efficiency*** in practicing operations and fulfilling tasks and the relations creating the ***sense*** of the activity as such. And, at the same time, to the necessity to make sure these relations are mutually interdependent. The activity levels can be laid out in the following table:

Activity	**Object**	**Function**
I. Molar activity: Algebraic transformations	**Motive** - mastery - aesthetic experience - to be good at maths	**Encouraging** (initiative-provoking) To persist in efforts to overcome obstacles and difficulties arising at level I and II
II. Tasks: - the calculation of functions of different types - the solution of a rider/theme - the solution of a system of equations etc.	**Goals** - to find the correct solution - to identify the value of the unknown etc.	**Orientation** - correct input analysis of the task - good “preparation” of the solution - the layout of steps, their sequence and time allocation etc.

III. Operations - multiplication, reduction - position record - the discrimination of symbols - managing operations using memory	**Means**, material tools, symbolic **instruments** including cognitive processes (memory, attention, arithmetic operations)	**Execution** - material traces (notes, schemas, auxiliary calculations…) - necessary technical support (infrastructure) of the operations

The relation between the quality of operations managed (III.) and the quality of the solutions to tasks (II.) expresses the **efficiency** of the cognitive activity/learning, usually in the form of microgenetic improvements (automatization, the repetition of invariants of an activity, is an exemplar which involves more abbreviated forms of an operation, its greater mastery – it opens the way for a higher level of generality of the operational concept used – and for the extension of the range of tasks which can be solved in the same domain).

The relation between the nature, frequency, complexity and above all interdependence (articulation) of tasks (II) and the essence of the activity expressed in its object (I) defines the **meaning** of learning.

Learning a mathematical concept is therefore a complexly structured activity which may involve such activities as memorizing definitions, routine practicing and consolidation of operations, as well as the difficult formulation of a hypothesis vis-à-vis a problem situation. The provision of pertinent tasks complemented only by verbal persuasion and model demonstration without the elaboration of activities on levels I and II cannot lead to success, since “meaning” cannot be enforced on the pupil from the outside; the pupil needs to possess tools to elaborate this meaning for himself. It is impossible to produce a motive of meaningful learning without efficient operations (including mental functions: attention, the memory of basic inference) and managed tasks. This efficiency alone, however, cannot ensure that pupils will find meaning in that which they may consider as an illogical chain of unrelated tasks or even as a purposeless drill of isolated operations. Even these may, in their turn, have a relatively positive effect – an effect of a functional solution of task situations; situations, which, nevertheless, fail to open the way towards the development of “intellectualization” (see above).

PERFORMANCE IN THE SITUATION VS. DEVELOPMENT

The difference between a *performance in the situation* (performance of a function) consisting in the repetition of invariants of an activity in a variety of situations on the one hand and *development* on the other is stressed by the French psychologist Yves Clot in his analysis of the activity of work (cf. for example Clot, 1999). Spontaneous learning first and foremost pursues efficient **performance of a function** in a situation whose boundaries are not transcended (calculate correctly a subtraction; more generally, “giving correct answers to the questions”; the finality of spontaneous learning is often preserved even within learning of scientific concepts at school and remains resistant to its requirements). However, this “information” – rather than knowledge – represents the basic prerequisite for the subsequent conceptual work. In Leontievian terms, we have to do with a level of operations (manipulating “tools”) and with that of tasks. Their incorporation into a routine is a sort of an organizational condition for the cognitive activity itself (this is especially true of the memory automatism regarding certain algorithms, e.g. arithmetic ones). However, such a „practical” learning (in regard to the school context) rarely goes beyond the level of the performance of a function in a situation. Hence, no opportunity is provided for the apprehension of the concept to open the way for development, for such learning fails to grasp the object of cognitive activity itself.

On the other hand, effectively mediated learning of the concept paves the way for **development** - of the pupil’s thinking and of his personality. This requires that routine tools be used in a variety of tasks (actions), that they pass through various situational contexts to have pressure put on the enrichment of their functionality (e.g. basic mathematical operators should be practiced in the context of calculus operating with both one-digit and double-digit numbers, in the context of a task in arithmetic and a task in geometry). Only such cognitive work – learning – enables a relevant generalization going beyond the limits of particular situations. Only thus could operations in decimal systems become – at least for some – a special particular instance of a more general set of conceptualizations. Learning which releases items of knowledge from their context without ignoring particular situations renders development possible: first of all the development of the child’s thinking; connected with this is the development of other psychological functions (e.g. we memorize better those things the inner logic of which we have apprehended) and finally the development of the personality of the pupil (he develops a feeling of mastering himself and his knowledge, he is harder to manipulate or less likely to fall victim to biased information).

For this reason, we should be warned against ill-considered preference of experience close to the child in education/learning and against the reduction of the mathematical activity to operations and tasks, to their attractiveness and playfulness. Of mathematics is this especially true. For it has an exceptional potential to contribute to the development of mental functions of the child and his/her personality; not merely to the broadening of his/her knowledge basis. The reason for this is that its programme is soon hardly reducible to “utility for life”. There is simply no “immediate” (non-mediated) connection between mathematical concepts or questions and social problems in the lives of people. It is futile to search for and incorporate this connection artificially into the education of maths under the pretext of its becoming more attractive. This connection exists only as highly mediated. For this reason, it is a key function of mathematics to contribute to the developmental emancipation of a young person by way of “intellectualization” (Vygotsky), as I explained above. In ordinary language, this function is referred to as "thought gymnastics". I believe, however, that there is more than its effect on cognitive development. What I wanted to emphasize is that mathematics and its didactics should not lose their developmental-psychological potential by accepting an unnecessarily reductionist version of the activating, constructivist and problem-situated attitude towards the education of the discipline.

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PME 1 TO 30 – SUMMING UP AND LOOKING AHEAD: A PERSONAL PERSPECTIVE ON INFINITE SETS

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This paper describes the development of the work on the learning and teaching of infinite sets within PME (mainly our studies). We start by reporting on students' ways of thinking and on the impact of related instruction. Then, we discuss the significant role of different representations in learners' reasoning with specific attention to consistency-related issues. We further present studies that examined the impact of several interventions on prospective teachers' intuitive reasoning and formal knowledge of infinity. We conclude by discussing the contribution of different theoretical models to understanding possible sources of learners' reasoning about infinity and to the repertoire of related instructional approaches.

In 1990 the first research synthesis of the work carried out by the International Group of the Psychology of Mathematics Education was published. In the concluding remarks to the introduction of this volume, Fischbein stated that:

> There are some basic directions to which attention ought to be focused in the future. First, we ought to enlarge the realm of "home grown" problems, particularly with questions related to concepts of advanced mathematics such as classes of numbers (especially irrational and complex numbers); infinity (dynamic and actual infinity, the concept of limit); the concept of function; the basic ideas of calculus (the concepts of derivative and integral), infinite sequences and series, probability and statistics; geometry (Euclidean and projective geometry, topology, analytic representations); and mathematical proofs and the axiomatic method.
>
> Fischbein, 1990, pp. 12

Fischbein's remark reflects the central role of mathematics in PME. Fischbein also suggested that PME as an organization should devote more time and effort to the study of cognitive issues related to advanced mathematical concepts in general and to the concept of infinity in particular.

In this PME (PME 30), the M that stands for mathematics is at the centre. Accordingly, we chose one advanced mathematical notion, the notion of infinity, to serve as a pivot in our plenary. This choice was made for several reasons:

(a) Mathematical significance and richness – Infinity is one of the central concepts in philosophy, science and clearly in mathematics. This concept has fascinated mankind since time immemorial. Moreover, there is a wide variety of meanings to the notion of infinity, including potential infinity (representing a process that could go on for as long as is desired), actual infinity (in the sense of the cardinal infinity of Cantor), ordinal infinity (also in the sense of Cantor, but this time representing correspondences between ordered sets), non-standard infinity (which admits all the

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 49-63. Prague: PME.

operations of arithmetic, including division to give infinitesimals). Furthermore, infinity is used in different mathematical domains, including analysis, probability, algebra and geometry.

(b) Psychological complexity – The concept of infinity itself and the theorems connected with it are surprising. There is a deep contradiction between this concept and our intellectual schemes, which are built on our practical, real life experiences and are naturally adapted to finite objects and daily events. For example, propositions like *the whole is equivalent to one of its proper parts* contradict our usual mental schemes. This conflict between experience in the finite world and some formal propositions about infinity provides an opportunity to examine various issues related to the learning and teaching of mathematics.

(c) Educational challenges – Teaching a central, mathematical notion that has a wide spectrum of meanings and is psychologically so complex is very challenging. It may serve as an appropriate occasion for evaluating different approaches such as the *cognitive conflict approach* and *teaching by analogy*.

The space allocated to this plenary can not do justice to the work on infinities within PME. We shall focus on our own research on infinities, referring mainly to studies on comparisons of infinite sets. Our presentation will follow the development of our research chronologically and summarizes our main findings.

PME 1 to PME 8: Students' Ways of Thinking

A glimpse at the first volumes of PME reveals that the study of the learning and teaching of infinities started with the exploration of students' ways of thinking. At PME 1 in Utrecht, the Netherlands (1977), David Tall, in a paper on "Cognitive conflicts and the learning of mathematics" already pointed at the conflicting meanings students allot to the notion of infinity (among other notions). He reported on students' own interpretations of key concepts, their use of words, the way they build concepts and the conflicts which occur as they restructure their schema. He stated that:

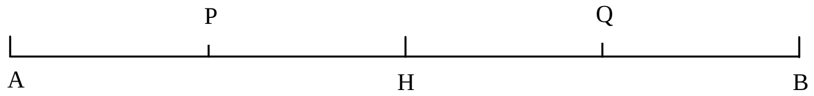
> Certain interesting phenomena occurred. In the first place we found an amazing variety of interpretations of well-known mathematical words, especially in terms of intuitive ideas before the words were "formally" defined in the lecture course… They included words like 'complex number', 'real number', 'limit', 'continuous', 'infinity', 'proof', 'some', etc. In many cases students gave conflicting explanations of words.

Tall, 1977, p. 4

At PME 2 in Osnabruck, Germany (1978), and in a subsequent more elaborate paper published in Educational Studies in Mathematics (Fischbein, Tirosh, & Hess, 1979), Fischbein et al. presented a study that focused specifically on students' intuitions of infinity. The aim was to determine the nature of intuitions of infinities and their development with age. This study reported on the intuitive ideas of 470 students in grades 5 to 9 regarding three infinity-related issues: (a) infinite divisibility; (b) transfinite cardinals, and (c) limits. Here are some examples of tasks:

Infinite divisibility

Task 1: We divide the segment AB into two equal parts. Point H is the midpoint of the segment. Now we divide AH and HB. Points P and Q represent the midpoints of the segments AH and HB, respectively. We continue dividing in the same manner. With each division, the fragments become smaller and smaller.



Question: Will we arrive at a situation when the fragments will be so small that we will not be able to divide further? Explain.

Transfinite cardinals

Task 4: Consider the set of natural numbers and the set of even numbers: N = {1, 2, 3, 4, …} D = {2, 4, 6, 8, …}.

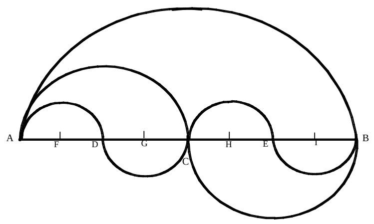
Question: Is the number of elements in set N equal/not equal to the number of elements in set D? Explain.

Task 7: Let us consider a line segment whose length is 1 cm. and a square whose side is 1 cm.

Question: Is it possible to find a one-to-one correspondence between the points of the line segment and the points of the square? Explain

Limits

Task 8: Construct a semicircle with segment AB as a diameter. Divide AB into two equal parts, AC and CB, and construct two semicircles on AC and CB. Continue dividing and constructing semicircles.



Question 1: What will happen to the length of the wavy line as we shorten the length of each sub-segment?

Question 2: What will happen to the sum of the areas determined by the semicircles as we shorten the length of each sub-segment? Explain.

The main findings were as follows:

1. In respect to infinite divisibility: Two types of responses were prominent, *infinitist* (accepting infinite divisibility) and *finitist* (rejecting infinite processes). About half of the students in each grade level provided each of these types of responses. These large discrepancies between infinitist and finitist reasoning suggested that the idea that a process of dividing a bounded line segment can go on forever is in itself contradictory.

2. Regarding transfinite cardinals: The most prevalent finding was the dominance and the stability of the inclusion consideration. That is, the vast majority of the students at all grade levels claimed that if one set is a proper subset of the other, it contains “less elements”.

3. Regarding limits: The most striking finding was that the percentage of wrong answers (that the sum of the areas of the semicircles remains constant) increases with age and with mathematical training. This indicates the overgeneralization of the conservation scheme.

During the discussions of these ideas at PME, the audience raised several issues, one of which related to measuring the intuitive acceptance of a mathematical statement. The issue at stake was: Is it possible to measure the feeling of “intuitive acceptance” experienced by a person when she or he is giving an intuitive solution to a problem? Consequently, a follow-up study was formulated (Fischbein, Tirosh, & Melamed, 1979; 1981). It was postulated that two dimensions of intuitive acceptance are to be considered and combined: the level of confidence in the solution, and its degree of obviousness. A questionnaire, including eight mathematical problems, seven of which referred to the notion of infinity, was developed and administered to 106 eighth and ninth graders. The participants were asked to solve each problem and explain their solution. After solving the problems they were asked to answer six questions tapping the intuitive acceptance of their own solution to each problem: three related to the level of confidence in the solution, and three to its degree of obviousness. A question addressing the level of confidence was, for example: *Do you have doubts regarding the correctness of your answer?* And the following question measured the degree of obviousness: *Is your answer self-evident for you?* The level of intuitive acceptance was calculated for each participant and for each answer. Then, the frequencies of the main types of solutions and the degrees of their intuitive acceptance were computed, yielding three categories of problems. The most demanding category, from an instructional perspective, was the one that included problems with high frequencies of typical, incorrect solutions accompanied by strong, intuitive acceptance. An example of such a problem is:

> Let us consider the following two sets:
> The set of natural numbers and the set of the points on a line l
>
> N= {1, 2, 3, 4, 5, 6…}   _ _ _ ______________________ _ _ _
> *l*
>
> Question: Is it possible to match each of the points on line l with one and only one natural number? (Each number and each point must be used only once).

All in all, the studies done until PME 8 indicated that students have stable intuitions regarding the comparison of infinite sets, that these intuitions are incompatible with the accepted, mathematical theorems and that the degree of intuitive acceptance students experience when giving inadequate intuitive solutions to comparison-of-infinite-sets tasks is high. Thus, the comparison of infinite sets seems to be a proper touch-stone for studying the complex relationship between intuitions and instruction.

PME 9 to PME 15: Instruction – Teaching Students about Infinite Sets

One of the challenges of mathematics education that kept arising during PME conferences concerned the development of research-based instruction that takes account of students' intuitive reasoning. Such an instructional unit, related to infinite sets, was designed, conducted and evaluated at Tel-Aviv University in 1981-1984 and was first presented at PME 9, 1985 in Utrecht, the Netherlands (Tirosh, Fischbein, & Dor, 1985). Later on, questions from participants in the various forums related to Advanced Mathematical Thinking at PME led to a more elaborate version (Tirosh, 1991) of this paper that appeared in a book edited by Tall (1991), on advanced mathematical thinking.

The main objectives of this instructional intervention were: (1) to identify the inner conflicts in students' intuitive understanding of actual infinity, and (2) to improve high school students' intuitive understanding of the notion of actual infinity through systematic instruction.

For the second aim of the study, a 20-lesson teaching program for tenth graders was developed, addressing: (1) basic notions of set theory; (2) the concept of cardinal number; (3) equivalence of infinite sets; (4) enumerable sets, and (5) non-enumerable sets. A special attempt was made, throughout the unit, to interact with students' intuitive background and to control their primary reactions. Several strategies were used to help the students overcome the inner contradictions in their intuitive understanding of actual infinity: (1) raising students' awareness of the inconsistencies in their own thinking; (2) discussing the origin of students' intuitions about infinity; (3) progressing from finite to infinite sets; (4) stressing that it is legitimate to wonder about infinity; (5) emphasizing the relativity of mathematics, and (6) strengthening students' confidence in the adequate definitions.

Two hundred and eighty students, from eight tenth grade classes, participated in this study (four experimental groups and four control groups). The main findings were:

1. As expected, inner conflicts were evident in the students' answers to the comparison-of-infinite sets tasks prior to and during instruction (e.g., a conflict between the view that all infinite sets are equivalent and the application of the part-whole consideration; a conflict between the statements: "a proper subset of a given set has a smaller cardinal number than the whole set", and "every infinite set has a proper subset which has the same cardinal number").

2. After instruction, about 70% of the students used only adequate procedures for establishing the equivalency of infinite sets (before instruction, none of the students applied such procedures), about 20% correctly solved only some of the problems whereas in respect to others they failed and resorted again to non-adequate intuitive techniques, about 10% of the students had not been able to free themselves from their primary intuitive constraints and used only non-adequate intuitive techniques for solving the problems.

The study indicates that an active didactical approach that illustrates some of the perplexing aspects of mathematics can enable students to (a) accept conclusions that at first appeared paradoxical; (b) recognize the coercive nature of intuitive thinking; (c) understand the need to control their primary intuitions; (d) refrain from responding intuitively, and (e) base their solutions on theorems and definitions.

The presentation of the sets that appeared in the studies described so far was horizontal, namely, the two sets were graphically presented in the same row, one beside the other. In the 1980s, the mathematics education community experienced a growing interest in the role of representations in students' mathematical reasoning (Duval, 1983; Janvier, 1987; Kaput, Luke, Poholsky, Sayer, 1987; Schwartz, 1987; Silver, 1986; Tirosh, 1990). In this spirit, we recognized that the representations of the infinite sets to be compared in the problem could have a significant impact on students' solutions. This insight opened new horizons to our explorations.

PME 16 to PME 22: Consistencies and Representations

In 1992 at PME 16 at Durham, New Hampshire, we presented a study addressing high school students' sensitivity to different representations of a comparison-of-infinite-sets tasks. In this study, the same two infinite sets {1, 2, 3, 4, 5…} and {1, 4, 9, 16, 25…} were presented in two ways, one of which was designed to encourage part-whole considerations (a numeric representation) and the other aimed at triggering one-to-one correspondence considerations (a geometric representation). This PME paper (Tsamir & Tirosh, 1992) describes our first attempt to explore the potential of such representations for evoking secondary school students' awareness of inconsistencies in their own thinking about infinity, and to examine their reactions when discovering inconsistencies in their responses. We found that most participants (20 out of 32) provided inconsistent responses, but only few noticed themselves that their responses were incompatible.

Two types of ideas were expressed by students who provided inconsistent responses: (a) *my [inconsistent] responses are legitimate*, e.g., "We are talking about two different mathematical systems: numbers and geometry… Each could have its own rules"; "The correctness of a mathematical system is determined by its usefulness in physics, chemistry etc., possibly these two methods that I used for comparing infinite sets, are useful in different domains or sub domains"; (b) *inconsistent responses are problematic*. Students who felt this resolved the contradictions in four ways: (b-i) *incomparability* e.g., "It is illegitimate to compare infinite magnitudes"; (b-ii) *singularity*, e.g., "All infinites are always equal, there is only one infinity"; (b-iii) *part-whole is the unique criterion;* (b-iv) *one-to-one correspondence is the unique criterion*.

The consistent responses addressed ideas of *singularity*, e.g., "All infinites are always equal, there is only one infinity"; *incomparability* e.g., "It is illegitimate to compare infinite magnitudes"; and *lexical arguments*, e.g., "Individual names were given to numbers representing different quantities; for instance, 7 indicates the same quantity of elements no matter what kind. Likewise, if there is more than one infinity, then there should have been matching names to describe each 'quantity' of infinity. But, as the number of elements in all those sets is called 'infinity', it must indicate a single magnitude" (Tsamir & Tirosh, 1992, Vol. 3, pp. 94).

In the 1992 paper we reported on the responses of 32 secondary school students to two representations of the same problem. This study was later on reported in the *Journal for Research in Mathematics Education* (Tsamir & Tirosh, 1999). A follow-up study was reported at PME 18, 1994, in Lisbon, Portugal (Tsamir & Tirosh, 1994). In the 1994 paper the sample included 189 secondary school students and four types of representations of infinite-sets-tasks were formulated (numeric-horizontal, numeric-vertical, numeric-explicit, geometric). Each student was asked to respond to two navigating problems, one encouraging one-to-one-correspondence considerations and the other part-whole considerations, and five problems were given in more than one representation (see Figure 1 on p. 56). The results of this study essentially confirmed the main result of the previous one: Students tended to provide representation-dependent responses. That is, the type of criterion they used for the comparison of the infinite sets depended on the specific characteristics of the representation: numeric-explicit and geometric representations elicited one-to-one correspondence considerations; numeric-vertical representations triggered single infinity considerations and numeric-horizontal representations encouraged inclusion considerations (see Table 1).

The studies presented so far provided us with information about the intuitive strategies that students tend to apply when comparing infinite sets. The data revealed that students tend to attribute the properties of finite sets and the strategies for comparing them to infinite sets. The comparison of infinite sets is a substantial part of the Cantorian Set Theory course. This course is usually included in the curriculum of prospective secondary school mathematics teachers in Israel and it is commonly presented formally, with little or no emphasis on students' intuitive tendencies to overgeneralize from finite to infinite sets. The course on Cantorian Set Theory is one of the demanding courses for prospective teachers. The accumulated, research-based knowledge on students' conceptions of infinite sets may well assist in designing a different kind of Cantorian Set Theory course, a course which takes into account students' ways of thinking and typical responses to different representations of the mathematical problems.

PME 23 to PME 27: Infinity in Teacher Education

A first stage in any attempt to develop a curriculum for a course is to explore students' conceptions regarding the topics to be taught. It is also essential to assess the impact of normative ways of teaching this course on students' knowledge. This seems to be true in general and in the case of infinite sets in particular. Thus, assessing the strategies that prospective teachers implement for the comparison of infinite sets before and after normative instruction was the aim of the initial stages of our work.

Arithmetic	Geometric

Single 1

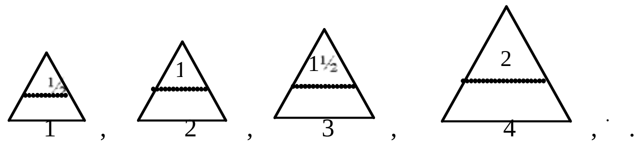
{1, 2, 3, 4, 5, 6, … } { … -3, -2, -1, 0, 1, 2, …}

Single 2 {1, 2, 3, 4, …}

(-1, -2, -3, -4, …}

Pair I A={1, 2, 3, 4, 5, 6, …}

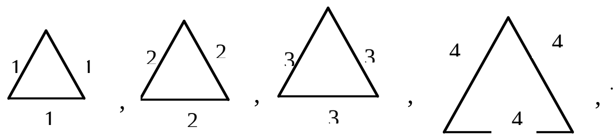
B={½, 1, 1½, 2, 2½, 3, …}



A: Sides of equilateral triangles
B: Midsegments of equilateral triangles

Pair II C={1, 2, 3, 4, 5, 6, …}

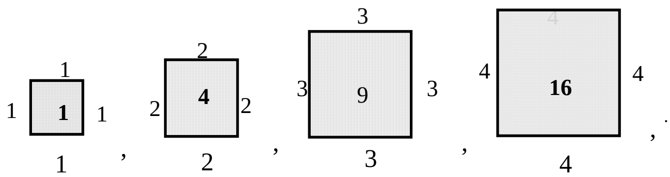
D={3, 6, 9, 12, 15, 18, …}



C: Sides of equilateral triangles
D: Perimeters of equilateral triangles

Pair III E={1, 4, 9, 16, 25, …}

F={4, 8, 12, 16, 20, …}



E. Areas of squares
F: Perimeters of squares

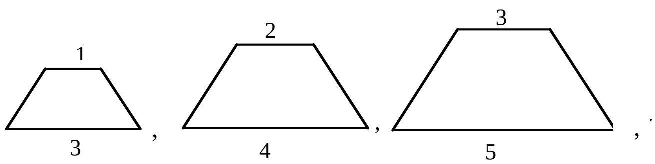
Triple I

G={1, 2, 3, 4, 5, … } H={3, 4, 5, 6, 7, …}

***

G={1, 2, 3, 4, 5, …}

H={3, 4, 5, 6, 7, …}



G. Upper bases of trapezoids
H Bottom bases of trapezoids

TRILE II

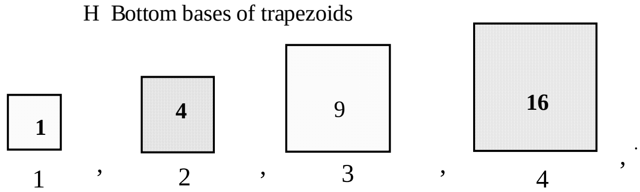
I={ 1, 2, 3, 4, 5, …}

J={1^2. 2^2. 3^2. $4.^2$, 5^2, …}

***

I={ 1, 2, 3, 4, 5, …}

J={ 1. 4. 9. 16, 25, …}



I. Sides of squares
J: Areas of squares

Figure 1. Schematic representation of the problems

The sets		Same number of elements (one-to-one correspondence)			
# of problem		Horizontal	Vertical	Explicit	Geometric
1.	{1, 2, 3, 4, ...} {...-2, -1, 0, 1, 2,...}	23(0)			
2.	{ 1, 2, 3, 4, ...} {-1, -2, -3, -4, ...}			98(73)	
3-4.	{1, 2, 3, 4, ...} {½, 1, 1½ , 2, …}		47(6)		80(45)
5-6	{1, 2, 3, 4, ...} {3, 6, 9, 12, ...}		52(9)		76(49)
7-8	{1, 4, 9, ...} {4, 8, 12, ...}		52(5)		82(37)
9-11	{1, 2, 3, 4, ...} {3, 4, 5, 6, ...}	25(0)	49(6)		75(33)
12-14	{1, 2, 3, 4, ...} {1^2, 2^2, 3^2, 4^2, ...}		61(13)	86(52)	85(33)
Average		**24(0)**	**53(8)**	**92(63)**	**80(39)**

(*) The frequencies of one-to-one correspondence responses are in parenthesis.

Table 1: Frequencies of “Same Number of Elements” (One-to-One Correspondence) in Different Representations

In 1999 at PME 23, in Israel, a first paper that discusses the strategies prospective teachers consider as appropriate for comparison of infinite sets before and after normative instruction of Cantorian Set Theory was presented (Tsamir, 1999a). Seventy one prospective teachers who had not yet studied the Cantorian Set Theory course and 110 who had completed the course three months before the study were provided with illustrations of 1-1 correspondence, single infinity and inclusion justification. They were then asked to determine whether each of these criteria seems suitable for comparing infinite sets and whether using each of them to compare infinite sets is acceptable. The findings indicate that the prospective teachers who had studied the normative course were significantly more inclined than those who had not to accept one-to-one correspondence and to reject the other criteria for the comparison of infinite sets. Still, in both groups most prospective teachers stated that more than one of these criteria for comparing infinite sets can be used, without noticing the resulting contradiction.

The accumulated data on prospective teachers’ typical solutions to comparison-of-infinite-sets tasks and on the impact of the normative course led to the development of several interventions (Tsamir, 1999a; 1999b; 2003). In these interventions several approaches that are commonly used in mathematics education were applied (e.g., cognitive conflict approach, teaching by analogy – in Tsamir, 2003). We therefore developed another intervention, based on Sierpinska's 1989 paper. In her study, Sierpinska first presented two students with the definition: "Two sets have as many elements as the other if their elements can be paired off, that is, if every element of the first of these sets has a pair in the second, and every element of the second has a pair in the first" (Ibid., p. 168). The definition was then applied, using collections of green and yellow counters, and negotiated. Then, the students were presented with a number of drawings, each consisting of a pair of geometrical figures (e.g., two segments, two circles). The students were asked if there are as many points in one figure as there are in the other. This approach was used to formulate an extended intervention aiming at strengthening the application of one-to-one correspondence and examining the prospective teachers' awareness of the inconsistencies that occur as a result of using more than one criterion for such comparisons (Tsamir, 1999a).

A more extensive intervention, an Enrichment Cantorian Set Theory Course, is described in Tsamir (1996; 1999b; 2000). The course related to prospective secondary mathematics teachers' tendencies to overgeneralize from finite to infinite sets and consequently to simultaneously apply the various available methods for comparing the number of elements of finite sets to those of infinite ones. The course consisted of twenty-four weekly class sessions of 90 minutes each. The first five sessions were devoted to discussing connections between mathematics and reality; the axiomatic, independent nature of mathematical systems and the crucial role consistency plays in determining mathematical validity. Research findings regarding inconsistencies in students' mathematical performance, possible reasons for their occurrence and suitable teaching methods which are suggested in the literature were discussed as well (e.g., Tall, 1990; Tirosh, 1990; Vinner, 1990). The remaining nineteen sessions of the course related to Cantorian Set Theory, discussing defined and undefined concepts, axioms and theorems. Primarily, various finite and infinite sets, the null set, relations and operations between sets were discussed. After comparing finite sets, infinite sets were compared and finally the powers of various sets were defined and discussed. The discussions of infinite sets followed Zermelo and Fraenkel's theoretical framework (for instance, Davis & Hersh, 1980/1990; Fraenkel, 1953/1961); various teaching methods were applied, taking into account historical aspects, students' primary intuitions and emphasizing the role of consistency in mathematics.

We present here a part of one activity that was used in the enrichment course, aiming to promote prospective teachers' awareness of the different criteria that they legitimately used for the comparison of finite sets, and overgeneralized to infinite sets. The activity unfolded in two main stages. Stage 1 aimed at increasing prospective teachers' awareness of their tendency to use at least four methods (counting, 1:1 correspondence, inclusion, and intervals) for comparing finite sets, and of the legitimacy of the simultaneous usage of all these methods. Stage 2 aimed at promoting prospective teachers' awareness that the application of different methods when comparing infinite sets leads to contradictory answers. The part of the activity that we present was included in Stage 2.

We first asked participants to individually solve several comparisons of infinite sets tasks. For instance,

> B = {1, 2, 3, 4, 5, 6, 7,....}   P = {100, 200, 300, 400, 500, 600, 700,...}
> The number of elements in sets P and B is equal / not equal. Explain

This task triggered the use of three different criteria (inclusion, one-to-one correspondence and single infinity).

The prospective teachers were then asked, once more, to individually solve the same problems. This time, however, the instruction was:

> Try to apply all the three methods:
>
> B = {1, 2, 3, 4, 5, 6, 7,....}   P = {100, 200, 300, 400, 500, 600, 700,...}
>
> Is '1:1' correspondence applicable? Yes / No
> If your answer is Yes-- Use this method to solve the problem
> Is the number of elements in set B equal to the number of elements in set P?
>
> Is 'inclusion' applicable? Yes / No
> If your answer is Yes -- Use this method to solve the problem
> Is the number of elements in set B equal to the number of elements in set P?
>
> Is 'all infinities are equal' applicable? Yes / No
> If your answer is Yes -- Use this method to solve the problem
> Is the number of elements in set B equal to the number of elements in set P?

The participants realized while performing this activity that the application of the three methods led to contradictory solutions. This created confusion and raised questions (for example, Why did it work with finite sets? What should be done in the case of infinite sets?). Participants were then asked to reflect on their responses in groups. The specific guiding questions were:

1. Is it OK to alternatively use these methods for the comparison of infinite sets? Why?

2. In your opinion, which (if any) of the various methods for comparing infinite sets is preferable? Why?

In the class discussion, after each group presented its approach, the participants reached the conclusion that the choice of only one method for the comparison of all infinite sets is essential, and this method must then be used exclusively.

The assessment of the enrichment course was done with reference to the normative course. The findings indicate that the highest rate of success in comparing the number of elements in infinite sets was among prospective teachers who had taken the enrichment course. These students were most consistent in their use of a single method and in expressing awareness of the need to preserve consistency within a given mathematical system. This suggests that when designing and teaching mathematics courses, attention should be given to the relations between formal and intuitive knowledge and to the conflicts which may arise in the mismatching applications of these different types of knowledge (e.g., Fischbein, 1987; Papert, 1980; Tall, 1980).

The work that we presented so far was mainly embedded in Fischbein's approach to mathematics learning and teaching (e.g., 1987; 1993, 1999, 2001). In the last decade we interpreted our studies on infinity by means of additional theoretical lenses. The following section briefly describes these attempts.

PME 24 to PME 30: Extending the Theoretical Approaches

The value of examining the same issues from different viewpoints can not be overstated. We experienced this when attempting to interpret our findings, using several theoretical models, including the intuitive rules theory (Stavy & Tirosh, 2000; Tirosh, Stavy, & Tsamir, 2000), the RBC model (Dreyfus &Tsamir, 2004; Tsamir & Dreyfus, 2002; 2005) and the conceptual change approach (Tirosh, & Tsamir, 2004). The different theoretical models that we applied to our data (and to some extended studies) provided us with an opportunity to reveal possible additional sources of learners' reasoning about infinity; equipped us with a more extensive vocabulary to discuss the phenomena observed; broadened our understanding of ways in which knowledge about infinity is constructed, and extended our repertoire of approaches to instruction. The intuitive rules theory, for instance, drew our attention to the substantial impact of salient yet irrelevant features of the task on students' responses. The RBC highlighted the fragility of knowledge constructed in the case of equivalency of infinite sets. The conceptual change approach offered a specific vocabulary to describe situations where intuitive and formal reasoning intermingle and also provided a coherent set of instruction design principles. These perspectives on the study of infinite sets were elaborated during various PME conferences. For instance, at PME 24, in 2000, during a project group on "Intuitive rules and mathematics learning and teaching" (Tsamir, Lin, Tirosh, de Bock, & Muller, 2000), at PME 26, in 2002, at a research Forum on "Abstraction: Theories about the emergence of knowledge structures" (Dreyfus and Grey, 2002). We carry on this line of enriching our understanding of the complex issues related to the learning and the teaching of infinite sets by applying different theoretical perspectives. In this spirit, at PME 30 in Prague, the work on infinite sets will be discussed in two research forums: one related to the conceptual change approach and the other on exemplification, i.e., the use of examples in mathematics teaching and learning.

Final Comments

We started this journey by quoting Fischbein from the first research synthesis book of PME, where he called for more research on issues related to advanced mathematical concepts. Throughout the paper we have shown how PME contributed to the development of our work on one such concept, that of actual infinity. A lot of work remains to be done in respect to the learning and teaching of this fascinating concept. PME will certainly continue to play a major role in evolving and determining new research avenues.

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PLENARY PANEL

Mathematics in the Centre

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Groves, Susie
Krainer, Konrad
Lins, Romulo (coordinator)
Rojano, Teresa

A CENTRE AND A MATHEMATICS

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In 1989 I attended my first PME. In Paris, 200 years away from the French Revolution, I was sitting at the chair next to my PhD supervisor and friend Alan Bell, during the general assembly. I remember there was a heated debate as to how much of didactics was to be acceptable at a conference named ‘Psychology of Mathematics Education’. Alan’s position was that it made no sense trying to be normative on that matter: if more people wanted to discuss didactical issues, that would be the direction taken by the group of people interested in attending those conferences.

Alan was not, of course, being careless: he’s never been. At that point my understanding was that maybe Alan was just manifesting his British roots, more precisely English and Welsh: pragmatism.

At PME Mexico, 1990, in one of the sessions of our ‘Algebra Working Group’, led by Ros Sutherland with the help of Teresa and others, a colleague vigorously protested against the many contributions which mentioned the History of Mathematics, Theory of Knowledge and Linguistics, saying that all of that had nothing to do with the *psychology* of mathematics education.

A number of years later, at the closing plenary during PME in Recife, Brazil, Kath Hart, then the PME president, ended her presentation urging all to honour what the name of the conference said: ***P***sychology of ***M***athematics ***E***ducation, with strong emphasis on all three words.

In view of those episodes, and having being absent from PMEs for a while, I couldn’t help but consider the possibility that, given the theme of this 2006 conference, ‘Mathematics in the centre’, this might be our last trench: mathematics. After 17 years — and I am not counting what might have happened before I joined the PME community — the inner centre seemed to have been moved to ‘mathematics’.

The key issue that troubled me even before I hit the ‘send’ button to reply to the invitation to organise this panel, was “what is ‘mathematics’ to be at the centre of?” The two simpler, general, answers were ‘at the centre of the psychology of mathematics education’ and ‘at the centre of mathematics education’. It seemed to me that the latter was more general, so I decided to dedicate my attention to it. If I could make sense of what it meant, making sense of the other would follow, in a very mathematical way.

I suppose we can — or should — begin by asking ‘what is mathematics education?’ I will not go into many details or shades. Roughly speaking, the camp is divided into ‘educating *for* mathematics’ and ‘educating *through* mathematics’. The former refers to processes through which people of all sorts become apt to practice mathematics as required in examinations, professions and everyday life (simple, daily, tasks). The other refers to processes through which people become apt to have a full status citizen life in a world in which mathematical models might be part — to a greater or lesser extent — of the governance of our lives, much as Ole Skovsmose and others have brought to our attention.

Unfortunately, given the existence of those two broad ways of understanding 'mathematics education', I was still left with the question "what is mathematics to be at the centre of"?

There is, indeed, a large amount of work done in Mathematics Education that would be de-characterised as such if one left proper mathematical content out of it. Teresa's paper, for instance, makes this point sufficiently clear and supported.

But there is also work, indeed quite interesting and relevant work, that barely mentions – or does not mention at all – specific mathematical topics (for instance: equations, fractions) or areas (for instance: geometry, algebra). I would like to mention the work of three colleagues in which I see this happening: Ole Skovsmose, already mentioned, Gelsa Knijnik, who is a key player in the Landless Workers Movement in Brazil, and Bob Moses, from the Algebra Project. Konrad's paper is another example and one close to us in this panel.

How can that be and what does it mean?

In my view, that is due to the fact that the very word 'mathematics' is something that, in our western or wersternalised cultures, floats above all of us or, better, it fills, in a sense, some cultural 'air' we are immersed in, something whose presence does not depend on the mention of any specific content or area.

When someone says "I hate mathematics" or "I love mathematics" or "mathematics is important to society", there is a sense in which that person is not referring any specific mathematical content. These are quite fuzzy statements if one tries to make sense of them in relation to school mathematics or to the mathematics of the mathematician. But, still, we are able and willing to accept those statements *prima facie*, as being about 'mathematics' and, so, related to 'mathematics'.

When Susie's paper mentions a keen interest in what she calls 'subject cultures', I think she is precisely acknowledging that there is a sense in which 'mathematics' in 'mathematics education' does not need to mean a reference to specific topics and the teaching and learning of those topics, although it may, of course, be meant in this way.

Also in Zahra's paper, one can sense a way in which 'mathematics' is present as a demarcation post in what can be characterized as a power struggle involving mathematics educators and mathematicians (and educators, although the mention to them is much less emphatic in the paper) and the negotiations to promote a pacific co-existence. Near the end of the paper she mentions the way in which a mathematician questioned a masters candidate, during the examination, about the 'mathematical identity' of the candidate's work. What could it be, in this case, that would add a 'mathematical identity' to her masters dissertation? I believe that from the point of view of the mathematician that would likely be 'mathematical content'. My reason for believing so is that, much more often than not, for the mathematician, professionally, talking about how people feel about mathematics and about what mathematics is, is not 'mathematics'. Generally speaking, asking a mathematician 'what is mathematics' may well produce an answer like '*this* is mathematics', pointing to an open mathematics book, and there is nothing intrinsically wrong with this; there are of course, cases in which a different kind of dialogue would follow.

Such questioning, in the context of Zahra's paper, gives us, I think, a quite clear example of how 'mathematics in the centre' can be part of an anchoring process that has not much to do 'properly' with the teaching and the learning of mathematics, but rather with different kinds of relations.

Let me offer, at this point, a metaphor that might help us to bring those and other aspects together in relation to the 'mathematics in the centre' issue.

Hurricanes have a quite disturbing characteristic: at their centre there is an intense calmness, no matter how violent things are closer to their edges. For those who know of that and find themselves taken by a hurricane, it is disturbing because despite the temporary quietness one knows that things might – and probably will – change at some point.

Let's now think of Mathematics Education — a field of professional activity —, as a hurricane. At the eye, its centre, everything is quiet and much as usual outside hurricane centres; should someone be magically transported straight to the centre, it would be possibly difficult to imagine something dramatic is happening around, apart, perhaps, from the dark sky in the horizon.

At the centre of Mathematics Education, then, people would not worry much about things being too different from what it uses to be in usual times. Unless, of course, they are aware of where they are and of what is happening around them.

Now, my metaphor forces me to bring together the ideas of time and space, because when I speak of 'usual times' I am also speaking of 'usual places (within the community)'. Here, again, Teresa's paper is enlightening, because it traces the issue of 'mathematics in the centre' both to tradition and to more recent concerns about the teaching and learning of mathematics. Tradition roots this issue in the transmission of mathematical knowledge in order to foster the science and in the emergence of the general idea of didactics; more recent concerns root it in the widely and socially perceived need for people who are mathematically proficient — as pointed by Susie, referring to Kilpatrick, Swafford and Findell's construct — and in the widely and socially perceived specificity of the teaching and the learning of *mathematics*.

I propose that, adopting the hurricane metaphor, we place tradition and current concerns at the eye of the storm. Teaching, development and research taking place there *naturally* has mathematics at its (inner) centre. Parts of what Teresa says in her paper somewhat agree with this application of the metaphor.

But that region of stability can be understood as being the eye of a hurricane as much as it can be understood as simply being a region of calmness with some strong rain somewhere around. In other words, it seems of interest to consider that such calm region might or might not be understood as the same as an equally calm region away from any hurricanes. For one thing, as I have already mentioned, those who are there and know it will most likely not be at ease with the possibly coming trouble, but, on the other hand, one may consider that what is happening at the edge of the hurricane (or should we say, in this case, in the visible horizon?) does not or will not significantly affect the calm region.

If we imagine a 'stationary' hurricane an argument could be made for both sides: it *does not* matter what is going on at the edge for us to understand what is happening at the centre; or, it *does* matter.

The key issue here is, I think, one of representation: how is the centred calmness represented, and, perhaps more crucially, by whom? And why so? I think these are questions that may help us to clarify the issues involved in the consideration of 'mathematics in the centre'.

The hurricane metaphor came to my mind almost through a naïve word association or meaning slip. But aided by it, examining the initially unsuspected — for me — complexity of 'mathematics in the centre', and considering the richness of elements and insights offered by the other four papers produced by the panel members, I decided that perhaps my best contribution would be towards offering a perspective from which our theme could be 'rephrased', so to speak, possibly allowing our discussion to illuminate as yet invisible corners of that issue. There are, of course, as all the four other papers make clear, many other corners which are already in the sunlight, for instance, the relationship between teachers' confidence with mathematics and their confidence to teach mathematics, and I certainly do not take issue with any of them.

That reflection led me to consider that instead of looking to 'mathematics in the centre' straight in the eyes, so to speak, I could rather deal with the issue of '*a centre and a mathematics*'.

On the one hand, the indetermination allows me to refuse assuming there is only one centre or even one that I should be taking as preferential here —thus allowing me not to engage in trying to determine what that centre is. On the other hand, that expression presents me with a useful degree of separation between the two elements in it.

With respect to this discussion, I will take *a centre* to be a region of stability, be it the eye of a storm or be it a nice day somewhere with clear sky. And I will take *a mathematics* to be a reason for 'mathematics' being mentioned. By doing so I can now argue that the issue of 'mathematics in the centre' can and should be understood both as the issue of what, within specific social practices of a specific culture, is perceived as mainstream in Mathematics Education — needed, recommended, natural, essential —, and the issue of the nature of *a mathematics* within those social practices and that culture.

Let me consider the latter issue first. When we use, in our professional considerations, the expression 'the nature of mathematics', what are we referring to?

Konrad's, Susie's and Zahra's papers tell me that we could be referring to the social and cultural nature of mathematics, here understood as an element in a culture that relates to other elements in possibly many different ways. Teresa's paper tells me that we could be referring both to the historical and to the epistemological natures of mathematics.

But if in Zahra's paper I see the most evident cultural aspect of 'mathematics' as that related to power structure and struggle, in Susie's the notion of 'subject cultures' blends a sociological view with an epistemological view, while Konrad's approach seems to me more definitely sociological.

Teresa takes us to the edge of the hurricane when she says that "*[…] the disciplinary boundaries that mathematics education shares with other disciplines […take us to the fact that] the place of mathematics in the field of mathematics education cannot always be well determined.*" Are there centres at which, contrary to this, we will find mathematics clearly dominant, as in the traditional views of mathematics teaching? (and, in my formulation, speaking of centres implies speaking of social practices and cultures)

When Konrad urges us on the need to produce "*[…] public relation activities in order to make the power and beauty of mathematics better understandable for our citizens.*", the nature of 'mathematics' is no different, in my view, from that of 'non-violence' or 'healthy life', and before the reader comes to the conclusion that I am saying this in a demeaning manner, let me clearly state that I fully agree with him, even to the extent of saying that those public activities could well involve mass-media public relations campaigns.

As to '*a centre*'.

What can be a centre for the mathematics education one practices? It could be reaching one of the top positions at a PISA table. It could be avoiding that pupils drop out of school before they get a certain number of years of schooling — thus avoiding that, not being in school, they stay somewhere else, perhaps engaging in totally undesirable activities (crime, harmful drug consumption). It could be to create an adequately prepared workforce — from the point of view of the needs of a society, from the point of view of the productive system or from the point of view of the Capital. It could be to help people to assume a full status citizenship (as necessary in highly technological societies, as Ole points out) or, quite on the contrary, to prepare people to be obediently disciplined. Or many more, and many shades and combinations of those. All that is not new, of course.

‘*Centres*’ as places. ‘*Mathematicses*’ (sic) as reasons for mathematics to be at a centre, that is, at a place. The hurricane metaphor helped me to understand that centres as regions of stability bring together the notions of tradition, cultural values and social demands, and that different reasons for mathematics to be at a centre may produce different natures for ‘mathematics’.

In no sense it was my intention to argue for or against the views represented in the four other papers related to this plenary panel — or any other views related to what I have said so far. In particular, the fact that I did not mention a number of points made in those four papers does not imply that I disagree with them, and given the emphasis I decided to have on this paper, I explicit mention all points in which the authors argue that mathematics should have a central place in mathematics education, although arguing that taking mathematics as the last trench in defense of our identity or specificity is a dangerous step.

So, my only claim is that it seems useful to approach our key question with a clear sense of ‘situatedness’. That is why I prefer to speak of *a mathematics* (in the sense I did) instead of speaking of (the) mathematics. It is not a matter of offering an alternative view of what ‘mathematics’ is, I leave this to the philosophers. And that is why I prefer to leave which centre we are talking about to those who are actually speaking about a centre.

The many possible combinations of the ‘*a mathematics*’ mentioned above (and many others not mentioned), with the various ‘*a centre*’ suggested, give, I think, at least a glimpse of the complexity of the issues we are dealing with in this panel.

If anything, I hope this paper can help us to keep this complexity present in our considerations about ‘mathematics in the centre’. And the same hope applies in relation to the *differences* that such a complexity and respect for it are bound to elicit within our Mathematics Education community.

THE NECESSITY OF COLLABORATIONS BETWEEN MATHEMATICIANS AND MATHEMATICS EDUCATORS: WAY TO HAVE “MATHEMATICS IN THE CENTER”

Zahra Gooya

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Mathematics education as a field of study in Iran is in its infancy. The proposal for master program of mathematics education was finally approved by the “mathematics branch” of the “supreme council of curriculum development” under the auspicious of the Ministry of Science, Research and Technology (Higher Education) in Iran in 1999. The process of the establishment of mathematics education in Iran is a good example of the collaborations between mathematicians and mathematics educators or education community in general. Thus, I have chosen to start with the developmental process of this field in Iran to discuss the necessity of collaboration between these two communities.

DEVELOPMENTAL PROCESS OF MATHEMATICS EDUCATION IN IRAN

A proposal for the mathematics education as a field of study in Iran was first proposed to the mathematical community in 1994. After long debates, a committee was set up at the Faculty of Education and Psychology of Tehran University to further discuss the proposal. This committee consisted of prominent mathematicians, educational psychologists, educators, and two mathematics educators (being the only ones in that time.) As a result of a meaningful collaboration among many stakeholders regarding the teaching and learning of mathematics, the committee prepared a revised proposal and presented it to the “Mathematics Council” of the Ministry of Higher Education. In this council, the members were insisted that the mathematics departments should be hosting any program related to mathematics and thus, the identity of such programs should be of that of mathematics departments. To cut the long story short, the master program of mathematics education received its final approval in 1999.

STARTING THE MASTER PROGRAM OF MATHEMATICS EDUCATION IN IRAN

After the final approval of proposed program for master of mathematics education in 1999, the ministry gave permission to those universities who have both faculty /department of mathematics and faculty of education to establish the program, and in this permission, it was indicated that the program should resides in the faculty of mathematical sciences.

In year 2000, I-as a member of mathematics department of Shahid Beheshti University- proposed to be the first host of this newly approved program in Iran, since I prepared the initial proposal in 1993, and I continuously strived and put

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 73-76. Prague: PME.

enormous effort for the establishment of this field in the country. In addition, up to this point, there are only 3 people having doctoral degree in mathematics education in Iran and two of them (including myself) are members of this department.

After the proposition, it took almost a year until the department agreed to start the program. What happened in this process and the developmental process of 1993 to 1999 is the point that I like to make about the nature of collaborations between mathematicians and mathematics educators in one hand, and about the sensitivity and expectations of mathematical community regarding mathematics education.

The main debate and heated discussions in the department was the epistemological and ontological ones regarding the nature of mathematics education, and its relation with mathematics as a discipline. Other concerns included the mathematical substance of the approved program; about mathematical content and mathematical proficiency of future mathematics educators. To give an example, in the revised program by the department, two mathematics graduate courses as obligatory requirement were added.

The collaboration and continual cooperation between mathematicians and mathematics educators is a one real challenge in the country. However, both communities are learning more from each other and both are making good progress towards bridging the gap between two by acknowledging the necessity of putting “math in the center.” In this way, mathematicians become more and more aware of the role of mathematics education in their real existence, and mathematics educators appreciate the richness of the problems that they receive from mathematics community. They both are trying hard to understand each other and together, pave the way for the enhancement of mathematics in our society.

POSSIBLE REASONS FOR A CHASM BETWEEN MATHEMATICS AND MATHEMATICS EDUCATION COMMUNITIES

Mathematicians were pioneers in thinking about the necessity of mathematics education for the enhancement of mathematics, and thus, mathematics education as a discipline and as a field of study was mainly started from mathematics departments (Kilpatrick, 1994). However, the reality of today’s practice shows that a chasm has been developed between mathematics educators and mathematicians for various reasons, and this event is dangerous for the healthy living of mathematics education. I thus, would like to mention a number of possible reasons that might have contributed to this great divide.

- *Mathematicians have many legitimate questions about the ways in which, mathematics education might help the mathematics community to overcome its problems regarding teaching and learning mathematics. The evidences prove that in many occasions, these questions have not had taken seriously by mathematics education community.*
- *Mathematics community deserves to be informed about the nature of research in mathematics education much more than what has been said so far.*

- *Mathematics community has enormous concrete problems regarding mathematics curricula at every level of education, mathematics teacher education, mathematical content, technology, etc, etc (Zangeneh, 2003.) The community needs to overcome many obstacles and is seeking both partial and immediate solutions as well as long standing solutions to its problems. Mathematics education community could and should be the best sanctuary for mathematics community which has not been the general case so far.*
- *Mathematicians and mathematics educators have not made enough effort to develop a common language to better understand their view points and thus, help to bridge the gap between two communities.*
- *Mathematicians in general, are not familiar with research paradigms in mathematics education, and therefore, do not pay enough attention to research findings coming from mathematics education community. This is especially important since many mathematicians are concerned about the educational process and they regularly discuss the educational issues in their own community. Thus, the establishment of trust between two communities is necessary for both, to understand each other and make them better able to develop a common language.*

THE NECESSARY REQUIREMENTS FOR COLLABORATIONS

Each one of the above cases needs to be taken seriously in order to pave the way for more meaningful collaborations among mathematics and mathematics education communities. To take concrete questions of mathematics community more seriously, both mathematics and mathematics education communities should establish more meaningful relationships, and set a stage for productive collaborations. A good starting point to implement research findings in mathematics education could be in mathematics curriculum development and textbook writing in which, both mathematics educators and mathematicians are accountable to public at large. In addition, mathematics education researchers need to be more sensitive about the practicality of their research, and about the ways in which, theory and practice could be bridged.

Before to end, I like to mention an event that happened when the first graduate student of mathematics education in Iran had her defense. The first student-Miss Mehrbani's research project was about "A model for the development of mathematics teachers' professional knowledge in Iran" in which, by using Krainer's model (2000), she developed a model for mathematics teachers in Iran. One of the very eminent Iranian mathematicians was invited to her defense as external examiner. After she presented her research, the examiner asked her that "what makes your finding specific to mathematics?" and he went on to ask that "you will be receiving a master degree in math education from math department. Your thesis should have mathematical identity. Otherwise, you could develop the same model for teachers in any other field. You need to show that your thesis could not been successfully accomplished by those who did not have their first degrees in mathematics, or have not taken graduate courses in mathematics. You need to show your mathematical view points, and present evidences to support your claims about mathematics teachers in particular." And he said that "that makes your research a mathematics education research and distinct it from theses in educational psychology, curriculum theories or such."

This first experience was extremely valuable to me, and helped me to better realize the sensitivity of mathematicians and setting up the necessary requirements for more meaningful collaborations among us.

Last, but not the least, I will like to end up saying that mathematicians are not from Mars and math educators are not from Venus! (Sultan and Artzt, 2005). They all could live together and collaborate with each other and live happily ever after, if they try to understand each others concerns, and if they all agree to have "math in the center" of their activities. Because I do believe that research findings of mathematics education community should have mathematical identity and have mathematics at their center stage.

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GENERIC VERSUS SUBJECT SPECIFIC PEDAGOGY — SHOULD MATHEMATICS BE IN THE CENTRE?

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Research into pedagogy and school change is a high priority in Australia and many other countries. This paper, which includes some preliminary findings from the Improving Middle Years Mathematics and Science: The role of subject cultures in school and teacher change[1] *(IMYMS) project, argues that, while there are key features that are common to quality learning environments across all subject areas, generic formulations of pedagogy fail to take account of the extent to which the disciplines being taught shape pedagogy or the contribution of Pedagogical Content Knowledge (PCK) to effective teaching — i.e. that there really is a need to put "mathematics in the centre".*

INTRODUCTION

> Our main game is and always should be pedagogy — teaching and learning in the face-to-face setting of classrooms. … At the same time, if we want to change student outcomes, … the three message systems — curriculum, pedagogy, assessment — need to be brought into proper alignment for us to get desired educational results and outcomes. (Luke, 1999, pp. 3–4)

Research into pedagogy and school change is a high priority in Australia and many other countries. Recent Australian initiatives such as Queensland's *New Basics Research Program* (see, for example, Education Queensland, 2000), the *Victorian Essential Learning Standards* (Victorian Curriculum Assessment Authority, 2005) and the Tasmanian *New Essential Learnings* framework (Department of Education Tasmania, undated) have attempted to break down the barriers between discipline areas by promoting generic formulations of thinking, learning, and pedagogy, as well as new ways of organising curriculum and new forms of assessment.

The ways in which such initiatives have dealt with the nexus between traditional discipline-based curriculum organisation and their new curriculum structures has varied, as has the extent to which disciplines such as mathematics have been seen as merely underpinning the new learning frameworks (for example, in the *New Basics*) or have been left relatively intact within a broader structure (for example, in the *Victorian Essential Learning Standards*).

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1 *Improving Middle Years Mathematics and Science: The role of subject cultures in school and teacher change* (IMYMS) is funded by an Australian Research Council Linkage Grant, with Industry Partner the Victorian Department of Education and Training. The Chief Investigators are Russell Tytler, Susie Groves and Annette Gough.

It is in this climate that the *Improving Middle Years Mathematics and Science: The role of subject cultures in school and teacher change* (IMYMS) project is investigating the role of mathematics and science knowledge and subject cultures in mediating change processes in the middle years of schooling.

This paper, which includes some preliminary findings from the IMYMS project, will argue that while there are key features that are common to quality learning environments across all subject areas, generic formulations of pedagogy fail to take account of the extent to which “the character of the disciplines being taught” shape pedagogy (Schoenfeld, 2004, p. 237) or the need to blend pedagogical knowledge and content knowledge into *Pedagogical Content Knowledge* (Shulman, 1986) — i.e. that there really is a need to put “mathematics in the centre”.

GENERIC PEDAGOGIES AND THE DISCIPLINE OF MATHEMATICS

> A mathematical proof is not the same as a scientific testing of a hypothesis, which is not the same as a historical account or comparison across accounts, which is not the same as a critique in the arts or literature. (Gardner, 2004, p. 234)

An investigation of non-mathematics specific pedagogical frameworks reveals much that resonates with views of what constitutes quality teaching in mathematics. For example, *Productive Pedagogies* — one of the three conceptual pivots of Queensland’s *New Basics Research Program* — focusses on four dimensions: Intellectual quality; Connectedness; Supportive classroom environment; and Recognition of difference. Within these, Intellectual quality is characterised by evidence of: Higher order thinking; Deep knowledge; Deep understanding, Substantive conversation; Knowledge as problematic; and Metalanguage (Education Queensland, 2000). All of these, except perhaps the last, would be seen as highly relevant to quality teaching in mathematics.

Similarly, although it is not a pedagogical framework, the notion of *Communities of Inquiry* — which underpins the *Philosophy for Children* movement — focuses on the development of skills and dispositions associated with good thinking, reasoning and dialogue; the use of subject matter which is conceptually complex and intriguing, but accessible; and a classroom environment characterised by a sense of common purpose, mutual trust and risk-taking. We have frequently argued (see, for example, Groves & Doig, 2002) that a desirable goal for mathematics education would be that mathematics classrooms function as (mathematical) communities of inquiry. However, mathematics and philosophy are quite different disciplines and the way a community of inquiry might look in a mathematics classroom is likely to be quite different from how it might look in a *Philosophy for Children* lesson.

For successful teaching to take place, there needs to be a clear view of what is meant by successful learning in a particular discipline. So, for example, Kilpatrick, Swafford, and Findell (2001, p. 116) define *mathematical proficiency* — their term for what they believe is necessary for successful mathematical learning to take place — as having five interwoven strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition.

However, while these aspects are likely to strike a chord with mathematics educators, they do not necessarily span the full spectrum of goals for mathematics teaching. For example, Lampert (1990) proposes a vision of classroom mathematics that differs from traditional practice by mirroring the key features of mathematics as a discipline. In a similar vein, Yackel and Cobb (1996) focus on classroom discourse and the socio-mathematical norms associated with achieving quality dialogue in mathematics classrooms. While pointing out the commonalities between mathematics and other disciplines, Schoenfeld (2004, p. 248) states “there are fundamental differences in content, processes, and epistemology between different fields. In consequence there will be differences in pedagogical goals, pedagogy, and the knowledge that underlies it (including pedagogical content knowledge)”.

Moreover, these differences in a discipline’s language and epistemology have a major influence on the way in which teachers “conceptualise the world, their roles within it, and the nature of knowledge, teaching, and learning” (Siskin, 1994, p. l52). In fact, according to Siskin, there is more commonality concerning issues of curriculum policy and teaching and learning between mathematics departments in different schools, than between departments in the one school.

Studies of effective teaching in mathematics have sometimes been equivocal about the value of content knowledge for teachers. However, two of the three major findings of the recent *Investigation of effective mathematics teaching and learning in Australian Secondary Schools*, involving almost 8000 students and over 200 teachers, were that:

2. Teacher knowledge and educational background is weakly related to teacher effectiveness. The more this education has to do with mathematical content and pedagogy, the more likely it is that teachers will be effective; and
3. The effectiveness of mathematics teaching in a school is related to the strength of professional community in the school's mathematics departments. (Ingvarson, Beavis, Bishop, Peck, & Elsworth, 2004, p. viii)

Thus, mathematics is central to the nature of the curriculum, pedagogy, teachers’ identity and allegiances, and their effectiveness in terms of cognitive and affective outcomes for their students.

THE IMYMS PROJECT

> Subject departments are not just smaller pieces of the same social environment or bureaucratic labels, but worlds of their own, with their own “ethnocentric way of looking at” things. They are sites where a distinct group of people come together, and together share in and reinforce the distinctive agreements on perspectives, rules, and norms which make up subject cultures and communities. (Siskin, 1994, p. l81)

The IMYMS project has its roots in the *Science in Schools* research project (SiS), which developed a strategy for improving teaching and learning science based on two major aspects: the SiS Components, a framework for describing effective teaching and learning in science, and the SiS Strategy, a strategic process for planning and implementing change (see, for example, Gough & Tytler, 2001).

Among the research questions being addressed by the project are the extent to which a generic "effective pedagogy" can capture the essence of teaching and learning in mathematics and science, and the links between teachers' pedagogies in mathematics and science.

A central part of the IMYMS project has been the extension of the SiS Components to produce the *IMYMS Components of Effective Teaching and Learning* in an attempt to describe effective teaching and learning in mathematics and science (for a full list of the IMYMS Components, see, for example, Groves & Doig, 2005).

This extension of the SiS Components to include mathematics as well as science has resulted in a number of distinct types of changes based on a review of the literature on effective teaching, interviews with "exemplary" mathematics teachers, and extensive discussions among members of the project team. Some of the SiS components were regarded as being equally applicable to mathematics, requiring only minor changes in wording (e.g. *Students are encouraged and supported to take responsibility for their learning*). Other changes, however, reflected the middle years focus of the project (e.g. *The teacher builds positive relationships through knowing and valuing each student*); the literature on effective teaching (e.g. *The teacher clearly signals high expectations for each student*); the teacher interviews (e.g. *Persistence and effort are valued and lead to a sense of accomplishment*); and our own previous research (e.g. *Subject matter is conceptually complex and intriguing, but accessible*; see Groves & Doig, 2002).

The changes were often vigorously contested within the project team and the differences between the "character" of mathematics and science were quick to emerge. Of course this was no surprise to the project team, as we were specifically seeking to identify what we are referring to as the role of subject cultures in teacher change.

As part of the project, teachers were asked to not only rate their own teaching in terms of the IMYMS components, but also to rate each component in terms of what they believe to be their importance for either mathematics or science, or separately for each when they taught both subjects (which was the case for all of the primary teachers and a minority of the secondary teachers). Data from late 2005 is just being analysed. However, preliminary analysis of data from 34 primary teachers and 22 secondary mathematics or science teachers in one of the four clusters of schools involved in the project, suggests that these teachers' views reflect some of the differences identified by the project team. In particular, teachers were more likely to rate the following as very important for mathematics than for science:

- Persistence and effort are valued and lead to a sense of accomplishment
- Students are encouraged and supported to take responsibility for their learning

- Subject matter is conceptually complex and intriguing, but accessible
- The teacher clearly signals high expectations for each student
- Learners receive feedback to support further learning
- Assessment practices reflect all aspects of the learning program.

The first, third and fourth items on this list were the ones most strongly contested within the project team. The second item, however, is somewhat surprising as it was one that was seen by the project team as more prominent in science than in mathematics teaching, while the last two were not seen as being particularly slanted towards either mathematics or science.

There was only one item that teachers were more likely to rate as very important for science than for mathematics — namely:

- Mathematics and science content is linked with students' lives and interests.

While this was again not surprising, it was surprising that there appeared to be very little difference in these teachers' views of the importance of the following statement, which the project team and earlier teacher responses had suggested were seen as more important in science:

- The learning program provides opportunities to connect with local and broader communities.

Further analysis of the full set of data will be carried out shortly. However, these examples are given here to illustrate the difficulties associated with attempting to produce generic descriptions of effective pedagogy for even the two areas of mathematics and science, which are frequently seen as being very closely aligned.

The project has generated significant amounts of data relating to teachers' beliefs and practice; students' performance, perceptions and attitudes; and the process of teacher change. It is apparent that the nature of mathematics and science, their purpose and role in both the community and schooling, and the quite different ways in which their curricula are constructed (at least in Australia) lead to quite different pressures on teacher pedagogy, and, for our purposes here, the need to "put mathematics at the centre".

CONCLUSION

Current calls to rethink curriculum and pedagogy based on cross-disciplinary "big ideas" and key elements such as inquiry and reflective thinking, have led to generic formulations of pedagogy and new curriculum structures that replace to varying degrees the traditional disciplines. While there is a need for such cross-disciplinary practices, it is important to take account of the extent to which a deep understanding of mathematical content and processes are central to effective pedagogy.

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HOW CAN SCHOOLS PUT MATHEMATICS IN THEIR CENTRE? IMPROVEMENT = CONTENT + COMMUNITY + CONTEXT

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Most schools have some mathematics teachers who have a strong mathematical background, profound pedagogical content knowledge and successfully engage their students in mathematical thinking and foster their interest in mathematics. Sometimes, these teachers have a relevant impact on other mathematics teachers' work. And in a few cases the school puts a special emphasis on mathematics in its school profile. However, at most schools, mathematics teachers are rather isolated and mathematics does not have the attraction as it could have. Links to other disciplines are rare, interdisciplinary projects often run without mathematics or only use it as a complementary tool. There is hardly any dialogue about the importance of mathematics or about goals and problems of mathematics teaching at the school. Schools need external support to improve this situation. It is suggested to put an emphasis on students' mathematical learning, but also to take social and organisational aspects into account. This leads to the importance of the three Co's – content, community and context. Examples from the Austrian reform initiative IMST are regarded.

MOST SCHOOLS DON'T HAVE MATHEMATICS IN THEIR CENTRE

The website of JRME (see https://math.byu.edu/jrme/) starts with the NCTM-statement *"More and Better Mathematics for All Students"*. I fully share this vision, although many students and adults would not support the words "more" and "all"; some people would even refuse to believe that mathematics could be learnt in a better way than they experienced at school. The gap between the vision and the reality is enormous. If we really want to improve the situation considerably, we need to change things at different levels, from new forms of teacher pre-service education to public relation activities in order to make the power and beauty of mathematics better understandable for our citizens. However, the most important thing is that we improve mathematics learning at our schools. And this is no mathematical problem, but one where content-related competence is intensively involved.

Sustainable improvement of mathematics teaching needs active teachers as change agents. In general, one single mathematics teacher can at best change teaching in his or her own classes. The assumption that an improvement of mathematics teaching in all classes can be achieved only by trusting the professional development of all individual teachers, centrally organised curricula changes or formulating standards, is naïve. Without communication and collaboration among mathematics teachers we will have no improvement on a larger scale. Only believing in the growth of

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 84-89. Prague: PME.

individual teachers – and generally always the same teachers come to professional development courses – is too narrow.

A group of mathematics teachers or a whole mathematics department, given all people share to a large extent the same vision of mathematics teaching, has the potential to change more. However, if they cannot communicate the importance of mathematics to other teachers (in particular to science teachers who are, in general, more likely to be interested in mathematical thinking), the principal, the parents and other stakeholders, the impact will be limited. A principal who hates mathematics or even a hostile "school climate" against mathematics can hinder or ruin the best competences, attitudes and efforts by mathematics teachers.

If the status of mathematics at schools and in society is low, it is a hard job for mathematics teachers to preach the importance, applicability etc. of this wonderful science. Thus, although mathematical competences of mathematics teachers are a very important precondition for good mathematics teaching, also other competences are highly relevant.

One competence deals with the ability to communicate the power and beauty of mathematics to other people, in particular to students and to colleagues of other subjects. This cannot be done simply by telling. We need to create situations where the other teachers themselves start to think about mathematics and to express their views on it. Starting points might be PISA items, SUDOKU puzzles, tasks of a final exam or other mathematical problems that take these people where they are. However, it is important to involve them in mathematics and in reflections about mathematics. Something that is not discussed in the public is not really considered as important or even existent, particularly nowadays living in an information society. This is true for the media, but also true for schools. If mathematics is not visible at a school (in exhibitions, conferences, competitions etc.), it is not in the centre of school life.

However, if teachers of other subjects get a better understanding of mathematics, and they possibly lose part of their fears and maybe realize some common issues between their subject and mathematics, it will be much easier to raise the reputation of mathematics at a school.

Research on "successful" schools shows that such schools are more likely to have teachers who have continual substantive interactions (Little, 1982) or that inter-staff relations are seen as an important dimension of school quality (Reynolds et al., 2002). The latter study illustrates, among others, examples of potentially useful practices, of which the first (illustrated by an US researcher who reflects on observations in other countries) relates to teacher collaboration and community building (p. 281): *"Seeing excellent instruction in an Asian context, one can appreciate the lesson, but also understand that the lesson did not arrive magically. It was planned, often in conjunction with an entire grade-level-team (or, for a first-year teacher, with a master teacher) in the teachers' shared office and work area. [Referring to observed schools in Norway, Taiwan and Hong Kong:]… if one wants more thoughtful, more* *collaborative instruction, we need to structure our schools so that teachers have the time and a place to plan, share and think."*

This example demonstrates not only that it is important to have a *"community"* of mathematics teachers at a school and that relevant communication must be centred on a meaningful *"content"* – in our case mathematics or students' learning of mathematics. In addition, this example also shows that the "content-focused community" needs a supportive administrative and organizational *"context"*: the community needs support from administrators, enough time, space and other resources; the community benefits from an educational system, a region or a nation that offers good general conditions, for example the existence of attractive professional development programs, content-related networks for teachers and schools, or a considerable autonomy and reputation of teachers.

In order to improve the situation of mathematics at schools *three Co's* and their interconnection are apparently playing a major role (see also Lachance, & Confrey, 2003; Krainer, 2003):

- *Contents* that are *relevant for all* who are involved (e.g. interesting mathematical activities for the students, challenging experiments, observations and reflections for teachers, constructive initiatives and discussions at schools);
- *Communities* (including small teams, communities of practice and loosely-coupled networks) where *people collaborate with each other* in order to learn autonomously but also to support others' and the whole system's content-related learning;
- *Contexts* (within a professional development program, at teachers' schools, in their school district, etc.) have *conducive general conditions* (resources, structures, commitment, etc.).

However, the recent situation of mathematics at schools is in stark contrast to the positive description of the three Co's:

Content: Most students don't find mathematics challenging, it is too far away from them. Mathematical topics seem to be isolated from topics in other subjects; very often links to everyday life or to other applications are missing or are at least rare. Even links within mathematics (e.g. between algebra and geometry) are not obvious to students. Mathematics appears as an unlinked body of facts and rules.

Community: Many teachers work as single fighters. Even within one subject – in our case example mathematics – communication about teaching is not very frequent. Links to colleagues of other disciplines are rare, interdisciplinary projects often run without mathematics or only using it as a complementary tool.

Context: There is hardly any school-wide dialogue about the importance of mathematics or about goals and problems of mathematics teaching. Only in a few cases schools put a special emphasis on mathematics in their school profile. Helpful external support for schools is rare.

Therefore, strategies for a sustainable improvement of mathematics teaching needs to start at different levels: students, mathematics teachers, teachers of other subjects, departments, principals, whole schools, teacher education institutes, networks of practitioners and theoreticians, the educational system, the society.

In the following, two cases within an Austrian reform initiative are sketched where the status of mathematics at a school is critically reflected on, in particular dealing with the relationship between mathematics and science.

TRYING TO PUT MATHEMATICS AND SCIENCE IN THE CENTRE

Following the poor results of Austrian upper secondary school students in the TIMSS achievement test, a research project (IMST1) was set up in which the results were analysed and additional investigations into the situation of mathematics and science teaching in Austria were started (for further details see e.g. Krainer, Dörfler, Jungwirth, Kühnelt, Rauch, & Stern, 2002).

For example, an analysis of web sites at schools in an Austria region showed that schools aim at convincing the public with regard to the quality of their work with a variety of initiatives. However, mathematics and science initiatives were extremely rare, whereas information technology and (predominantly English) language initiatives seem to attract much energy by students, teachers and principals. This concern has been underlined in a workshop with principals at secondary schools who pointed out that mathematics and science teachers in general don't belong to the „powerful groups" of teachers. This has a magnifying impact on many questions, for example, whether a school decides to set a focal point on mathematics and science teaching or to put an emphasis on sports, language, arts etc. In addition, some upper secondary schools reported that increased school autonomy led to a decreasing numbers of students choosing branches with mathematics and science. All these findings were interpreted as an indicator for the decreasing status of mathematics and science teaching at schools on a large scale (apart from some good counter examples due to very engaged single teachers or partly teams of teachers).

As a consequence of the research project, the reform initiative *IMST2 - Innovations in Mathematics, Science and Technology Teaching* (2000-2004, later on followed by IMST3) was launched. It aimed at supporting teachers' efforts in raising the quality of learning and teaching in mathematics and science. IMST2 comprised four priority programs according to the challenges sifted out in the research project.

In three of these priority programs, a university team supported on average about 10 schools each year. In a forth program about 20 further mathematics and science teachers or teacher educators were supported in a less intensive way. Evaluation shows that the project was well received by the participants and led to improvements at several levels (see e.g. Krainer, 2005).

With regard to the topic of this panel, in particular two results are interesting.

a) All schools which put an emphasis on implementing or further developing a mathematics and science branch, established interdisciplinary practical studies and laboratory teaching. However, often mathematics was not really integrated or even included. For example, one school wrote in their report reflecting on their first year in this project: *„Our joint goal was to find contents in the curricula of our subjects in order to work on them in an interdisciplinary way. … It was most difficult to find interconnections between mathematics and the three natural sciences [biology, chemistry and physics]. Very often mathematics was only treated as a complementary science."* This corresponds with an IMST study which showed that science teachers ascribe school development a much higher role than mathematics teachers. What are the reasons for these phenomena? Does the high status of mathematics make teachers (partially) reserved against serving as a complementary science? Do mathematics teachers or mathematicians, more than others, want to stay among themselves? Are there spoors of the great Greek history where only a mathematical élite had entry to the temple? If we want to claim "mathematics for all" we need to open our doors.

b) When mathematics teachers critically reflected their own teaching, shared their experiences with other colleagues and took consequences perceptible to students, most initiatives were highly successful. In a lot of cases, the initiatives had also an impact on other teachers or even the whole school. In contrast to the former example, such a considerable change at an IMST2 school is briefly described (for more details see e.g. Jungwirth, 2002 and 2005; Benke, 2006; Krainer, 2005). Here, two secondary mathematics teachers started investigating their own teaching practices and this lead, for example, to mutual classroom visits in a team of three mathematics teachers. They also shared their experiences with other mathematics and science teachers within the IMST2 project and at their school as well as with members of the school-board. Finally, this had consequences for the whole school: all teachers decided to either use instruments to evaluate their own teaching or to join a peer-group where teachers visit each other in classrooms. Another outcome was the introduction of a new subject putting an emphasis on laboratory teaching in science. Thus here the initiative started from two mathematics teachers and led to a whole school development.

In a recent series of interviews (about two years after the school's participation in IMST2), one of the two mathematics teacher stressed: *"And I regard that as a sensational success, since it began with a small questionnaire … and it ended up with a school development program with 120 teachers enthusiastically involved, and now these two years are gone … We are working now on a school development program 2, where it is considered to retain certain elements like the peer-groups and to support it. In particular, young colleagues regard that as a chance to observe senior ones and to ask for further information."* He also stressed that *"the principal's readiness or positive attitude towards our participation at IMST2"* always was very positive, and that they *"reported continually in conferences"*. For him, this led to the fact that *"the importance of such participations is raised"*, that *"one is not smiled at by the teaching staff"*, and that *"the other teachers might see a benefit for themselves"* (translated from Benke, 2006).

In the second example, all three dimensions *content*, *community* and *context* were highly developed. In order to achieve *more and better mathematics for all students* we need a strong focus on mathematics and students' mathematics learning. However, we also need to take into account social and organisational issues. Only then we will be able to put mathematics in the centre.

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MATHEMATICS IN THE CENTRE: THE CASE OF SPECIFIC DIDACTICS

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Works devoted to characterize mathematics education as a discipline have significantly contributed to defining its tasks, methods and theoretical foundations (see for example: Freudenthal, 1973; Sierpinska & Kilpatrick, 1998; Lerman, Xu & Tsatsaroni, 2002; Boaler, 2002, among many others). But in all these attempts, it has been recognized that the disciplinary boundaries that mathematics education shares with other knowledge areas such as psychology, history, epistemology, semiotics or sociology are not always clear (Rojano, 2004). Even more, in some survey studies this issue has been raised in relation to mathematics itself (see for example Goldin, 2003 and Dörfler, 2003). That is, the place of mathematics in the field of mathematics education cannot always be well determined. This means that in particular, it not always can be said that mathematics is in the centre. Nevertheless, the emergence of specific didactics in the domain of educational research seems to favour a better understanding of this disciplinary limits issue and to keep a focus on the subject area.

SPECIFIC DIDACTICS

One could state that the 80s was the decade in which specific didactics flourished. The leap was taken when in previous decades people realized that general pedagogical recommendations were not very precise at the particular level of each specific curricular subject (for instance mathematics, science or language subjects). And hence we saw the emergence of themes, currents, trends in educational research that shifted the spotlight toward interaction between the subject and the object of knowledge, and in which certain aspects of the latter's very nature were incorporated into the analysis of that interaction. This in turn gave rise to the appearance of very specific educational disciplines, such as mathematics education and science education. Since then the object of knowledge in our field (i.e. mathematics) has been on centre stage.

Locally within the field of mathematics itself, the phenomenon of specific didactics arose as well and one can recently read specialized literature and find sections devoted to the didactics of fractions, the didactics of geometry, of algebra, etc. The common denominator in the research dealing with the specific didactics of mathematics is that the subject matter that provides the specificity must remain in the centre, for example in the case of the didactics of algebra, matters algebraic must remain at the core (the set of papers in Stacey, Chick & Kendall, 2004 illustrates this issue). The foregoing made it possible to delve deeply into issues such as the difficulties faced by students when dealing with particular conceptualizations (such as ratio, number or variable) or those of developing certain skills.

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FROM THE OUTSIDE-IN INFLUENCES

The theoretical currents that conceive of learning as a social process of knowledge construction and that have recently greatly influenced research on mathematics education have also wielded influence on the specific didactics, and aspects such as the context and interaction among subjects have played leading roles in those studies. Nonetheless how geometry or algebra or probabilities, etc., are learnt continues to be a major concern.

Another great influence has come from incorporating usage of technology learning environments into the varying didactics. Research dealing with teaching and learning with technology can find within the framework of the previously mentioned socio-cultural perspectives, "customized" theories, given that issues such as the role of the tool and of student interaction with the tool, with the teacher and with other students are matched by well characterized elements in those theories (see for instance the work of Artigue, 2002 and Noss & Hoyles, 1996). Despite the great emphasis that has been placed on studying such aspects, when developing educational software, the specificity of mathematical content is quite marked. And this is illustrated by developments such as dynamic geometry packages (Balacheff & Kaput, 1996; Mariotti, 2001) and use of graphers and spreadsheets (Sutherland & Rojano, 1993).

Today there is absolutely no doubt as to the how much the afore mentioned influences have broadened and ramified the thematic universe of research on mathematics education in general, over and above the ramifications of specific didactics. And it is within this context that the latter have also taken on new shape. What is most interesting is how the mathematical core in each one has also been transformed. In the particular case of incorporating technology environments, reference is made to undertaking a *computational transposition* (Balacheff, N. & Kaput, J., 1996) that consists of how the (mathematical) knowledge is transformed during the process of implementing educational software due to computational constraints. For instance, in view of those constraints decisions must be made regarding the structure and type of representation for any given knowledge or the types of algorithms that are going to be applied or the type of description of the objects to be manipulated. According to Balacheff, the consequences on knowledge of such a *technological transposition* are as crucial as those of the *didactic transposition*, as formulated by Y. Chevallard (Chevallard 1985) with which we are already familiar.

FINAL REMARK

This brief reference to the evolution of specific didactics in view of the changes in theoretical outlook or the incorporation of new learning tools is an invitation to once again raise the issue of "mathematics in the centre". Yet those endeavours would not be aimed at questioning whether the heart of our educational research continues to be mathematics and its teaching or mathematics and its learning, but rather to delve into precisely how mathematics is transformed or "transposed" during the course of above mentioned evolution.

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RESEARCH FORUMS

RF01 *Teachers researching with university academics*

Coordinators: Novotná, Jarmila & Zack, Vicki & Rosen, Gershon & Lebethe, Agatha & Brown, Laurinda & Breen, Chris

RF02 *Exemplification: The use of examples in teaching and learning mathematics*

Coordinators: Bills, Liz & Mason, John & Watson, Anne & Zaslavsky, Orit

RF03 *The conceptual change approach in the learning and teaching of mathematics: An introduction*

Coordinator: Vosniadou, Stella

RF01: TEACHERS RESEARCHING WITH UNIVERSITY ACADEMICS

Coordinators: Jarmila Novotná, Vicki Zack, Gershon Rosen, Agatha Lebethe, Laurinda Brown, Chris Breen

A PME working group, *Teachers as Researchers*, first met in 1988, and then met annually for nine years. This working group was based on the belief that classroom teachers could and should carry out research concerned with the practice of teaching mathematics. This theme, based on contributions from members of the group, led to the publication of a book (Zack, Mousley & Breen, 1997). What was the role of the university academic in supporting or challenging teacher-researchers in the chapters in this book? Was there an academic acting as leader or facilitator? Do teacher-researchers aim to become independent of their mentors?

What is meant by teacher research? In Anderson & Herr (1999) the following characterisation is given:

> By practitioner research we refer to a broad-based movement among school professionals to legitimate knowledge produced out of their own lived realities as professionals. This includes an ongoing struggle to articulate an epistemology of practice that includes experiences with reflective practice, action research, teacher study groups, and teacher narratives. (Note 1, p. 20)

The role and status of 'knowledge' in teacher research is an object of sharp and vivid debate not only in the field of mathematics education (Metz & Page, 2002). Breen (2003) comments,

> On the one hand, there is a growing movement for more teachers to become involved in a critical exploration of their practice through such methods as critical reflection, action research, and lesson studies. The contrasting position makes the claim that these activities have done little to add to the body of knowledge on mathematics education. (Abstract, p. 253)

Jaworski (2005) believes that one way to add to the body of knowledge is through 'co-learning partnerships',

> The action research movement has demonstrated that practitioners doing research into their own practice […] learn *in* practice through inquiry and reflection. There is a growing body of research which provides evidence that *outsider* researchers, researching the practice of other practitioners in co-learning partnerships, contribute to knowledge *of* and *in* practice within the communities of which they are a part. (p. 2)

Is academic research useful to practising teachers of mathematics or is it generally inappropriate? What happens when teacher-researchers seek to validate their work through studying for a post-graduate degree? What forms of research collaboration between university academics and teachers of mathematics exist? What are their advantages and limitations? The contributors to this research forum will focus our explorations on the theme of teachers researching with university academics through addressing some or all of the following questions:

> *Question 1*: Who are we? What are our connections with teacher-researcher collaboration? How did we start our work in this area? How do we work?
>
> *Question 2*: Why do we engage in teacher-researcher collaboration? What is it for? Who is it for? *e.g.*, developing theory about teaching and learning; personal transformation; making a difference in classrooms?
>
> *Question 3*: Who speaks for whom, to whom and for what purposes (balance of the roles for issues of voice, power, reciprocity and identity)?
>
> *Question 4*: What can we do in such a cooperation that could not be done only by teachers or only by researchers?

We believe that the research done in collaboration between teachers and university academics is a powerful tool for improving both theory and its implementation in practice under the condition that respect is given to the roles of all participants. The diversity of research theories and experiences of the contributors to this forum range through academics approaching teachers and working with them in formal projects, seeking to be as equal as possible; an academic approaching a teacher and working together in a mutually negotiated way; a teacher approaching an academic and working in a way driven by the teacher; an autonomous teacher calling on academics when necessary; a group of teacher-researchers studying for a higher degree becoming independent from the university academic.

What are the lessons that we can take from these various interactions about the way in which academics and teachers can work together collaboratively and mutually successfully and at the same time allow the teacher's voice to flourish? Do some situations provide a greater possibility for this to happen?

SEEING MORE AND DIFFERENTLY – TELLING STORIES: COLLABORATIVE RESEARCH ON MATHEMATICS TEACHING AND LEARNING

Laurinda Brown and Alf Coles

Question 1: We, Laurinda Brown and Alf Coles, met in 1995. At the time, Alf was beginning his career as a mathematics teacher and Laurinda was working in a university education department with student teachers of mathematics on a one-year postgraduate course. We have therefore worked together for just over ten years.

When we met, Laurinda was particularly interested in how new teachers of mathematics develop their teaching styles and strategies and what her role might be in that process. Having taught mathematics herself for fourteen years she saw herself as a teacher and a researcher with no conflict between these roles: "there is only that which I bring to whatever context I am in – I cannot help but bring those perspectives to the range of activities in which I engage" (Brown (with Coles), 1997, p.103). In working with her student teachers to support them in developing a range of teaching styles beyond their initial images of how they were taught, Laurinda began theorizing about what she called ‘purposes’ (Brown and Coles, 2000a). These were not ‘tips for teachers’ (behaviours to implement), nor philosophical positions (beliefs related to mathematics or teaching mathematics). ‘Purposes’ were in a middle position, motivations to act, such as ways of finding out what their students know from which student teachers can develop a range of teaching strategies. Laurinda was looking to find a teacher, new to the profession, to work with who had not done her course and was finding teaching challenging. As we began working together, Laurinda was the researcher and Alf the teacher but rapidly our frames merged. We starting looking in the same direction as co-teachers and co-researchers.

Alf: Reflecting back on my first year of teaching had produced in me a feeling of inadequacy akin to despair – looking back over all that time, looking for the lessons which had been ‘good’ from which to start to build next year they had seemed rare. No lesson really seemed to match up to my ideal image of what seemed possible and there was a strong sense of a gap between where my philosophy lay and the day-to-day practice of what was actually happening in the classroom.

Laurinda: Alf and I discussed the possibility of working together. Alf asked me: what do you want to do? and the only answer was that if the work were to take place the agenda would emerge from conversations. What seemed crucial was that the agenda for the work was Alf’s. My investigation would be subordinate to his agenda. (Brown (with Coles), 1997, p.106)

Laurinda asked whether Alf could bring to mind particular moments or times during a part of parts of lessons that had felt closest to his ideal. This provoked two ‘brief-but-vivid’ (Mason, 1994) anecdotes. Without any prompting from Laurinda, Alf made a connection between the two incidents, saying, energetically: ‘It’s silence, isn’t it? It’s silence.’ Silence was recognized to be a purpose by Laurinda. This was something that we could work with, exploring strategies for using the silence of the teacher within the mathematics classroom. The work that we have done has supported our personal transformation as teachers.

How do we work together? At most once a week, we spend time together in Alf’s classroom. Dependent on our focus we might use videotape for data but mostly Laurinda takes observation notes against the current issue. We stay with the detail of what has happened in discussions after the lesson, ‘What did we notice?’, allowing patterns and differences over time to emerge that become the foci of what we work with – critical incidents noticed by one or other of us. Foci have been, for example, using the questions ‘what's the same’ and ‘what's different?’ as a teaching strategy (Brown and Coles, 2000b). Part of our work is writing together and our first joint paper ‘The Story of Silence’ appeared as a PME paper (Brown and Coles, 1996).

Question 2: These struck us as being really good questions. Why do we engage in this research collaboration? It has always been clear to us that we are personally transformed by the process and changes are apparent over time within Alf’s classroom and in Laurinda’s work with student teachers. As we engage in (often the same) activities, such as researching and teaching in classrooms or viewing videotapes, we are literally aware of seeing more – in the sense of what seems like a finer mesh to look through. As we collaborate with each other and with others interested in the teaching and learning of mathematics, we also see things from more perspectives. We have engaged theoretically; through reading and applying the work of other's (particularly Bateson, 2000, 2002; Varela, 1995, 1999; Maturana, 1994, 2004 - these authors talk about using 'difference' as a natural way of learning (a difference that makes a difference) and with David Reid we ran a discussion group at PME 26 to focus attention on the similarities and differences between these (and other) authors; within an enactivist frame (Varela, 1999); through developing our own theories-in-action.

It is clear that the writing process helps us but why would any of these stories of our developing awarenesses of teaching and learning in one classroom be of interest to anyone else? In 2003, a review was written of the British Society for Research into Learning Mathematic (BSRLM)'s work, through consideration of its day conference proceedings (three each year) over seven years. The author of this review, Marilyn Nickson, commented on our corpus of work presented in that community:

> '… worldwide research projects in the development of teaching in mathematics education tend to encourage models of critically reflective practice leading to the development of communities of enquiry together with critical intelligence in them. This type of research is well illustrated by the work of Coles and Brown […]. [Their] initial paper, relating to their ongoing study, includes a reflection on what it is like for teachers and researchers to work together. […] As well as positive outcomes in terms of classroom learning, the study in its entirety is a very good example of the benefits of collaboration over time between a teacher and a colleague for whom research is part of his or her professional life. The fact that the BSRLM community as a whole gains from it is an added bonus to the profession as a whole.' (Nickson, 2003, pp. 63-4)

So, there is something that the UK community values about the process of us sharing our work over that time and we learn through the process of writing those stories. We have also shared our work through research papers in the PME community (1996, 1999, 2000) and in the work of the PME Teachers as Researcher Working group (in Zack, Mousley & Breen, 1997). In research writing more generally, our use of story, "the pattern which connects" (Bateson, 2002, p. 7) tries to convey that "little knot or complex of that species of connectedness which we call relevance" (Bateson, 2002, p. 12). In the process of writing, we make more connections and tell more (different) stories often about the same sequence of data – this again allows us to see more.

From Alf's perspective early in our collaboration:

> One discipline that has also come out of the work with Laurinda is that of staying with the story. In my notes on teaching in the first year - the observations are in general distant - about whole classes - with observation and analysis all mixed in. What I have been working on this year is forcing myself to hold back the analysis and stay just with stories about individuals or groups of individuals. Analysis (or synthesis) from this data then has the possibility of throwing up something I had not been aware of before…previously

… mixing …analysis and observations …meant that I was never surprised … There was little chance of my accessing those things I did that I was unaware of - but which yet had profound effects. (Diary, Alf, 10/95)

Question 3: In working with the silence of the teacher, Alf started exploring the effects of offering images to students, in silence, inviting their silent responses or asking them to describe what they see. Alf is not the one to whom students listen for explanations but Alf becomes the listening teacher. The children are working on the mathematics and it is their work on the mathematics that Alf is interested in learning about. This situation is parallel to Laurinda being interested in what Alf is working on, whilst Alf concentrates on his learning about teaching. So, the balance of the roles means that we are always both learning, sometimes about different things. Alf has explored issues of silence, listening and hearing through an MEd in Mathematics Education, presenting his dissertation at PME25 (Coles, 2001).

Question 4: There is a reflexivity built in to this co-operation. Laurinda continues to learn about life as a teacher in a classroom and develop as a teacher, which is important to her role as a teacher-educator. Alf learns about being a researcher and has been a named participant in a successful bid for research funding, completed his Master’s degree and built his practice through researching in action. We provide mirrors for each other as co-researchers, sometimes co-teachers that allow us to reflect deeply about the teaching and learning of mathematics, specifically the development of mathematical classroom cultures. Neither a researcher with their own agenda nor a teacher perhaps inarticulate about their practice would be in the position to add the component of collaborative writing – learning through outer speech and responding to each other’s questioning – that has allowed the weaving of stories in acts of meaning. What we seem to be dealing with over the years is the cultivation of awarenesses of awarenesses, learning about learning, where the other provokes another meta-layer of awareness in ourselves as we work to provoke second level awarenesses for our various students. And mathematics is the vehicle in which we both work – doing the mathematics together.

JOINT REFLECTION AS A WAY TO COOPERATION BETWEEN RESEARCHERS AND TEACHERS

Alena Hošpesová, Jana Macháčková and Marie Tichá

Question 1: We started to give attention to reflection through studying the basic competences of the teacher (Hošpesová & Tichá, 2004). Many authors (e.g., Jaworski, 2003; Schön, 1983) cite a competence of qualified pedagogical reflection (the teacher’s analysis of their own thinking and ways of dealing with students suitable for planning their own lifelong education) and consider it to be a determining feature of each teacher’s professionalism. Other authors (e.g., Svec, 1996; Steinbring, 2002) assume that reflection creates space for the transition from intuitive to conscious and justified action. We agree with Czech educationist Slavik that: “It is possible to treat reflection connected with interpretation of teaching/ learning situations as the best way to develop the teachers' professional way of thinking and to present practical didactical theory". (2004, our translation)

On the intuitive level, reflection is present in all human activities and thus in teaching, too. However, if we want to speak about a qualified pedagogical reflection (which includes observation, contemplation and consideration) then we also take into account description and analysis of key elements, evaluation or revaluation, ways of explanation, accepting decisions and determining a new strategy (Slavik & Sinor, 1993). We must consider conscious reflection on our own teaching from the point of view of goals and content of the teaching, and methods of work and their realisation. Knowledge of content is assumed as a given.

We understand reflection not only as a retrospective act but also as part of the whole process of teaching, penetrating preparation, realisation and evaluation. "Joint reflection" seems to be a contradiction. Reflection is often seen as something personal or individual. But if we observe a teaching episode within a group of other people expressing their views freely, our reflection is influenced and changed (cf, Cobb et al., 1997). Joint reflection in a group of teachers and researchers influences positively the improvement of teacher's competences (Hošpesová & Tichá, 2004) and can be seen as a form of cooperation between teachers and researchers.

Our cooperation with a group of primary school teachers (all fully engaged only in teaching) started during a four-year project within the international Comenius project "Understanding of mathematics classroom culture in different countries". The general aim of the project was to contribute to the search for ways to improve the quality of continuous in-service education of primary school teachers and so to support the development of teachers' competence (Hospesova & Ticha, 2003). We initially assumed that we would examine the different approaches coming from different countries. The co-operation within the project itself led to the amendment of our initial intentions, and we focused our attention more deeply towards the preparation of teacher training courses promoting qualitative changes in classroom culture; the development of a more sensitive approach by teachers to pupils' ways of thinking; the ability to use this in lessons; and an awareness of situations that could be valuable from the point of view of the pupils' learning processes. We started to aim at the cultivation of teachers' competences through self-reflection and joint reflection (Scherer & Steinbring, 2003).

How do we work? What are the forms of efficient cooperation in our case? The key feature of cooperation was the equal status of all members of the team in all areas of work, i.e., when preparing, carrying out and analysing instructional experiments. Usually a more active role for researchers and a more or less passive role for teachers is expected. Researchers are supposed to determine and evaluate teachers' work, the teachers are in the role of people putting into practice the ideas of someone else. We gradually persuaded the teachers that we all have the same level of responsibility, although our roles and interests are different. During our cooperation we prepared several teaching experiments realized by the teachers. The cooperation gradually established itself in the following form:

- At the beginning of our work on the mathematical topic of the teaching experiment, we discussed (and when necessary the researchers summarised for the teachers) its mathematical background and its possible didactic elaboration.

- After discussions amongst the whole team, the teachers independently prepared experimental lessons for their classes to be part of the usual school teaching.

- The experimental lessons were video-taped (25 recordings in all) by the researchers. The teaching was as close to “ordinary” as possible.

- The teacher who taught the lesson chose, according to her opinion, the most interesting teaching episodes, usually discussing the selection with a researcher. The members of the team then reflected on the video-recordings individually (including the performer or observer of the action). This meant that each member of the team had at his/her disposal video-recordings of a chosen episode or episodes to analyse and assess, aiming to be prepared for subsequent joint reflection.

- Chosen episodes became the core of joint reflection in the meeting of the whole team. These discussions were usually audio- or video-taped so that it was possible to study the level of reflections of all participants. The level of reflections developed over time, growing in quality. We perceived several mutually-connected levels; a simple dialogue with conversations aimed at intuitively-understood observations such as “I liked/disliked this” in which teachers generally spoke about their feelings; looking for effective methods of teaching for specific mathematical content which aimed to improve teaching; a deep analysis of teaching from the point of view of goals, methods and content, which led in turn to the preparation and realisation of the teacher’s own instructional experiments.

It is obvious, that all teachers in our group did not reach the last level. For some teachers (regardless of their age), it is very difficult to take part in discussions and to express their opinion. Apparently, they need more time to think the situation over and, say, study literature. Their low self-evaluation of their teaching and uncertainty in their own mathematical understanding may be impeding their progress.

Question 4: The teachers from early on in the relationship realised that joint reflection facilitates their personal motivation.

> Diana: For me, the self-reflection and help of other colleagues are important. In some situations I would not be able to change by myself even if I wanted to.

> Betty: The opportunity to communicate about problems in teaching is a huge ‘driving engine’ for me.

The teachers also gradually grasped the need to videotape the lessons, because it enabled them to balance their involvedness in education and acquire the critical distance, *i.e.*, to follow (their own) education from “outside” and to fall into the role of reflective teacher and teacher-researcher.

> Ann: The video recordings, which are authentic, are excellent and allow me to observe my work from a different standpoint, from the position of an observer of the efficiency and quality of my teaching – verbalization, correctness and

accuracy of formulation of the tasks, and the quality of my communication with pupils.

After the third year of cooperation, the teachers themselves formulated the benefits from the work on the project as follows:

> The work on the project brings me a lot of new things. Some topics (of school mathematics) I do not understand quite precisely and the discussion of the background and didactic elaboration, *i.e.*, about theory and practice, helps me a lot. ... My responsibility has grown. I think more about what I do in education, what the children do, whether all the children understand...Even in the project I realized the importance of the personality of the teacher.

We perceived the development of more sensitive teachers' approaches to pupils' ways of thinking and of the ability to use knowledge in this area in teaching. Teachers became more interested in the pupils' understanding. *i.e.*, what does it mean to understand certain school mathematics topic?

> Cecily: Thanks to the project, I have an opportunity to see the teaching of mathematics more deeply. As soon as I realised what we had learned thanks to the joint reflection and self-reflection, how our approach towards the teaching of mathematics had changed, I began to realise that the changes did not affect only my mathematics lessons. I started to ponder on the idea that the method of self-reflection and joint reflection can be useful in other lessons, too. I asked myself the question: If self-reflection and joint reflection lead to the improvement of mathematics lessons in terms of both mathematical and didactic aspects and force a teacher to work on him/herself and educate him/herself, why could it not work in the Czech language or in geography or in any other lesson?

Joint reflection brought a shift (an improvement) in the researchers' knowledge in various ways. It deepened their understanding of various aspects of mathematics education, opening ways to recognise causes of failure for some teachers; why problems appear in teaching and how to remove them; understanding of processes going on during mathematics teaching in the social context of the classroom and helping to show the possibilities of using this knowledge for person- centred education and the strengthening of the constructivist approach.

Analysis of reflection could be used as a diagnostic instrument by the researchers, allowing them to recognise shifts and improvements in teachers' professional competence, opinions and approaches on the basis of external representations (Duval, 1995), seeing which aspects (subject, didactic and pedagogical) should be emphasised in the education of future teachers and practising teachers. During the meetings and joint reflection of chosen episodes from teaching mathematics lessons, the researchers had an opportunity of influencing practising teachers indirectly, informing them about actual research results in mathematics education.

Pupils indirectly profit from the fact that their teachers pay them more attention and discover both misunderstandings and problems, and abilities that are not necessarily apparent in everyday teaching. (We have not done the analysis of the video recordings with pupils so far.)

We consider that joint reflection can bring about an improvement in the mathematics classroom culture, *e.g.*, studying key phenomena for assessment of processes going on during mathematics lessons; following changes in approaches and behaviour of participating teachers; studying the impact on students' knowledge. We need to do a longitudinal study to study these areas.

Acknowledgement This research is supported by the grant GACR 406/05/2444, and Institutional research plan AV0Z10190503.

OPENING THE SPACE OF POSSIBILITIES: TALES FROM THREE TEACHERS

Agatha Lebethe, Neil Eddy and Kendal Bennie

We are three teachers who were brought together because we shared similar strong feelings about our work as teachers. All three of us were drawn to an advertised Masters programme in Teaching which foregrounded the practice of teachers (see Breen, 2002). All three of us decided that we wanted to take the themes of a particular module, which introduced us to the ideas of complexity science and enactivism (Maturana and Varela, 1986; Davis, 1996) as well as those of the Discipline of Noticing (Mason, 2002), into our research dissertation. This proved to be an extremely complex experience as our first hurdle was to convince the traditionally-minded academics who were presenting a generic Research Methods course that our planned research was legitimate and acceptable. Despite the difficulties that each of us experienced along the way, we were pleased when each of us received recognition from the academy that our work was worth the award of distinction.

Our presence as part of this Research Forum stems from our growing belief that our experience is generalisable. Not enough is being done to ensure that the authentic and embodied voices of teachers are being heard and respected to a sufficient degree in an undomesticated form.

Agatha: I spent 10 years working as a mathematics in-service field worker at the University of Cape Town. I took a very long time to begin my Masters Studies. I was waiting for a Masters programme that would respect and acknowledge my Practice and would allow for the voice of the teacher, the teacher educator and the researcher to be heard. My earlier experience with the way most Academics supported teachers in Research made me determined to engage in a form of research that would allow me to open up and reflect upon my experiences as a teacher educator; tell of my discomfort, my turmoil, and confusion and also explore the complexity in trying to make sense of my teaching. By weaving my voice into the research I wanted to understand my practice and describe my journey through the research as I unravelled the layers of habitual actions of my teaching. This meant that I could not disembody my multiple voices from the text and this necessitated writing in the "I". I needed to interact with the text, with the research as a living medium and therefore my intention was not to seek answers but rather to understand the journey through research and to appreciate my interaction with my environment and so embrace possibilities.

I wanted the research to be a text that was honest and authentic. However, crafting such a text meant challenging the existing genres of dissertations at my University. One of the ways I did this was by insisting that my existence in the research process and in the dissertation was that of the hermeneutic inquirer. As an hermeneutic inquirer I was not seeking a truth and I was not a detached observer in the inquiry and therefore the research was grounded in interpretation. This was evident in the methods I employed for both the data collection and analysis.

I believe that there is a space for teacher researchers like myself to make the process of inquiry systematic and public. We need to uncover our own beliefs, assumptions and our biases. We need to make explicit our own theorizing both in our everyday practice and in formal research. I would like to see us examining how these theories have influenced the nature and structure of our work.

I would also like to see University postgraduate courses for teachers place less of a heavy emphasis on being pre-packaged, pre-determined and linear. Teachers experience needs to be validated and this can be done by researching how we live within worlds (our teaching world) of interpreted meaning.

For van Manen (2001):

> … to do research is always to question the way we experience the world, to want to know the world in which we live as human beings. And since to know the world is profoundly to be in the world in a certain way, the act of researching – questioning– theorising is the intentional act of attaching ourself to the world, to become more fully a part of it, or better, to become the world. (2001, p. 5)

Neil: I am a teacher. Initially I flourished in the world of the academy – the world of immutable truth obtained from data analysed objectively. I then began teaching in my classroom where the faces in the desks staring at me were neither numbers nor objects.

It was my practice here that forced me back to the academy, with the hope of finding a means to understand my practice, from within my practice. I was not comfortable with research that required me to isolate – I needed a method that embraced wholeness. I heard of a Masters course offering a different philosophy. In the first year, I became aware of people using methodologies that promoted the improvement of one’s practice and privileged teacher experience.

In the second year I endured a compulsory course on research methodology. Suddenly the exclusionary walls of the academy, so effectively deconstructed, were now bulldozed back into place – twice as oppressive now that I had a view beyond. “Choose a small question, for which data is easily obtained, write it up and submit”. “Get the cloth on your back, prove that you can research, then you can start asking the questions that truly matter to you.” I put together a proposal, but somewhere lost the track and disappeared from the Academy. I never did get my proposal passed, but, after a while, my practice forced me back to continue the research.

I chose to expose my research through the metaphor of a science fiction story by Clarke (1956) and the story's hero, Alvin. In the story, the city of Diaspar reproduces itself in an unchanging cycle. The comfort of the citizens is ensured by removing the pain that is associated with change.

Through this story I was able to tell my story of change from a deeply scientific and disembodied paradigm to one more deeply rooted in the "I". It led through emerging deep ecology works onto a set of statistical techniques that I used to open questions.

In a parallel story, I am concerned with the voice of the teacher and the lack of acceptance of this voice. I am convinced that the teacher has a story to tell – one rich in knowledge and a truth – but we lack a language with which to converse. The university academic has a language, a voice, a firm grasp of the status quo – and therefore owns the truth. The researcher can, in the status of "doctor", diagnose problems in practitioners, operate on them to correct these, and extract the truth, thus building their power. I feel the teacher is often left with the numbed feeling of the anaesthetic and a distrust of the academy of which he or she is an integral, but voiceless part.

My supervisor allowed me the space to develop a voice. Very little came from him on how things should be done and this, although difficult, allowed me to sit with the data and find my way, while he kept the academy away. I ended up not only with a dissertation, but also with a more mindful practice.

Kendal: My aim was not to contribute to a body of knowledge outside of myself but rather to contribute to my own (en)active knowledge, to improve my teaching practice on a daily basis and through this have a greater impact on those that come into contact with me, especially learners. It started and continues with me wanting to be(come) a better teacher. It has to do with a belief about the need for personal responsibility. If each of us could make a commitment to improving ourselves, the change in the world would be phenomenal.

> The universe changes when something as miniscule as a thought changes - because that thought is not merely in the universe, it is part of the universe. (Davis, 1996, p.14).

The nature of this research meant that I was not starting with a question that I would answer. The questions were to emerge from the research (process) and the answers were unlikely to be simple solutions but more likely to create more questions. It was to be a step out of a universe of binary questions and answers and into a multiverse of awareness of the uncertainty with which we live. I would not be following the well-worn path of illusory objectivity but rather riding in a subjective ocean where the ground beneath me was not solid and the path in front nonexistent.

Education is "about sensitivity to and transformation in others. The only certain place to stand is in the most unlikely place: ourselves." (Mason, 1994, p. 5). And yet when I did find the sea, a part of me cried out for the security of land and a path to follow.

I proposed to research and write about my learning while using the analogy of surfing to distance myself and help me understand and analyse my experiences. My research proposal presenting this approach was rejected by the academy because my research method came before the questions I was to answer. Views I invoked were called heretical and in the minority.

While most research has a frame that researchers work towards, the vagueness of my frame (plan) was the most exciting part. My frame (or lack thereof) didn't formalise proceedings, it opened space for (r)evolution. To learn required me being aware, (ob)serving, listening and (re)visiting writing.

Capra (1991, p. 51) explains how Eastern religions use mythology, metaphor and paradoxes to explain reality better than language, in its linear fashion, is able to.

All through the writing I found myself tempering a flair knowing the sharks didn't like me being in the water. Fortunately I was not scared out of the water and the rewards are still reverberating.

DIVERSE ROLES, SHARED RESPONSIBILITY

Jarmila Novotná and Alena Pelantová

The scientific aim of the cooperation of teachers and university academics is to accomplish the research necessary for advancement of knowledge of the mathematics education phenomena. In this contribution, a model involving a limited number of staff in school, university researchers and teacher trainers is presented. The focus is on the different types of participants' involvement and responsibilities as well as on the scientific and practical results of such cooperation.

Our cooperation developed from being significantly unbalanced with most responsibility put in the hands of the university academic, towards real cooperation with a clear division of responsibilities. As the basis for the characterization of participants' involvement and responsibilities, the organization of the COREM (le Centre d'Observation et de Recherche sur l'Enseignement des Mathématiques) school (Salin & Greslard-Nédélec, 1999; Novotná, Lebethe, Rosen & Zack, 2003) is used. The benefits and limits of cooperation as well as the differences and similarities with COREM as a representative of a whole institution working on the basis of cooperation form the framework for our contribution.

Questions 1, 2 and 3: Alena is qualified for teaching mathematics and geography to pupils aged 11-15. At present she is the head of a school in Prague. Jarmila is a University teacher training future teachers of mathematics. She is involved in research in the domain of mathematics education cooperating intensively with researchers from abroad.

We met for the first time in 1992 when Jarmila was the coordinator of the project *Integration* between basic school and general upper school. The project represented something new at that time, breaking the uniformity of the educational system. It represented a challenge for the participating teachers, an attempt to improve the organisation of education offering more responsibility and professional freedom to the school and its teachers. Alena was the teacher of mathematics and geography in the school and actively participated in the project. In the first period of our cooperation, proposals of what to do and plans were elaborated by researchers from the Faculty of Education of Charles University. Teachers implemented these ideas.

The cooperation with Alena continued after the end of the project and after she became head. Our roles and responsibilities in the cooperation have gradually changed. Our roles are different but our present position in the cooperation is balanced with clear division of responsibilities.

We will illustrate this division of our roles and contributions by one episode from our cooperative research. We can identify three different roles of Alena and Jarmila: Alena in the role of a teacher (we will label it as Alena-teacher) and a researcher (Alena-researcher). Jarmila acts here as a researcher only (including the role of an observer) - labelled simply Jarmila.

We dealt with solving word problems, dealing with the division of a whole into unequal parts with 12-year old students. The long-term practical experience of Alena-teacher confirmed by Jarmila's (and not only hers) investigations and research (*e*.g., Novotná, 2003) and their discussions with other teachers signalled the didactical demands of the topic for students before and after being taught school algebra.

We decided to focus in our cooperation on this type of word problems at pre-algebraic level. Our experience confirms that word problems dealing with the division of a whole into unequal parts belong to those school mathematics domains where we can clearly see that the arithmetical and algebraic processing of the problems impose different solving strategies (Bednarz & Janvier, 1994). In school mathematics, algebra is often presented to students as a new and more efficient tool for solving problems. But students at the elementary (pre-algebraic) level have already experiences with arithmetic solving strategies and they can profit from them when starting to use algebraic procedures.

Jarmila proposed the framework of the theory of didactical situations in mathematics (Brousseau, 1997) in which we started to look for teaching strategies that could help students to overcome the difficulties that they face when solving this type of word problem. Jarmila and Alena-researcher performed the *a priori* analysis of the type of word problems including possible solving strategies (correct and incorrect) including the level of mathematical thinking required, necessary knowledge and possible obstacles. In this period, Alena-teacher was not too active; her role of researcher was much stronger. Nevertheless, her experience as a teacher was irreplaceable. She helped to keep a "realistic" platform in our plans and proposals. Based on this analysis, Jarmila, Alena-researcher and Alena-teacher prepared the didactical unit to be realized in the classroom (Pelantová & Novotná, 2004). The didactical unit was designed as a sequence of adidactical situations of action, formulation and validation and the following institutionalization. The lesson was taught by Alena-teacher and video-recorded and observed by Jarmila. We used the same division of roles for the lesson as was used in COREM – Jarmila did not intervene, the whole responsibility and all decisions during the lesson were left to Alena-teacher. The next step, the *a* *posteriori* analysis of the realized didactical unit was done by Jarmila and Alena-researcher with Alena-teacher's explanations and other ideas mainly focusing on the reasons for her decisions to modify the prepared course of the lesson.

All "three" of us, Jarmila, Alena-researcher and Alena-teacher prepared the test which was assigned to students three months later, the aim of which was to see the stability of knowledge built in the didactical unit. The results were surprisingly good. Even children who behaved rather passively during the adidactical situations showed good command of applying the acquired knowledge.

The final part of the described activity, done mainly by Jarmila, was the integration of the results into the broader research framework. Jarmila and Alena-researcher disseminated the results at various scientific events.

Question 4: In Novotná, Lebethe, Rosen & Zack (2003), the following questions are posed: "Does the teacher need the direct presence of a researcher during his/her teaching?" Answer, "No"; "Does the researcher in education need the direct cooperation with one or more teachers?" Answer, "Yes". The reasons for the answers are given there, together with the benefits of such cooperation for a teacher.

For Alena (Alena-teacher), the cooperation offers access to theoretical frameworks and research that she would not acquire without our cooperation. The career of Alena-researcher was born in our cooperation. Alena's position is that described by Brousseau (2002):

> When I am acting as a researcher, the interpretation of each step of teaching begins with a systematic questioning of everything, a complex work of a priori analyses, of comparisons of various aspects of the contingencies, of observations first envisaged and then rejected, etc. … When I am a teacher, I have to take a number of instantaneous decisions in every moment based on the real information received in the same moment. I can use only very few of the subtle conclusions of my work as researcher and I have to fight with starting to pose myself questions which are not compatible with the time that I have, and that finally have the chance to be inappropriate for the given moment. I react with my experience, with my knowledge of my pupils, with my knowledge of a teacher of mathematics which I am treating. All these things are not to be known by the researcher.

At the theoretical level, Jarmila's research questions can be treated independently from the school reality. However, to find answers to her research questions she needs the direct contact with Alena and the access to the teaching reality. Thus, she can avoid the danger of producing superficial answers to research questions, results not having "real roots" and with a doubtful applicability in the school reality.

Our cooperation influences significantly not only the teaching and learning of mathematics in the school but several other domains of the life of the school. The following list is not exhaustive but illustrates this impact:

- involvement in international cooperation (Socrates Comenius 1) began with advice and contacts gained from Jarmila's institution;

- other possibilities of learning about other good practice in teaching various subjects and the space for presenting good practice of Alena's school during visits of foreign colleagues from institutions training future teachers and teacher students;

- access to student teachers, facilitating the recruitment of young qualified teachers for the school with new, fresh ideas about the teaching/learning process;

- involvement of the school in various surveys (not only about teaching mathematics) with the outputs enlarging the horizons of staff knowledge about new trends in education;

- access to information about conferences, seminars and summer schools focusing on education and the possibility for active or passive participation at these events. The result is not only the increasingly good reputation of the school but also the staff's increasing knowledge of new educational trends.

And, finally and importantly, the key to successful cooperation is the harmonious cooperation of all partners involved, which is true for us, Jarmila and Alena.

RESEARCH WITH TEACHERS: THE MODEL OF COLLABORATIVE RESEARCH: STUDY OF JOINT DEVELOPMENT MECHANISMS FOR AN APPROACH TO THE TEACHING OF MATHEMATICS TO INUIT CHILDREN IN KINDERGARTEN AND PRIMARY GRADES 1 AND 2

Louise Poirier

In the spring of 2000, the Inuit community and the Kativik School Board were pondering over the difficulties encountered by students in mathematics and the measures that could be taken to help them. One significant fact that could help explain these difficulties is that Inuit students learn Inuit mathematics in their own language in the first three years of their schooling and then go on to study in either French or English. Having heard of the work I was doing at that time in Montreal with immigrant children learning mathematics in French as a second language, members of the Kativik School Board of Nunavik (Northern part of the province of Québec) asked for my help.

In the Fall of 2000, I visited several Inuit villages in order to observe classrooms, to meet teachers and their students. Those visits prompted several remarks:

Mathematics and language: The Inuit children start school (kindergarten, first and second grade) in Inuktitut. The first concepts they learn in math are learned in Inuktitut. Then, at that time in third grade (the situation has changed this year) they would switch either to French or to English and they would pursue their learning of math in that second language.

Mathematics and culture: Until recently, mathematics was seen as a universal language but this view is now questioned. Inuit children learn mathematics in Inuktitut but they also learn Inuit mathematics. And Inuit mathematics is quite different from the "southern maths", the mathematics taught in Montreal, which they will learn in third grade and up in the second language. For example, when children learn how to count, they are using a base-20 system and not the base-10 system they will use when they change to French or English. It would seem that for these students two separate and distinct universes are cohabiting: the world of day-to-day life and the “southern” mathematical world. Furthermore, the first world, the world of every day life, has nothing to do with the second one, the one of mathematics done in school. Mathematics is not perceived as helpful in day-to-day life.

Spatial capacities: The students I met are very good at spatial representation and geometry. Unfortunately, the present curriculum does not put enough emphasis on these strengths.

Mathematics and teaching methods: Teaching methods used by some teachers up North (paper/pencil exercises) are not “natural” methods of learning for these Inuit children. Traditional teaching and learning are done through observation and listening to stories or enigmas.

Faced with this dual phenomenon of first learning mathematics in Inuktitut and then in French or English, the instructional situation becomes highly complex: how can these two cultures be combined and accommodated in mathematics teaching situations? The main purpose of our project is to study the joint development process of mathematics teaching situations adapted to Inuit classrooms.

For this project, we have two theoretical frameworks: the studies done in ethnomathematics to help us better understand the impact culture has on the learning of mathematics and collaborative research that guides us in our work with teachers (Bednarz, Poirier, Desgagné et Couture, 2001; Desgagné, Bednarz, Couture, Poirier et Lebuis, 2001).

The social dimension of mathematics has grown in importance in the teaching of mathematics (*e.g.,* Lakatos, 1976; Ernest, 1991). If mathematical knowledge is a social construction, the community and the culture of the learners will play an important role in their learning. According to Bishop (1988), we are more and more concerned by what he calls the cultural interfaces in the teaching of mathematics:

> In other countries, like Papua New Guinea, there is criticism of the ‘colonial’ or ‘Western’ educational experience, and a desire to create … an education which is in tune with the ‘home’ culture of the society. The same concern emerges in other debates about … Lapps and of Eskimos. In all of these cases, a culture-conflict situation is recognized and curriculum are being re-examined. (Bishop, 1988, p. 179)

The Inuit community of Québec is no exception. If we want to re-examine the Inuit curriculum and develop learning activities adapted to the Inuit culture, the researcher who is not a member of that community can not do that alone. The risk of developing activities that will not be suitable, or well-adapted, is too great.

When a researcher develops such teaching situations the question of the validity of those situations for the school rises (Artigue, 1990; Arsac, Balacheff and Mante, 1992; Desgagné, Bednarz, Couture, Poirier et Lebuis, 2001). The teachers will use these situations according to their environment and their conceptions about teaching and learning. This process can have an impact on the learning situations and the researchers sometimes do not recognize the situations they have created. On the other hand, teachers sometimes have great difficulties reproducing what the researchers have put on paper: the environment and the context are not the same. How can we bridge the difference between these two worlds?

The development of learning situations, in our view, has to go through the understanding that the teacher has of the environment and his conception of teaching and learning. This seemed particularly important in the context of teaching mathematics in the Great North to Inuit children. It was then necessary to integrate people of the Inuit community in the development of the learning situations. Our team included 4 Inuit teachers, 3 Inuit teacher trainers and myself. This group helped us get the triple point of view that we felt necessary: Inuit culture, the teachers' experiential knowledge and didactics.

The collaborative research framework seemed an interesting path to follow since it implies that the teacher's actions and the rational behind these actions are part of the data of the research). The aim is not only to develop interesting didactical situations that will help students acquire certain knowledge (what didactical analysis would help) but those situations must be viable in context, in the classroom for which they are meant. This can only be achieved, in my point of view, with the help of the teachers' experience and in this particular project, with their knowledge of the Inuk culture. Cooperation between the researcher and teachers in creating adapted teaching situations is given concrete expression in reflective practice (Schön, 1987). It involves a planned alternation of situation development, classroom experimentation, and feedback. This planned alternation looks like this: Team meeting to elaborate learning situations – experiment with these situations in the classrooms – discussion of those experiments and development of new situations – experiment with these new situations and so on.

In order to start the discussion, we used Bishop's framework. Thus Bishop (1988) recognizes that mathematics is a cultural product and as such has been developed in several different ways depending on the culture. However, he has recognized 6 domains that are present in the different cultures. These domains seam to be necessary to the development of mathematical knowledge (number, localization, measurement, design, games and explanation). It is interesting to note that these domains constitute the mathematical content of primary school. One way, according to Bishop, to diminish the gap between the phenomena of enculturation and acculturation would be to develop a bi-cultural strategy:

> One possible way is to use as a structural framework the six activites… If these activities are universal and if they are both necessary and sufficient for mathematical development, then a curriculum which is structured around those activities would allow the mathematical ideas from different cultural groups to be introduced sensibly. It is indeed possible by this means to create a culturally-fair maths curriculum. (Bishop, 1988, p. 189)

During our meetings we discuss one or several of these domains: how were they dealt with in the Inuk tradition? How are they taught in the Kativik curriculum? How are the teachers teaching them to the students? These discussions give us the opportunity to refresh the teachers' memory about these mathematical concepts. Some of these teachers are not trained. Experimentation and analysis thus take place in two phases: analysis of the meetings between the teachers and the researcher, and analysis of classroom experimentation. This project is about half way through. But already we can see some benefits of that type of research but also some drawbacks.

In September 2002, after having read the project, Betsy Annahatak who is in charge of the Curriculum Development Department of the Kativik School Board wrote this about the collaborative research:

> The collaborative research project that you described have some important elements that will help Kativik on the process of developing a Math Curriculum for Kativik School Board. Although the subject being addressed is on Math, as a curriculum developer I am very interested in this research because I expect to see elements and factors extracted form this research that will help us structure other programs and help us develop a culturally responsive curriculum for Kativik School Board. This research proposal is also a unique project in the history of KSB research specifically addressing curriculum questions in a minority, bicultural, and bilingual situation. As described in your paper, the dual phenomena with two cultures in contact in a learning environment, and in a school setting using the subject of math, is like an unexplored expedition to a foreign area of the universe of learning. (Betsy Annahatak, Curriculum development department, Kativik School Board, September, 2002).

The Inuit People, one of the most studied in the world, unfortunately do not get much back from research. Collaborative research, involving people from the community, gives a better assurance that it will bring something back to the community. Being part of a research group such as this one is a great learning experience for all participants. But it raises several challenges that will be discussed further in our presentation: difficulties of bringing people from different cultures together (teaching culture and research culture; Inuk culture and Quallanat culture), the language issue (several mathematical words and expressions simply do not exist in Inuktitut and when they do, some may induce erroneous conceptions, for example, a rhombus in Inuktitut is said as the " square from the playing card"), the teacher training…

DEVELOPING A VOICE

Gershon Rosen

Question 1: I am a full time teacher in a secondary comprehensive school in Israel, teaching mathematics as well as other subjects. Being a full time teacher, I see my role in this RF as representing the practitioner in the school situation, trying to make our voices heard. I am in the privileged position of being on the front line on a day-to-day basis, coping with all the frustrations as well as enjoying the highlights of educating our youth, not just teaching them mathematics. I am also a link between the practitioner and the researcher as I am in regular contact with those in Israel who research mathematics education and produce texts and materials for the classroom. My role for many years has been that of disseminating, interpreting and adapting current research for the teachers through workshops, in-service courses and working with the teacher in the classroom as well as bringing back to the researcher the comments of the teachers and adaptations made in order to “make things work”.

Results from my own research as a practitioner have been published mainly, and on a regular basis, in professional journals in the UK and in Israel. When I started teaching mathematics all that was required was a Bachelors Degree in Mathematics. No teaching certificate was needed. My first teaching post? - an all boys comprehensive in inner city London. I was the new boy who was given the classes that no one in the department wanted to teach, mostly non-English speakers, in a lecture theatre - the boys sitting up there looking down on me. This was my first realization that not everyone should be taught maths in the “traditional way”. Without being aware of it I had already started to research my practice. Forty years of accumulating practical knowledge: what works in what situations; when to give pupils an answer to their questions; when to help them discover the answer for themselves; when is a pupil ready for a mathematical proof? and when to leave a proof to a later date in order not to interrupt the thought process driven by intuition. This kind of research is not driven by a specific question but varies from lesson to lesson, class to class and from year to year and is influenced by so many external factors. There is no possibility of a clinical or quasi-clinical investigation, and in any case, such a “laboratory” investigation has very little relevance to what is going on in the classroom. The priorities are different.

Over the years I learned to try out different approaches, adapt them, re-write them. As I look back and forward, I see that what I have done in my own classroom and in my work with other teachers has emerged from questioning established wisdom, in both curriculum practices and research practices. I found conventional ways of doing mathematics as prescribed in official textbooks were not working for me in my classroom, and I was driven by the need to search for new ways. Thus I have looked closely at what I am doing while teaching and learning, studying it, seeing what works and what doesn’t and trying to find out ‘why.’ I have felt the need to share with other teachers, especially with those who are living with challenging school situations in order to share with them what has worked for me, and to help them explore their own ways of doing mathematics both for themselves and with the children. In my work with them, I encourage them to build upon their own life experience as a learner as a model for ways into mathematics. I will illustrate with the following vignette:

> I was recently asked by my colleagues, a group of experienced and successful teachers but very traditional in their approach, to give a workshop on teaching probability. They implied that they had never learnt it for their degree and were hesitant of teaching the topic. One member of staff said that she had once solved a very simple question with her class. It had to do with drawing two coloured balls from a bag with replacement. She was not sure that she handled it correctly. I encouraged her to describe how she proceeded

> and we would work from there. What was of particular interest to me was that, for example when teaching geometry, this teacher endeavoured to ensure that the students could quote definitions of the various geometrical figures and could set out formal proofs even if it meant rote-learning. When it came to solving the probability problem she drew pictures and tree diagrams and said forcibly: "I don't use any formal words. I just draw pictures and use elementary procedures like counting. I know that there exist formulae but I don't know which ones to use - I need to see the full picture - That's how I understand it and that's how my students will understand it."

There are two main elements in this teacher's response which are key features of my research and which have guided me in my work with my students in my classroom, as well as in my work with other teachers. These elements are: "I need to see the full picture" and "I don't use any formal words. I just draw pictures and use elementary procedures like counting. I know that there exist formulae but I don't know which ones to use." The idea of "using your own words" is a crucial one: keep close to your own way of doing. Seeing the full picture is another vital idea. I take issue, as I will state below, with research that breaks things down into small entities, with the result that the whole picture is lost. My theory about teaching is that with *less* we can do *more*, and I have expanded upon my theory elsewhere (see Rosen, 2003, pp. 91-96). Put very briefly, I submit that we can often achieve an understanding of a task using more primitive methods than the textbooks prescribe. Globally we consider the world we are about to explore mathematically. *With less* we find an elementary technique with which to explore and with *do more* we explore as much of that world as possible with that elementary technique . . .

For the first ten years of my teaching career, including the years in which I attained both a teaching certificate and a master's degree in mathematics, I took little account of research in mathematics. The only personal contact I had with researchers in the UK was with the late Edith Biggs who was a practitioner–researcher and in many ways has been a role model for me. Since coming to Israel and taking a course at the Weitzmann Institute, I have regularly collaborated with many of the researchers both there and at other academic institutes in Israel. Through recent encounters with educational researchers in Israel and the UK, I have been introduced to some forward-looking possibilities.

However, it has seemed to me that generally there is too much research for research sake with little connection to the realities of the classroom situation; looking at pupils' mistakes and misunderstandings and concluding with the feeling that teachers should "do something about it". Many maintain that mathematics is hierarchical and that a mastery of the basics is required before moving on to higher levels. I have read learned papers that break down a topic, such as word problems, into levels of difficulty and formats concluding that these formats should be worked on by the teacher. I have argued (as you will see in regard to my theory *with less do more*) that this type of breakdown leads to the writers of material and the teachers of the mathematically less gifted, taking ever decreasing steps until pupils loses interest because they feel they are not making any progress, or more importantly, lose sight of the whole because the little pieces have become discrete and thus meaningless.

Researchers could do more to connect their work with life in classrooms, adapting their research papers to appear in journals that have a teacher audience, and show how their research work applies in practice. They might suggest other articles, which would, for example, point out theoretical frameworks that ground their study and provide links to further reading, thus enabling readers to extend their understanding of the article.

Question 2: I am driven by the need to make a difference in classrooms, for the non-academic students I teach. In addition, I would like the teachers with whom I work to see that they can make a difference in their classrooms. I have developed my own theory about teaching and learning (*with less do more*). The work I have done with my children and with other teachers has transformed me as a person and as a teacher. Thus my answer to the question is yes to all three points. My aim is to empower the people with whom I work, the children in my classrooms, and the teachers with whom I engage in workshop sessions. I endeavor to elicit from them/show them how they can succeed. Dilemmas regarding how to go about teaching curriculum topics designated by the Ministry are a key focus in my discussions with the teachers. I will present one example. It is one of a number of dilemmas that have arisen in discussion with the teachers. In this case, the teacher remembered that when she herself had learned arithmetic series in school she substituted in a formula and solved equations, but her pupils couldn't handle even simple algebraic manipulations. How was she to proceed?

> I opened the book at random and pointed to the following question:
>
> Given the arithmetic progression 11, 14, 17, . . . how many terms must be added together to reach the sum 861?
>
> She said that she couldn't remember the formula. I said that she didn't need to, just use any knowledge she had as this would be the way she would have to work with her pupils. I produced a calculator, paper and a pencil and told her to start writing. She used the calculator to continue the series down the page. We didn't even define arithmetic series. She started to add the column of figures until she reached the required sum. She then counted how many numbers and wrote down the answer. Here was a case in point of the two basic principles, that of *with less do more* and never losing sight of the generality or globality of the problem involved. To get to a particular sum was a blip in the generality. The sum could have been any number reached before or after the designated sum. At the same time it was also clear which totals could not be achieved by summing this series. The control of the question was in her and hopefully in her pupils' hands. I said that now she and her pupil should see how many of the questions they could solve using the calculator as a tool and being in control of the problem.

I contend that the strategies I share with/elicit from these teachers are ones useful not only in their classroom work with non-academic students, they can inform the teaching and learning of mathematics by all learners.

Question 3: An essential focus of this paper is that of developing voice, power and identity in regards to working with mathematics. My attention is threefold: to help the less-mathematically-gifted non-academic student develop his or her mathematical voice and identity by helping the non-mathematics graduate develop his or her voice as a mathematics teacher. As a bonus, develop my voice as a researcher. My belief in my students is paramount. My work with them enhances their self-esteem. Helping them develop their mathematical voice, even if their mathematical vocabulary is limited, is a vital goal. In my interaction with the teachers, I am encouraging them to question, to re-shape and re-invent their practice, and to try to determine what works for them, and why. The strategies I emphasize with the teachers – strategies which I elicit from them, such as saying things in one's own words, looking at the big picture, drawing pictures and using elementary procedures - foster the emergence of the teachers' voices, and through them, their students' voices. The students, and at times the teachers, are re-shaping their identities as doers of mathematics, and as people who can engage with it in strength, or if not with strong positive feelings, as least not with avoidance or fear. I continue to develop my voice as researcher by questioning, by studying, by learning with and from others in a reciprocal way.

LEARNING ABOUT MATHEMATICS AND ABOUT MATHEMATICS LEARNING THROUGH AND IN COLLABORATION

Vicki Zack and David Reid

Question 1: We are Vicki and David. Vicki is an elementary school teacher and a researcher of her own practice for the past twelve years. David is a university educator and researcher interested in teaching. We first met in 1995 during the PME conference in Portugal, a surprise given that we had both lived and worked in Montreal for many years, but had somehow never run into each other. Our collaboration began eight years ago when Vicki invited David to help with an inquiry that had stumped her and her students that year (1996-1997). Since then our work together has evolved as we have explored, individually and together, ways of stimulating and studying children's learning and our own learning.

Our collaboration has taken several forms. Vicki's research has generated a corpus of video and written data recording her student's interactions in solving mathematics problems in small groups. We sometimes view videos together and discuss what we see through the filter of our own research interests. At other times we watch separately, and discuss by email or phone. Sometimes our research focus arises from an interesting episode, and at other times we wish to explore a general phenomenon in more detail and choose specific episodes to study that are suitable. At times David has taken on the role as a guest teacher in Vicki's classroom and this provides us with additional video and written data from a different context. Quite often, as we will describe below, we see something that puzzles us and having a second person to view, analyse and discuss the data helps us to move our understanding forward. At other times we are theorising together and our work with the data grounds our discussions.

Question 2: In this section we will use our individual voices to address the question of why we engage in teacher-researcher collaboration. Vicki speaks first.

Vicki: Through close study (research) and at times with crucial input from David, I have learned more about the children's ways of thinking and more about the mathematics, and this in turn has affected my practice, in a continuing cycle. I will reflect below on the diverse and vital roles David has played in my learning: David as resource, as catalyst, and as collaborative partner as we explored questions about mathematics and about how one comes to understand mathematics. For me, engaging in teacher research work alone and in collaboration with David has resulted in personal transformation, in making a difference in classrooms, and in developing theory about teaching and learning.

David as resource person: I will begin by discussing David’s role as an invaluable resource and support. I have enlisted David’s help on a number of occasions when aspects in the mathematics have puzzled and intrigued me. My background in mathematics is weak. At times I feel vulnerable when I do not understand, and I will only seek help if I feel I can trust the other person to not make me feel inept.

In one instance, about which I have written and spoken previously at PME (*e.g.*, 1997), I was startled to discover that the children and I could not construct an algebraic formula for the Count the Squares task (a variant of the chessboard problem), which I had assigned to them (Zack, 1997). I was stuck. The ‘non-obvious’ algebraic expression which was available in a mathematics journal and which I showed the students, $n(n+1)(2n+1) \div 6$, was of interest to many of the students in my class, but they wanted to know why it worked as it did. During the 1997 PME conference in Finland, I appealed to members of PME to see if any could suggest a way to make this formula -- $n(n+1)(2n+1) \div 6$ -- meaningful to 10-11year olds. A number of people with whom I spoke shared their individual understanding of the proof but were perplexed in regard to how they would make it meaningful to fifth graders, and one wondered why I would even pursue this endeavour. David took on the challenge, and worked for a number of years, trialling a number of approaches with various cohorts of my students (1995-1996, 1998-2001), with the goal of showing the children how the non-obvious formula works (see Zack & Reid 2003, 2004 for an example of one of the visual proofs David constructed).

In another instance, again in regard to a component of the Count the Squares task, in response to an idea proposed by two of their team members (Ted and Ross), three students in the five-member team offered counterarguments embedded in everyday language, but which upon closer analysis revealed a complex mathematical structure. In considering the children’s arguments, I asked David to use a mathematician’s phrasing to express the children’s ideas; as a result I and others were better able to appreciate the complexity inherent in the children’s ideas (Zack, 1999). In yet another instance, in a situation in which I had asked all the children in the class to consider the Ted-Ross idea heard a number of years before and to see if they agree/disagree, and to state why, one child, Jake, offered a counterargument which was startling and clever. In a follow-up interview I asked Jake to explain his thinking. I, however, could not understand what Jake was saying, and appealed to David to explain Jake’s thinking to me (Zack, 2002). Only then could I appreciate the power of Jake’s pattern, and understand why it worked as it did. Later I was startled to realize that the pattern Jake constructed was the same pattern which Mason, Burton and Stacey (1982) present in the book Thinking Mathematically. Thus, in the above-mentioned examples and other instances, due to close study and essential input from David at critical junctures, I have grown in my understanding of the children's ways of thinking and of the mathematics with which we are engaged.

David and Laurinda (Brown) as catalysts: On the idea of doing the "same problem" again and again: I will share here an instance in which David and Laurinda served as catalysts to me, asking that we all consider the question of what happens when we assign the 'same problem' again and again with different groups. In deliberating upon what I gained by re-visiting the 'same problem', I noted that the first year gives me a feeling for the preliminary framework. In subsequent years, most of the learnings which emerge are common (though never commonplace) to my classes over the years, but it is the unusual pathway(s) and the resultant learnings which have been of particular interest to me.

Laurinda has suggested that the teacher's 'noticing', which becomes more finely tuned with each encounter, "has everything to do with 'what is possible to see and hear' (Brown, Reid, & Zack, 1998, p. 55).

David and I as collaborative partners exploring together the idea of "good-enough understanding": For the past few years David and I have been discussing how one comes to understand complex ideas. Our interest arose as a result of our in-depth study of the thinking of the fifth-graders in my classroom, and as a result as well of our reflections on our own learning. The episodes focal to our discussion of "good-enough understanding" were the ones during which David met with my fifth-graders during one week in May (1995-1996, 1998-2001) to discuss with them the visual proof he had constructed. The discussions led to the two of us theorizing about how one learns complex ideas (Zack & Reid, 2003, 2004). I feel odd to be speaking about theorizing since my feeling had always been that theories were woven by philosopher-academics and handed down to teachers who then tried to understand them. And yet here I am theorizing. We will briefly explain our thinking. Learning mathematics is often portrayed as sequential; complete understandings of underlying concepts is assumed to be necessary before new concepts can be learned. However, we contend that all learners operate with good-enough understanding. When confronted by many complex ideas the first time through, learners (children and adults alike) make many tentative, temporary decisions and keep a number and sometimes contradictory possibilities 'in the air', waiting at times to the end to make sense of what has happened. Opting for a temporary decision which is 'good enough for the time being' is not only a good move, it is one we make all the time when in the midst of learning. In the everyday use of the term, some have equated the ideas of 'good enough' and 'making do' with laziness. However, we submit that good enough is the best we can do when doing our best, that is, when putting in maximum effort. As we show in our two-part article (Zack & Reid, 2003, 2004), the students press to make sense of complex ideas. The untidy and inevitably partial nature of the students' work is part and parcel of coming to understand. The students' disposition to proceed on the basis of an incomplete grasp is, we contend, an essential component in complex problem solving. The evolution of our thinking about "good-enough understanding" could not have happened without our longstanding work together.

David: Most of the work I do, I do in collaboration. This is an extension of my belief that learning is a social process, so I pursue my own learning through research in social contexts. Because I am interested in learning more about the way students reason in the specific context of school mathematics classes, much of my collaboration occurs with teachers. My collaboration with Vicki is unusual as she had already made her classroom a research site before I met her and began collaborating with her. This means that she brings a rich theoretical background, a commitment to teaching through problem solving, and an unusually rich data set to our work. I benefit from all these. Vicki's expertise in communication and discourse offers an alternative to my more psychological and mathematical perspective. She has taught me a great deal about this way of viewing mathematical activity. Vicki's commitment to teaching through problem solving results in a classroom context that is (unfortunately) unusual in Canadian schools. Not all teachers have this commitment, or the background to create such learning contexts. As I am interested in observing the reasoning that takes place in such contexts my collaboration with Vicki gives me access to data that is otherwise hard to come by. And because Vicki was already researching in her classroom before we met, the data she has gathered stretches back in time and covers a wide range of children with different styles of approaching problems. Vicki also has a phenomenal memory of individuals and events and can usually locate examples of similar or contrasting behaviour by other children in other classes in other years.

Question 3: Vicki: One of my goals has been to show others the power of children's thinking. The children know that what they say matters to me, that I am listening and observing closely because I am genuinely interested in them and in their thinking. In regard to aspects of proving, in particular in regard to counterarguments (refutations), the children have pointed the way. They formulate generalizations about observed regularities in regard to diverse patterns they have detected (NCTM, 2000, p. 262) and use this reasoning in situations which are real and meaningful to them, to prove or disprove mathematical claims. My role is to study provocative instances, work to understand them, go at times to David for help which further deepens my appreciation of the children's thought processes and their relationship to the history of mathematical thought, and work with the children to make explicit to them the power of their reasoning. I have shown them how singular their work is, and that at times they have engaged in the problem-solving process with ideas which reflect original thought. I want to be sure that my voice is heard and that through me the children's voices are heard, and so I write. Knowing that I share their ideas with teachers and researchers through conference sessions and publications is powerful for the children. Our identities as mathematics learners with important ideas to share and pursue are established.

David and Vicki: In our collaboration we sometimes speak together, to the community of mathematics educators (*e.g.*, Zack & Reid, 2003, 2004; Brown, Reid & Zack, 1998). But we also speak separately at times, writing papers independently but reading and commenting on each other's writing throughout the process. These papers are also directed to the mathematics education research community (Reid, 2001), as well as to mathematics teachers and other educators (*e.g.*, Zack, 1999).

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RF02: EXEMPLIFICATION: THE USE OF EXAMPLES IN TEACHING AND LEARNING MATHEMATICS

Co-ordinators: Liz Bills, John Mason, Anne Watson, Orit Zaslavsky
Additional Contributors: Paul Goldenberg, Tim Rowland, Rina Zazkis
For this and related papers see http://mcs.open.ac.uk/jhm3/PME30RFmain

This Research Forum will focus on the contribution that attention to examples can make to the learning and teaching processes, taking as background the ways in which examples are construed within different theories of learning (details are in the following background paper). Thus the forum addresses issues at the very heart of mathematics education. Much of the forum time will be allocated to participants engaging with mathematical tasks and classroom data. Our hope is that juxtaposition of previous experience with experience of the tasks offered will stimulate them to develop and express their own theoretical understanding of exemplification, leading to a synthesis and re-expression of perspectives and directions for further research.

Goals of the research forum

to acquaint PME members with existing knowledge and experience on teachers' and learners' use of examples in mathematics and with issues involved and research findings associated with exemplification;

to raise the profile of this field as an important domain of research;

to bring the issues associated with exemplification in mathematics education into a coherent articulation from which future directions for research may be formulated.

The Research Forum has been designed around some key questions:

What makes an example effective for learning mathematics, and in what context?

What things do teachers consider when selecting or constructing instructional examples?

What factors influence learners' perception of examples, and how do we deal with the tension that arises due to 'mis-match' between teacher's intention and learners' attention?

What is entailed and revealed by the process of constructing examples and how does construction of examples promote mathematical understanding?

Structure of the research forum sessions

Session 1: Examples, a teacher's perspective	*Session 2:Examples, a learner's perspective*
Setting the stage (Orit Zaslavsky)	Third task, in small groups (Anne Watson)
First task, in small groups, (Orit Zaslavsky)	Commentary (Paul Goldenberg)
Commentary (Tim Rowland)	Fourth task, in small groups, (Rina Zazkis)
Second task, in small groups (Liz Bills)	Commentary (Paul Goldenberg)
Central issues and future research questions (John Mason)	Issues and future research questions (Orit Zaslavsky, John Mason)

EXEMPLIFICATION IN MATHEMATICS EDUCATION

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BACKGROUND

There is evidence from earliest historical records that examples play a central role in both the development of mathematics as a discipline and in the teaching of mathematics. It is not surprising therefore that examples have found a place in many theories of learning mathematics. Many would argue that the use of examples is an integral part of the discipline of mathematics and not just an aid for teaching and learning. The forum takes as its background both the variety of ways in which examples are construed within different theories of learning and the contribution that attention to examples can make to the learning and teaching processes. Consequently the forum can be seen as addressing issues at the very heart of mathematics education, both drawing upon and informing many other research topics. We argue that paying attention to examples offers both a practically useful and an important theoretical perspective on the design of teaching activities, on the appreciation of learners' experiences and on the professional development of mathematics teachers.

The importance of these ideas does not actually depend on the framework used for analysing teachers' intentions, nor on any terms used to describe forms of teaching, such as: 'analytic-inductive' or 'synthetic-deductive', 'traditional' or 'reform', 'rote-learning' or 'teaching for understanding', 'authentic' or 'investigative'. Issues in exemplification are relevant to all kinds of engagement with mathematics.

This paper positions exemplification on the research agenda for the community by giving a historical overview of the way examples have been seen in mathematics education; an account of associated literature; an exploration of how exemplification 'fits' with various perspectives on learning mathematics; accounts of issues relating to teachers' and learners' use of examples; and directions for future research.

WHAT IS A MATHEMATICAL EXAMPLE?

The word *example* is used in mathematics education in a wide variety of ways. This section offers a brief overview of the scope of our use of the term and points to some useful distinctions that can be made between different uses.

Examples in the form of worked solutions to problems are key features in virtually any instructional explanation (Leinhardt 2001) and examples of all kinds are one of the principle devices used to illustrate and communicate concepts between teachers and learners (e.g. Bruner *et al.* 1956, Tall & Vinner 1981, Peled & Zaslavsky 1997). Diagrams, symbols and reasoning are all treated as particular yet thought about (by the teacher at least) as general. Examples offer insight into the nature of mathematics through their use in complex tasks to demonstrate methods, in concept development to indicate relationships, and in explanations and proofs. The core issue is whether learners and teachers are perceiving the same (or indeed any) generality.

An important pedagogic distinction can be made between examples of a concept (triangles, integers divisible by 3, polynomials etc.) and examples of the application of a procedure (finding the area of a triangle, finding if an integer is exactly divisible by 3, finding the roots of a polynomial etc). Sowder (1980) tried to avoid this confusion by distinguishing between 'examples' and 'illustrations'. However, within the category of 'examples of the application of a procedure', or 'illustrations' we distinguish further between 'worked(-out) examples', in which the procedure being applied is performed by the teacher, textbook author or programmer, often with some sort of explanation or commentary, and 'exercises', where tasks are set for the learner to complete. The worked-out example has been the subject of a body of research within psychology (e.g. Atkinson *et al.* 2000, Renkl 2002).

Of course, these distinctions are neither precise nor clear cut. Gray & Tall (1994) underline the fact that the same notation may be viewed as signifying a process or an object, so that, for example, a teacher may offer a representation of the function $y = 2x + 3$ as an example of a linear function, but the learner may see it as an example of a procedure (for drawing a graph from an equation). There is a good deal of 'middle ground' between exercises and worked examples, for instance when a teacher 'leads' a class through the working out of a typical problem using questions and prompts.

Across these broad categories of form and function of examples there are three special descriptive labels: 'generic example', 'counter-example' and 'non-example'. Generic examples may be examples of concepts or of procedures, or may form the core of a generic 'proof'. Counter-examples need a hypothesis or assertion to counter, but they may do this in the context of a concept, a procedure or even (part of) an attempted proof. Non-examples serve to clarify boundaries and might do so equally for a concept, for a case where a procedure may not be applied or fails to produce the desired result or to demonstrate that the conditions on a theorem are 'sharp'. In fact all three labels have to do with how the person (teacher or learner) perceives the mathematical object in question, rather than with qualities of the object itself.

The term *example* here includes anything used as raw material for generalising, including intuiting relationships and inductive reasoning; illustrating concepts and principles; indicating a larger class; motivating; exposing possible variation and change, etc. and practising technique (Watson & Mason 2002a, 2002b). *Exemplification* is used to describe any situation in which something specific is being offered to represent a general class to which learners' attention is to be drawn. A key feature of examples is that they are chosen from a range of possibilities (Watson & Mason 2005 p238) and it is vital that learners appreciate that range.

EXAMPLES FROM A HISTORICAL PERSPECTIVE

The whole point of giving worked examples is that learners appreciate them as generic, and even internalise them as templates so that they have general tools for solving classes of problems. Unfortunately their use in lessons is often reduced to the mere practice of sequences of actions, in contrast to a more investigative approach (Wallis 1682) in which learners experience the mathematisation of situations as a practice, and with guidance abstract and re-construct general principles themselves.

Whereas mathematical investigations and the use of ‘authentic or ‘modelling’ approaches appear to be a relatively recent pedagogic strategy, there are historical precedents. The earliest mathematical records (Egyptian papyri, Babylonian tablets and later copies of lost Chinese manuscripts) all use context-based problems with worked solutions to illustrate procedures, or what came to be called *rules* and then later *algorithms* in medieval and renaissance texts. They sometimes point specifically to a generality with comments such as ‘thus is it done’ or ‘do it thus’ (Gillings 1972, p. 232), and ‘this way you may solve similar problems’ or ‘by the same method solve all similar problems’ (Treviso Arithmetic 1478, see Swetz 1987, p. 151).

By the 16th century European authors of mathematical texts had begun to justify the presence of examples in their texts, commenting explicitly on the role that examples play for learners. Girolamo Cardano (1545, see Witmer 1968) used phrases such as:

> We have used a variety of examples so that you may understand that the same can be done in other cases and will be able to try them out for the two rules that follow, even though we will there be content with only two examples; It must always be observed as a general rule … ; So let this be an example to you; by this is shown the *modus operandi* in questions of proportion, particularly; in such cases (Witmer 1968, pp. 36-41).

By the late 19th and early 20th century, pedagogic principles become more and more explicit in some cases, if only to attract teachers to ‘new’ pedagogic approaches. For example a textbook from Quebec (MacVicar 1879) claims that:

> The entire drill and discussions [examples] are believed to be so arranged, and so thorough and complete, that by passing through them the pupil cannot fail to acquire such a knowledge of principles and facts, and to receive such mental discipline, as will prepare him properly for the study of higher mathematics. (piv)

Some authors scramble different types of problems, or different looking problems, presumably to engage the learner in recognizing the type, while others collect exercises according to the technique needed, perhaps to promote a sense of the general class of which the exercises are but particulars but more probably to focus on fluency of performance. For example, the expansion by the schoolmaster Iohn Mellis (Record 1632) of John Dee’s extension of Robert Record’s original arithmetic (Record 1543/1969) offers collections of worked examples which offer a variety of differences in what is given and what is sought, so as to draw attention to a wider class of problem type that can be solved by the same method or ‘Rule’.

The design of sequences of examples is a central issue in their instructional use that influences both the inductive and deductive aspects of learning. For example George Pólya (1962) provided long sequences of exercises building up generalisations from a simple starting idea. He ended one such a chapter with a final task:

> Devise some problems similar to, but different from, the problems proposed in this chapter – especially such problems as you can solve. (Pólya 1962, p. 98)

The idea that creating your own examples and questions can aid learning is not new. Record has his scholar in dialogue with the author constructing examples, and Cardano invites the reader to construct their own examples of questions.

Historically there have been two main approaches to the use of examples, distinguished in the 18th century by the terms *analytic* and *synthetic*. The difference amounted to whether general rules were presented before or after worked examples (or even not at all). In the early 19th century Warren Colburn instituted in the USA the inductive method advocated by Johann Pestalozzi (1801):

> The reasoning used in performing these small examples is precisely the same as that used upon large ones. And when anyone finds a difficulty in solving a question, he will remove it much sooner and much more effectively, by taking a very small example of the same kind, and observing how he does it, than by [resorting] to a rule. (Colburn 1826, pp. 141-142)

Herbert Spencer (1878), developed the ideas further, expecting learners to infer the general from carefully presented particulars

> Along with rote-teaching, is declining also the nearly-allied teaching by rules. The particulars first, and then the generalizations, is the new method … which, though ‘the reverse of the method usually followed, which consists in giving the pupil the rule first’ is yet proved by experience to be the right one. Rule-teaching is now condemned as imparting a merely empirical knowledge – as producing an appearance of understanding without the reality. To give the net product of inquiry without the inquiry that leads to it, is found to be both enervating and inefficient. General truths to be of due and permanent use, must be earned. … While the rule-taught youth is at sea when beyond his rules, the youth instructed in principles solves a new case as readily as an old one. (Spencer 1878, pp. 56–57)

Alfred Whitehead summarised this approach as

> To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought. (Whitehead 1911, p. 4)

Pólya asserted:

> [in doing mathematics]… we need to adopt the inductive attitude [which] requires a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations. (Pólya 1945, p. 7).

Even more important than the distinction between inductive and deductive, between ‘general first’ or ‘general later’, are finer distinctions and hybrid approaches which will emerge in later sections. Both inductive and deductive approaches are compatible with constructive accounts of learning and rely on exemplification: inductive learning implies that the learner is making some generalisations about actions or concepts while working with a range of examples (seeing generality through particulars); deductive learning implies that the learner is able to make personal sense of a definition or general principle, and adapt it for current and future use (seeing particular instances in the general).

Examples can be useful stimuli for prompting self-explanation leading to understanding. Cardano acknowledges that sometimes it is too confusing to state a general method, and suggests that examples provide explanation. This sentiment is reflected in a wide range of text authors over the centuries, and by Richard Feynman:

> I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go (Feynman 1985, p. 244).

By contrast, Zazkis (2001) observes that starting with more complex problem situations and more complex numbers not only provides an opportunity for learners to simplify for themselves in order to see what is going on before returning to the more complex, but also provides an opportunity for learners to appreciate more fully the range and scope of generality implied by the particular exemplars. Furthermore, learners are not deceived by the attraction of doing simple computations with small numbers rather than attending to underlying structure.

This survey illustrates a diversity of approaches to examples in learning and teaching. In some cases the succession of examples is the important feature of their use. Their explicit and implicit similarities and differences, the number and variety exhibited, and their increasing complexity can all be used to promote inductive learning. In other cases a single example is intended as a generic placeholder for a completely general expression of a concept, object or process to support deductive thinking.

EXAMPLES FROM A THEORETICAL PERSPECTIVE

Examples play a key role in various classes of theories of learning mathematics. Social and psychological forces and situational peculiarities influence and inform both the examples and the concept images to which someone has access at any moment. The notion of a *personal example space* nicely complements the notion of a *concept image* in this respect. Thinking in terms of variation highlights the importance both of the succession of examples and the aspects which are varied in that succession in affording learners access to key features of a concept or technique.

The role of examples in doing mathematics

Various mathematicians have written about the importance of examples in appreciating and understanding mathematical ideas and in solving mathematical problems (e.g. Pólya, Hilbert, Halmos, Davis, Feynman). Whenever a mathematician encounters a statement that is not immediately obvious, the 'natural' thing to do is to construct or call upon an example so as to see the general through intimate experience of the particular (Courant 1981). When a conjecture arises, the usual practice is alternately to seek a counter example and to use an example perceived generically to see why the conjecture must be true (Davis & Hersh 1981).

Often a mathematician will detect and express a structural essence which lies behind several apparently different situations. Out of this arises a new unifying concept and an associated collection of definitions and theorems. Sometimes a particular example will suggest some feature which can be changed, leading to a richer or more unifying concept, or at least to an enriched awareness of the class of objects encompassed by a theory. It is not examples as such which are important to mathematicians, but what is done with those examples, how they are probed, generalised, and perceived.

The role of examples in theories of learning mathematics

The importance of encounters with examples has been a consistent feature of theories and frameworks for describing the learning of mathematics This section offers a very brief overview of different ways in which theories of learning have used examples.

How people abstract or extract a concept from examples has been specifically studied in psychology from the point of view of how examples and non-examples influence the discernment of concepts (e.g. Bruner 1956, Wilson 1986, 1990, Charles 1980, Petty & Jansson 1987). In Artificial Intelligence attention on default parameters (expectations and assumptions) for triggering frames (patterns of behaviour) were used to try to reproduce concept acquisition (e.g. Winston 1975, Minsky 1975).

Genetic epistemology (Piaget 1970, see also Evans 1973) assumes that individuals actively try to make sense of their world of experience, supported by social groupings (Confrey 1991) in which they find themselves. It underpins many current theories of mathematics learning, by assuming the impact of new examples on existing mental schema through assimilation and accommodation. Piaget's notion of *reflective abstraction* (Dubinsky 1991) implies experiences and actions performed by the learner through which abstraction is possible.

Building on Piaget's notion of *schema*, Skemp (1969) wrote about the learning of mathematical concepts through abstraction from examples, which meant that the teachers' choice of which examples to present to pupils was crucial. His advice on this topic includes consideration of *noise*, that is the conspicuous attributes of the example which are not essential to the concept, and of *non-examples*, which might be used to draw attention to the distinction between essential and non-essential attributes of the concept and hence to refine its boundaries. Once a concept is formed, later examples can be assimilated into that concept (Skemp 1979) and a more sophisticated *concept image* can be formed (Tall and Vinner 1981). Vinner (1983, 1991) describes a gap between learners' concept image and the concept definition: concept images can be founded on too limited an exploration of the examples encountered so that features of the examples which are not part of the concept are retained in the concept image, a process recognised and elaborated on by Fischbein (1987) as *figural concepts*. Concept images are therefore often limited to domains with which learners are most familiar and so may be too limited to be useful. A considerable part of research results on wrong, alternative and partial conceptions can be convincingly interpreted in this way. Thus improving learners' conceptions amounts to reducing the gap between their concept images and the concept definition. Tall and Vinner point to the importance of the examples in closing this gap.

Thorndike *et al.* (1924) followed a behaviourist line in using examples as stimuli to provoke learning responses, and Gagné (1985) developed this into a hierarchy of behaviours of increasing complexity. Dienes (1960) used cleverly constructed games and structured situations as examples of mathematical structures in which to immerse learners so that they would experience examples of sophisticated mathematical concepts through their own direct experience. Others follow historical precedents in trying to describe what it is like for learners to make sense of new concepts (Davis 1984) and worked examples (Anthony 1994).

Marton and colleagues (Marton & Booth 1997, Marton & Tsui 2004) developed the notion of varied examples as a way to encounter concepts noting that what is needed is variation in a few different aspects closely juxtaposed in time so that the learner is aware of that variation *as* variation. Marton even formulates a definition of learning as becoming aware of one or more dimensions of variation which an example could exhibit. Since teacher and learner may not appreciate the same dimensions of variation, Watson & Mason (2005) expanded this to appreciating a particular concept as being aware of *dimensions of possible variation* and with each dimension, a *range of permissible change* within which an object remains an example of the concept.

Recent articulations which connect the genesis of mathematical knowledge with the processes of coming to know also clarify the central role of examples as the raw material for generalizing processes and conceptualizing new objects. Sfard (1991) follows Freudenthal (1983) in seeing learners moving from an operational to a structural understanding of concepts through a process of *interiorisation* and *condensation* leading to *reification*. Interiorisation and condensation are slow, gradual processes, taking place over time and through repeated encounters with examples. Dubinsky and his colleagues (see Asiala *et al*. 1996) have introduced a theory of the development of mathematical knowledge at undergraduate level which they call APOS theory (actions, processes, objects, schemas). Again the theory predicts that encounters with examples will be part of the process by which learners will move from action to process and then to object conceptions. The Pirie & Kieren (1994) onion model of the growth of understanding focuses on image construction and folding back between states, yet still recognizes that it is direct experience of examples which contribute to the formation of personal images and knowings.

Another aspect of the relationship between examples and concepts or processes centres on the notion of generic example, or prototype. A generic example:

> involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff 1988, p. 219)

Freudenthal (1983) describes examples with this potential as *paradigms*. A strand of psychological research beginning with Rosch (1975) has explored how these *prototypes* (representatives of categories) are used in reasoning. Hershkowitz (1990) drew attention to the tendency to reason from prototypes rather than definitions in mathematics, and the errors that this kind of reasoning can produce. Often learners' concept image is largely determined by a limited number of prototype examples (e.g. Schwarz & Hershkowitz 1999) so it is important to go beyond prototypes using non-typical examples to push toward and beyond the boundary of what is permitted by the definition, becoming aware of that boundary during the process (the range of permissible change). Approaches to helping learners expand their reasoning beyond prototypes have been described in a number of specific areas of mathematics.

Dreyfus (1991) discusses the role of examples in abstraction, and in particular the different uses that might be made by learners of single examples and collections of examples. He suggests that, for a relatively sophisticated mathematical learner, a definition and a single example may be sufficient, whereas less experienced learners may need large numbers of carefully selected examples before they can abstract the properties of the concept.

The theory of personal example spaces

The collection of examples to which a learner has access at any moment, and the richness of interconnection between those examples (their accessible *example space*) plays a major role in what sense learners can make of the tasks they are set, the activities they engage in, and how they construe what the teacher-text says and does. Zaslavsky & Peled (1996) point to the possible effects of limited example-spaces accessible to teachers with respect to a binary operation on their ability to generate examples of binary operations that are commutative but not associative or vice versa.

Watson and Mason (2005) formulated the notion of a *personal example space* as a tool for helping learners and teachers become more aware of the potential and limitations of experience with examples. They identify two principles:

> Learning mathematics consists of exploring, rearranging, gaining fluency with and extending your example spaces, and the relationships between and within them.
>
> Experiencing extensions of your example spaces (if sensitively guided) contributes to flexibility of thinking and empowers the appreciation and adoption of new concepts.

A personal example space is what is accessible in response to a particular situation, to particular prompts and propensities. Example spaces are not just lists, but have internal idiosyncratic structure in terms of how the members and classes in the space are interrelated. Their contents and structures are individual and situational; similarly structured spaces can be accessed in different ways, a notable difference being between algebraic and geometric approaches. Example spaces can be explored or extended by searching for situationally-peculiar examples as doorways to new classes; by being given further constraints in order to focus on particular characteristics of examples; by changing a closed response into an open response; by glimpsing the infinity of a class represented by a particular.

Summary

While there is a long history of attention to the provision of suitable examples intended to indicate the salient features which make examples exemplary, recent developments indicate that social and psychological forces and peculiarities play a central role in both the personal example space to which learners have access and the concept image which they develop. Particular attention needs to be paid to the succession of examples and both the dimensions of possible variation and their associated ranges of permissible change to which learners are afforded access.

EXAMPLES FROM A TEACHER'S PERSPECTIVE[1]

The treatment of examples presents the teacher with a complex challenge, entailing many competing features to be weighed and balanced, especially since the specific choice of and manner working with examples may facilitate or impede learning. Note that the aspects mentioned here are interrelated, not disjoint.

Examples as tools for communication and explanation

Examples are a communication device that is fundamental to explanations and mathematical discourse (Leinhardt 2001). The art of constructing an explanation for teaching is a highly demanding task (Ball 1988; Kinach 2002a, 2002b), as described by Leinhardt *et al.* (1990):

> Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. […]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases. (p. 6, *ibid*).

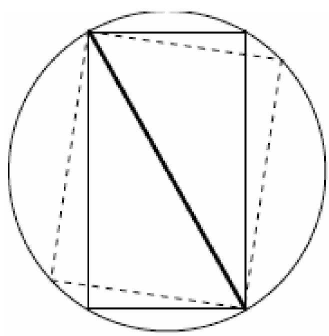
Leinhardt & Schwarz (1997) claim that when teaching meta-skills

> The purpose of an instructional explanation is to teach, specifically to teach in the context of a meaningful question, one deserving an explanation. (p. 399, *ibid*).

That is to say that the meaningful question, the example, plays a key role in the instructional explanation.

Peled & Zaslavsky (1997) distinguish between three types of counterexamples suggested by mathematics teachers, according to their explanatory power: specific, semi-general and general examples. They assert that general (counter)-examples explain and give insight regarding the reason why a specific conjecture is not true and strategies to produce more counterexamples.

The conjecture that two rectangles with the same diagonal must be congruent, is false. The diagram (taken from Peled & Zaslavsky 1997) can be regarded as a general counter example because it communicates an explanation of why the conjecture is false without reference to particular values. Furthermore, inherent to this example is the notion that there are an infinite number of different rectangles with the same diagonal.



With respect to communication, a teacher must take into consideration that an example does not always fulfil its intended purpose (Bills 1996; Bills & Rowland 1999). Mason & Pimm (1984) suggest that a generic example that is meant to demonstrate a general case or principle may be perceived by the learners as a specific instance, overlooking its generality. What an example exemplifies depends on context as well as perceiver.

Attributes which make an example ‘useful’ include:

Transparency: making it relatively easy to direct the attention of the target audience to the features that make it exemplary.

Generalisability: the scope for generalisation afforded by the example or set of examples, in terms of what is necessary to be an example, and what is arbitrary and changeable.

Examples with some or all of these qualities have the potential of serving as a reference or model example (Rissland-Michener 1978), with which one can reason in other related situations, and can be helpful in clarifying and resolving mathematical subtleties. Clearly, the extent to which an example is transparent or useful is subjective. Thus, the role of the teacher is to offer learning opportunities that involve a large variety of 'useful examples' (yet not too large a variety that might be confusing) to address the diverse needs and characteristics of the learners.

To illustrate some of the distinctions mentioned so far, consider the following examples of a quadratic function (these examples and the subsequent elaboration appear in Zaslavsky & Lavie 2005, submitted):

$$y=(x+1)\cdot(x-3); \qquad y=(x+1)^2-4; \qquad y=x^2-2x-3$$

These are three different representations of the same function. Each example is more transparent about some features of the function and more opaque with respect to others (e.g., roots; position of the vertex and minimum value; *y*-intercept). However, these links are not likely to be obvious to the learner without some guidance on how to read or interpret the expressions. Moreover, it is not even clear that learners will consider all three as acceptable examples of a quadratic function, since, for example the power of two is less obvious in the factored form, and a quadratic may have been defined to look like the third expression. A teacher may choose to deal with only one of the above representations, or s/he may use the three different representations in order to exemplify how algebraic manipulations lead from one to another, or in order to deal with the notion of equivalent expressions.

Each different representation communicates different meanings and affords different mathematical engagement, but there are further possible differences in perception. What a learner will see in each example separately and in the three as a whole depends on the context and classroom activities surrounding these examples, and her own previous experience and disposition. A learner who appreciates the special information entailed in each representation may be informed by them to be alert to their differing qualities in the future, even to the extent of effectively using them as reference examples or reference forms when investigating other (quadratic) functions.

To an expert there are some irrelevant features, such as the use of particular letters yet, a learner may regard *x* and *y* as mandatory symbols for representing a quadratic function. Another irrelevant feature is the fact that in all three representations all the numbers are integers. A learner may implicitly consider this to be a relevant feature, unless s/he is exposed to a richer example-space. Learners may also generalise and think that all three representations can be used for any quadratic function.

None of these considerations need be conscious; even the learner who is not *deliberately* making sense of what is offered is still becoming familiar with a particular range of examples which create a sense of normality. Hence, the specific elements and representation of examples, and the respective focus of attention facilitated by the teacher, have bearing on what learners notice, and consequently, on their mathematical understanding. Paul Goldenberg (personal communication) pointed out that sometimes an example can be too specific to be useful; learners and teachers need to be aware that the shift to seeing examples as ‘representative and therefore arbitrary’ is non-trivial and may need classroom discussion.

Uses of examples for teaching

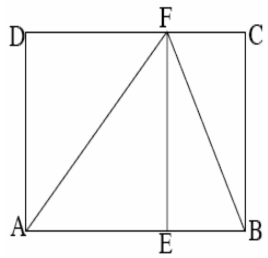
Some authors have categorised examples according to the use for which they are particularly suited. Notable amongst these, Rissland-Michener (1978) distinguishes four types of examples (not necessarily disjoint), which have epistemological significance: start-up (which help motivate basic definitions and results, and set up intuitions in a new subject), reference (which are used as standard instances of a concept or a result, model, and counterexample and referred to repeatedly in the development of theory), model (which are paradigmatic, generic examples) and counterexamples (which demonstrate that a conjecture is false and are used to show the importance of assumptions or conditions in theorems, definitions and techniques).

Rowland and Zaslavsky (2005) distinguish between providing examples of something as raw material for inductive reasoning, as particular instances of a generality, and providing an environment for practice. For example, in order to teach subtraction by decomposition, a teacher might work through say, 62-38 in column format; for practice a collection of well-chosen subtly varying particular cases might be set as an exercise. In the case of concepts, the role of examples is to facilitate abstraction. Once a set of examples has been unified by the formation of a concept, subsequent examples can be assimilated by the concept.

Another kind of use of examples in teaching, more often called ‘exercises’, is illustrative and practice-oriented. For us, exercises are examples, selected from and indicative of a class of possible such examples. Typically, having learned a procedure (e.g. to add 9, to find equivalent fractions, to solve an equation), the learner rehearses it on several such ‘exercise’ examples. This is first in order to assist retention of the procedure by repetition, then later to develop fluency with it (Rowland & Zaslavsky 2005). When the teacher repeatedly demonstrates how to perform on these practice exercises, the learning mechanism that is facilitated may share some characteristics of the learning from worked-out examples (see section 6b).

Hejný (2005) notes that the focus of attention needs to be not only on what can be generalised from one example, but also on a structured set of tasks which may direct learners to find a general or abstract idea. For example, he suggests helping learners in primary grades discover a formula for the area of a triangle, by offering them a rich problem situation, from which a general relationship can be induced.

> Divide a given rectangle ABCD by a segment EF to make two rectangles AEFD and EBCF. These rectangles are divided by diagonals AF and BF into four right-angled triangles. Consider eight shapes: five triangles: AEF, AFD, EBF, BCF, ABF and three rectangles: ABCD, AEFD, EBCF.



> Given the area of two of these shapes, find the areas of all the others;
>
> Given the length of three segments from the following: AE, EB, AB, DF, FC, DC, AD, EF, BC find the areas of all the triangles.
>
> What do you have to know to find the area of triangle ABF?

Most of the studies that deal with sets of example suggest that the specific sequence of examples has an impact on learning. In particular, it is recommended to combine examples and non-examples within a sequence of examples, in order to draw attention to the critical features of the relevant examples. There is an argument for examples to be 'graded', so that learners experience success with routine examples before trying more challenging ones. However, it should be noted that sequencing examples from 'easy' to 'difficult' is not always effective (Tsamir 2003). Exercises designed for fluency are likely to be differently structured to exercises designed to promote or provoke generalisation (Watson & Mason 2006).

Leron (2005) uses the term *generic proof* to refer to what Movshovitz-Hadar (1988) calls a *transparent proof* or *pseudo proof*. Leron illustrates generic proofs with reasoning to justify the fact that every permutation can be decomposed as the product of disjoint cycles. As a simpler example consider the proof that the sum of two odd numbers is an even number. One can use two 'general' odd numbers that are not special in any obvious way, e.g. 137 and 2451, and present a 'proof' for these two numbers, e.g.: $137 + 2451 = (136 + 1) + (2452 – 1) = 136 + 2452$. This form of presentation can be read generically as justification that the sum of *any* two odd numbers is equal to the sum of two even numbers and hence even. However, learner attention has to be directed appropriately in order to have this effect. The specific choice of examples together with the transparency with respect to the main ideas of the proof both play an important role.

Finally, examples (or exercise examples) can be used for assessment of learners' performance and understanding in a broad sense. The more conventional way would be to present learners with examples of problems or mathematical objects and ask them to follow certain instructions (e.g., solve the problem, compare the objects etc.). In this, the teacher assumes that these examples are cases of a more general class of problems or objects, and considers learners' performance with these examples as a representation of their knowledge. In a way, several researchers use carefully selected examples to investigate learners' schemes (e.g., Dreyfus & Tsamir 2004; Peled & Awawdy-Shahbari 2003). Section 7 elaborates on researchers' use of examples.

Another approach that some teachers (as well as researchers) use for revealing learners' conceptions and ways of thinking is by asking learners to generate their own examples of problems and of objects (e.g. van den Heuvel-Panhuizen *et al.* 1995, Zaslavsky 1997, Hazzan & Zazkis 1999, Watson & Mason 2005).

Teachers' choice of examples

Research on teachers' choice of examples is rather scarce. Ball *et al*. (2005) maintain that a significant kind of mathematical knowledge for teaching involves specific choices of examples, that is, considering what numbers are strategic to use in an example. Similarly, Rowland & Zaslavsky (2005) note that the choice of 62-38 in column format to teach subtraction by decomposition is not a random choice: the digits are all chosen with care because constructing examples is not an arbitrary matter, though there is usually some latitude in the choice of effective examples. The 8 could have been a 9; on the other hand, it could not have been a 2. It could have been a 4, say, but arguably the choice of 4 is pedagogically less effective than 8 or 9, because subtracting 4 from 12 would lead some pupils to engage in finger-counting, distracting them from the procedure they are meant to be learning. Attending to the range of change of digits that is permissible without changing the learners' experience (Watson & Mason 2005) is essential in choosing instructional examples.

Novice teachers' poor choices of examples have been documented by Rowland *et al*. (2003) who considered the way in which student teachers give evidence of their subject knowledge in their teaching of mathematics to primary school children, one aspect being the choice of examples. The authors present instances of choices which, in their words, 'obscured the role of the variable' (p. 244): reading a clock face set at half past the hour by using the example of half past six; using as the first example to illustrate the addition of nine by adding 10 and subtracting one, adding nine to nine itself. Often the unintentionally 'special' nature of an example can mislead learners.

In selecting instructional examples it is important to take into account learners' preconceptions and prior experience. In particular, careful construction of examples could enable teachers to identify and help learners cope with the effect of previous knowledge and existing schemes (implicit models) on the construction of new knowledge. Research findings on learning could serve as a rich source for teachers' selection of effective examples for this purpose. For example, Peled & Awawdy-Shahbari (2003) suggest asking learners to compare carefully selected pairs of decimal or common fractions, in order to identify the implicit models by which they operate. An effective example for decimal fractions would be to ask learners which number is bigger: 2.8 or 2.85. Some learners claim that 2.8 is bigger "because tenths are bigger than hundredths". Similarly, in comparing $\frac{5}{6}$ and $\frac{3}{5}$, some will say that $\frac{3}{5}$ is larger because fifths are larger than sixths, because they focus on the size of the fractional part and ignore the number of parts. Similarly, the study by Tsamir & Tirosh (1999) regarding learners' tendencies to address inclusion considerations when dealing with comparisons of infinite sets informed the choice of examples Tsamir and Dreyfus subsequently presented to learners (Tsamir & Dreyfus 2002).

In secondary school the considerations in selecting specific examples seem to be far more complex than in primary school. Zaslavsky and Lavie (2005, submitted) and Zaslavsky and Zodik (in progress) discuss teachers' considerations underlying their choice of examples. Issues that came up in their study include: the tension between the teachers' desire to construct 'real-life' examples and the mathematics accuracy they feel they are 'sacrificing' when doing so; the dual message of randomly selected examples since the randomness may convey the generality of the case, however it may also yield impossibilities or inadequate instances; the visual entailments of examples in geometry, and the ambiguity regarding what visual information may be induced and what should not. A classic instance is that when a 'general' triangle is sketched, some learners rely on the relative magnitude of length of its sides, leading to examiners asserting with every diagram 'not drawn to scale'.

Summary of teacher perspective

The use of examples in the classroom is an essential but complex terrain. It involves careful choices of specific examples which facilitate the directing of attention appropriately so as to explain and to induce generalisations. Desirable choice of examples depends on many factors, such as the teaching goals and teachers' awareness of their learners' preconceptions and dispositions.

It has been proposed (*e.g.*, Tall & Vinner 1981, Chi *et al*. 1989, Chapman 1997) that the key feature of learning is not what is presented but rather what is encoded in the learner's mind, what is constructed by the learner, what practices are internalised.

EXAMPLES FROM A LEARNER'S PERSPECTIVE

The crucial factors for appreciating and assimilating concepts, and for learning techniques are the form, format and timing of examples encountered, and experience of ways of working with and on examples. When invited to construct their own examples, learners both extend and enrich their personal example space, but also reveal something of the sophistication of their awareness of the concept or technique.

Concept formation

Davis (1984) described mathematical objects emerging from specific experiences:

> When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practiced, the procedure itself becomes an entity - it becomes a thing. […] The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun. (pp. 29-30, *ibid*).

In the process of concept formation, the operational conception (focussing on the process) is often first to develop, gradually moving towards a structural approach (focusing on the object) (Rumelhart 1989). Gray and Tall (1994) use the example '2 + 3' to illustrate how a symbol sequence or expression may be conceived either as aa process (add) or a concept (sum). A learner might perceive an example either as a process, or as an object, or both (*proceptually*). For example, if a learner's only experience of equations is of being shown how to solve them, with the language only of ‘doing’, then it is unlikely that a conceptual understanding will be formed easily.

Charles (1980) argues that while for ‘easy’ concepts a sequence of examples from which to generalise may be sufficient, for more ‘difficult’ concepts non-examples are also necessary to delineate the boundaries of the concept. Wilson (1986) points out that learners can be distracted by irrelevant aspects of examples, so the presence of non-examples provides more information about what is, and is not, included in a definition. Since examples are far more effective than formal definitions in appreciating concept (Vinner 1991), learning might be enhanced by contact with a rich variety of examples and non-examples. Paul Goldenberg (private communication) observed that there is a big difference between noticing for oneself a salient feature in a collection of examples and then naming it, and being given a new word followed by a sequence of objects which are supposed to illustrate its meaning.

How rich and in what variety needs careful study however. Bell (1976) reported that school learners often do not recognize the significance of counterexamples and would not necessarily alter their conjectures or proofs if a counterexample did crop up, and this is reflected in the observation that undergraduates also tend to monster-bar (MacHale 1980) rather than modify their concept image. It is fairly obvious that a limited experience of examples and non-examples may lead to a restricted concept image, but it is also the case that limiting mathematics to sequences of examples ‘to be done’, rather than sets of examples to be understood, may induce learners to focus on completing their tasks rather than on making sense of the tasks as a whole (Watson & Mason 2006). A succession of examples does not add up to an experience of succession. Not attending to the whole may result in an overly restricted understanding of the nature of mathematics.

Learning from worked-out examples

Several studies point to the contribution of worked-out examples for learning to solve mathematical problems (e. g. Reed *et al.* 1985; Reimann & Schult 1996; Sweller & Cooper 1985). However, providing worked-out examples with no further explanations or other conceptual support is usually insufficient. Learners often regard such examples as specific (restricted) patterns which do not seem applicable to them when solving problems that require a slight deviation from the solution presented in the worked-out examples (Reed *et al.* 1985, Chi *et al.* 1989). Note however that the immensely insightful mathematician Ramanujan was, while a student, able to treat a book of summarised generalities as a sequence of particular examples!

Watson & Mason (2002a, 2002b) suggest that worked-out examples might even inhibit learners' ability to generalise apart from recognition of the syntactical template. One explanation of this phenomena was given by Reimann & Schult (1996), based on Artificial Intelligence literature. They claim that the information captured and attention drawn in worked-out examples is mostly the solution steps, which limit matching and modification processes. Furthermore, Reimann & Schult (*ibid*) assert that it is important to specify in a worked-out example the steps that were taken and the reasons for taking them, that is, how attention is directed. This is consistent with the findings of Chi *et al.* (1989) and Renkl (2002) who emphasise the importance of learners' self-explanation of the worked-out example, and also with the work of Eley & Cameron (1993) who found that learners considered an explanation to be better if it included the 'trigger' for each step. Worked-out examples *may* enhance learners' learning, and in particular their problem solving performance, but only if they are used in ways which encourage explanation and reasoning.

Much of the research in this area has been directed towards a view of learning as measurable by performance of techniques and solution of word problems, rather than of learning as conceptual understanding or mathematical enquiry. The role of worked-out examples in conceptual understanding deserves further research.

The role of examples in mathematical reasoning and problem solving

Examples can play a role in facilitating non-routine problem solving, a process in which reasoning about the situation allows the learner to apply and adapt sequences of techniques whose purposes need to be understood. If this is seen as a process of applying known techniques, the relevant worked-out examples which the learner has experienced need to be sufficiently different, and sufficiently explained, for the purpose of the techniques used to be understood. If, on the other hand, problem-solving is seen as a process of modelling a situation and tackling it heuristically, a learner needs to have some knowledge of similar situations in order to be successful.

One of the main processes of reasoning about novel situations is reasoning by appealing to similarity (Rumelhart 1989). Rumelhart refers to a continuum, moving from 'remembering' a suitable example to 'analogical reasoning'. Another central kind of mathematical reasoning that necessitates generation of examples is proving by refutation. Addressing learners' difficulties in producing and using appropriate counterexamples is another challenge for teachers' use of examples (Zaslavsky & Peled 1996; Zaslavsky & Ron 1998). Pólya (1945, 1962) elaborates on the processes of inductive (example-based) reasoning, generalization, and analogical reasoning, all of which greatly depend on examples.

It seems that all learners who are even only partially engaged try to generalise from sequences of examples, implicitly or explicitly, and that this is done by the natural process of discerning differences and similarities in what is available to be perceived. What they choose to stress and ignore, and what they 'get from it' is highly variable. Discerning invariance and variation explains many standard misconceptions in mathematics: learners generalise inappropriately, but in ways which can be seen to be the products of mathematical reasoning, given their experience. Thus learners are always engaged in mathematical reasoning whenever they are exposed to a set of examples of anything, although this may not be recognised or made explicit.

There are many unresolved issues. For example, Hejny (personal communication) questions whether 'natural' generalisation is always the same kind of process, or whether it differs according to whether one is encountering a concept, a process, etc..

Novices and expert mathematicians alike depend on experiences with a single rich generic example, or else, as with most novices, numerous examples, in order to get some intuition about the situation and then try to generalise and reason from them. (Bills & Rowland 1999, Zaslavsky & Lavie 2005). This mixture of logical-based reasoning (using deductive mechanisms) and example-based reasoning (Lakatos 1976) characterises mathematical competence at every level.

Weber & Alcock (2004, 2005) documented how undergraduates learning to prove use examples in reasoning and constructing proofs. They recognised that professional mathematicians switch fluently between examples (specific cases) and formal definitions, so they asked how learners make the transition to this fluency, if this shift has not been made explicit for them. They found that example use for such learners is often illustrative and empirical rather than general and deductive. Where their reasoning failed, they were more likely to self-correct errors to do with the individual example than errors to do with the underlying rationality. Alcock & Weber (in press) then distinguished between two learners who used a *referential* approach to proof and a *syntactical* approach. The learner who used referential approach rejected examples as a tool for developing structural understanding and may have needed help in describing examples more formally, to see how doing so might offer the structure for a formal proof. The learner who approached the task of proof construction as if it were solely a manipulative exercise might have benefited from using specific examples to give her work some meaning, but self-generation of appropriate examples is not trivial for learners who are unused to doing so.

The role of learner generated examples in learning

Learning is an activity which requires initiative and intention. Getting learners to construct their own examples proves to be a highly effective strategy for transferring initiative from the teacher to the learner (e.g. Zaslavsky 1995, Niemi 1996, Dahlberg & Housman 1997, Hazzan & Zazkis 1999, Zazkis 2001, Watson & Mason 2005).

The current shift from teacher-centred to learner-centred pedagogical environments in order to foster mathematical classroom discourse, fits with encouraging learners to construct their own examples, which in turn enables teachers to detect the kinds of understandings reflected by learners' examples (*e.g.* Watson & Mason 2005, and as suggested by Zaslavsky 1995). Creation of an example is a complex task that calls upon conceptual links among concepts (Hazzan & Zazkis 1999). Dahlberg & Housman (1997) showed that learners who generated examples as a strategy of learning were more likely to understand new concepts. 'Give an example of …' tasks prove very useful in assessing learners’ understanding (Niemi 1996).

When learners have been asked to create their own examples, they experience the discovery, construction or assembly of a space of objects together with their relationships. Whereas Rissland-Michener (1978) saw *example spaces* as canonically objective, construction is often idiosyncratic, combining modifications of conventional and familiar objects to construct new objects, to recognise new relationships, and to enjoy new meanings and personal understandings.

Easily-available *canonical* spaces, such as those teachers and textbooks commonly use, form suitable starting points for further extension, just as in any learning the learner can only start from what is already known, which may be a proper subset of what is relevant. In other words, through construction, learners become aware of dimensions of possible variation and corresponding ranges of permissible change within a dimension, with which they can extend their example spaces.

From a mathematical perspective it may be possible for an expert to see a large potential space of examples, or at least to have past experience of a large space, but what comes to mind in the moment may only be fragments of that potential. Spaces are often dominated by strong images, some of which may be almost universal. What is accessible in one situation may not be so readily accessible in another. The experience of constructing examples for oneself can contribute to increased sensitivity in future, triggering richer example spaces.

Summary of learner perspective

Examples play a crucial role in learning about mathematical concepts, techniques, reasoning, and in the development of mathematical competence. However, learners may not perceive and use examples in the ways intended by teachers or textbooks especially if underlying generalities and reasoning are not made explicit. The relationship between examples, pedagogy and learning is under-researched, but it is known that learners can make inappropriate generalisations from sets of examples, or fail to make any conceptual inferences at all if the focus is only on performance of techniques. The nature and sequence of examples, non-examples and counter-examples has a critical influence of what opportunities learners are afforded, but even more critical are the practices into which learners are inducted for working with and on examples.

The relationship between examples and logical deduction in proof, or analogical reasoning in problem solving, cannot be assumed to be assimilated or even accommodated by learners without explicit support and provocation. It is valuable for learners to create their own examples, since this process requires complex engagement with concepts and mathematical structures

Learners naturally perceive variation and invariance in what they experience, and make generalisations from this activity, developing example spaces whose contents may be triggered in future situations. How these contents are structured and inter-related is the outcome of past experience and with ways of working with examples.

EXAMPLES FROM A RESEARCHER’S PERSPECTIVE

From a researcher’s perspective the role of examples in mathematics education research concerns choices based instructional design, in research on learning, and the role of case studies, considered as research examples, in theory development in mathematics education. The three points will be illustrated by means of examples from a research project, in which they are prominent without, however, being explicit.

Research-based design

Research findings depend critically on specific properties of examples just as much as teaching and learning. For example, in the study by Dreyfus & Tsamir (2004); and Tsamir & Dreyfus (2002, 2005), which deals with the comparison of the cardinalities of infinite sets, the task set initially was to compare the numbers of elements in the set of natural numbers with the number of elements in the set of perfect squares. Two representations were used: numeric and geometric. In the numeric representation, the sets were represented on three cards:

Card A {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, …}

Card B {1, 4, 9, 16, 25, 36. 49, 64 81, 100, 121, 144, 169, 196, 225, …}

Card M was identical to Card A. The inclusion relationship was highlighted by asking learners to choose and mark the perfect squares on Card M.

Card M {$\boxed{1}$ 2, 3, $\boxed{4}$ 5, 6, 7, 8, $\boxed{9}$ 10, 11, 12, 13, 14, 15, $\boxed{16}$ 17, …}

The geometric representation used squares and the correspondence between side length and area (see Tsamir & Dreyfus 2002, for a detailed description).

The examples in this first task were chosen with attention to research findings (Tsamir & Tirosh 1999) regarding learners' tendencies to think in terms of inclusion when presented with a numeric representation of the task, and to identify the one-to-one correspondence in reaction to the geometric representation of the same task. Consequently, learners may be expected to reach contradictory answers.

After several more tasks using either or both representations as well as algebraic correspondence rules between the sets, the task in the third session was to compare the set A of natural numbers to a set V, which was given numerically as {0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, ...} (Tsamir & Dreyfus 2005). Here the algebraic rule of the one-to-one correspondence is not easily apparent, nor is the establishment of such a correspondence geometrically. Without adequate preparation, a learner could thus be expected to use only inclusion considerations.

When designing a sequence of tasks, whether for the purpose of teaching or research, the characteristics of each specific example need to be taken into consideration. These characteristics include the different representations in which the example can be cast, and whether the example triggers certain types of reasoning, such as analogy or cognitive conflict. Whereas in teaching not all examples will usually be determined ahead of time since inspired and creative teaching involves sensitivity to the flow of events and on the spot decisions by the teacher, in research, on the other hand, researchers usually do plan all examples in advance; nevertheless, decisions to add or omit examples in a specific stage of the research may be made on the basis of the analysis of previous stages or exigencies in the moment.

Research on learning

Learners' abstract mathematical constructs usually emerge from their occupation with specific cases, i.e. examples. This becomes particularly clear in the research mentioned above, which analyzes the case of one learner, Ben, addressing the comparison of powers of infinite sets. How and what exactly learners may or may not learn from examples only becomes clear after detailed, careful and controlled observation, and analysis of the observations, by researchers.

As an example, consider what Ben did (not) learn from the two tasks presented above. When presented with the first task, Ben claimed, as expected, that the number of elements in set A was larger than the number of elements in set B, explaining that "set B is actually part and I mean REALLY part of set A", and that "it is easy to notice that the further I go [in set B] the larger the intervals". Over the next two sessions, Ben gained insight into the problematic aspects of using inclusion and correctly solved this and all other tasks presented to him by using one-to-one mappings between infinite sets in numeric/algebraic and geometric representations. He reached what the researchers interpreted as consolidated in-depth constructs allowing him to solve such tasks, and it seems that this was on the basis of a carefully designed sequence of tasks. For example, with respect to the comparison of set A above with the set of natural numbers greater than 2, he explained:

> "The two extra, unmatched elements stand out and trigger the conclusion that here we have infinity and here infinity plus two, which SEEMS larger. Instead of matching numbers at the same ORDINAL place [pause]. I mean, assuming that if for each place *n* there is one and only one element in each the two sets, then they go on hand in hand, corresponding, and extra elements are just in our imagination. The infinite nature makes it possible that no matter which number you chose in one of the sets, at the same ordinal place there is a matching specific number placed in the other set. It cannot be that the numbers in the second set are finished and cannot provide a matching element, because the set is infinite, and this behavior of plus two goes on, like, forever."

In the third session, Ben was asked to compare the sets A and V (see above). This example, which was intended to introduce more challenging tasks, turned out to provide the researchers with insight into the complexity of what had been interpreted as Ben's consolidated knowledge about the comparison of countable infinite sets. For over 20 minutes, Ben assiduously tried to establish, geometrically or algebraically, a one-to-one correspondence between A and V. He even noticed that there is a one-to-one correspondence between set A and the set of differences between successive elements of V. But then he ended up concluding,

> "The differences between successive elements get larger and larger. Wow! REALLY larger. I see. Set V consists of fewer elements. REALLY fewer."

Even insistent questioning by the interviewer did not sway his opinion. The interviewer remarked:

> "You once told me that using inclusion and correspondence leads to contradiction. And then you read that only equivalence correspondence should be used for comparing infinite sets. Right?"

To this, Ben replied that yes, indeed, using inclusion and one-to-one correspondence may lead to a contradiction, and that he had not used inclusion except to prove that there exists no one-to-one correspondence.

Based as it was on careful choice of a sequence of examples, this research has advanced our understanding of the important characteristics of consolidation (Dreyfus & Tsamir 2004). Equally interestingly, the choice of the introductory example to the third session also turned out to have an important, though unplanned role in the research because it led to modification of our conception of consolidation.

Research on learning is necessarily based on examples because all learning is either fundamentally based on examples, or at least strongly supported by examples. The choice of examples thus influences research on learning, and possibly research results. Are such research results reliable? Not quite. An example was found where Ben's supposedly consolidated knowledge broke down. Without this example, conclusions about Ben's consolidation of knowledge about the comparison of infinite sets would have been exaggerated.

There are two ways researchers can counterbalance this influence of examples: One is to be acutely aware of it, and attempt to analyze it, thus recognizing the influence, and the possible ensuing limitations of any specific piece of research; and the other is to carry out several parallel research studies using different sets of examples, the subject of the next subsection.

Theory building

It is generally agreed that theory building is one of the aims of research. In mathematics education, researchers' theoretical constructs about X (e.g. a specific learning process such as consolidating) tend to emerge from observation of a few, sometimes of a single example of X, combined with theoretical reflection on X. The small number of examples is a necessary limitation, due to the fact that examples are often "large" in the sense that they may require weeks of detailed observations and subsequent painstaking analysis of the observations.

Research on constructing and consolidating knowledge is a case in point. Learners can be given opportunities for constructing knowledge – but they cannot be forced to construct; researchers thus provide learners with opportunities, and hope they can observe what they are looking for. Consolidating recently constructed knowledge, by definition, is an ongoing process that may last hours or years. Dreyfus & Tsamir (2004) have proposed characteristics of consolidation on the basis of a single, albeit detailed and very carefully analyzed, but still only a single example, namely the example of Ben constructing and consolidating his knowledge about the comparison of infinite sets.

In a similar vein, the entire ‘RBC theory’ made up of *Recognizing*, *Building-with* and *Constructing* (Hershkowitz *et al.* 2001), within which the consolidation research is located, has been proposed on the basis of a single example, a 9th grade learner learning about rate of change as a function. Again, one example has served to propose an entire theory. Subsequently, the same and other researchers have shown that the theory is applicable to many other contexts, possibly after suitable modification. The theory has thus been strengthened and validated. It is important to stress that this validation is based on examples as well. In this sense, examples play a central and crucial role in the establishment of theory, the other basic element of theory building being theoretical reflection.

Summary of research perspective

The choice of examples, and their sequencing, is crucial in instruction. Examples may be chosen for using specific representations and they may be sequenced to go from easy to difficult for triggering analogy, or from difficult to easy for triggering cognitive conflict (Tsamir 2003). Consequently, research on learning mathematics is necessarily based on examples as well, and the choice of mathematical examples may influence research results. Researchers can counterbalance this influence by being aware of it, by taking it into account when drawing conclusions, and by carrying out parallel research studies using different sets of examples.

Moreover, there is a second level of example use in research. A research study, such as the one about Ben, may itself serve as an example that forms the basis for theory building. Additional examples of research studies are a tool for validating the theory.

FOR FURTHER RESEARCH

Particular attention needs to be paid to

the sequencing and timing of a succession of examples, and both the dimensions of possible variation and their associated ranges of permissible change to which learners are afforded access.

ways of directing learner attention so as to perceive exemplariness;

ways of drawing teachers’ attention to the importance of the choices of examples they make when working with learners;

the role of worked-out examples in concept formation;

ways of directing learner attention so that sets of exercises are pedagogically effective.

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[1] Parts of this paper, particularly Section 5 - “Examples from a Teacher’s Perspective”, are based on research supported by the Israel Science Foundation (grant No. 834/04).

RF03: THE CONCEPTUAL CHANGE APPROACH IN THE LEARNING AND TEACHING OF MATHEMATICS: AN INTRODUCTION

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THE CONCEPTUAL CHANGE APPROACH: FOUNDATIONAL ISSUES

The conceptual change approach has its roots in Thomas Kuhn’s account of theory change in the history of science (Kuhn, 1970) and has served for many years as a source of hypotheses about the learning and teaching of science (Posner, Strike, Hewson & Gertzog, 1982). According to the classical conceptual change approach, students form misconceptions in science which present formidable alternatives to the scientific theory and which need to be replaced. This process of replacement was seen as a rational process of theory replacement which can be promoted through cognitive conflict and can take place in a short period of time.

This approach became the leading paradigm in science education until it became subject to several criticisms. Among other things, it was pointed out that it provides a rather simplistic view of misconceptions, ignoring their complex interrelations with other concepts, as well their interaction with context. It was argued that conceptual change happens in larger situational, educational, and socio/cultural contexts, that it is affected by motivational and affective factors and that it takes a lot of time to be accomplished (Caravita and Halden, 1994). The instructional practice that has emerged form this framework, namely cognitive conflict, was also criticized as not having a sound constructivist basis and as ignoring students’ productive ideas (Smith, di Sessa, & Rochelle, 1993).

In response to these criticisms we have developed a re-framed conceptual change approach to learning which is known as ‘the framework theory approach’ (see Vosniadou, Baltas, & Vamvakoussi, in press). The framework theory approach to conceptual change that we propose meets all the criticisms of Caravita & Halden (1994) and Smith et al. (1993). First, misconceptions are not viewed as unitary, faulty conceptions but as parts of a knowledge system consisting of many different elements organized in complex ways, and thus, conceptual change is not a sudden, gestalt-like restructuring, but a gradual theory change that takes a lot of time to be accomplished. Second, we make a distinction between the learner’s initial framework theory of physics, (a naïve physics) prior to systematic instruction, and misconceptions that are produced after instruction. Research with infants has shown that the process of constructing a naïve physics starts soon after birth. By the time children go to school they are deeply committed to an ontology and causality that distinguishes physical from psychological objects and which forms the basis for the knowledge acquisition process (Carey, 1985; Vosniadou, 1994; 2001). Naïve physics facilitates further learning when the new, to be acquired information is consistent with existing conceptual structures. However, when children start to be exposed to scientific explanations, explanations which may be radically different from what they already know (i.e., different in their structure, in the phenomena they explain, as well as in the very concepts that comprise them), naïve physics may stand in the way of learning science.

The framework theory approach to conceptual change predicts that new information which is incompatible with what is already known is more difficult and time consuming to be learned than new information that can enrich existing structures. Moreover, it explains the formation of misconceptions as 'synthetic models' resulting from learners' attempts to simply add the new information to existing but incompatible knowledge structures.

This re-framed conceptual change approach is a constructivist position -- it argues that learners actively construct knowledge on the basis of what they already know – and it is capable of predicting when prior knowledge can stand in the way of learning something new. Although it focuses primarily on cognitive aspects of conceptual change, it is complementary and not contradictory to other approaches that deal with metaconceptual, motivational, affective and socio/cultural factors (Vosniadou, 2001; Vosniadou & Vamvakoussi, in press). In particular we emphasize the importance of of *metaconceptual awareness* because we believe that students are not aware of their prior beliefs and additive learning mechanisms that can sometimes result in the distortion of new information, of *motivational factors* because students must want to change, and of the broader social and cultural context that can provide the educational background and the appropriate tools to facilitate conceptual change.

THE FRAMEWORK THEORY APPROACH TO MATHEMATICS LEARNING AND TEACHING

The learning of mathematics has many similarities to the learning of physics. As it is the case that students develop a naïve physics on the basis of everyday experience, they also develop a "naïve mathematics", which appears to consist of certain core principles or presuppositions (such as the presupposition of discreteness in the number concept) that facilitate some kinds of mathematical learning but may inhibit others (Gelman, 2000). Such similarities support the argument that the conceptual change approach could be fruitfully applied in the case of learning mathematics.

Mathematics educators have been reluctant to adopt this approach to mathematics learning and teaching because mathematics is not considered to undergo revolutionary theory changes similar to physics. Thomas Kuhn himself exempted mathematics from the pattern of theory development and theory change in science. This is the case because, unlike science, the formulation of a new theory in mathematics usually carries mathematics to a more general level of analysis and enables a wider perspective that makes possible solutions that have been impossible to formulate before (Corry, 1993).

However, Kuhn's radical incommensurability position has been criticized both within the philosophical and psychological circles (see Vosniadou, Baltas, Vamvakoussi, in press). In the context of the learning sciences, a number of researchers have pointed out that even in the case of the natural sciences conceptual change should not be seen in terms of the replacement of students' misconceptions with the "correct" scientific theory, but in terms of enabling students to develop multiple perspectives and/or more abstract explanatory frameworks with greater generality and power (e.g., see Driver, Asoko, Leach, Mortimer, & Scott, 1994; Spada, 1994). It thus appears that the theory replacement issue may not be an issue in the case of science and mathematics learning.

IMPLICATIONS FOR THE DESIGN OF MATHEMATICS INSTRUCTION

Many mathematics educators have noticed that prior knowledge can hinder the acquisition of some mathematical concepts. Fischbein (1987) was one of the first to notice that intuitive beliefs may be an important contributor to students' systematic errors in mathematics, a fact also noted by Vergnaud (1989) and Sfard (1987). The importance of the conceptual change approach is that it can provide a basis from which such widespread findings can be systematized and explored for the purpose of designing more effective curricula and instruction (Vosniadou & Verschaffel, 2004). It can be used as a guide to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors and misconceptions, to provide student-centered explanations of counter-intuitive math concepts, to alert students against the use of additive mechanisms in these cases, to find the appropriate bridging analogies, etc. In a more general fashion, it highlights the importance of developing learning environments that foster intentional learning and the development of metacognitive skills required to overcome the barriers imposed by prior knowledge (Schoenfeld, 1987; Vosniadou, 2003).

AIMS AND SCOPES OF THIS RESEARCH FORUM

The purpose of the Research Forum is to present empirical evidence in support of the argument that the framework theory approach to conceptual change can be helpful in the design of curricula and instruction in mathematics. More specifically, the six papers included in this Research Forum address one or more of the following questions.

What does the conceptual change predict about the development of mathematical concepts?

In most cases, learning is successfully accomplished by adding new information to what is already known. However, when the new, to be acquired, information comes in conflict with what is already known, the use of additive mechanisms results in the distortion of the new information and the creation of synthetic models. So far, the creation of such synthetic models has been shown in the case of science (e.g., models of the earth such as the 'hollow sphere' and the 'dual earth'). In this research forum, Biza and Zachariades, Van Dooren, De Bock and Verschaffel, Vamvakoussi and

Vosniadou present evidence showing that students create synthetic models of mathematical notions, such as the tangent line and the rational numbers intervals, as predicted by the conceptual change framework.

What is the explanatory power of the conceptual change approach in the context of mathematics learning?

Practically all contributors highlight the fact that the conceptual change framework can offer persuasive explanations of about certain students' misconceptions and systematic errors that are well documented in the literature coming from mathematics education research. We point out the paper by Van Dooren, De Bock and Verschaffel that seeks to explain students' tendency to over-use the notion of proportionality in terms of the conceptual change framework.

Is the conceptual change approach applicable in the case of more advanced mathematical concepts, or is it the case that it can only explain learning of elementary concepts?

Most of the papers included in the Research Forum refer to studies conducted with participants at level of secondary or university education. Merenluoto and Lehtinen, Vamvakoussi and Vosniadou present evidence showing that some of the fundamental presuppositions of students' initial explanatory frameworks about numbers, like the idea of discreteness, may constrain students' understanding of rational and real numbers, continuity, and limit, until the last grades of high school and even at the university level. Biza and Zachariades deal with the concept of tangent line, which is introduced in instruction as the circle tangent line and is not related –at least, not directly- to experience. They show that students' initial understandings of the notion of the tangent line constrain students' further understanding of the tangent line on a curve.

Given that the use of symbolic notation is an important component of mathematics learning, can the conceptual change approach make meaningful predictions about students' difficulties in using mathematical symbols?

Christou and Vosniadou show that students' experience with natural numbers influence the way they interpret the use of literal symbols in algebra. More specifically, students tend to substitute literal symbols with natural numbers only. Vamvakoussi and Vosniadou propose that presuppositions related to the symbolic notation of numbers make part of students' explanatory frameworks of numbers.

Is conceptual change a "cold cognition" approach, or does it take into account factors other that cognitive?

Although originally a cognitive- oriented approach, the conceptual change framework can take into account motivational, affective, situational and other factors, such as epistemological beliefs, that influence learning (Sinatra & Pintrich, 2003; Vosniadou, 2003). In this research forum, Merenluoto and Lehtinen argue that students' certainty about their answers in mathematics tasks is related to their level of understanding and is predictive of their willingness to change their beliefs. They find that students who are in a transition phase feel less certain about their answers than those who rely firmly on their intuitive but inadequate knowledge.

What can mathematics education gain from a conceptual change approach to mathematics teaching?

The conceptual change approach proposes specific principles that can serve as guidelines in designing instruction (see for example, Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001). Tirosh and Tsamir make use of these principles in order to reflect on their successful teaching interventions aiming at university students' understanding of the comparison of infinite sets. They argue that they constitute a valuable framework for instruction and research on instruction.

Beyond these specific principles, however, an essential aim of this Research Forum is to sensitize researchers and educators to the problem of conceptual change. As Resnick points out in her commentary, it took some time for the mathematics education researchers to realize that initial mathematical understandings may not always be supportive of further mathematical learning, and that in some cases they may in fact inhibit further learning. In order for mathematics educators to make good use of instructional design principles, like the meta-principle proposed by Greer in his commentary, it is important that they take a wider perspective on the role of prior knowledge.

CONCEPTUAL CHANGE IN MATHEMATICS LEARNING: THE CASE OF INFINITE SETS

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Research in mathematics education indicates that in the transition from given mathematical systems to wider ones learners tend to attribute all the properties that hold for the former to the latter. In particular, in the context of Cantorian Set Theory, students and prospective teachers have been found to attribute properties of finite sets to infinite ones – using various methods to compare the number of elements in infinite sets. These methods, which are acceptable for finite sets, lead to contradictions with infinite ones. In this presentation we show that the conceptual change approach constitutes a valuable framework for analyzing and reflecting on students' reasoning and on instructional interventions related to the comparison of infinite sets.

The conceptual change theory has been widely used to interpret students' solutions in a series of developmental studies referring to science education (e.g., Carey, 1985; Posner, Strike, Hewson, & Gertzog, 1982; Vosniadou & Brewer, 1992). This theory was mainly used to explain knowledge acquisition in specific domains, while describing the significant role of reorganization of existing knowledge structures in some processes of learning.

Vosniadou, Ioannides, Dimitrakopoulou, and Papademetriou (2001) argued that scientific explanations of the physical world often run counter to fundamental principles of intuitive knowledge, which are confirmed by our everyday experience. Consequently, in the process of learning, new information interferes with prior knowledge, resulting in the construction of synthetic models (or misconceptions), and this shows that knowledge acquisition is a gradual process during which existing knowledge structures are revised slowly. Similarly, when studying mathematics, in the course of accumulating mathematical knowledge, the student goes through successive processes of generalization, while also experiencing the extension of various mathematical systems. For instance, the move from one number system to a wider one preserves some numerical characteristics, adds some others while yet others are lost. For example, the transition from natural numbers to integers enables one to solve a problem like 5-7 (closure under subtraction). Yet, at the same time it becomes impossible to generalize that subtraction “always makes smaller” and the system no longer has a smallest number.

In recent years several researchers have attempted to explore the promises of the conceptual change framework to mathematics learning and teaching (e.g., Vosniadou & Verschaffel, 2004). In this presentation we discuss the applicability of the conceptual change approach to the learning and teaching of the Cantorian Set Theory. We focus on one major aspect of this theory: the equivalency of infinite sets. The terms “comparing infinite sets", "comparing infinite quantities" and "determining the equivalency of infinite sets" are used interchangeably to account for the comparison of the cardinalities of these sets.

The Cantorian Set Theory is the most commonly used theory of infinity today. Yet students face great difficulties in acquiring various properties of the equivalency of infinite sets (Borasi, 1985; Duval, 1983; Fischbein, Tirosh & Hess, 1979; Lakoff & Nunez, 2000; Tall, 1980; 2001; Tirosh, 1991; Tsamir, 1999). It was reported that when asked to compare the numbers of elements in two infinite sets students at different grade levels used methods that are adequate only for the comparison of the number of elements in finite sets. For example, students expect that the number of elements in a set which is the union of two distinct, non empty sets is larger than that of the number of elements in each of these sets. This however is true for finite sets, but not for infinite ones. It seems evident from the related research findings and also from the historical development of the Cantorian Set Theory that the acquisition of various aspects of the theory in general and the equivalency of infinite sets, in particular, necessitates radical reconstruction.

In the course of the last twenty years we designed and evaluated several methods of teaching the Cantorian Set Theory (Tirosh, Fischbein & Dor, 1985; Tirosh, 1991; Tsamir & Tirosh, 1999; Tsamir, 1999, 2003). The major principles that guided the development of these instructional practices (as described, for instance, in Tirosh, 1991) were: identifying the intuitive criteria students use to compare infinite quantities; raising students’ awareness of inconsistencies in their own thinking; discussing the origins of students’ intuitions about infinity; progressing from finite to infinite sets; stressing that it is legitimate to wonder about infinity; emphasizing the relativity of mathematics, and strengthening students' confidence in the new definitions.

We evaluated the impact of traditional courses with little or no emphasis on students' intuitive tendencies to overgeneralize from finite to infinite sets, and of courses that were developed in line with the principles listed above, on high-school students and on prospective mathematics teachers' intuitive and formal knowledge of Cantorian Set Theory. Our findings in the different studies indicate that instruction that implemented these principles led to promote the reconstruction of knowledge structures (Tirosh, 1991; Tsamir, 1999). These interventions promoted the learners' awareness of the differences between finite and infinite systems, and of the contradictions that result from interchangeably applying different criteria when comparing infinite sets. Looking at these instructional interventions through different lenses could provide additional insights into their pros and cons.

In the Research Forum we shall show that the instructional design principles deriving from the conceptual change approach (as presented by Vosniadou et al., 2001) offer a valuable framework for analyzing and reflecting on instructional interventions in mathematics. More specifically, we focus on the mathematical notion of equivalency of infinite sets, using the instructional design recommendations of the conceptual change approach to analyze and reflect on related learning environments.

ASPECTS OF STUDENTS' UNDERSTANDING OF RATIONAL NUMBERS

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In this paper we examine students' understanding about the dense structure of rational numbers from a conceptual change perspective. We argue that students' difficulties reflect the fundamental presuppositions of their explanatory frameworks about number, which are tied around their understanding of natural numbers.

NATURAL VS. RATIONAL NUMBERS

Research in mathematics education has shown that prior knowledge of natural numbers gives rise to numerous misconceptions that pertain to both conceptual and operational aspects of rational numbers (e.g. Moss, 2005). Gelman (2000) argues that, even before instruction, children form an understanding about numbers, which is based on principles pertaining to the act of counting. This view is very close to a key assumption of the conceptual change theoretical framework that we adopt (Vosniadou, 1994; Vosniadou & Verschaffel, 2004), according to which children form initial explanatory frameworks about numbers, which are tied around their understanding of natural numbers. These frameworks facilitate thinking and learning about rational numbers, when new information is compatible with the underlying presuppositions, but cause difficulties and systematic errors when new information comes in contrast with what is already known.

In what follows, we will focus on students' understanding about the dense structure of the rational number set.

THE SET OF NATURAL VS. THE SET OF RATIONAL NUMBERS

The set of natural numbers consists of discrete elements which share a similar form, in the sense that any natural number is represented as the combination of a finite number of digits. Contrary to the natural numbers set, the set of rational numbers is dense, since between any two rational numbers there are infinitely many rational numbers. On the other hand, from a mathematical point of view, the set of rational numbers also consists of elements that share a common form in their symbolic representation, since any rational number is represented as the ratio of two integers. Alternatively, any rational number can be represented in decimal form, either as a simple or as a recurrent decimal. For the trained mathematician, it may be easy to move from one representation to the other, or even entertain both representations simultaneously, without losing the sense of the rational numbers set being a homogenous set, with dense structure. However, it is well documented that students have many difficulties moving flexibly and effectively among the various forms of rational numbers (e.g. Moss, 2005). We claim that students draw on symbolic notation to treat natural numbers, decimals and fractions as different, unrelated sorts of numbers. This claim is supported by evidence coming from research in various domains showing that novices tend to group objects on the basis of superficial characteristics (see for example Chi, Feltovich, & Glaser, 1981). Students' tendency to group numbers on the basis of their form may be enhanced by the fact that there are considerable differences between the operations, as well as the ordering of decimals and fractions.

We assume that the particular characteristics of the natural numbers set mentioned above (discreteness, elements with unique symbolic representation, homogeneity of forms) are key elements of students' initial "theories" about numbers and are bound to constrain students' understanding of the dense structure of the rational numbers set. Prior research has provided evidence that the idea of discreteness is indeed a barrier to the understanding of density for students at different levels of education (Malara, 2001; Merenluoto & Lehtinen, 2002; Tirosh, Fischbein, Graeber, & Wilson, 1999). In addition, Neumann (1998) reported that 7th graders had difficulties accepting that there could be a fraction between two decimals, indicating their belief that decimals and fractions are unrelated sorts of numbers.

Based on the above remarks, we assume that the development of the concept of density requires conceptual change. We expect that students form synthetic models of the structure of rational numbers intervals, reflecting the constraints associated with their initial explanatory frameworks about numbers, as well as the assimilation of new knowledge into their incompatible knowledge structures.

These hypotheses were tested in two empirical studies with 9th and 11th graders (Vamvakoussi & Vosniadou, in press, 2004). In the second study, we also investigated the effect of the number line on students' responses to tasks regarding density. Following an ongoing discussion in the conceptual change literature about the effect of external representations (e.g. Vosniadou, Skopeliti, & Ikospentaki, 2005), we assumed that the effect of the number line is rather limited and may disappear, when the number line is withdrawn. According to our results, the presupposition of discreteness was strong in 9th grade and remained robust up to 11th grade. As expected, the presence of the number line did not facilitate students to a significant extent. Students' accounts of the rational numbers intervals reflected the expected constraints, as well as new knowledge and techniques pertaining to rational numbers –in this sense, they can be termed as *synthetic models*.

Adopting the conceptual change approach, we traced key elements of students' explanatory frameworks about numbers that may hinder further learning about rational numbers. We suggest that this approach could lead us, through a systematization of widespread findings on students' difficulties with rational numbers to a more clear picture of their explanatory frameworks about numbers and help to make detailed predictions about the barriers imposed by students' prior knowledge.

ACKNOWLEDGMENTS: The present study was funded through the program EPEAEK II in the framework of the project "Pythagoras - Support of University Research Groups" with 75% from European Social Funds and 25% from National Funds.

CONCEPTUAL CHANGE IN THE NUMBER CONCEPT: DEALING WITH CONTINUITY AND LIMIT

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In our earlier studies 538 upper secondary level students (age 17-18 years) answered to a questionnaire concerning the density of numbers on the number line, explained the concepts of limit and continuity using their own words and estimated their certainty on their answers. 272 of these same students answered to the same questions half a year later. The results indicate that the majority of students based their answers on discrete numbers, everyday thinking of continuity and limit as a bound. In this presentation students' problems with these concepts is explained by the radical nature of conceptual change in the number concept, and by the dynamics of motivational and cognitive factors in conceptual change.

INTRODUCTION

Extending the number concept from natural numbers into the domains of more advanced numbers requires a radical change in prior thinking of numbers, a conceptual change. Research findings suggest that the majority of students have some kind of difficulties in this change. For example, in our large surveys 37 % of upper secondary students (n = 538), 67 % of student teachers (n = 62) and 14 % of university students (n = 71) majoring in mathematics systematically used the rules and logic of natural numbers while working with the rational and real numbers (Merenluoto & Lehtinen, 2004). These problems of mistaken transfer are widely known among mathematic educators (cf. Fischbein, Jehiam & Cohen, 1995). Even at the higher levels of education, students seem to be unaware of their thinking about numbers or of the fundamental conceptual difference between natural, rational and real numbers. Typical to these kinds of problems is that students' prior knowledge interferes or even restricts learning of new conceptual knowledge.

The small natural numbers and the conception of always having the "next" number are among those special concepts which have a high unconditional certainty attached to them, and it is therefore difficult even to see any reason to change one's thinking. These conceptions have a high intuitive acceptance attached to them as being self-evident, self-justifiable or self-explanatory, and this easily results in over-confidence (Fischbein, 1987).

This kind of certainty seems to be derived from at least three different sources: firstly, from innate cognitive mechanisms related to numeral reasoning principles; secondly, from everyday experiences of counting and the linguistic operations; and thirdly from the formal mathematical instruction in learning the notion of natural numbers. Conceptual change means leaving the previous "safe" environment and stepping into a new unknown territory in one's thinking and reasoning.

In this presentation our aim is to explain students' problems with the concepts of limit and continuity by the difficulty of conceptual change in the number concept and by the dynamics of motivational and cognitive factors in conceptual change.

Method

Participants in the first measurement were 538 students' (age 17-18 years) on upper secondary level; and in the second measurement hall a year later 272 of these same students. The questionnaire was presented to the students in ordinary classroom conditions, in which the students answered to questions about the density of numbers on the number line. In addition, they where asked to describe in their own words what is meant with the concepts of continuity and limit of a function. They were also asked to estimate their certainty while answering to the questions.

Students' answers were scored on *primitive level* if the students clearly used their prior knowledge of whole numbers on the domain of rational /real numbers, or if they exclusively based their answers on their everyday concepts and/or experiences. On the *level of partial identification* the students' answers showed fragmented recognition of some details of the question. The level where the students explained their answer with clear operational arguments was scored on the *level of operational understanding*. In some answers there were hints of some structural understanding and we scored those on the *beginning structural understanding level*. In this scoring the level of partial identification and operational understanding represent the "synthetic" models in the process of conceptual change. The level of *beginning* *structural understanding* represents a transition level to more radical conceptual change.

RESULTS AND DISCUSSION

The main result from the study was that the majority of the students based their answers on thinking of discrete numbers, on everyday thinking of continuity, and the limit as a bound in both measurements. Only less than ten percent of the students gave answers that could be categorised on the level of beginning structural understanding. Significant to these answers was their reduced level of certainty which also seems to be an indication of a transitional level.

The students' achievement level in mathematics had a significant relation to the high sensitivity to the cognitive demands and to high estimation of certainty that seems to be optional for the conceptual change. However, the moderate operational understanding of the concepts with high certainty estimations seems to have a tendency to prevent the students' from noticing the cognitive conflict. The results also suggest that situations demanding conceptual change are coping situations for the students. They need to give up their earlier confidence based on familiarity of natural numbers and move into a different kind of mathematical thinking; where different kind of rules and operations are dealt with. In order to success in this process, they need to tolerate the inevitable feeling of ambiguity which comes from newly learned operations and concepts while enough certainty has not yet gained to cope in this new environment.

In general, we assume that the difficulties students have in the acquisition of new areas of mathematical knowledge, like extensions of number concept and the concept of limit, are not only due to the increasing complexity of the knowledge but also to situations where prior knowledge systematically supports the constructions of misconceptions. The students might benefit from a new kind of approach in the teaching of these concepts, an approach where the differences between everyday and mathematical thinking are explicitly emphasised, where students' metaconceptual awareness is developed and their metacognitive skills of dealing with seemingly conflicting concepts is supported.

THE LINEAR IMPERATIVE: SEARCHING FOR THE ROOTS AND THE IMPACT OF THE OVER-USE OF PROPORTIONALITY

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The mathematics education literature repeatedly mentions students' tendency to over-use certain properties of the linearity concept. By means of three research-based examples, we explain how conceptual change theory may shed more light on the origins and on the persistence of this phenomenon.

INTRODUCTION

Linearity or (direct) proportionality is an important idea in mathematics and sciences. Accordingly, it receives a lot of attention in elementary and secondary education. When using the terms 'linear' and 'proportional', we refer to functions of the form f(x)=ax (with a≠0). Such relations have various interrelated characteristics. For instance, their graphical representation is a straight line through the origin. Other examples of properties of linear functions are that any two ratios within the function yield a proportion (a/b=c/d), that f(x+y)=f(x)+f(y) and that f(kx)=k f(x).

Throughout the curriculum, students repeatedly encounter such properties (sometimes formally, sometimes informally), and when tackling linear problems, they may apply any of them. For instance, when judging whether 6 litres of water mixed with 500 grams of sugar taste equally sweet as 15 litres of water with 1200 grams of sugar, students can check whether 6/500=15/1200. Or when confronted with the missing-value problem "12 eggs weigh 720 grams, what is the weight of 36 eggs?" they might apply the f(kx)=k f(x) property: 3 times as many eggs weigh 3 times as much. (Less skilled proportional reasoners may calculate the weight of 12+12+12 eggs, which is essentially the same.)

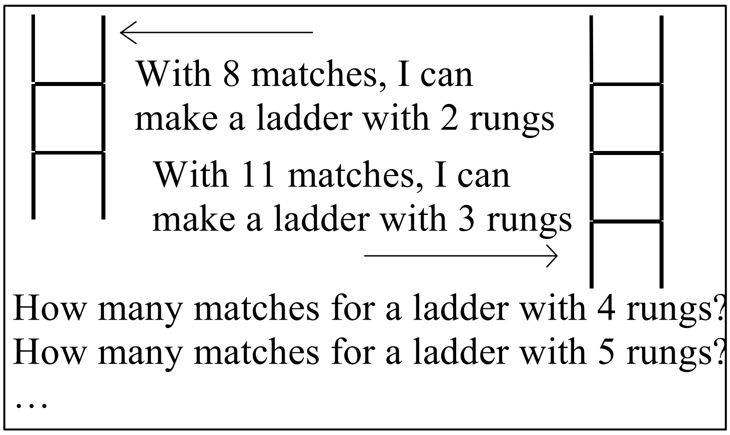
The research literature, however, points out that students of various ages tend to apply some of these properties also when this is inappropriate, and do this in various mathematical domains. Below, we explain by some research-based examples how conceptual change theory (CCT) may shed new light on the origins and on the persistence of this phenomenon.

SOME MANIFESTATIONS OF THE OVER-USE OF LINEARITY

Each of the properties of linear relations mentioned above can also be applied by students in non-linear situations, and thus lead to errors. From the wide variety of examples, we selected three research-based cases to discuss briefly here.

Example 1. "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Kim has run 5 rounds, Ellen has run 15 rounds. When Kim has run30 rounds, how many has Ellen run?" Van Dooren et al. (2005) administered this problem – along with various other non-linear missing-value word problems – to 3rd to 8th graders. In 3rd grade, 30% of the non-linear problems were answered linearly (i.e., for the cited problem: 90 rounds), and this increased considerably until 51% in 5th grade (this went perfectly in parallel with students' acquisition of missing-value proportional reasoning skills), with a decrease thereafter to 22% in 8th grade.

Example 2. Stacey (1989, p. 148) offered the task shown on the right to 9-13-year olds. The most frequent erroneous answers were due to assumptions of proportionality: Students used the $f(x+y)=f(x)+f(y)$ and $f(kx)=k\ f(x)$ properties of linear functions (e.g., "one needs 8+11 matches to make a ladder with 2+3 rungs"), or combinations of both.



Example 3. "*The Carmel family has 2 children, and the Levin family has 4 children. Is the probability that the Carmels have 1 son and 1 daughter larger than/equal to/smaller than the probability that the Levins have 2 sons and 2 daughters?*" Stavy and Tirosh (2000, p. 58) presented this problem to 7th to 12th graders. Percentages of students who erroneously answered "equal to" increased with age from 33% in 7th to 62% in 12th grade. The authors interpreted this as evidence for students' tendency to use the 'Same *A*-same *B*' intuitive rule: The ratio boys/girls is the same (1/2) in both families, so students reason that the probability is the same. Similar and other linear errors in probabilistic reasoning can be found in Van Dooren et al. (2003).

CCT AS AN INTERPRETIVE FRAMEWORK

To what extent can we interpret the above-described phenomena in terms the assumptions made in CCT (see, e.g., Vosniadou, 1999)? A first key assumption is that some important knowledge elements are already acquired through preschool and out-of-school experiences. This seems true for linearity: In their most simplistic form, linear relations are experienced from early on (even the acts of counting and measuring implicitly assume linearity), and they are frequently confirmed in everyday life. Additionally, much of the elementary school math curriculum consists of further reinforcing and enriching the existing (proportional) conceptual structures: Experiencing their validity in new contexts, and discovering new, abbreviated or more abstracted problem solving procedures and representations. As such, the linear idea starts acting as a broad 'framework theory' about mathematical relations, and proportional method becomes a panacea. Consider, e.g., the findings of Van Dooren et al. (2005): The over-use of linearity was already present to some extent as early as in 3rd grade, but it became considerably more influential throughout elementary school, in parallel with students' originating proportional reasoning skills.

CCT furthermore assumes that the presuppositions within such a framework theory generally are unavailable to conscious awareness and deliberate hypothesis testing Again, this seems true for the over-use of proportionality: De Bock et al. (2002) have shown that students' choice for a proportional method occurs very quickly, and that students are strongly convinced about the correctness of their (proportional) solution method, while they hardly can explain how it works and why it is correct for a particular task. It can also be suspected that students applying the $f(x+y)=f(x)+f(y)$ property in the study by Stacey (1989) were even not aware that this property (only) holds for linear functions, and that they were, consequently, assuming linearity.

A final important issue in CCT is that when students encounter new information that is incompatible with prior knowledge (in the context of a framework theory), learning will be more difficult, and misconceptions may originate as 'synthetic models' resulting from learners attempts to assimilate the new information to existing conceptual structures. Take the 'Carmel' probability problem mentioned above (Tirosh & Stavy, 2000) as an example. The concepts of 'chance' and 'proportion' are very strongly related – an idea that is grasped intuitively even before formal instruction in probability (see Van Dooren et al., 2003). For example, simple proportional reasoning easily shows that it is equally likely to draw a white ball from an urn with 20 white balls out of 30 as from an urn with 200 white balls out of 300. Van Dooren et al. (2003) observed, however, that students apply this type of reasoning to any other parameter in a probabilistic situation (e.g., in a task like "The likelihood of getting heads at least 20 times when tossing a coin 30 times is smaller than/equal to/larger than the likelihood of getting heads at least 200 times out of 300"). Even after instruction in the law of large numbers and in the binomial distribution, most students answered "equal to". Some of their protocols clearly showed synthetic models, e.g., the application of 'linear formulas' like "(300×)/200" to calculate the probability of ge tting at least 200 heads in 300 coin tosses (each having a likelihood of), although such methods were never dealt with during formal instruction.

CONCEPTUAL CHANGE IN ADVANCED MATHEMATICAL THINKING: THE CASE OF TANGENT LINE

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Tall and Vinner (1981) introduced the term concept image in order to describe the cognitive structure in the individual's mind that is associated with a given concept. Students, in their mathematics education, study some concepts in upper high school or in university that have already been taught at an elementary level. In these cases the concept images which are created at the early studies have to be generalised in order to be applicable in the broader context. According to Harel and Tall (1989) there are different kinds of generalisation which depend on the individuals mental construction. They call an *expansive generalisation* one which extends the student's existing cognitive structure without requiring changes in the current ideas. They also call a *reconstructive generalisation* one which requires reconstruction of the existing cognitive structure.

In the *reconstructive generalisation* the old concept image has to be radically changed so as to be applicable in a broader context. Many students cannot achieve this generalisation and as a result various misconceptions can occur. Some of them are caused by the student's effort to assimilate the new information in their existing knowledge, although these two are incompatible. These misconceptions could be predicted and investigated through a conceptual change approach as *synthetic models* (Vosniadou, 1994). In particular, in this paper we investigate students' synthetic models regarding the tangent line of function graph.

TANGENT LINE

The concept of tangent line is introduced to students in the Euclidean geometry context as the circle tangent. Two properties that characterise the circle tangent are the following: the line has only one common point with the curve; the line has a common point with the curve and leaves it on the same semi – plane. In this context, students create concept images which are based on the above properties. After that, students are taught the tangent line in the broader context of function graphs, in Calculus. In this case students have to reconstruct the previous concept images in order to broaden their applicability range; therefore, a reconstructive generalisation is needed. Many students fail to make this reconstruction effectively. They act under the influence of the circle tangent and create irrelevant concept images (Tall, 1987; Vinner, 1982, 1991). We claim that this influence causes synthetic models and we will try to investigate them.

Methodology

Data reported in this paper was collected from a questionnaire administered to 182 first year university students (97 female) of the Mathematics Department of University of Athens. All participants had been taught about the tangent line in Euclidian geometry, in Analytic geometry and elementary Calculus courses during the 10^{th}, 11^{th} and 12^{th} grade, respectively, but not at the university level. The questionnaire included tasks in which the students had to define in their own words the tangent line; to describe some properties of it; to recognise if a drawn line is a tangent line of the corresponding curve; to construct the tangent line of designed curves at a specific point; and, to write the formula of the tangent line of the curve of a function in general and in concrete cases. Our data analysis was based on a latent class analysis (LCA) using the software MPLUS (Muthen & Muthen, 2004).

Results

Four groups of students were formed using LCA. The first group consisted of 54 students that generally accepted or sketched the right tangent line. The second group consisted of 56 students. The majority of these students accepted as a tangent a line that has more than one common point with the curve; they recognised that there is no tangent at the points in which the derivatives from the left and the right exist without being equal, which we term "edge points"; and they rejected a tangent line either when it coincides with a part of the curve close to tangency point or at inflection points. The third group consisted of 44 students who rejected the tangent lines which cut the curve at other points than this of tangency and generally accepted only the lines that leave the whole curve at on the same semi – plane. In addition, they did not accept a tangent line at the "edge points". The last group consisted of 28 students who answered correctly in the same tasks as the students of the third group except in the cases of "edge points". It would appear that the students of the last two groups applied the properties of circle tangent in the cases of function graph without enriching their concept images with new information. We will concentrate on the students of second group who, although they enriched their concept images with new elements, they remained under the influence of circle tangent. From the above description of the characteristics of the majority of this group, it seems that these students kept the visual representation of the circle tangent and apply its properties at a neighbourhood of the point of tangency. Their concept images were effective for many cases of graphs but they were not adequate in general. A student of this group wrote: "tangent line is a line that 'touches' a locus without 'going through' it at the tangency point"; she rejected a line at an inflection point by writing: "This isn't (a tangent line) because it 'goes through' C_f"; in the case the curve coincides with the tangent line close to the tangency point, she responded: "This isn't (a tangent line), because it doesn't touch (the curve) at only one point but at infinitely many ones". This student had no problem in symbolic manipulation; in symbolical context, she calculated correctly the equations of tangent lines similar to those that she rejected as tangents in the graphical context.

Conclusions

The majority of the students of the second group, like the one mentioned above, used the circle tangent properties in their answers in a more sophisticated way than these of the last two groups. They applied them at a neighbourhood of the point of tangency. Their concept images were adequate in many cases but not in general and these students, through their answers to the tasks, demonstrated consistency in their successful / unsuccessful responses. It would appear that students have assimilated the new information about the tangent line of graphs to the existing knowledge of the circle tangent although these were incompatible. Their misconceptions could be interpreted, through the conceptual change approach, as synthetic models created by students in their attempts to deal with the tasks of the questionnaire.

Acknowledgments:

The present study was funded through the programme EPEAEK II in the framework of the project "Pythagoras II – Support of University Research Groups" with 75% from European Social Funds and 25% from National Funds.

STUDENTS' INTERPRETATION OF THE USE OF LITERAL SYMBOLS IN ALGEBRA – A CONCEPTUAL CHANGE APPROACH

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The main focus of this paper is to examine student's difficulties with literal symbols in algebra. We present the results from an empirical study where the theoretical framework of conceptual change was used.

THE CONCEPTUAL CHANGE APPROACH AND THE USE OF LITERAL SYMBOLS IN ALGEBRA

The way students interpret the use of literal symbols in algebra has been the subject of a great deal of research worldwide. Early research on students' understanding the algebraic notation was influenced by Piaget's theory. For example, Collis (1975) categorized the way students interpreted the use of literal symbols in algebra in three levels according to the well-known Piagetian stages. Within a Piagetian framework, Kuchemann (1981) also argued that students tended to interpret literal symbols to stand for specific numbers only. They could not understand the use of letters as 'generalized numbers' which is a symbol that could take on multiple values. In addition, other researchers found that students did not always interpret literal symbols to stand for numerical values, but could interpret them to stand for objects or labels of objects (e.g., Stacey and MacGregor, 1997; Booth 1988).

In the present study we adopted the conceptual change approach (as defined by Vosniadou & Vershaffel, 2004) in order to better predict and explain students' difficulties with literal symbols in algebra. According to this theoretical framework, students develop a naïve mathematics based on their experience with natural numbers only. Because natural numbers are the primary elements from which concepts of other numbers are constructed, many students even in secondary education, think that all numbers share the same properties with the natural numbers. This seems to be one of the reasons why misunderstandings and difficulties appear when numbers other than natural, for example fractions (see Gelman, 2000) or rational numbers (see Vamvakoussi & Vosniadou, 2004) are introduced in the mathematical curriculum.

The finding that students think that literal symbols correspond to specific numbers only is consistent with the conceptual change approach, and can be explained to be derived from students' prior knowledge of natural numbers (where every natural number has a unique symbolic representation). However, the conceptual change approach that we propose also makes predictions about the kinds of numbers students think that can be represented by literal symbols. More specifically, we predict that when students assign numbers to literal symbols there will be a strong tendency to use natural numbers only.

In previous studies, Christou and Vosniadou (2005a, 2005b, submitted) investigated this hypothesis. For example, in Christou and Vosniadou (2005a) two questionnaires were given to fifty-seven 8^{th} and 9^{th} grade students. Questionnaire A (QR/A) asked the students to write down numerical values they thought could be assigned to a series of algebraic expressions such as ‘a’, ‘-b’, ‘4g’, ‘a/b’, etc. Because in QR/A all responses can be considered as correct, since all values can be assigned to each algebraic expression, a different questionnaire, Questionnaire B (QR/B) was also designed. Questionnaire B asked the students to write down the numerical values that they thought could not be assigned to the same algebraic expressions. In this latter questionnaire, only the scientifically correct response, namely that all numbers can be assigned to each algebraic expression, is correct.

The results showed that when students were asked in QR/A to write down numerical values they thought could be assigned to the given algebraic expressions which contained literal symbols, they tended to substitute only natural numbers for the literal symbols. Very few students responded with numbers other than natural numbers, as for example decimal numbers or fractions. Even though these responses could be considered as correct, they were nevertheless different from the responses expected from a mathematically sophisticated participant and demonstrated students’ tendency to think of literal symbols as only standing for natural numbers.

In QR/B, where students were asked to write down numerical values they thought could not be assigned to the given algebraic expressions, students tended to assign numbers where the literal symbols were replaced by negative whole numbers (e.g. -1, -2, -3). For example, students responded with numbers such as 4(-1), 4(-2), etc. as numbers that could not be assigned to ‘4g’, or with numbers such as (-2)/(-3), (-3)/(-4), etc. as numbers that could not be assigned to ‘a/b’. Because negative whole numbers are the additive inverses of the natural numbers, we interpreted these responses as students’ tendency to change the sign that the given algebraic expression appears to have in order to give numbers that could not be assigned to it, while at the same time replacing the literal symbol itself only with natural numbers.

A one-way ANOVA compared the two questionnaires but no statistical differences were obtained on the use of natural numbers to replace literal symbols. In both questionnaires, students appeared reluctant to assign numbers other than natural to the given literal symbols. These findings support our hypothesis that students’ prior knowledge of natural numbers influences students in interpreting literal symbols to represent mostly natural numbers and most rarely fractions or real numbers, and in this way it can stand in the way of learning algebra.

Acknowledgments

The research was supported by a grant from the Program "Pythagoras - Support of University Research Groups" (EPEAEK II), 75% from European Social Funds and 25% from National Funds.

THE DILEMMA OF MATHEMATICAL INTUITION IN LEARNING

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The intellectual landscape in mathematics learning has shifted in the last several years. For many years it appeared that mathematics learning was not troubled by difficulties of misconceptions, such as had been documented for concepts in physics and certain other physical sciences. Detailed research (e.g., by Gelman & Gallistel, 1978; Greeno, 1983; Resnick, 1982, 1986; Vergnaud, 1982) on children's learning of counting, addition and subtraction suggested that children's intuitive knowledge of number actually supported their acquisition of mathematically correct forms of reasoning. Furthermore, similar forms of mathematically correct, but essentially intuitive forms of reasoning were shown to exist among children and adults with very minimal amounts of schooling (Nunes, Schliemann & Carraher, 1993). Some demonstrably effective programs in early mathematics education (e.g., Resnick, Bill, Lesgold & Leer, 1991) were built around the hypothesis that children's intuitive mathematical understanding could be recruited as the foundation for a (socio)constructivist form of mathematics teaching.

Perhaps the most ambitious and best documented approach to education based on children's intuitive mathematics knowledge was the Cognitively Guided Instruction (CGI) program developed by Carpenter and his colleagues (Carpenter, Fennema, & Peterson, 1987; Carpenter & Moser, 1984). CGI was actually not a program for children at all, but rather a system of professional development in which teachers studied the research on young children's developing mathematical ideas and learned how to interpret children's problem solutions. This was a "double-constructivist" approach: Depend on children to invent and justify mathematical solutions based on their intuitively developed knowledge; and depend on teachers to invent specific teaching strategies that respond to children's mathematical ideas and struggles. It worked very well for primary grade teachers and students, producing significant gains in student performance on several different kinds of assessments. But its effects were less strong and reliable for older students.

For a long time, the weaker effects of the CGI approach for older students was attributed to the lack of available research on students' conceptions of more advanced topics in the curriculum. A largely unarticulated assumption reigned—that approaches to teaching students and preparing teachers would not be fundamentally different as more advanced mathematical content came into play. Many mathematics educators (e.g., Davis & Maher, 1993; Kaput, 1987; Schwartz & Yerushalmy, 1992; Schwarz, Nathan, & Resnick, 1996) worked to build representational systems that would make more advanced concepts "visible" or "intuitive."

What most of the intuition builders did not anticipate, however, was that intuitively grounded mathematics might, in some cases, make it *harder* to learn more advanced mathematical ideas. Put another way, ideas that are well-founded for some early-learned domain of mathematics could actually get in the way of learning mathematical domains that were introduced later. The papers prepared for this forum show that we have to give up the simple assumption that earlier learning in mathematics always supplies a positive groundwork for later learning. The papers show that the concept of number that derives from principles of counting and enumeration works for the positive integers, but not for rationals, where an infinity of numbers exists between any two numbers (Vamvakoussi & Vosniadou), for principles of continuity and limit (Merenluoto & Lehtinen), or for Cantorian set theory and the comparison of infinite sets (Tirosh & Tsamir).

The papers also document situations in which early-taught formal mathematical concepts—such as proportionality—or the use of literals to "stand for" numbers in algebraic expressions overgeneralize and make it hard for students to appreciate more advanced concepts such as non-linear functions (Van Dooren, DeBock & Verschaffel) or algebraic expressions that represent non-natural numbers (Christou & Vosniadou). This kind of interference also extends to university-level geometric reasoning, where the concept of tangent learned in plane geometry with respect to circles requires radical, and cognitively difficult, revision when applied to function graphs in calculus (Biza & Zachariades). The conclusion is inescapable. Mathematics knowledge sometimes gets in the way of mathematics knowledge. What is not at all clear is what to do about it. This should hardly be a surprise. Even in physics, where we have known about unbudgeable misconceptions for a long time, thoughtful educators have not found a foolproof way of teaching certain core concepts. But we can engage in some grounded speculation, perhaps leading to empirical investigations. I will do this in the form of a few questions for our mutual consideration.

Is there a "best" developmental sequence for teaching mathematical concepts that will maximize positive effects of prior mathematical learning and minimize interference?

Building school mathematics on the foundation of intuitive concepts of number, addition and subtraction has proved powerful for the primary grades. Is there a way of formulating and sequencing later mathematics instruction to provide some of the same intuitive foundation (or at least less interference) for learning advanced concepts?

Is there a role for "direct instruction" in the competition between earlier mathematics concepts and the newer ones being introduced?

If a new concept is competitive with an earlier-learned concept, it might be sensible to tell students that they are now entering a "strange" world—much as they do in science fiction—and ask them to actively set aside their initial assumptions as they learn new rules and definitions and apply them to new situations. Students would still be actively "constructing" their understanding of the new concept, but they would know that they should actively suppress certain "old knowledge" in order to successfully play a new cognitive game.

Who should be the primary target of our interventions? Students directly – or teachers?

CGI worked by developing *teachers'* understanding of basic mathematics concepts and teaching them to interpret student responses in new ways. Could the same approach work for domains of mathematics in which student intuitions are discontinuous with the new concepts to be learned? How hard would it be to teach teachers themselves the new, discontinuous concepts? Is it best to encourage students and teachers to explore new domains together? What kinds of tools will they need?

Can mathematical formalisms be taught in ways that support cognitive development in mathematics?

Several of the conceptual change problems discussed in forum papers seem to revolve around brittle use of mathematical notations and formalisms. We used to think, when we were focused on initial mathematics learning, that the solution was to avoid formal notations during learning. Yet, for at least some domains of mathematics such as algebra, the formalism actually is the referent – for example, the generalized meaning of a literal expression is no number at all, no particular expression, but actually the literal itself. How might we teach mathematical formalisms so that they become part of students' attempts to understand new concepts, rather than barriers?

DESIGNING FOR CONCEPTUAL CHANGE

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Four design principles for teaching mathematics for conceptual change are proposed, with examples illustrating why they are needed. All are subsumed under the metaprinciple that a long-term perspective is essential.

Galileo, despite being a genius untrammelled by conventional thinking, could not comprehend how the number of points in a line could be considered equal to the number in a longer line (though he did have the considerable insight that there was something puzzling that he did not understand). Yet a student of today is expected to understand the resolution of this apparent paradox. As Sinclair (1990) pointed out, the challenge of mathematics education is that we expect children to master in a few years complexities that historically took millennia to evolve through collective conceptual change.

Of course, a fundamental difference between the historical development and that of the student is that the latter negotiates conceptual change under instruction. Thus, argues Freudenthal (1990, p. 48):

> Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now.

I propose some design principles for mathematics education for conceptual change, and one meta-principle. The first principle derives from an observation Thorndike made persuasively some time ago, that students learn inductively on the basis of the examples to which they are exposed. Indeed, in interpreting documented weaknesses such as many of those represented in this Research Forum, it might be sensible to consider this simple explanation routinely before looking for more complex ones. A further implication is that, when conducting research on conceptual change, attention should be given, as far as possible, to the instructional histories of the students.

The difficulties that children have in extending the additive and multiplicative conceptual fields beyond the natural numbers are amply documented (e.g. Greer, 1994). Arguably, a contributory factor in this case is that for several years in school (though not out of school) they encounter only (relatively small) natural numbers. As a result, the properties of natural number (such as that multiplication makes bigger, division makes smaller) become firmly entrenched.

A similar narrowness is pervasive in the numbers that appear within algebraic expressions (Christou and Vosniadou). Have you ever, for example, seen a quadratic equation like this: 2.67x2 – 3.86x – 12.23 = 0? Yet, with a calculator, there is no reason why a student should not be able to solve it.

On the basis of a multiplicity of such cases where, as David Tall once put it "We make it too easy for children to understand", I propose:

Design Principle No. 1: Given a space of examples, do not choose too narrowly.

A particular case to which this principle is applicable is the "illusion of proportionality" (Van Dooren, De Bock, Verschaffel). This term refers to the tendency of students to respond to word problems that have surface features suggesting proportionality by inappropriately applying proportionality, a tendency with strong historical echoes. In this case, the narrowness of sampling takes the form of exposing students only to cases where proportionality genuinely or supposedly provides an appropriate model, without any discrimination training using counterexamples.

A second design principle brings to mind Poincaré's definition of mathematics as "the art of giving the same name to different things". Consider the range of mathematical objects (still under development) to which people have attached the name "number". Or contemplate the many and varied usages of the word "tangent" (Biza & Zachariades). Accordingly, I propose:

Design Principle 2: Where possible, anticipate later expansions of meaning.

This principle addresses the problem raised by Fischbein (1987, p. 198):

> A certain interpretation of a concept or an operation may be initially very useful in the teaching process as a result of its intuitive qualities (concreteness, behavioral meaning etc.). But as a result of the primacy effect that first model may become so rigidly attached to the respective concept that it may become impossible to get rid of it later on. The initial model may become an obstacle which can hinder the passage to a higher-order interpretation – more general and more abstract – of the same concept.

This principle applies not only to words, but also to symbols. For example, there has been a wide body of work showing that children whose only experience of the use of the equals sign is interpretable as "makes" face later difficulties in algebra when another meaning, namely equivalence, is essential. In this case, it is easy to anticipate later expansions by including, from very early on, expressions such as $5 + 7 = 8 + 4$ and $7 = 2 + 5$. Indeed, given the multiplicity of other ways in which the equals sign is used in mathematics, other forward-looking moves of this nature might be advisable. This example also illustrates that, despite frequent assertions to the contrary, mathematical language, notation, and representations are inherently ambiguous. This ambiguity is an advantage to those who are familiar with it, can effortlessly disambiguate through contextual cues, and understand the underlying conceptual linkages, but for the novice it can be a major source of confusion.

Pre-emptive action may not always be possible. At first glance, it's not clear how one could prepare in advance for the conceptual change necessary to understanding the density of the real numbers or the basis for comparing infinite cardinalities (Tirosh & Tsamir, Merenluoto & Lehtinen, Vamvakoussi & Vosniadou). A second glance suggests that, in the former case, suitable software could enhance the intuition of continuous growth of a quantity and, in the latter, the method of comparing finite numbers through one-to-one correspondence could be given more and earlier prominence.

The next proposal is the obvious:

Design principle 3: Identify the points at which conceptual change is necessary and look for bridging devices.

For example, various such devices for expanding arithmetic operations beyond the natural numbers are sketched by Greer (1994), following the suggestion by Semadeni (1984, p. 381) of the "concretization permanence principle":

> Rather than forcing problematic concretizations for the extended operations or proceeding formally, the teacher is advised to start with some sound concretization which is familiar to the children within the original number range and which is capable of extension.

As has been pointed out by Fischbein (1987) and others (see Greer, 2004, p. 545) the need for radical conceptual restructuring cannot be avoided. One recourse suggested by Fischbein amounts to:

Design principle 4: Discuss with the students what is going on.

Thus, Fischbein's colleagues Tirosh and Tsamir have taught students Cantorian Set Theory while emphasizing identification of student intuitions and discussion of their origins, raising awareness of contradictions within the students' thinking, drawing attention to the trickiness of all attempts to characterize infinity, and deepening student's insights into the role of definitions.

As illustrated and recommended by Fischbein (1987) and others, there are many powerful ways in which the historical record can be mined to illuminate discussion of the nature of cognitive obstacles and the conceptual changes needed to overcome them. For example, as late as 1831 an eminent mathematician could write that "3 - 8 is an impossibility, it requires you to take from 3 more than there is in 3, which is absurd" (De Morgan, 1910 [originally 1831], pp. 103-104).

All of the above principles may be subsumed under:

Design metaprinciple: Take a long-term perspective.

One of Jim Kaput's many enduring legacies is his emphasis on this metaprinciple, on which he elaborated, in particular, in relation to calculus (Kaput, 1994) and arithmetic/algebra (Kaput, 1999). In relation to calculus, he stated (Kaput, 1994, p. 78) that:

> I look closely at the origins of the major underlying ideas of calculus for clues regarding how calculus might be regarded as a web of ideas that should be approached gradually, from elementary school onward in a longitudinally coherent school mathematics curriculum.

Countervailing forces to this perspective include the lingering effects of behaviorism in folk pedagogy, such as a belief in the obviousness of the principle of monotonic and incremental movement along a simple/complex dimension, and the short-termism engendered by the desire to maximize scores on the next test.

As sketched above, elements in implementing this metaprinciple include careful sampling from the valid range of examples, the anticipation of expansions of meaning, the search for bridging devices when such expansions are necessary, and the open discussion with students of why conceptual change is necessary, and often difficult, which takes them deep into historical and philosophical analyses.

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DISCUSSION GROUPS

DG01 *Abstraction in mathematics learning*

Coordinators: Mitchelmore, Michael & Ozmantar, Mehmet Fatih & White, Paul

DG02 *Indigenous communities and mathematics education: Research issues and findings*

Coordinators: Baturo, Annette & Amit, Miriam & Lee, Hsiu-Fei

DG03 *Participation, thought and language in the context of mathematics education (2nd year)*

Coordinators: da Rocha Falcão, Jorge Tarcísio & Frade, Cristina & Lins, Romulo & Meira, Luciano & Winbourne, Peter

DG04 *Facilitating teacher change*

Coordinators: Hannula, Markku S. & Sullivan, Peter

DG05 *Troubling learners' and teachers' relationships with mathematics: Theoretical perspectives on researching learning mathematics*

Coordinators: Cotton, Tony & Hardy, Tansy & Mendick, Heather & Povey, Hilary & Walshaw, Margaret & Hanley, Una

DG06 *Towards new perspectives and new methodologies for the use of technology in mathematics education*

Coordinators: Lins, Bibi & Giraldo, Victor & Carvalho, Luiz Mariano & Edwards, Laurie

DG07 *The mathematics textbook – A critical artefact?*

Coordinators: Pepin, Birgit & Grevholm, Barbro & Straessser, Rudolf

DG08 *Argumentation and mathematics education: Some unifying thoughts*

Coordinators: Schwarz, Baruch & Boero, Paolo

DG09 *Mathematics and gender: Setting research agendas*

Coordinators: Steinthorsdottir, Olof Bjorg & Rossi Becker, Joanne & Forgasz, Helen

DG10 *Learners learning through their own activity: multiple perspectives*

Coordinators: Simon, Martin A. & Dougherty, Barbara

DG11 *Mathematics education and society - MES*

Coordinators: Gates, Peter & Zevenbergen, Robyn

DG01: ABSTRACTION IN MATHEMATICS LEARNING

Michael Mitchelmore — Mehmet Fatih Ozmantar

Macquarie University, Sydney — Gaziantep University, Turkey

Paul White

Australian Catholic University, Sydney

This discussion group is a continuation of the group of the same name that first met during PME-29 in Melbourne. The RBC model of abstraction (Hershkowitz, Schwart & Dreyfus, 2001) and the Empirical Abstraction model (Mitchelmore & White, 2004) will provide the theoretical background. It will be assumed that participants are already familiar with these two models.

The aim of this year's meeting is to follow up some of the points raised last year, for example:

- What is the role of *contexts* in the abstraction process?
- What is the role of the *task* in leading students to the formation of abstractions?
- Is the abstraction process different in elementary and advanced mathematics?
- How can we design research so as best to study abstraction in action?
- What are the educational implications of the two models for the design of classroom teaching and learning activities?

A brief introduction will focus on the main features of the two models under discussion and clarify the questions to be discussed. A small number of invited speakers will then each briefly discuss one these questions, after which the group as a whole will react. Group members will be expected to contribute to the discussions insights or questions arising from their own research.

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DG02: INDIGENOUS COMMUNITIES AND MATHEMATICS EDUCATION: RESEARCH ISSUES AND FINDINGS

Coordinators:

Annette Baturo, QUT, Australia

Miriam Amit, Ben-Gurion University of the Negev, Israel

Hsiu-Fei Lee, National Taitung University, Taiwan

The aim of this Discussion Group is to build a community of PME members who have researched Indigenous mathematics education issues (or who would like to undertake research in the field but are unsure of the protocols involved) in order to support Indigenous mathematics learning students' mathematics outcomes and refine methodologies appropriate for the variety of Indigenous communities.

Research in Indigenous mathematics education has complexities that go beyond that of mainstream mathematics education. Smith (1999) argues that research should focus on improving the capacity and life chances of Indigenous peoples and that such research should be community-driven, collaboratively planned, executed and analysed in order to promote real power-sharing between the researched and the researcher. This second Discussion Group would like to focus on one or more of the following issues:

- the building of Indigenous community capacity through the development of mathematics programs;
- transition to school and early childhood mathematics for Indigenous children and their families;
- culturally-based mathematics programs for Indigenous children - advantages and disadvantages;
- the development of mathematical literacy in Indigenous children;
- respect for Indigenous knowledge / desire for school knowledge

DG03: PARTICIPATION, THOUGHT AND LANGUAGE IN THE CONTEXT OF MATHEMATICS EDUCATION (2ND YEAR)

Coordinators:
Jorge Tarcísio da Rocha Falcão (Universidade Federal de Pernambuco, Brazil),
Cristina Frade (Universidade Federal de Minas Gerais, Brazil),
Romulo Lins (Universidade Estadual Paulista/RC, Brazil),
Luciano Meira (Universidade Federal de Pernambuco, Brazil),
Peter Winbourne (London South Bank University, UK)

The works of this DG started in PME-29, Melbourne, 2005, under the coordination of Jorge Falcão, Cristina Frade and Steve Lerman (LSBU, UK). A range of theoretical perspectives on thought and language were discussed including those of Etissier, Piaget, Vergnaud, Vygotsky, Polanyi, Bakhtin, Wells and Edwards. The group kept the discussion grounded by giving examples from some coordinators' researches. A number of participants expressed a desire to continue the group's work and suggested, for PME-30, to focus on an important aspect of the issue of thought and language, that of ***the differences and similarities between practical knowledge and theoretical knowledge***. To launch the discussion of this thematic the DG coordinators/PME members propose the following questions:

1. Under what perspective (or perspectives) is pertinent to talk on practical knowledge, theoretical knowledge and transfer of learning/knowledge?

2. Regarding the methodological point of view what would be the crucial elements to be considered for a semiotic exploration of tacit/practical knowledge?

3. What is the theoretical status of non-explicit pragmatic abilities of illiterate mathematical users (e.g. carpenters dealing with geometrical concepts of area and/or perimeter, third world "children of the streets" dealing with money in real business contexts)?

4. Different scenarios and contexts of mathematical activity make up for different 'language games', as performed by agents who themselves continually reinvent the games' own particular nature. How to describe the dynamics that contingently governs the process of approximating and distancing language games usually played in disparate scenarios and contexts, as in the case of mathematical activity in- and out-of-school?

5. What are the relevant consequences of this discussion to mathematics education?

With the aim to discuss these questions, the group will focus both on some theoretical perspectives (Gerard Vergnaud, Michael Polanyi, Paul Ernest, Jean Lave, Etienne Wenger, Basil Bernstein, Ludwig Wittgenstein, Mikhail Bakhtin, among others) concerning the conceptualization of practical knowledge, theoretical knowledge and other related concepts, and on some classroom and out-of-school data to be examined, according to these perspectives.

DG04: FACILITATING TEACHER CHANGE

Markku S. Hannula[1] and Peter Sullivan[2]

[1]University of Helsinki, Finland and University of Tallinn, Estonia
[2]La Trobe University, Australia

The intention of both pre-service and in-service teacher education as well as that of many interventions in schools is to promote some kind of change in teachers. This change can be an increase in knowledge and skill, but also it can be changes in the (student) teachers' emotional disposition, beliefs or classroom actions. Various case studies suggest that it is possible to influence knowledge, attitudes and/or practices of (student) teachers and many educators have developed their own techniques for changing (student) teachers. For example Smith, Williams and Smith (2005) and Senger (1999) have constructed models for teacher change. This discussion group will consider the nature of such changes and processes for measuring and reporting on such changes.

We can distinguish, for example, professional development and a 'therapeutic' approach as types of approach to facilitating teacher change. There are several practical problems in facilitating such changes, especially if changes require a radical conceptual change (e.g. in teaching philosophy) or a change in psychologically central parts of the affective domain (e.g. identity). There are also ethical questions about the appropriateness of imposing a change that has not been initiated by the (student) teachers themselves. There are also methodological considerations about ways of measuring and reporting on changes, recognising that self report, especially after some intervention, may be unreliable.

This discussion group will continue the discussions initiated at PME29, where we identified, for example, different catalysts and inhibitors for teacher change. The aim for the discussion group is to bring together educators with different theoretical approaches to teacher change, in order to allow 'theoretical triangulation'.

We invite people to share their own experiences of and views about facilitating and researching teacher change. To allow maximum involvement of participants we will alternate between whole-group and small-group discussions.

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DG05: TROUBLING LEARNERS' AND TEACHERS' RELATIONSHIPS WITH MATHEMATICS: THEORETICAL PERSPECTIVES ON RESEARCHING LEARNING MATHEMATICS

Tony Cotton[1)], Tansy Hardy[2)], Heather Mendick[3)],
Hilary Povey[2)], Margaret Walshaw[4)], Una Hanley[5)]

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A growing number of researchers working within mathematics education are interested in exploring the nature of the relationships that learners form with mathematics, productive and unproductive, and the reasons why some people appear not to form the kind that would enable them to derive pleasure and success from the subject. This work has drawn on socio-cultural theories and poststructuralist framings of identity to make sense of people's developing relationships with mathematics. This discussion group brings together some of these researchers to put their work in conversation with each other and, in so doing, to generate a discussion with others interested in processes of identity production in mathematics education. In previous years discussion groups at PME have engaged with critical perspectives on the social and political in mathematics education and its research. The PME 26 discussion group 'identified the need for us to contribute to a shift in the dominant discourse of the mathematics education research community itself'. This work has continued in subsequent conferences. This proposed discussion group is intended to explore further how the shifts the researchers in this group have made in their theoretical framings can offer alternative, more multivalent understandings of learners' relationships with mathematics. In doing so we seek to generate a conversation about which new theoretical resources can substantially change the way we think about the relationships between people and knowledge in mathematics education, and mathematical identity in particular.

The contributors to this discussion group feel that mathematics offers a valuable site for exploring and developing such theories and for investigating their value for education research. Extracts from our work will be used in the 1st session to explore how different theoretical approaches act to constrain the ways in which we see learners of mathematics and how we imagine both their and our own possibilities for action. This will draw on theoretical perspectives including feminist theory, psychoanalysis and post-structuralism, and will include experiments with narrative/storying as an analytic methodology. In the 2nd session we will develop the discussion to consider changes in our understandings of relationships with mathematics to explore the question, "How do the theoretical frameworks we have brought into operation reconfigure the power relations circulating in mathematics teaching and learning situations?" Through this question we will explore reasons why some learners do not form productive or enriching relationships with mathematics. We will discuss what we - as teachers, teacher educators, and researchers - might want and be able to do about it.

DG06: TOWARDS NEW PERSPECTIVES AND NEW METHODOLOGIES FOR THE USE OF TECHNOLOGY IN MATHEMATICS EDUCATION

Bibi Lins	Victor Giraldo	Luiz Mariano Carvalho	Laurie Edwards
UNICSUL	UFRJ	UERJ	Saint Mary's College

At PME 28 we started the discussion group aiming to initiate a dialogue that moves away from current methods and frameworks to new perspectives and new methodologies for considering the use of technology in mathematical education. Three general questions led the discussion:

- What perspectives are used to investigate the use of technology in Mathematics Education in different countries?
- How would new perspectives allow us to re/think the role of users of technology?
- What new methodologies would enable us to investigate difficult issues concerning teaching and learning situations in microworlds environment?

The first session went as freely as possible for encouraging the participants to speak about their own work, own perspectives and views about Technology: its use and the role of its users. There were about 20 participants who vivid engaged in the discussion while listening to each other's views. We spent most of the session on this discussion, leaving the last five minutes to decide what "we" would be doing about the second session. The "conversation" was very fruitful for all participants as a way of knowing where each of us come from in terms of perspectives and methodologies. This session served as a background to what this discussion group could come to be and what direction it could take.

In the second session, Lins was asked to present some of the known approaches about Technology and introduced the approach of treating Technology as Text and users as readers from an Anti-Essentialist viewpoint (Lins 2002, Woolgar 1997) to be discussed within the group. The discussion was about four different approaches to Technology: technological determinism, social shaping, actor-network and technology as text.

As it came to be a quite stimulating discussion, the coordinators were strongly asked to carry on the discussion group to the PME 29 and gradually to build up what "we" would like to do and to take from it. Unfortunately this could not happen.

For the PME30 we hope we can carry on the DG and build up what we had in mind at PME28.

DG07: THE MATHEMATICS TEXTBOOK - A CRITICAL ARTEFACT?

Birgit Pepin — Univ. of Manchester, UK
Barbro Grevholm — Agder Univ. Coll., Norway
Rudolf Straesser — Giessen Univ., Germany

Acknowledging the importance of textbooks in the school environment, the purpose of this discussion group is to bring together researchers, students and practitioners at national and international level who are interested in exploring mathematics textbooks and their use in the classroom.

Investigations in several countries reveal that textbooks are still dominating the work in many mathematics classrooms (Valverde et al, 2002). Teachers use them as a guiding artefact in planning for long- and short-term purposes. The textbook has been regarded both as the authority in terms of mathematical knowledge and as the de facto national curriculum (Mayer et al, 1995). Students may use them as an instrument to indicate that they fulfilled the work needed.

Whilst there have been many large scale mathematics textbook studies in the US (e.g. Valverde et al, 2002), there have been relatively few studies in the European context. At secondary level Pepin and Haggarty (2001) investigated mathematics textbooks and their use in English, French and German classrooms. Recent studies in Sweden inquire into textbooks as the potentially implemented curriculum and investigate differentiated tasks (Johansson, 2003; Brandstrom, 2005).

Recognising the importance of textbooks we believe that there is not enough research about the inner structure and the way textbooks are developed and used. Research needs to explore the dependency or autonomy of textbooks on curricula, into the way they influence, if not control the teachers' and students' everyday work in class or at home. We have created a network of researchers and doctoral students interested in textbook research and identified a number of questions for discussion. We would like to open our discussion to a wider audience through a Discussion Group at PME-30.

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DG08: ARGUMENTATION AND MATHEMATICS EDUCATION: SOME UNIFYING THOUGHTS

Organizers: Baruch Schwarz and Paolo Boero
Hebrew University, Jerusalem and University of Genova

Argumentation has been used in mathematics by many researchers as a vehicle of shared understanding through which learning emerges. Argumentation is rooted in divergent philosophical traditions, though. Also, it has been used by psychologists with different theories on learning and development. Research in mathematics education and argumentation is mature enough to reflect on central issues on which there is agreement and disagreement. Leading researchers have agreed to take part in this discussion group in two 90 minutes long sessions. The discussion will focus on several issues

1) The definition of argumentation: Several divergent definitions have been given by argumentation theorists such as Perelman, Toulmin, van Eemeren and Antaki in the last decades. Mathematics educators have used some of these definitions and neglected others. A thorough discussion is needed to understand the definitions and to reflect on some conflicting results that the choice of definitions has implied.

2) The goals and functions of argumentation: accommodating divergent views, understanding, convincing, wining, etc. We will

3) The role of argumentation in construction of knowledge and in learning in general. For example, we will focus on classroom discussions and small group collaborative problem solving in which different types of argumentative talk may emerge. An overview of empirical results in mathematics education will be presented. This overview will be undertaken in light of a reflection on the definition of argumentation adopted in the studies reviews and the goals and functions of argumentation.

4) The relation between argumentation and rationality (according to Habermas' definition of "rational behaviour"), especially related to "proving", but in the perspective of more general issues (concerning intercultural studies). We will stress the gap between argumentation in informal settings (conversations, dinner, etc.) and argumentation at school

5) Methodological issues concerning the study of the role of argumentation in construction of knowledge, learning and development. Since argumentation is generally a social activity during which participants

6) Argumentation and the design of activities. Since argumentation cannot be considered as a manipulation imposed on students but rather as a natural commitment to accommodate divergent or different views, we will discuss the issue of the design of activities that invite students to engage in argumentation: confronting students with different mental models, the use of hypothesis testing devices, the role of the teacher in classroom discussions, the presentation of controversies, etc.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 194. Prague: PME.

DG09: MATHEMATICS AND GENDER: SETTING RESEARCH AGENDAS

Olof Bjorg Steinthorsdottir, University of North Carolina, USA/Iceland
Joanne Rossi Becker, San José State University, USA
Helen Forgasz, Monash University, Australia

In 2005 we had a lively discussion group that centered on three areas of interest: intervention strategies that might be used in countries such as Korea with large extant gender differences in achievement; how to study linkages among gender, ethnicity and socio-economic status; and setting a research agenda for future work on gender and math. The group felt that our work on these important topics had barely begun, so we propose to continue the discussion group with the aim of setting some agendas for research in various parts of the world where such work is most pertinent relative to these three themes.

Two students, Crystal Hill and Beverly Bower Glienke, at UNC Chapel Hill, did a search for articles in mathematics education that included gender and interaction between gender and factors such as ethnicity in a collection of eighteen peer-reviewed and research-based journals from 1990 to present. Their results were staggering: out of eighteen peer-reviewed and research-based journals, only twenty-six mathematics articles focused on gender differences. Studies that study specifically the interaction of gender and ethnicity were more scarce. Out of twenty-six articles, there were five articles that focused specifically on African American students. Despite that their emphasis focused on African American students, they noticed that other ethnic groups had limited representation. Their results was presented at the annual meeting of the NC Association for Research in Education, in March 2006.

<u>Activities:</u> We will begin with brief introductions and a short presentation on the paucity of work that has studied the interaction of gender, ethnicity and socio-economic status to stimulate discussion. Depending on the size of the group, we may break into small groups to discuss critical questions such as those posed below or others that emerge from the participants. Small and large group discussions will be synthesized into key ideas for continued discussion, possible joint research, or future action.

How do we develop research alliances that allow for the study of gender, ethnicity, and socio-economic status?

What inhibits or encourages data being collected across all of these dimensions?

Who has influence at the state and/or national level on the mathematics curriculum and/or the assessment program? Is gender a factor here? How can we impact such programs?

What methodological approaches and theoretical framework(s) would enable us to investigate difficult and unresolved issues concerning gender especially as they relate to ethnicity and socio-economic status?

DG10: LEARNERS LEARNING THROUGH THEIR OWN ACTIVITY: MULTIPLE PERSPECTIVES

Co-ordinator: Martin A. Simon
Penn State University

Asst. co-ordinator: Barbara Dougherty
University of Mississippi

Vygotsky and Piaget made a similar distinction between scientific concepts and spontaneous concepts. It is learning of the former that poses the challenge for mathematics education. Learners seem to learn mathematics through their own mathematical activity and through participation in discourse with other learners and more knowledgeable others (e.g., teachers). How do we understand these processes? How can we plan for learning opportunities that will maximize their learning of important mathematics?

This discussion group will focus on how students learn from their own mathematical activity. It is in understanding better the processes by which learners learn through their activity that mathematics educators can develop a more scientific approach to the design, selection, sequencing, and modification of mathematical tasks for student learning.

The coordinators originally approached this issue from different theoretical orientations. Simon and his colleagues worked on this problem based on the Piagetian constructs of assimilation and reflective abstraction. Dougherty and her colleagues based their work on Davydov's approach to teaching primary-school mathematics grounded in activity theory (deriving from the work of Vygotsky). Despite the different theoretical starting points, Simon and Dougherty have noticed an important confluence of ideas. This discussion group can provide a venue for further exploration of these issues.

The aims of the discussion group are to:

- Foster among an international group a discussion of work that is contributing to or has the potential to contribute to a collective understanding of the processes by which learners learn from their own activity.
- Discuss implications of this emerging understanding for elaborating a pedagogical framework on the design, selection, sequencing, modification, and implementation of mathematical tasks for student learning.
- Consider the confluence and contrasts with respect to these issues that result from different theoretical perspectives, including but not limited to activity theory and constructivism.

Each of the co-ordinators will lead one of the sessions, providing a brief set of ideas and questions to be considered and an example or two from recent research data to provide a common focus for discussion.

2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 196. Prague: PME.

DG11: MATHEMATICS EDUCATION AND SOCIETY - MES

Peter Gates and Robyn Zevenbergen

University of Nottingham, UK / Charles Sturt University, Australia

This discussion group will be an important opportunity for PME participants to meet and discuss work in a broad set of domains which until recently have not been within the remit of PME but which many PME members have worked for some time. With the change in aims of PME at PME29, it now becomes possible for PME conference activities to engage with research in the social and political dimensions of mathematics education. Mathematics education is a key discipline in the politics of education acting as a gatekeeper to employment and further education. Thus, managing success in mathematics becomes a way of controlling the employment market. Mathematics education also tends to contribute to the regeneration of an inequitable society through undemocratic and exclusive pedagogical practices which portray mathematics and mathematics education as absolute, authoritarian disciplines.

There is a need for discussing widely the social, cultural and political dimensions of mathematics education; for disseminating research that explores those dimensions; for addressing methodological issues of that type of research; for planning international co-operation in the area; and for developing a strong research community interested in this view on mathematics education. The First International Conference on Mathematics Education and Society took place in Nottingham, Great Britain, in 1998; subsequent Conferences were held in Portugal (2000); Denmark (2003); Queensland (2005). On all four occasions, participants from around the world had the opportunity of sharing research, perspectives and theoretical orientations concerning the social, political, cultural and ethical dimensions of mathematics education and mathematics education research. This discussion group aims to bring together such researchers to offer a platform on which to build future collaborative activity within PME and beyond.

Drawing on work and collaboration already undertaken at the four previous conferences, the MES Discussion Group will focus on a central discussion theme: "*Power, politics and research*". There is much interest in many countries over the ethics of educational and social research. There is a sense of needing to be ethical in research, but there is an uncertain line between 'ethical action/thought" and compliance. As conservative agendas take over more and more of educational practice and policy, the implications for the conduct of research are becoming more profound. How does one 'ethically' research aspects of power, disadvantage etc.

In addition, the group will consider the future of the MES group both outside and inside PME.

WORKING SESSIONS

WS01 *Designing mathematical research-situations for the classroom*

Coordinators: Knoll, Eva & Ouvrier-Buffet, Cécile

WS02 *COSIMA – An opportunity for mathematical communication*

Coordinators: Cockburn, Anne D. & Peter-Koop, Andrea

WS03 *What is effective mathematics teaching? East meets West*

Coordinators: Cai, Jinfa & Perry, Bob & Ying, Wong Ngai & Kaiser, Gabrielle & Maass, Katja

WS04 *Gesture, multimodality, and embodiment in mathematics*

Coordinators: Edwards, Laurie & Robutti, Ornella & Bolite Frant, Janete

WS05 *Teaching and learning mathematics in multilingual classrooms*

Coordinators: Barwell, Richard & Civil, Marta & Díez-Palomar, Javier & Moschkovich, Judit & Planas, Núria & Setati, Mamokgethi

WS06 *Complexity research and mathematics education*

Coordinators: Davis, Brent & Simmt, Elaine & Sumara, Dennis

WS07 *Purposeful professional development for mathematics teacher educators*

Coordinators: McMahon, Teresa & Sztajn, Paola & Ghousseini, Hala & Loewenberg Ball, Deborah

WS08 *Intuitive vs. analytical thinking: A view from cognitive psychology*

Coordinators: Ejersbo, Lisser Rye & Inglis, Matthew & Leron, Uri

WS01: DESIGNING MATHEMATICAL RESEARCH-SITUATIONS FOR THE CLASSROOM

Eva Knoll[1] and Cécile Ouvrier-Buffet[2]

[1]Mount Saint Vincent University, Halifax, Canada
[2]Didirem Paris 7, France

> "Freudenthal claimed that mathematical practice often led to a didactical inversion in which the genetic sequence was reversed in exposition. While the psychological processes are those that create the product, the ordinary conventions of lecturing offer the product first and then expect the genetic processes to take place as the succeeding exercises are attempted" (Burn, 2002, p. 21).

In several parts of the world, there has been a trend towards the creation of learning environments which attempt to reverse Freudenthal's inversion by engaging students in a manner similar to that experienced by active research mathematicians. Examples of this include NSF sponsored programmes for undergraduate students, and the French "Math.en.Jeans", which runs programmes involving schools at various grade levels. Fundamentally, the offered experience is intended to expand student awareness of mathematics as a discipline, and of its practice.

The aim of this working group is to assemble researchers interested in investigating the educational outcomes of these experiences, and in developing criteria for their design and evaluation, geared towards students at various levels of schooling.

The proposed sessions will begin with the enactment of an exemplary situation, representative of the learning experience under discussion, offering participants a direct experience of mathematical research as practiced by professional researchers. In the remaining available time, the situation will serve as the basis for a forum on educational considerations stemming from such experiences. These considerations will include the heuristic strategies that were used in the research situation, the construction of mathematical knowledge that can emerge from these experiences, and epistemological and didactical considerations of programmes that propose these experiences.

Participants will be invited to establish a community of scholars interested in an ongoing dialogue focusing on these issues and propose contexts for the experimental evaluation of the resulting theoretical frameworks.

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WS02: COSIMA – AN OPPORTUNITY FOR MATHEMATICAL COMMUNICATION

Anne D. Cockburn
University of East Anglia, UK

Andrea Peter-Koop
University of Oldenburg, Germany

This working session arises from the 3-year cross-cultural COSIMA project (http://www.cosima-project.org/) whose primary aims are:

- to highlight the importance of developing classroom communication processes in socio-constructivist mathematics teaching and learning.
- to increase teachers' awareness of the need for communication strategies in the primary mathematics classroom not only through verbal and written explanations but also extending beyond these

The workshop sessions are intended to trial and discuss activities designed to:

- enhance teachers' ability to use communication to increase pupils' autonomy when learning mathematics through the use of challenges and meta-tasks.
- provide 'ready for classroom use' ideas, contexts, tools and classroom materials for learning environments in primary mathematics that foster the communication of students' strategies.

Participants will engage in learning environments ranging from traditional subtraction (to explore how teachers might expand their repertoire and thus communicate key underlying concepts more effectively); to games (to improve children's ability to classify); to computer simulations (to visualize, explore and communicate functional relationships as well as the structures of 3-dimensional objects) and paper folding (to develop children's understanding of shape and space.).

The goal of the workshop is to evaluate and critically assess the proposed learning environments in terms of their role in teacher pre- and in-service education.

WS03: WHAT IS EFFECTIVE MATHEMATICS TEACHING? EAST MEETS WEST

Jinfa Cai	Bob Perry	Wong Ngai Ying	Gabriele Kaiser	Katja Maass
University of Delaware	University of Western Sydney	Chinese University of Hong Kong	University of Hamburg	University of Education Freiburg

Cultural beliefs about teaching do not directly dictate what teachers do, but it is often argued that teachers do draw upon their cultural beliefs as a normative framework of values and goals to guide their teaching. Although there is no universal agreement about what effective mathematics teaching should look like, no one questions the idea that the teachers' instructional practices are influenced by their cultural conceptions of effective teaching. Thus, a fundamental question is: What is effective teaching for teachers in different countries?

The coordinators of this Working Session have gathered interview data from outstanding elementary/middle school teachers in Australia, Mainland China, Hong Kong SAR-China, and United States of America on what the teachers perceive an effective mathematics teacher to be and do. These four regions/countries were chosen to represent a spectrum of East and West cultures. The coordinators are keen to share these data with a group of PME participants who are willing to bring their own perspectives to the data, to extend interpretations of the data and to consider ways in which the findings might be used to further understand similarities and differences in mathematics teaching in the East and West.

The two working sessions will be devoted to gaining a greater understanding of the cultural nuances of mathematics teaching and how these might affect practice.

In the first session, the data will be shared with all participants and carefully analysed in terms of both what they say about teaching mathematics and how they might relate to cultural issues and histories in the countries of origin. This analysis will then be applied to the experiences of the participants in order to extend the group's understanding of cultural beliefs about teaching mathematics.

The second session of this working group will: (1) engage participants by sharing their research and experience about effective teaching in respective countries/regions and (2) invite participants to reflect on the research methodology and data analysis from the first session. Participants will be encouraged to collect data using similar approaches in their respective countries/regions. The working session will conclude with the coordinators outlining their plans for a special issue of *Zentralblatt für Didaktik der Mathematik (ZDM)* on the topic of the working session. The coordinators of the working session will also engage with participants to make specific plans for potential collaboration on the topic for another special issue of *Zentralblatt für Didaktik der Mathematik (ZDM)* or a book.

WS04: GESTURE, MULTIMODALITY, AND EMBODIMENT IN MATHEMATICS

Laurie Edwards
St. Mary's College of California, USA

Ornella Robutti
Università di Torino, Italia

Janete Bolite Frant
Pontificea Universidade Catolica do Rio de Janeiro, Brasil

The goal of the Working Session is to deepen the investigation of mathematical thinking, learning, and communication by considering the variety of modalities involved in the production of mathematical ideas. These modalities include gesture, speech, written inscriptions, and physical and electronic artefacts. The central purpose will be to examine how basic communicative modalities such as gesture and speech, in conjunction with the symbol systems and social support provided by culture, are used to construct mathematical meanings. In addition, the role of unconscious conceptual mappings such as metaphors and blends will be investigated in relation to gesture and the genesis of mathematical concepts. Relevant theoretical and empirical work has been carried within cognitive linguistics (Lakoff & Núñez, 2000), semiotics (Radford, 2002) and psychology (McNeill, 1992). Themes and questions to be addressed include:

- How do gestures relate to speech, writing (eg, of formulas), drawing, graphing and other modalities of expression?
- How do gestures condense/manage information during social interaction?
- How are conceptual metaphors and blends involved in students' cognitive processes while learning and doing mathematics, in different settings?
- How do gestures and unconscious conceptual mechanisms relate to external representations and technologies used in mathematical activity?

The Working Session will consist primarily of small groups working together to: (1) make progress in answering one of the above (or a related) question; and (2) engage in collaborative analysis of videotaped or other data showing the use of various modalities in mathematical activity.

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WS05: TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS

Richard Barwell, University of Bristol, UK

Marta Civil, University of Arizona, USA

Javier Díez Palomar, University of Arizona, USA

Judit Moschkovich, University of California, Santa Cruz, USA

Núria Planas, Universitat Autonoma de Barcelona, Spain

Mamokgethi Setati, University of the Witwatersrand, South Africa

Multilingualism is a widespread feature of mathematics classrooms around the world. The place of multilingualism in education and in wider society is, however, an intensely politicised issue. In North America and Europe, fierce, often acrimonious debates continue as to whether school students should be able to use minority languages in their learning of mathematics and other subjects. In Africa and Asia, meanwhile, a similar debate arises concerning the different possible roles of local and regional languages and English as a language for schooling. The place of mathematics in research in multilingual classrooms is subtly bound up with these debates. Thus, for example, analysis of multilingual mathematics classroom interaction is sometimes accused of overlooking mathematics. The aim of this working group, therefore, is to explore the links between mathematics and the politics of language use in multilingual settings.

The two working sessions will be devoted to analysing video and transcript data concerning the teaching and learning of mathematics from a range of contexts, including the USA, South Africa and the UK. We will offer various ideas from discourse analysis and socio-linguistics as analytic tools. The activities will be organised around three key questions:

- How does an analysis of language use reveal anything about mathematics?
- How can analysis of mathematics classroom interaction take account of broader linguistic forces, through which, for example, some languages may be silenced?
- How does language use empower or disempower students and teachers and what implications does this have for the learning of mathematics?

By exploring these questions and the relationship between them, we hope to highlight more general underlying issues of relevance to mathematics classrooms around the world.

WS06: COMPLEXITY RESEARCH AND MATHEMATICS EDUCATION

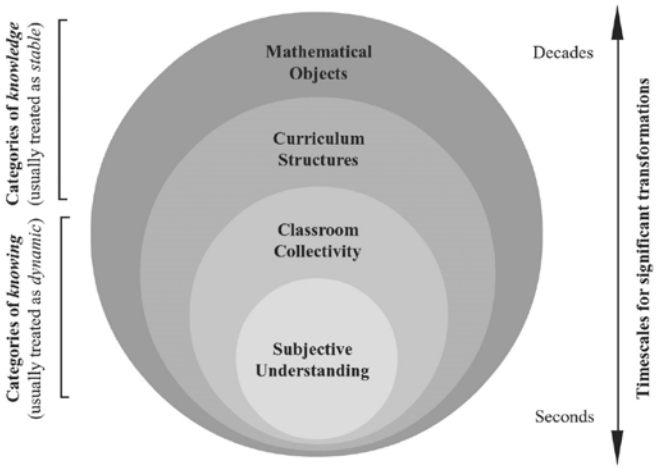
Coordinators: Brent Davis, Elaine Simmt, & Dennis Sumara
Assistants: Mary Bieseigel, Helena Miranda, Elizabeth Mowat, & Jérôme Proulx

University of Alberta

AIMS

The purpose of this working session is to explore the utility of an explicit complexivist theoretical frame in mathematics education research, principally as a means to interrogate junctions/disjunctions of contemporary discourses. Complexity theories are concerned with self-organizing (emergent), adaptive forms and/or events—that is, with phenomena that might be described in terms of 'learning systems' or 'learners.' With regard to mathematics education, examples of complex phenomena include individual knowers, bodies of knowledge, and social systems.

Among mathematics education researchers, the theoretical frames that have been adopted and adapted to study complex phenomena include subject-centred constructivisms and sociocultural theories. Oriented by the principle that complex phenomena cannot be collapsed into one another, but must be studied at the levels of their emergence, these theories might be described as specific examples of complexity discourses. The session will be organized around a connected series of discussions of how complexity thinking might be used to 'reach' across current frames and theories. We also attend to perhaps under-represented cases of emergence (see, e.g., the diagram, in which class & curriculum dynamics are identified as possible *learners*.)



PLANNED ACTIVITIES

While we will offer a brief introduction to complexity discourses, participants are encouraged to visit the complexityandeducation.ca website prior to the conference. Video clips and brief interpretive presentations will be used to frame and illustrate discussions of varied complex phenomena (e.g., as indicated in the diagram, above). Small- and whole-group interactions will be principally concerned with raising questions, offering critiques, and locating topics in current research literatures. During break-out discussions, at least one member of the team will participate in each group, taking notes and crafting summary posters of key issues and insights. These posters will then be used to focus attentions during a final plenary discussion of the utility of complexity thinking for mathematics education research and practice.

 2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 206. Prague: PME.

WS07: PURPOSEFUL PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHER EDUCATORS

Teresa McMahon, Paola Sztajn,* Hala Ghousseini, & Deborah Loewenberg Ball

University of Michigan and University of Georgia*

Examining a professional development experience for mathematics teacher educators, we discuss theories of design, identify five features used to enhance participants' ability to study teaching, and explore participants' interactions with these learning opportunities. We will study video of practice during this session.

OVERVIEW AND SIGNIFICANCE

Just as teachers need opportunities to develop skills to teach mathematics that is useful to students, mathematics teacher educators need purposeful opportunities to develop skills to teach mathematics that is useful for teaching. Van Zoest (2005) reports how difficult it can be for accomplished school mathematics teachers to teach prospective teachers in mathematics methods courses; their experience teaching students does not necessarily equip them with the explicit knowledge or skills to teach teachers. Without deliberate opportunities, teacher educators are not fully prepared to work with teachers. In June 2004, the Center for Proficiency in Teaching Mathematics—an NSF-funded initiative—designed and studied a week-long institute. Sixty-eight teacher developers—mathematicians, mathematics educators, and school-based teacher educators—participated. A core component of the curriculum was a laboratory class of 18 prospective elementary teachers, which provided a practice-based focus for the experience. The education of mathematics teacher educators or the development of professional developers is just beginning to be conceptualized (Cochran-Smith, 2003; Stein, Smith & Silver, 1999). In this session, we begin to conceptualize what mathematics teacher educators need to know and how they might best learn it.

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WS08: INTUITIVE VS. ANALYTICAL THINKING: A VIEW FROM COGNITIVE PSYCHOLOGY

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The gap between intuitive and analytical thinking is of fundamental concern for mathematics education research and practice, and has been studied by several researchers, including Fischbein, Skemp, Stavy & Tirosh, and Leron & Hazzan. In this Working Session we'll introduce and discuss a related influential research paradigm from contemporary cognitive psychology, called *Dual Process Theory* (*DPT*). (e.g. Kahneman's 2002 Nobel Prize lecture in economics, accessible from the link below.) According to this theory, our cognition operates in parallel in two quite different modes, called *System 1* (*S1*) and *System 2* (*S2*), roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts in the brain, and have different evolutionary origins (S2 being evolutionarily more recent and, in fact, largely reflecting *cultural* evolution). S1 processes are characterized as being fast, automatic, effortless, "cheap" in terms of working memory resources, unconscious, and inflexible. In contrast, S2 processes are slow, conscious, effortful, and relatively flexible. In addition, S2 serves as monitor and critic of the fast automatic responses of S1, with the "authority" to override them when necessary. In daily life, the two systems mostly work in concert, but research in cognitive psychology has exhibited many situations in which S1 produces quick automatic *non-normative* responses, while S2 may or may not intervene in its role as monitor and critic.

The work of the WS will consist of discussing and analysing – in small groups and in plenum – examples from the mathematics education research literature and from the participants' experience. These examples will be used to illustrate the use of DPT as a theoretical framework for interpreting empirical data in mathematics education research. In the process we will consider the following questions:

- How does DPT compare to the familiar intuitive/analytical distinction?
- What new perspectives are offered by DPT on human cognition in general and on mathematical cognition in particular?
- What new perspectives can DPT offer on mathematics education *research*?
- What new interesting research *questions* may be suggested by DPT?
- What new interesting research *methods* may be suggested by DPT?
- Can DPT offer new insights for mathematics education *practice*?

For more information on the discussion group, and relevant references, see http://www.warwick.ac.uk/go/pme-ws-dpt

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SHORT ORAL COMMUNICATIONS

THE MATHEMATICS TEACHERS' CONCEPTIONS ABOUT THE POSSIBLE USES OF LEARNING OBJECTS FROM RIVED-BRAZIL PROJECT

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The present study investigates the Mathematics teachers' conceptions about the possible uses of learning objects from RIVED-Brazil project as potential aided resources in the Mathematics presential teaching and learning process using computational environments.

In order to identify such conceptions, we assumed a qualitative approach through case study, of RIVED-Brazil project, adopting semi-structured interviews as our data collecting technique based con TRIVIÑOS (1987) perspective.

Interested in this reflection, we analyse the current practices of the interviewed teachers, their intentions and ideal expectations about the tools, resources, technologies and environments as well as, after these points of view, we present the two educational modules selected for this research, intending identify which are the possibilities that, for these Mathematics teachers, can immerge from the use of them.

Our aim is to study the potential contributions that can occur from the integration among the use of Mathematics learning objects, which belong to the educational modules selected for this research, and the expectations and current teaching practices of the interviewed teachers. In order to do that, three Mathematics teachers were interviewed and two educational modules were selected from RIVED-Brazil project, analysing all of that under some aspects related to Activity Theory based on ENGESTRÖM (1999) perspective, mainly focusing on the expansive cycle definition. The teachers had identified new chances of education situations thus making possible better integration between practical and theory.

Keywords: Mathematics Education; Learning Objects; Mathematics Teachers' Conceptions; Technological Education; RIVED-Brazil.

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ON THE WAY TO UNDERSTANDING INTEGRATION

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In order to promote learners' appreciation of mathematical topics and to enrich their experience of learning mathematical concepts to the fullest, it is necessary to construct opportunities for them to make use of their powers to contact important mathematical structures and to become aware of dimensions or aspects of the concept. Here I report on a study which probes learners' awareness of the mathematical topic of integration. Twelve pairs of students studying engineering, mathematics and education have been invited to construct relevant mathematical objects meeting specified constraints. This is just one component of the development and exploitation of a rich framework for what it means to understand and appreciate a mathematical topic.

CONSTRUCTING MATHEMATICAL OBJECTS

Learners' engagement in activities that reveal something about their awareness can inform both teachers and researchers about how learners' understanding is currently structured and can inform appropriate teacher intervention. This presentation reports on a phenomenographic study which probes learners' awareness of the mathematical topic of integration. Twelve pairs of students studying engineering, mathematics and education were interviewed regarding their understanding of integration and were invited to construct relevant mathematical objects meeting specified constraints, following Watson and Mason (2005). The study aims to reveal how learners' understanding is structured, which contributes to the development and exploitation of a rich framework referred to as 'structure-of-a-topic' (Mason, 2002) for what it means to understand and appreciate a mathematical topic. Learners seem to focus their attention mainly on one aspect of the topic, namely techniques and overlook other important connections and associations. Constructing mathematical objects that meet certain constraints can reveal learners' awareness in discerning with dimensions of possible variation and enrich their example space not only in terms of its content but also in the relationship among elements. I conjecture that encouraging learners to construct mathematical objects serves both to reveal learners' awareness and to assist them in gaining deeper understanding of the mathematical concept.

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THE IMPACT OF GRAPHIC-CALCULATOR USE ON BEDOUIN STUDENTS' LEARNING FUNCTIONS

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This research examined the impact of graphic-calculator use on comprehending mathematical concepts and thinking processes during problem solving. Short-term and long-term influences were examined. The research population was comprised of 9th graders from among the Negev Bedouin, a group considered to be among the weakest populations in many domains, especially the domain of education. It was hypothesized that introducing graphic calculators into their classes would help them improve their knowledge and understanding. We compared an experimental group (n = 95) that used graphic calculators to a control group (n = 89) that studied the same subjects without calculators. Questionnaires based on the standard curriculum were developed that could be answered with or without graphic calculators and administered identically to both groups. The pupils' responses were qualitatively analyzed to characterize patterns and categorize answers and then quantitative methods were applied.

Short-term influence: 1. The experimental group exceeded the control group in accomplishing tasks requiring mathematical inference, demonstrated meaningful learning and avoided pseudo-conceptual answers (Vinner, 1997). These findings are consistent with Demana & Waits (1990), and with expectations of improvement resulting from the introduction of graphic calculators into the classroom. 2. Use of graphic calculators was only slightly favored over the regular method for developing visualization of "function behaviour" and for developing the ability to extrapolate graphs beyond the range given. This finding contradicted expectations that graphic technology would develop these abilities.

Long-term influence: Two years after the interventive experiment ended, the short-term results were tested to gauge their impact on the study of advanced function-related concepts. No significant differences were found between the experimental group and the control group and no positive influence of graphic calculators use was preserved. The explanation for this will require further study.

Conclusions – Even though graphic calculators only had a short-term impact, they did expose the pupils to a meaningful learning experience, and to a genuine mathematical investigation and the advanced-technology experience blunted these Bedouin pupils' sense of deprivation.

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HOW TO PUT IT ALL TOGETHER?
REDESIGNING LEARNING ACTIVITIES TO MEET THE NEEDS OF ELEMENTARY TEACHER CANDIDATES

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One of the main challenges in the preparation of elementary teachers is focusing teacher candidates on the nature of the thinking processes, supporting pupils' construction and understanding of mathematical concepts. In addition, it is important to keep in mind the candidates' beliefs about mathematics and mathematics teaching, which in many cases clash with inquiry approaches to teaching and learning mathematics. To deal with this challenge, I designed and implemented a new course, titled *Mathematics Investigations.* The main purpose of this course is to deal with current demands of ICT integration and inquiry-based approaches to teaching and learning mathematics. The challenge is to design learning experiences that successfully amalgamate all of these concepts (Ball, 2000).

The theoretical framework of *Mathematics Investigations* is spanned by generative learning theory, autonomous learning and representing knowledge in multiple ways. *Mathematics Investigations* provides an experiential background for redefining mathematics learning activities in an environment that simultaneously accommodates for inquiry-based learning and ICT integration, and challenges candidates' beliefs about mathematics and mathematics teaching.

The course consists of five components of unequal grading weight: Problem sets, Reflections, Self Evaluations, Readings, and the Final Presentation. An appropriate rubric is provided for each of these elements (except readings, which are incorporated as part of all the other components). The course product is a Digital Resource File (DRF) that consists of five problem sets, the final presentation, and possible additional resources that students consider relevant for their future work. More details can be found at http://www.education.wichita.edu/alagic/319spring06/319spring06.asp.

The longitudinal mixed (qualitative and quantitative) method study examines the *Mathematics Investigations* initiative. The sociocultural theoretical perspective encompasses the naturalistic research paradigm as its theoretical framework (Erlandson, Harris, Skipper, & Allen, 1993). The focus of this presentation is the students' results and perceptions about *Mathematics Investigations*.

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READING MATHEMATICS TEXTBOOK AS A STORYBOOK

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As part of a national program to promote the use of mathematics for national education and development, *Math*NED, a non-governmental organization, has initiated the formation of mathematics clubs in all schools in Ghana. Through these clubs, hands-on outdoor mathematics activities that are to motivate students to see the fun in studying mathematics and the relevance of mathematics to daily life endeavors are promoted. One of such activities is to get students in small groups to take turns in reading their mathematics textbook as a storybook.

Two schools have been selected to participate in the pilot study. Under the guidance of a mathematics teacher the students, in groups of four, spend one hour each week to read their mathematics textbooks. One person reads a page of the textbook and all four discuss the contents and make sure they understand the content. The students are to take all the initiatives and the teacher only acts as a facilitator of the process.

Preliminary results from the study indicate that 90% of the 500 students participating in the study would like outdoor activities to be made part of their mathematics lessons. Some of the comments from the students include the following: “So mathematics can be interesting”, “I like it because there is no exam”, “How can I be asked to read mathematics textbook? Ah, that should be boring…yeah, but now I am learning”, “Now, I can understand.”

I will like to share details of the study with participants during the conference.

NAMING AND REFERRING TO QUANTITIES WHEN SOLVING WORD PROBLEMS IN A SPREADSHEET ENVIRONMENT [1]

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We are developing a research project that studies what students learn when they are taught to solve verbal arithmetic-algebraic problems in the spreadsheet environment. The teaching model used is based on a competence model that we have called Spreadsheet Method (SM) (Arnau y Puig, 2005). This model means, among other things, teaching the convenience of naming the quantities used when solving a problem. As part of this research, we have carried out a case study where the performances of six pairs of 1st year secondary students (11-12-year-olds) were observed. We report some remarks regarding whether they name the quantities they use or not and regarding the name they make up for these quantities. We also show how they refer to the quantities involved in the problem-solving process, when they introduce formulae in the spreadsheet, since naming the quantities is also useful to refer to them. Thus, it is observed that they avoid assigning names to the unknown quantities that don't appear in the problem statement, doing mental operations or using formulae in which more than one arithmetic operation appears. When unknown quantities are labelled, the students assign out-of-context names such as "extra".

We have identified four different ways of referring verbally to quantities when formulae are being written, such as: "Reference to cell position as the intersection of row and column", "Reference to the name of the quantity", "Reference to the cell with some gesture support" and "Reference to the value of the quantity". Nevertheless, when the students solved problems using the SM, they didn't use the "Reference to the value" to refer to unknown quantities; however, when they did not use the method taught, solving the method arithmetically instead, they did sometimes use it. This seems to point out that the students, who used the method taught, were aware of the fact that the numerical values that appear in cells during the solving process can't be used as names for unknown quantities, because they are provisional values that are made to vary till the solution of the problem is reached.

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[1] This research was supported in part by a grant from the Dirección General de Investigación del Ministerio de Educación y Ciencia de España, ref. SEJ2005-06697/EDUC.

IF IT DIVIDES BY 4, IT MUST DIVIDE BY 8: ARGUMENT AND ARGUMENTATION IN ELEMENTARY SCHOOL MATHEMATICS CLASSROOMS

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This study considers whether there is any evidence of argument or argumentation occurring in ‘ordinary’ primary mathematics classrooms and if so, what conditions support such constructive classroom interaction. It examines whether there are specific teacher behaviours or strategies that are linked with these occurrences. Adopting a socio-cultural perspective of mathematics, its teaching and learning, it considers the role that the teacher plays in guiding her pupils into participation in mathematical activities. In this process the talk of the mathematics classroom plays a vital part and the study explores the ways in which teachers use talk to induct their pupils into mathematical discourse. There has been considerable research into the role of discussion in learning mathematics and this study builds on this body of research. It analyses transcripts of classroom talk gained through audio recording and field notes arising from extended periods of participant observation. The transcripts were analysed in terms of the function and form of the different contributions and their connections with mathematical reasoning and thinking. The comparison of sections of transcript taken from two different lessons with two different teachers considers whether there is any evidence of argument or argumentation occurring in these settings. Arguments were identified through a thorough investigation of the data based on Toulmin’s (1969) analyses of the structure of argument.

The first lesson in which elements of mathematical argumentation could be identified used appealing mathematical problems involving finding the factors of large numbers. The children were able to engage with these problems as they had been involved in earlier investigations into the characteristics of multiples of the different numbers up to 10. The context of searching for factors of three digit numbers challenged the boundaries of their prior knowledge and forced them to extend their mathematical reasoning into unfamiliar territory. In contrast the lesson in which the talk was predominantly linked to the task of carrying out the procedure of counting on and back along a number track, showed no evidence of children using techniques of argumentation. Neither the teacher nor the pupils appeal to grounds, warrants or backings for the claims that they make. The two contrasting examples show that, if mathematical discussion is to be facilitated, there is the need to define suitable problems for discussion as well as to create a classroom context in which children can express their own mathematical meanings and understandings.

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UNIVESAL AND EXISTENTIAL MATHEMATICAL STATEMENTS - SECONDARY SCHOOL TEACHERS' RESPONSES

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Proof is an essential entity of mathematics. The processes of examining the validity of conjectures are at the core of any student's mathematical development. Thus, a major task of mathematics teachers is to communicate to their students the spirit of mathematics as a science of conjectures and proofs. This task implies that teachers lead and encourage their students to rationally examine conjectures, to assess their validity and to prove or refute them. It is therefore vital for teachers to be familiar with both formulating conjectures and reacting to arguments that purport to prove or refute mathematical conjectures. A number of researchers have studied teachers' conceptions of proofs, and almost all of them described teachers' knowledge of valid universal statements (e. g., Dreyfus, 2000; Knuth, 2002a, 2002b).

The major aim of our project is to explore practicing secondary school teachers' conceptions of proofs. The study focuses on valid and invalid, universal and existential statements and addresses Subject Matter Knowledge and Pedagogical Content Knowledge issues. This paper focuses on the teachers' reactions to adequate and inadequate justifications to valid and invalid universal and existential statements. The statements are taken from elementary Number Theory, and the given justifications include numerical examples, algebraic arguments and non-formal generalizations.

Twenty-two secondary school teachers answered written questionnaires, and then participated in a one semester research-based-intervention that dealt with proofs and refutations of different types of mathematical statements. The findings show that all teachers responded correctly to those justifications, which include numerical examples - they accepted them as proofs when adequate and rejected them when inadequate. However, with regard to algebraic arguments and non-formal generalizations, approximately half of the teachers' responses were unsatisfactory - they accepted invalid justifications or rejected valid justifications. In the presentation, these and other results will be described.

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SITUATIONS, LINEAR FUNCTIONS AND THE ‘REAL WORLD’

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In this presentation, I will discuss constraining relations between different representations (real world model, table, equation, graph) modelling a linear functional relationship in a physical device. The theoretical background of my study is situated cognition (Lave, 1988), situation theory (Devlin, 1991) and the theory of information flow (Barwise & Seligman, 1997). My discussion is based on the detailed case study of a pair of 8th grade students (out of 9) working through a number of problems asking them to produce and interpret a table, equations and graphs. The student work session was videotaped. In a subsequent analysis, the interaction was transcribed and each (topical) utterance was then modelled within situation calculus. Each of the representations was considered as a self-contained discursive situation, which was explored by the students. Particular attention was given to reference relationships and constraining relations between situations.

Constraining relations turned out to be of one of four types: a) Simple constraints between states of affairs in a situation, b) Constraining relations as elements of modelling causal (and temporal) relations within the statement of events, c) Constraining relations between different situations and d) Constraining relations between different objects of different situations.

I hypothesize that mathematical conceptual understanding frequently operates with constraints of the last type (d). Such constraints afford the statement of a given situation to infer the existence or properties of other objects in other situations. Stated this way, it seems not surprising that students frequently have difficulties “reversing” interpretations. E.g. when generating an equation from some situation students may find it difficult to interpret the equation as model for the situation. I will argue that a reversal requires students to break down the original total situation in relevant features (Martin & Schwartz, 2005), and to attune to what elements may carry mathematical significance. I will discuss this using concrete examples from my material and draw implications for mathematics teaching.

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THE RELATIONSHIP BETWEEN HIGH SCHOOL MATHEMATICS AND CAREER CHOICES AMONG HIGH ACHIEVING YOUNG WOMEN[1]

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A predicted shortage of science, technology, engineering, and mathematics [STEM] majors, especially at graduate levels, is of US national concern. Efforts directed at improving women's participation in STEM careers are seen as a solution to the problem. We have conducted a longitudinal study of high achieving young women from middle school to college. The design of grounded theory frames the research, testing the conjecture that the choice of the advanced mathematics track through high school is critical to one's choice of career. Many view Calculus as a "critical filter" for achieving one's career goals, especially in STEM. Data from student records and phone interviews conducted in January 2006 informed this study of 86 subjects who are 17-20 years old. More than half of these girls are planning to or have committed to major in STEM careers, and 60% have taken calculus 1 or beyond, Only four (not represented) are undecided as to their career choices [See Table 1].

Math Taken	STEM Careers	Other
Alg 2 – Calc1	23	29
AP Calc, Calc2,3	20	10

Table 1: Mathematics Courses Taken in High School and Choice of Careers

Collapsing categories we conducted a Chi Square test [$\aleph^2_{.05.1}= 3.897$] to compare advanced math and STEM careers. Results indicate that high school girls' choice of mathematics maybe driven by career choice rather than success or failure in high school calculus. Further analysis finds that none of these girls are interested in majoring in computer science or physics. One of the 86 intended to major in mathematics and one in teaching mathematics. When asked to list strengths they will bring to the workplace, they describe themselves as hardworking, determined, organized leaders who are good with people. Only one girl described herself as "smart." While girls are taking more advanced mathematics in high school, it appears that these choices have little impact on high achieving young women's perceptions of self. Perhaps it is still true that young women are not recognized for their intellectual contributions, especially in mathematics (Chipman, 1996).

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[1]This work is supported in part by NSF Grant #0204222 and IBM SUR program.

STUDENTS' UNDERSTANDING OF AMBIGUITY IN SYMBOLS

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Using symbols to communicate quantitative information is an important component of reform mathematics (Cobb et. al, 2000). However recognizing that symbols can be ambiguous may not be intuitive for young students. Here we present three studies looking at how students understand this important concept.

Students often do not realize that symbols can be more or less ambiguous. Fifty-one 4^{th}-graders determined the source of error in two stories. In one, a mother gave her child an ambiguous message about which purse to grab. The child picked the wrong purse, although one consistent with an interpretation of the message. In the other, a student read an ambiguous note another child had written that said: "2 3" for how much fruit the class had. Based on the note, the student said there were 23 pieces, when actually there were 2 apples and 3 bananas. In both stories, a child received an ambiguous message, and made a "wrong" interpretation. However, in the purse example, 94% of children blamed the ambiguous message for the error, while in the fruit example, only 44% blamed the ambiguous representation. The rest blamed the interpreter of the message because she "got the wrong answer", or because she should have known what it meant. In the quantitative example, the exact same students were less likely to see the symbolic message as potentially ambiguous.

To help students appreciate ambiguity in symbolic representations, two studies used Teachable Agents (TA). TAs are programs where students learn by teaching the computer (Schwartz et. al, 2001). Fifth graders taught their agent by inventing symbolic codes for fractional amounts. High school seniors created symbolic codes to characterize statistical distributions. When students' codes were ambiguous, the TA produced multiple fractional amounts (or distributions) that were consistent with different interpretations. Post-tests indicated that the younger students improved in their abilities to think about errors as possible misinterpretations. The older students became more precise in the representations they created and more likely to think in terms of specificity and ambiguity.

Together these studies suggest that recognizing ambiguity in symbols is not automatic for students, but this concept can be developed with moderate intervention.

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REFORM-ORIENTED TEACHING PRACTICES AND THE INFLUENCE OF SCHOOL CONTEXT

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The study used a combination of survey and interview to explore the understandings and use of reform-based teaching approaches of three New South Wales (Australia) primary teachers. A survey was initially used to determine the extent to which teachers' beliefs and self-reported practices reflect those advocated in reform-oriented curriculum materials produced locally and internationally. In particular, it focused on specific teaching strategies associated with each of the five processes of the Working Mathematically strand in the *Mathematics K-6 Syllabus* (BOSNSW, 2002). Data from the survey were used to develop initial 'profiles' of teachers so as to determine how closely their beliefs and reported practices aligned with those recommended by reform-oriented documents. Interestingly, the three teachers with profiles indicating strongest support for reform-oriented practices taught at the same school. Together, their profiles and interview data provide insight into the socio-systemic factors (Jaworski, 2004) that facilitate such practices in teachers within the school context.

The interview data confirm the existence of several key factors that have been linked to environments supportive of changes to teaching practices. In summary, these factors included: collegial interaction (Taylor, 2004) characterised by contextual factors such as leadership style, the provision of time for teams of teachers to interact on collaborative projects and to discuss philosophical as well as practical aspects of their teaching. Interestingly, it seems that the same structures supporting innovative practice can also foster misunderstandings in teachers' knowledge about reform-based practices. This was the case with the three teachers' understandings of *questioning* and *reflection* within the Working Mathematically strand of the syllabus. This is a significant point and raises questions about professional learning that is exclusively situated within the workplace.

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APPROACHING LINEAR (IN)DEPENDENCE WITH EXAMPLE-GENERATION

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This study is part of the growing research on students' understanding in undergraduate mathematics education. Linear algebra has become one of the required undergraduate courses for many disciplines. As research showed many students leave the course with limited understanding of the subject (Dorier, 2000; Carlson et al, 1997). Hazzan & Zazkis (1999), and Watson & Mason (2004) advocate that the construction of examples by students contributes to the development of understanding of the mathematical concepts. Simultaneously, learner-generated examples may highlight difficulties that students experience.

This study examines how and in what way example-generation tasks can inform about and influence students' understanding of linear algebra, focusing on the concepts of linear dependence and independence. Further, it discusses students' difficulties with constructing examples in linear algebra, and also explores possible correlations of students' understanding with the generated examples. The APOS theoretical framework was adopted in this study to interpret and analyse students' responses (Asiala et al, 1996).

This study provides a finer and deeper analysis of students' understanding of linear algebra. The results of the study showed that the example-generation tasks reveal students' (mis)understandings of the mathematical concepts. The results suggest that these tasks can serve as an effective research tool for undergraduate mathematics education as well as a pedagogical tool. This study also offers a variety of example-generation tasks for implementation in linear algebra course.

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METAPHORS IN TEACHER’S DISCOURSE: GRAPHING FUNCTIONS

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The purpose of this paper is to present an analysis of a phenomenon that is observed in the dynamic processes of teaching and learning to graph functions at high school: the teacher uses expressions that suggest, among others, (1) orientation metaphors, such as “abscise axe is horizontal”, (2) fictive motion, such as "the graph of a function can be considered as the trace of a point that moves over the graph". (3) ontological metaphors and also (4) conceptual blending. Fieldwork included classroom episodes and interviews. This presentation is part of a larger research where we have tried to answer four questions: What metaphors are used/produced by a teacher while explaining graphical representation of functions in a high school class? Is teacher aware about the metaphors he uses in his discourse and, if so to what extent is this awareness? What is the effect that produces about the understanding of the students? What is the role that metaphors play in meaning negotiation? We will be delivering partial results on the three first questions.

In recent years, several authors (e.g., Johnson, 1987; Lakoff & Núñez, 2000; Núñez, 2000, Radford 2002, Robutti 2004) have pointed out the important role that metaphors play in the learning and teaching of mathematics. We start assuming metaphor as understanding one domain in terms of another. According to Lakoff and Núñez (2000), metaphors generate a conceptual relation between a source domain and a target domain by mapping and preserving inferences from the source to the target domain. Partial results shown that metaphors play an important role also in negotiating meaning in classrooms, and we propose a model that takes into account the dynamic of the interplay of discourses. Metaphors in classrooms may have two different directions. In one hand there are metaphors that teachers use believing they are facilitating learning, and on the other hand there are students’ metaphors. Metaphors, as seen here, play an important role also in negotiating meaning in classrooms, and we will show during presentation a proposed model that takes into account the dynamic of the interplay of discourses.

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DESIGNING INSTRUCTIONAL PROGRAMS THAT FACILITATE INCREASED REFLECTION

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Early efforts to create and assess "communities of learners" focused on increasing participants' opportunities to interact. However, various research reports indicated that although the amount of communication can be increased via programmatic imperatives, the degree of critical engagement may not. We report on efforts to shift both design and research foci for instructional programs from facilitating increased communication to facilitating increased critical reflection.

COMMUNITY AS MEANS FOR SUPPORTING LEARNING

Early efforts to design instructional programs around communities of learners led to the development of checklists for documenting the existence of learning communities and using them as design criteria for developing new ones. Critiques of this approach included (1) no checklist can ever be complete, (2) counting interactions obfuscates the quality and depth of critical reflection, and (3) there are many definitions for "community." For example, do communities transcend particular members and time periods, or does any group of people engaged, even for a short and specified time period, in a shared practice or endeavour constitute a community (Barab, King, & Gray, 2004; Hsu and Moore, 2005).

A Shift in Design and Research Foci

Our efforts to design and research instruction programs (both online and in-person) define community as broadly as possible; we eschew constraints on specific time periods, physical proximity, or member role. One result of this conceptualization of community is that we design for, and assess programs based on, a goal of enhancing participants' depth of reflection as the primary avenue for supporting learning. This is accomplished by facilitating "spot on" communications between interested parties instead of broad communication among all members. This has led to some broader research constructs for assessing instructional programs and some broader design criteria that aim to maximize personal rather than collective trajectories.

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CONFORMISM IN TEACHING MATHEMATICS

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Psychologists define conformism as a person's blind following of other people's opinions. Teachers of mathematics frequently encounter conformism in the classroom. Therefore, the teacher's comprehension of the phenomenon is extremely important for the students' development of mathematical thinking. Conformist behavior is known to play a double role in a person's socialization. On the one hand, if the majority opinion is more valid, it assists in correcting the erroneous opinion or behavior while protecting the individual's psycho. On the other hand, conformist behavior impedes the assertion of independent behavior or opinion and reduces the development of critical and creative thinking. Mathematics teachers notice that very often students arrange their correct solutions and answers to the wrong answers, given by the teacher or published in their textbooks. The objective of this study was to determine how much influence the teacher's authority and textbook answers have on students' opinions. Seventy seventh-grade students were presented with tasks and answers, some of which were incorrect. Results showed a rather high level of "confidence in answers", which indicated that critical thinking in students was inadequate. The findings also provided evidence of the negative influence of conformism on the students.

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CONSTRUCTING MULTIPLICATION: CAN PUPILS DEVELOP MODELS FOR THINKING?

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Number sense has been considered one of the most important components of elementary mathematics curriculum. The development of personal strategies of calculation and its implications to solve problems in real situations are recommended by both international literature (Fuson, 2003) and Portuguese curricular documents.

The project *Number sense development: curricular demands and perspectives* aims to study the development of number sense in elementary school (5-12 years old). We developed 6 qualitative case studies. Each case study analyzed the implementation in a particular classroom of a related sequence of 3 or 4 tasks – *task chain* - and covered different grades from kindergarten (5 years) to 5th grade (11 years).

We will present a discussion based on the analysis of one of the case studies developed by the project and implemented in a second grade classroom (7 years). The focus of the *task chain* experimented in this case, is a learning trajectory to develop the multiplication concept. More precisely it deals with relation between some table products and the understanding of specific properties of multiplication. This learning trajectory was also foreseen to introduce the double number line model and to enhance the concept of multiplication relating it with the rectangular model.

In our curricular tradition formal procedures are introduced prematurely in elementary school. International standards claim that teacher should foster the pupils' reflection on their mathematical procedures and ideas providing the construction of personal and shared understanding on numbers and operations (Gravemeijer & Galen, 2003). Our data show a progressive awareness of multiplicative strategies. In the first task some pupils used repetitive addition and others had trouble with the task. In the last task different procedures appeared showing a great diversity and flexibility. In our presentation we will discuss the progression from first to the forth task and the flexible way how some models may help pupils to develop number sense. In particular we will analyze how some pupils use number relationships, how they formalize these relations and how they use them with proficiency.

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THE TRIGONOMETRIC CONNECTION: STUDENTS' UNDERSTANDING OF SINE AND COSINE

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This study concerns the portion of trigonometry which moves from right triangles to the coordinate plane, and then establishes sine and cosine as functions. The goal of the study was to explore students' understanding. Its theoretical framework was based on the model of Schoenfeld, Smith and Arcavi (1993), which gave a four-tier structure for a microgenetic analysis of a particular content topic. Two of the tiers involved aspects that are important for building a coherent understanding: (1) fundamental background concepts and (2) issues related to context. In the research reported here, a content framework for this portion of coordinate trigonometry was developed, and then applied in a case study of a group of students at the end of their work with this topic. The results were used to create a model of students' understanding of sine and cosine and also to refine the content framework. As part of the framework, a set of foundation concepts that underlie the subject of coordinate trigonometry was set forth. These ideas are called here the Trigonometric Connection.

The case study involved 120 honors students. The major data sources were a written test administered to the group and follow-up interviews with seven students. Both quantitative and qualitative methods were used to analyze the test scores. The qualitative analysis of the interviews provided insight into the thinking of the students as they answered the test questions. The study revealed that many students had an incomplete or fragmented understanding of the three major ways to view sine and cosine: as coordinates of a point on the unit circle, as the horizontal and vertical distances that are the graphical entailments of those coordinates, and as ratios of sides of a reference triangle. In addition, several cognitive obstacles were identified. Some involved issues specific to the study of trigonometry. These included a fragile conception of rotation angle and unit, and a failure to connect a rotation on the unit circle to a point on the graph of the cosine or sine function. Other obstacles involved more fundamental topics, such as an inability to relate the coordinates of a point on a graph to the horizontal and vertical segments that connect it to the axes. A difficulty related to context was the nature of sine and cosine as both ratios and numbers.

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STUDENT BELIEFS AND ATTITUDES FROM POETRY WRITING IN STATISTICS

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In previous work (Bulmer & Rolka, 2005; Rolka & Bulmer, 2005) we have developed a model of student beliefs and attitudes towards statistics through an analysis of pictures created by the students to express their view of "statistics" and what it means to them. This has served the dual role of a pedagogical approach for providing an inclusive environment in large undergraduate classes (see Bulmer & Rodd, 2005) while also forming the basis of a research project which extends the literature of mathematical beliefs (Törner, 2002) to world views of statistics.

In this presentation we will show the results of an application of this methodology to the analysis of poetry written by statistics students. The same categories that had been developed for the picture analysis were applied to the thematic content of the poetry. Additionally, the poetry provided the opportunity for an analysis of language and style which provided a new dimension to the embodiment of student beliefs in their creative expression.

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THE TEACHING OF PROOF IN TEXTBOOKS : A FRENCH-GERMAN COMPARISON

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We describe and explain some similarities and differences observed in textbooks in the teaching of the proof in France and Germany.

THEORETICAL FRAMEWORK

Knipping (2001) studies the proofs of Pythagoras' theorem observed in classroom situations in France and Germany. We extend this study to the teaching of proof in French and German textbooks. We adapt Toulmin (1958)'s theory on arguments of plausibility and necessity to Chevallard (1992) s' anthropological theory of didactics : the validation of mathematic teaching are the double transposition of proofs from the mathematical institution producing mathematical knowledge (where only arguments of necessity are used to prove) and validations of others institutions (like the "daily life" where arguments of plausibility could be used to validate).

RESULTS OF OBSERVATION

In both countries proof appears in the curriculum as an object to be taught. We observed in both countries textbooks where proof is taught in a special lesson : the functions of proof influence the different types of tasks (discovering, controlling, changing registers, …).

The study of validations of class theorems shows similarities about combining different types of arguments as well as different types of functions. Differences are observed on the types of technology and technique involved in the validation and on the weight given to different types of arguments and registers used, with an explanation related to the institutional conditions (moment of introduction, didactical contract, function assigned to the validation, educational system, …).

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CLASSROOM: A LEARNING CONTEXT FOR TEACHERS

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Classroom is often associated with the formal learning context of students but it is also a rich context for teachers development of professional knowledge. In this paper I refer to an investigation project aiming to study curricular practices of mathematics teachers. I describe a classroom episode of students working with graphic calculators in an particular situation and analyse the factors that contributed for the development of their teacher conceptions about the potentialities of this technology.

AN EPISODE FROM MARGARIDA CLASSROOM

Margarida is a Portuguese secondary mathematics teacher. She uses graphics calculators in her classroom to confirm the results students obtain from analytic procedures she teaches them in advance (Canavarro, 2004). But one day, during a formal assessment test, she saw her students working autonomously in a very different way she recommended. And they were able to produce a correct response about functions using only the graphic calculator. Margarida got very surprised.

DEVELOPING NEW CONCEPTIONS AND KNOWLEDGE

Recognising the success of the work of the students, Margarida changed her view about graphics calculators from a “drawing instrument of graphs of functions” for a powerful “resource for mathematical thinking”. Four factors contributed for that: (1) Having the opportunity for observing students mathematical productions and recognize their validity (Brown & MacIntyre, 1993); (2) Keeping the capacity of surprising herself with the world around (Shön, 1983); (3) Being able of reviewing personal didactical knowledge (Day, 1998); (4) Reflecting about the meaning of classroom practices (van der Berg, 2002).

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AN INVESTIGATION OF DIFFERENCES IN PERFORMANCE IN MATHEMATICS BETWEEN PARALLEL STUDENTS AND NORMAL ENTRY STUDENTS AT THE POLYTECHNIC – UNIVERSITY OF MALAWI

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INTRODUCTION AND BACKGROUND

Since 1965 when The University of Malawi was established, university education in Malawi has been mainly public in ownership as well as in operational control. Over the past decade or so, The University of Malawi has been receiving less financial allocations from the government than the estimated expenditure. Consequently, the cost of staff, learning and research materials, catering and accommodation services made it difficult to sustain the operations of the university. The implications of such scenario was the increasing debt burden that threatened to compromise the quality of learning. Hence the University of Malawi started enrolling a category of students on economic fees basis. In Malawi, this category of students who pay the full tuition fees are referred to as “parallel students” while the category of students who are sponsored by the government are referred to as “normal entry” students. The “parallel students” are responsible for their accommodation and meals while the “normal” entry students are accommodated and fed by the University. This paper investigated whether these categories of students perform differently in mathematics.

End of year mathematics grades for 2003/2004 academic year for some first year programmes at the Polytechnic, a constituent college of The University of Malawi were analysed. The findings indicated that the mean score for normal entry students was higher than that of the parallel. Although boys performed better than girls in general, the normal entry female students outperformed male parallel students. The results indicate that being parallel had a negative effect on the students’ performance in mathematics.

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A STUDY ON ELICITING THE FORMULA FOR THE AREA OF TRIANGLE FROM STUDENTS' STRUCTURING OF TILE ARRAYS AND FIGURE RECONSTRUCTIONS

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The rectangular array model is important for mathematics learning because of its use to model multiplication, to represent fractions and to serve as a basis for the area formula (Outhred & Mitchelmore, 2000). The model was used to investigate 3-6 grades students' performance in drawing, counting, and measuring tasks. These tasks involved in representing arrays of units that were given in different perceptual cues, in calculating the numbers of elements in arrays, and in construction of arrays in correct dimensions when no perceptual cues was given. These skills focus on linking the unit (in this case, a tile), on the iteration of this unit to cover a triangular figure, and on the lengths of the sides of figure. Strategies on array-based tasks were inferred from drawings.

Four distinct strategies of covering were identified: (1) incomplete covering, (2) covering using visual estimation, (3) covering using concrete unit, and (4) finding the number of units that would fit along at least one dimension. There were four levels of cognitive development in structuring of 2D arrays of tiles. At *level 1*, students could not find the components that characterized tile and triangle. They were unable to draw array given units along two adjacent sides of a right triangle. At *Level 2*, students structured arrays as one-dimensional paths. This reflective decomposition requires students to abstract their path-creating movements so that they can reflect on, locate, and coordinate the movements within a 2D frame of reference. The transition to *Level 3* requires the student to conceptualize the array as filled by copies of row or column composites. This shift occurred as mental actions that coordinated tiles within rows and columns in a way that enabled students to see the array as a set of congruent rows. The transition to the third level involves a curtailment of the explicit construction of rows-as-composites within the iteration process. Instead of re-presenting the square-by-square construction of each row composite, students take the tiles in a column as indicators of row composites. Finally, at *Level 4*, the movement to row-by-column structuring without perceptual material, requires students' use of separate, combine, move and supply to transfer a triangle into a rectangle, and interiorize the entire process of indexed iteration of a row-as-composite.

These findings provided us with the notion of teaching for the conception of area formulation of triangle and directions for future research study on this model.

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DECISION MAKING AT UNCERTAINTY: MOVING ON A PRIME LADDER

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This report examines prospective elementary school teachers' use of prime numbers when asked to simplify a "large" fraction in a clinical interview setting. Participants' approaches to the task are interpreted through the framework: heuristics and biases of thinking under uncertainty.

Studies on learning number theory have paid specific attention to students' understanding of prime numbers and prime decomposition (Zazkis and Campbell, 1996, Zazkis and Liljedahl, 2004). This report reflects a corresponding interest in identifying potential sources that influence students' beliefs and approaches. The task analysed in this report invited students to simplify the fraction 448188/586092.

Tversky and Kahneman (1974) introduced a framework, of Heuristics and Biases associated with Subjective Probabilities (HBSP), to explain judgements people make during uncertain events. As per the framework, people evaluate phenomena according to: *representativeness*, *availability of instances* and *adjustment from the anchor*; from which, each heuristic leads to specific biases. For example, a*vailability* refers to the ease with which a person can bring to her or his mind instances of occurrences of an event. Tversky and Kahneman (1974) stated that since words are easier searched for by their first letter than by their third letter, people judge that more words start with the letter *r*, than words that have *r* as the third letter of the word, which is incorrect. The result, that words that can be searched for more effectively are deemed to be more abundant, was designated: *biases due to the effectiveness of a search set.* Similar results were found in our task.

While using 2 and 3 as a starting point is reasonable, we found that these numbers were "overused" by participants. Alternatively stated, students were more readily able to recall simplification by the "first" few prime numbers 2 and 3. Further analysis of the task was conducted, with the other heuristics and biases of the framework, and the results suggest that participants' struggles associated with elementary number theory may originate in the use of intuitive probability. We introduce the notion of *prime ladder* to refer to an individual's list of primes.

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A STUDY ON IMPLEMENTING INQUIRY-BASED TEACHING TO FACILITATE SECONDARY SCHOOL STUDENTS' LEARNING IN THE RETAKING MATHEMATICS COURSE

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This paper presents a preliminary study on implementing inquiry-based teaching to facilitate the secondary school students' learning in the intensive retaking mathematics course. The research subjects were twenty-seven first–year students who failed to pass the mathematics course in the academic year and were required to retake a twenty-four hours (six hours a day) intensive mathematics course. Because they were obviously low achievers with negative learning attitude towards mathematics under the traditional lecture-based instruction, the inquiry-based instruction was adopted as a new teaching strategy since there is evidence that inquiry-based instruction enhances student performance and attitudes, and facilitates student learning (Haury, 1993; Jarrett, 1997). The purpose of the study is to examine the effects of the new teaching strategy applied to such an intensive retaking mathematics course.

The instruction design is based on the framework of Speer's inquiry cycle which includes *exploration*, *invention* and *expansion* (Speer, 2003), integrated with the strategies of peer's cooperative learning and teacher's posing probing questions. Through the pre-test and post-test of a questionnaire for examining the student's understanding, it is concluded that their cognition had been improved since there exists significant difference between the means of the pre-test and post-test. Besides, most of the students directly expressed in the interview that they could reflect on their conceptions, probe for misconceptions, and even develop concept change through discussion with peers and the teacher. In addition, the students became able to communicate their ideas with others, which might be beneficial to deepen their understanding. In contrast with the traditional lecture-based teaching in which teachers are sole purveyors of knowledge and students passive receptacles, inquiry-based instruction provides the student with more chances to become an active learner who can construct her/his own understanding.

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GIRLS EXCELLED BOYS IN LEARNING GEOMETRIC TRANSFORMATION USING TESSELLATIONS

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Tessellations are the pattern of iterations of geometric transformation. We can find them in the works of Escher who is the famous Dutch artist. In the article, the figures of patterns we present are a pig, a frog, Tchiucheonwhang (the mascot of Korean football supporters), including Escher's, which are constructed using the computer geometric program, GSP(Geometer's Sketchpad). We want to talk about the gender differences on students' achievement and disposition toward mathematics related to tessellations. If they are supported with this kind of interesting figures constructed by their own hands, especially female students will have more interest in learning geometric figures.

INTRODUCTION

This study is to investigate how differently female students, compared to male students, learn geometric transformation through an experimental study with the treatment of tessellations so that we could be more aware of the characteristics of females' learning style of mathematics and apply the implications of the study to teaching them mathematics. Also, the study reports the correlation between sub components of Mathematical disposition students displayed at a middle school in Korea after the experiment. The experimental study was executed in one middle school in Kyunggi province, Korea, from September to November in 2005. Arbitrarily 4 classes were selected for the data collection, which belong to the lowest level of students and were homogeneous in math achievement (2 classes for the control group, 2 classes for the experimental group). The teacher provided the instructional materials including 8 lessons about tessellations and gave the pre and post tests (MAT and MDT) to collect the data before and after these lessons. MAT and MDT were administered according to the instructions as the manual to the 4 classes.

Through analysing the data, the results showed that these male and female students differed significantly in achievement of learning geometric transformation when they were guided with tessellations using a computer. The female's disposition could positively be influenced more when they were supported in this kind of a learning environment. This requires that the teacher need to be balanced in teaching students with their instructional materials that concern about giving a fair chance to both genders according to learning characteristics of different genders.

A NATIONAL SURVEY OF YOUNG CHILDREN'S UNDERSTANDING OF BASIC TIME CONCEPTS

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The purpose of this study is to realize the perception of the basic time concepts by young children in Taiwan. A random sample of 1100 children was tested by TBTC (Test of Basic Time Concepts). Results indicated most of children are possessed of the attributes of succession and irreversible but are not possessed of the attribute of cyclical daily events.

INTRODUCTION

As many researchers have shown the concept of time is complex and obscure. The assessment of young children's understanding the time concept appears to especially difficult. Up until now, the assessment of the understanding of the concept of time has been achieved, on the basis of several theoretical models that were constructed by various researchers. However, the research in this field has rarely been discussed by using Taiwanese population.

METHODOLOGY

A sample survey was employed in the study. A nationwide sample of 1100 school age (7-9 years old) children was randomly drawn from elementary schools in Taiwan. Based on the theories of child's conception of time (Friedman, 1977, 1978 &1986; Piaget, 1969), the TBTC (Test of Basic Time Concepts) was developed to measure the children's understanding of basic time concepts. The collected data was analysed by several item analyses and statistical methods.

CONCLUSIONS

The study found that most of children are possessed of the attributes of succession and irreversible but are not possessed of the attribute of cyclical daily events. Results of this study could provide the substantive evidence for researchers, instructors and curriculum developers in the related areas.

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AN EXPLORATION OF THE MATHEMATICAL LITERACY OF IRISH STUDENT PRIMARY SCHOOL TEACHERS

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At the previous PME meeting, the plenary panel discussed the significance of PISA, with its innovative framework, interlinking mathematical skills, content and context (Jones, 2005). The study reported here seeks to explore the mathematical knowledge and skills which Irish primary teachers bring to the teaching of mathematics. Problem-solving is germane to the Irish primary mathematics curriculum and mathematical skills development is an explicit requirement. For these reasons the eleven items released for public use following the PISA 2000 assessment (OECD, 2002) were used to investigate the mathematical strengths of prospective primary teachers. Using IRT scaling PISA ranked these items in levels of difficulty from one to six, the higher levels requiring higher order reasoning and thinking skills.

Raw scores of second year student teachers (N=71) on the eleven items were mapped to the six proficiency levels. Some student teachers did very well on PISA with one student scoring 100% and a further three scoring in excess of 90%. But there was a ceiling at proficiency level 4 for up to 80% of students. The teaching objectives of the primary mathematics curriculum resonate strongly with the PISA framework yet the necessary underlying mathematical skills are not highly developed among many of the student teachers in the study. Nor are the limitations evidenced in mathematical literacy confined to process skills. If teachers subscribe to

> the application of mathematics in a wide variety of contexts [which] gives people the ability to explain, predict and record aspects of their physical environments and social interactions

then they must be able to

> recognise situations where mathematics can be applied, and use appropriate technology to support such applications (Government of Ireland, 1999, p. 2).

Many of the student teachers in this study may have difficulty in fulfilling this curricular expectation.

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MATHEMATICS TEACHERS' KNOWLEDGE AND PRACTICE: A SURVEY OF PME RESEARCH REPORTS

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This paper discusses a survey of PME research reports involving teachers' knowledge and practice for 1977-2005. It focuses on the nature and distribution of the categories of studies identified and what this suggests for future work in this area.

This paper discusses a survey of PME research papers involving teachers' knowledge and practice for 1977-2005. The intent of the survey was to identify and discuss such papers in terms of objects of study, theoretical emphases, methodological approaches, results, and other issues, and to suggest possible directions for future research. In this paper we focus on the nature and distribution of the categories of studies we identified and what this suggests for future work on the mathematics teacher.

We reviewed 335 research reports that fitted this area of studies. Our preliminary analysis produced the following ten categories:

Teachers' knowledge of mathematics
Teachers' practice
Teachers' beliefs and conceptions
Teachers' knowledge of mathematics teaching
Teachers' development/change/self-development
Teachers' researching
Teachers' attitudes and other affective aspects
Teachers' thinking/metacognition/reflection
Teachers in community
University teachers

These categories are interrelated, and many papers fit in more than one. We classified a paper according to its major problematic. Our basis of identifying these categories involved theoretical perspectives of teacher knowledge, different contexts in which teacher activities can be situated, and the teacher working alone or cooperatively with other teachers or researchers. We discuss the first four of these categories in great details in PME 2006 handbook. For these four categories, that include 68% of the papers, we found that the studies cluster in three periods: 1977-1985, 1986-1994, and 1995-2005. Period 1 contains very few papers. Period 2 contains a large amount of papers dealing with teachers' knowledge of mathematics and teaching, and beliefs and conceptions. Studies on teachers' practices began to appear in period 2 and grew at an amazing rate in period 3. To further our understanding of the mathematics teacher's practice and professional identity it may be worthwhile to discuss issues raised by the full set of ten categories of papers.

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PSYCHOLOGICAL ASPECTS OF STUDENTS THINKING AT THE STAGE OF GRAPHICAL REPRESENTATION IN THE PROCESS OF INVESTIGATION OF FUNCTIONS

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In this presentation I am going to discuss some psychological aspects of students thinking at the stage of graphical representation in the process of investigation of functions, in a calculus course for engineering students. As is known, the development of students' abilities in the investigation of functions and sketching their graphs is one of the main purposes of calculus studies, essential in the formation of the so called "function-sense" (Eisenberg & Dreyfus, 1994).

Investigation of functions is a complicated process of a student's thinking, which includes recognizing basic properties of functions, solving equations and inequalities, performing calculations and symbolic manipulations, and returning to the world of visual perceptions of obtained analytical results by interpreting them on the sketch of the graph of the function. One could say that in the above mentioned process, the student takes a "journey through the different worlds of mathematic thinking" (Tall, 2004).

The standard way of investigating functions is to follow a long scheme that includes many items with algebraic and symbolic calculus operations, at the end of which the student must draw the graph sketch of the function. Our long-term experience reveals that this crucial stage of visual interpretation in analytical investigation is the most difficult for many students. It appears that the students fail to put their results into graphical images. In order to improve this situation, we organized an experimental group in which, at the end of each step of the investigation, the student must say to him/herself (in accordance with "Thinking and Speech Ideas" (Vygotsky, 1987)): "what is the graphical meaning of this analytical result?" and to draw the proper lines on the coordinate system. By this approach, the student draws the graph element by element and corrects it step by step. The qualitative and quantitative, direct and indirect effects and comparison of these studies will be presented.

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FORMATIVE FEEDBACK AND MINDFUL TEACHING OF UNDERGRADUATE MATHEMATICS

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Our interest is in instructors of undergraduate mathematics asking questions of students that guide the instructors themselves to be mindful of instructional design: questions that stimulate self-regulation in, and transformation of, instructors in relation to curriculum and instructional design. Black & Wiliam (1998) examined approximately 250 studies and found that gains in student learning resulted from a variety of methods all of which had a common feature: formative assessment. Cronbach (1957) saw formative assessment as part of the process of curriculum development. Roos & Hamilton (2005) emphasize the intellectual debt of formative assessment to cybernetic theory, via the critical notion of positive feedback. They link formative assessment to a mindful approach to teaching, curriculum development, and instructional design. The critical intellectual basis of formative assessment, is that positive feedback provides a stimulus to the activities of self-regulation and transformation – essential elements, in Piagetian epistemology, for generating higher intelligence.

Royall (1997) formulates three questions that have wide applicability to the interpretation of all data: (1) What do I believe now that I have this answer? (2)What do I do now that I have this answer? (2) What is this answer evidence for? Utilizing Royalls' data analysis questions applied to formative assessment leads to the following general feedback model: (1) A teacher asks questions of students, in class or in tests. (2) Students answer questions, verbally or in writing. (3) The teacher analyses student answers using one or more techniques of data analysis. (4) The teacher reflects on changing beliefs, need for action, or evidence for or against an existing assumption. (5) The teacher modifies and redesigns curriculum as a response to the analysis of student answers.

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MATHEMATICS EDUCATION IN THE SOUTH & WESTERN PACIFIC: BUILDING LOCAL CAPACITY TO SUPPORT TEACHERS OF MATHEMATICS

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Over the past seven years, two National Science Foundation (NSF) funded projects (DELTA & MENTOR) identified and trained mentor teams drawn from and working on remote Pacific islands. During the past four years these mentor teams, composed of college mathematics instructors, district mathematics specialists, and expert classroom teachers, supported some 400 novice teachers of mathematics.

The ability of the Projects to achieve their stated purpose of nurturing effective mathematics instruction in novice and experienced teachers was dependent upon the achievement of a number of goals, including but not limited to: (1) developing experienced mathematics educators' understandings of their roles and responsibilities of mentors, as well as of effective mentoring processes; (2) developing experienced mathematics educators' skills as mentors, as well as their abilities to design and implement professional development models that foster professional growth in teachers; (3) increasing mentors' and novice teachers' mathematical content knowledge, as well as their understanding of associated pedagogy; (4)increasing novice and experienced teachers' ability to plan, implement and assess instructional sequences that reflect an understanding of the principles of standards-based mathematics learning and teaching; and (5)developing novice and experienced teachers' and mentors' abilities to reflect critically on their practices and on their growth as mathematics teachers and educators.

In addition to individual and focus group interviews, observations, and audio and video conferences, impact was measured using two instruments designed for the projects: a mathematics content test administered yearly to the novice teachers, and biennially to the mentors, and a pre-post attitude questionnaire given to both mentors and novice teachers. These measures indicate modest growth in content knowledge, and a growing appreciation and understanding of standards-based instruction on the part of the novice teachers. The mathematical knowledge of the mentors has grown significantly. Questionnaire data for the mentors is not available until the conclusion of the projects in 2007. Anecdotal data from a few of the island communities indicates growth in student achievement taught by the novice teachers. Improved understanding and appreciation of the richness of their diverse island cultures is extensive among both mentors and novice teachers.

Acknowledgment This material is based upon work supported by the National Science Foundation under Grant numbers ESI 9819630 & ESI 0138916.

TEACHER MEDIATION OF TECHNOLOGY-SUPPORTED GRAPHING ACTIVITY: A VIDEO-BASED CASE STUDY

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This paper describes a case study of technology use in a secondary mathematics classroom in England. The study is part of an ongoing wider research project ('T-MEDIA', funded by ESRC) employing digital video to explore teacher mediation of subject learning supported by projection technology. Our primary aim is to illuminate effective pedagogy by helping teachers to make explicit their underlying rationale.

One reflective practitioner was observed and videoed over a sequence of 6 lessons on straight line graphs. Interview data from teacher and students, teacher diary, lesson materials and student work were also collated. A collaborative approach was taken in analysing the video and other data, closely involving a subject colleague of the teacher and two researchers. A key aim was to compare and integrate multiple perspectives, collaboratively developing a grounded socio-cultural theory of the processes through which teachers mediate student activity with technology. The unique methodology complements other recent studies involving collaborative video review (e.g. Armstrong *et al.* 2005; Moyles *et al.* 2003), e.g. by building a form of intermediate theory (Cobb *et al.* 2003). The ultimate goal is to develop dissemination DVDs that characterise the key strategies emerging, using accessible language.

In this presentation we will share our preliminary findings concerning the strategies that the teacher used in this context. She catered for her mixed ability class by blending a wide range of technological resources and activities for whole class, pair and individual activity – including data projector, dynamic graphing software on laptops, interactive online materials, a classroom performance voting system, worksheets and a card matching, graph recognition task. Lessons were highly participative and carefully structured to harness the visual affordances of the technologies to facilitate understanding of gradient and intercept. The stepwise guided discovery approach was underpinned by a dialogic style of interaction, by continually responding and adjusting assistance to pupils, and by a collaborative learning culture which supported pupils' conceptual engagement.

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A CATEGORIZATION OF DIFFICULTIES ENCOUNTERED BY 13-TO-15-YEAR-OLDS WHILE SELECTING INVERSE ALGEBRAIC OPERATION

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Demby (1993) found that many 11-year-old pupils were not able to evaluate expressions such as 28+75–75 and 14·15:14 without performing the successive operations. Bills et al. (2003) studied forms of reasoning and difficulties of 12-year-olds with simplifying expressions such as $(a–b)+b$ and (x+x+x): 3.

The present paper also concerns students' understanding of the relationship between the inverse operations and applying it. Observation of mathematics lessons in Grades 5-8 has revealed difficulties of students who were to name the inverse operation(s) needed in the procedure for finding x from a given equation. Dealing, e.g., with the equation 3x=7, some students thought that subtracting 3 will give x. Dealing with the equation 3x–5=4, some students proposed that one should "divide by 3 and then add 5" or "divide by 3 and then subtract 5".

The purpose of this study was to diagnose thoroughly such difficulties. Students of grade 7 were given a paper-and-pen test followed by individual interviews. They were to select an operation, or a sequence of operations, which would give x when applied to an expression such 5+x, 5–x, 5x, 5x–2, x:5 (no equation was given). Students were first shown an example (it was the expression –4x+1 which was not given in the task) and were told that they were expected to say, e.g., "subtract 1, and then divide by –4". At the end of Grade 8 the same students were given an analogous, more advanced test (with fractional coefficients).

Student's difficulties have been characterized by identifying the following eight types of errors or no-answers in their written tests: (1) Lack of x-leading orientation in the student's answer; (2) Pseudo-operation; (3) Operation acting on fragment of the expressions only; (4) Incorrect inverse operation; (5) Incorrect order of operations; (6) Error in performing algebraic operations; (7) Partial solution; (8) No answer.

Surprisingly, both in Grade 7 and in Grade 8 the easiest were expressions of the type ax, where a was a given number (easier than expressions of the type $a+x$).

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‘IT’S INFINITY’: MATHEMATICAL INSIGHT IN A PRIMARY CLASSROOM

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Abstraction and student engagement have been central to recent discussions on insight in mathematics education. Herschovitch, Schwartz and Dreyfus (2001) suggest that the three epistemic actions that constitute abstraction are ‘recognising’, ‘building with’ and ‘constructing’ (sometimes termed the ‘RBC model of abstraction’). Dreyfus and Tsamir (2004) maintain that the construction of a new structure is followed by a consolidation phase. Consolidation allows the student to make flexible and confident use of the abstract notion in a variety of situations. Associations also exist between positive affect in students and the creative development of a new cognitive structure (Williams, 2002).

While learners at secondary and tertiary levels have been the focus of such studies, little research has been done on the construction of mathematical insights by primary school students. The purpose of this paper is to explore the way in which pupils aged 10 – 11 years developed an understanding about the infinity of rational numbers in an interval of the number line. During a mathematics lesson, pupils were asked how many fractions there might be between zero and a half. Initially the pupils counted fractions displayed in their textbooks but, as the conversation progressed, Dan, a mathematically gifted student, exclaimed ‘it’s infinity’. He had constructed this idea by recognising that ‘there’s more fraction words than there is in that book’. Other pupils built on Dan’s insight to develop novel structures of their own. There was evidence of high levels of engagement at all phases of the discussion. Follow-up written work showed that while some students seemed to have consolidated their new understanding, the thinking of others was still at a fragile stage.

Preliminary findings of the research suggest that the RBC model of abstraction is a valuable tool for the analysis of mathematical insight in primary education and that links exist between engagement in a task and the collaborative construction of a new mathematical structure.

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WHAT IS TO BE KNOWN? A SHORT PRESENTATION OF EPISTEMOGRAPHY

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The piece of theory we present here is an attempt to generalize and conceptualise findings about knowledge we made during previous researches, which belong to two quite different domains of mathematics education: Algebraic Thinking and Mathematical Discussion. According with many authors we found that symbolic and linguistic knowledge plays a central role in Algebraic Thinking. And we faced the following question: to what extent is this knowledge, *mathematical*? Letters and symbols are not mathematical objects in the same way that numbers or sets or functions are; but on the other hand they are equally necessary to do mathematics. Mathematical Discussion Situations too involve knowledge that is not strictly mathematical: logical knowledge, knowledge on how to participate to a mathematical discussion, and more generally knowledge on how to do mathematics (what is a proof, what requirements must meet a statement to be accepted, what is the role of a counterexample etc.). Roughly, this kind of knowledge is on statements on mathematical objets rather than just on mathematical objects. **Epistemography** is a description of the organization of what the subjects *have to know* in order to actually do mathematics (and not to *pretend* to do mathematics!). Epistemography is not about what is in the subject's mind: it is not a branch of cognitive psychology (neither of genetic epistemology). Epistemography is based on the following assumption: doing mathematics involves operating on *signs*, which *represent* mathematical objects and relationship, with material or mental *instruments*. Then, we developed a three-order knowledge model. **1st-order knowledge** can be divided in: **Conceptual** (about mathematical objects and relationship between these objects), **Semiolinguistic** (about semiotic and language systems) and **Instrumental** (about the use of material or semiotic instruments). Students have to know, too, how signs represent objects and objects relationship, how operations on signs can be tools and how mathematical properties can be tools (which are the sides of the triangle made of Conceptual, Semiolinguistic and Instrumental Knowledge). **2nd-order knowledge** is about the principles of the mathematical game or, in other words, about what is *legitimate* to do and what is not; for instance, can one say that he or she proved a statement just by giving an example? Pierce, Frege, and Wittgenstein are invaluable guides to clarify this extremely complex relationship between objects, signs, practices and rules. **3rd-order knowledge** allows the subjects to *identify* if what one is doing is mathematical or not (or to identify to what domain of mathematics it belongs). Epistemography is by no way a learning or a teaching theory; but it allows us to analyse in depth the mathematical knowledge that is to be learnt or to be taught.

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"THE MOST NORMAL PATH": HIGH ACHIEVING GIRLS CHOOSING MATHEMATICS COURSES

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Our longitudinal study of high-achieving girls found significant differences between test results and grades, up to Algebra II, between the groups of girls who took mathematics beyond Algebra II, and those who did not (Wilson, Mojica, Slaten, & Berenson, In review). Past grades and test results may have a strong influence on girls' mathematics pathways choices. This paper reports on follow-up interviews with a sample of girls, explicating past grades and results' role among other factors. During interviews, past results always came up as an *indirect* factor in girls' mathematics pathways decisions, filtered through the following *direct* factors:

- College requirements: *"It is recommended for good colleges"; "It makes more sense for my future"; "I can get into a better college this way"*
- Teachers: *"My teacher said I would be good at it"; "My teachers recommended me to take these courses".*
- Peers and school norms: *"This is the most normal path people take at my school"; "This is the standard course of study direction at our school"; "That's what people who did not get very good grades [in Algebra II] take".*
- Family: *"My parents are encouraging me to take these classes"; "My mom will get me a tutor so I can take pre-calculus"; "I want to be a chemical engineer. People say I might need college calculus for that. My aunt and uncle have a lot of chemical engineer friends."*

The secondary nature of past grades and test result influence on pathways choices is especially apparent in the cases of conflict. Statistically, lower results correlate with the "no math beyond Algebra II" path. However, interviews revealed that strong family and peer support and strong college preferences outweigh the short-term "I would be good at other courses" reasoning. Our data indicate that college preferences, in turn, are strongly influenced by family (Droujkova, Berenson, Rindos, & Tombes, 2005) Even a girl unhappy with her grade, her teacher and her whole experience in mathematics classes up to Algebra II is likely to take mathematics classes beyond, given her "mom will find a tutor" and people around her think she "can get into a better college this way."

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EXPLORATORY MATHEMATICS TALK IN FRIENDSHIP GROUPS

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Background to studies in small group work

The benefits to learning of working in groups have been known for some time. Good, Mulryan and McCaslin (1992, p167) describe "clear and compelling evidence that small group work can facilitate student achievement as well as more favourable attitudes towards peers and subject matter". They advocate a future focus for research on the *socially situated learning* which occurs in small groups. These authors argue that research on small groups has gone beyond a need to justify its benefit through improved learning outcomes. They emphasise the need for work on the factors which affect discourse *processes* as well as factors which affect achievement outcomes.

Findings from empirical work in a secondary mathematics classroom

All the groups studied across the age and ability range demonstrated evidence of exploratory talk. There was a direct relationship between the length of time groups had worked together and the amount of exploratory talk identified. Some groups demonstrated a 'talking aloud' means of connecting everyone's talk which served to keep everyone engaged with the task and acted as a means of maintaining cognitive cohesion. Even groups which exhibited little exploratory talk, still had a 'way of working' together that was positive. This enabled each member to function in an atmosphere of trust and a familiarity of 'unwritten rules'. One of the factors in this study which separates it from almost all other studies of small group work in mathematics education is the study of self-selecting groups on the basis of friendship. Some of the findings are reported in Edwards and Jones (1999) and support those of Zajac and Hartup (1997) who found that friends are better co-learners than non-friends. They suggest reasons for this include the fact that knowing each other well means knowing similarities and differences, so that suggestions, explanations and criticisms are more likely to be more appropriately directed to each other. Mutual commitment generates particular expectations which support collaborative means of working. Security with friends means more activity in novel problem-solving situations. This presentation reports on a subsequently (2005) funded research project on friendship groups in secondary mathematics classrooms in the south of England.

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CONCEPTUAL BASIS OF PROOF: EVIDENCE FROM LANGUAGE AND GESTURE

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This short oral will present preliminary results from an investigation of proof as a conceptual "object;" in other words, it will address what kind of "thing" a proof is, conceptually. The central question is how an understanding of proof and proving may be grounded in unconscious conceptual mappings and communicated through particular language choices and spontaneous gestures. Given that proving is the defining practice of mathematics, it is important to investigate the conceptual capabilities that allow people to create and understand proof This investigation is distinguished from prior work on the teaching and learning of proof (cf, Hanna, 2000) because its goal is to determine the conceptual underpinnings of proof, relating it to other kinds of thinking, rather than to examine proof solely within the context of mathematics. The theoretical framework for the research comes from work on conceptual mappings (Fauconnier & Turner, 2002) and gesture studies (McNeill, 1992). Consistent with constructivism, this work holds that our understanding of new domains and processes is derived from our existing knowledge, but posits new mechanisms (e.g., conceptual blends), revealed in our language to explain how new knowledge is constructed. Gesture studies examine gesture as an integral component in both communicating and constructing ideas.

The data comprise 1 1/2 hours of videotaped interviews with three university faculty responsible for teaching prospective secondary mathematics teachers about proof. The participants were asked about their approaches to teaching proof, areas of difficulty for students, and their own memories of learning about proof. The tapes were transcribed and linguistic evidence for unconscious conceptual blends and metaphors identified, as well as gestures related to descriptions of proof and proving. The data, both linguistic and gestural indicate that ideas of proof are grounded in several different existing conceptual domains (inputs to the blend), including the idea of a construction and a journey. The oral report will provide video examples of the evidence for this analysis, as well as additional findings from the study.

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IN-SERVICE EDUCATION UNDER MARKET CONDITIONS

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This study investigates in-service education with the methodology of design research. The results from the study demonstrate a clash between two goals of the teacher-educator: The goal of meeting the participating teachers' expectation and the goal of helping them to learn to reflect on their own professional behaviour. In general, in-service education is moving towards more process orientation in which teachers are encouraged to act as reflective individuals (Cooney and Krainer, 1996). This tendency has been observed in Denmark as well together with a more open competition in the offering of in-service courses. In-service education has moved into market conditions, where the target group is the voluntary teacher-participants and the headmasters of their schools. Working with reflections often requires changes at a personal level, which may course anxiety (Fullan, 1985; here Pinar et al. 1995, p. 702). If this anxiety grows to be unpleasant, the teachers can easily choose not to take the in-service course. For more reasons, it is an advantage for everybody that the participating teachers are happy with the course, but is it always possible or desirable?

In the research, I followed teachers through an in-service course – designed and ran by me – back into their classroom, teaching open practical problems in mathematics. The results from this study showed a discrepancy between expectation from me and the way the teaching in the classroom afterwards was practiced. Thus, the teachers were satisfied with their output from the course, while I was less satisfied. It turned out that the teachers were inspired by the in-service education and changed their teaching, but without adapting one of the main goals of the in-service course, which was reflective communication in the classroom when working with open practical problems.

These experiences made me re-design the in-service course to focus more on communication and reflection. The main idea in the new design builds on the assumption that the classrooms in the two situations – at the in-service course and in the schools – have many elements in common. The interpretation of the results from the re-design shows how working with reflection can maintain a balance between causing anxiety stimulating professional development.

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STATEMENTS OF PROBLEMS AND STUDENTS´ CHOICES BETWEEN LINEAR OR NON-LINEAR MODELS

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The phenomenon of overgeneralization of linear models to non-linear contexts has been extensively documented with primary and secondary school students (Van Doorem *et* al, 2005). In studies conducted over the last five years (Villarreal *et al*, 2005), we have documented the presence and persistence of this phenomenon with students studying to be agronomists and mathematics teachers at the University of Córdoba. In these studies, a series of word problems were posed to the students for them to solve. Based on the analysis of the written solutions and the students' strategies, the need emerges to question the manner in which the problems are stated, in terms of their comprehension and interpretation in relation to the possibility of recognizing the subjacent mathematical model.

In this communication, we will focus on the analysis of the statements of the problems posed in relation to the decision of the students' choices regarding different models (linear or non-linear) to resolve them. These statements may be classified as well-structured from the perspective of the person proposing them (the researcher). According to Kilpatrick (1987) in a well-structured problem "*the pertinent information needed to solve the problem is contained in its statement, the rules for finding a correct solution are clear, and you have definite criteria for a solution*" (p. 133). Nevertheless, from the perspective of the person solving them (the student), this may not be the case. Consequently, the student should build strategies to be able to solve the problem or, if necessary, reformulate it. Based on this, we propose to analyse the conceptual and procedural means, and the practices in the schools, that these students use to question the problems presented and decide which subjacent model can lead them to a solution, correct or not, and its validation.

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EQUITY AND QUALITY MATHEMATICS EDUCATION: FINDINGS FROM TIMSS DATA FOR GHANA

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The mathematics education reform in many nations including, Ghana, envisions schools where all students, irrespective of their background characteristics have the opportunity to successfully learn mathematics. To achieve this vision, we expect the schooling systems to function in such a way that students' success learning mathematics do not depend on their diverse backgrounds. That is, if the processes within the schooling system ensure quality outcomes for all students, then we would expect equity in mathematics outcomes within their high achieving schools. This research employs TIMSS data for Ghana to explore the quality of mathematics education in relations to equity.

EQUITY AND MATHEMATICS EDUCATION IN GHANA.

In 2003, about 50 countries including Ghana participated in Trends in International Mathematics and Science Study (TIMSS). TIMSS is designed to help participating countries collect educational achievement data at the fourth and eighth grades with the objective of providing information about trends in their students performance over time. This information includes extensive familial and school background characteristics. The analysis of this information is expected to inform policy development intended; to identify and monitor areas of progress or decline in students' achievement, and to address concerns for equity and the quality of mathematics education. This is the first time that Ghana has participated in an international mathematics assessment that has provided the reliable data to examine equity and the quality of mathematics education within schools in Ghana. In the TIMSS assessment, the average Ghanaian student scored 276 compared to the international average of 466. This dismal performance indicates that there are quite a number of Ghanaian students who are less successful learning mathematics. This paper employed a multilevel framework to explore the schooling processes that provide opportunities for Ghanaian students to successfully learn mathematics. Our analysis indicates that there is a significant variation among schools in their students' mathematics outcomes. That is, students are more success- ful learning mathematics in some schools than others. The most successful students are males with highly educated parents. These students have high academic expectation, like mathematics and are confident learning mathematics. They attend schools that are often located in towns but not in villages or remote areas. In these schools, the parents of students are highly educated, and the mathematics teachers do not frequently use calculators but often provide students the opportunity to explain their mathematical ideas. These findings demonstrate that efforts to improve the quality of mathematics education and ensure equity in mathematics outcomes would demand a comprehensive and holistic approach involving; students, parents, and teachers.

IS SENSITIVITY FOR THE COMPLEXITY OF MATHEMATICS TEACHING MEASURABLE?

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Teaching is acting and deciding in a complex system (e.g. Jaworski & Gellert, 2003). And if the teacher attempts to meet the didacts' requirements for a stronger problem orientation of mathematics instruction, teaching will become even more complex, specially regarding to mathematical and cognitive aspects. So for a long-term successful teaching a sufficient *sensitivity* for the complexity of problem oriented mathematics instruction (POMI) seems necessary. Special demands on teacher education arise from it. At least we should try to *sensitize* teacher students for the complexity of POMI as soon as possible.

Some preparatory works for this purpose could be finished. In (Fritzlar, 2004) I reported on a provisional operationalization of sensitivity for complexity of POMI and an interactive realistic computer scenario as a diagnostic tool for the user's degree of sensitivity. This scenario has some important advantages in comparison with real situations, but there are also some disadvantages, e.g.: While the user works on the scenario her/his willingness or ability to verbalize is often rather low and if so, her/his reflections can be hardly reconstructed. That's why I provide as a supplemental diagnostic tool an interview about decision situations in teaching which are close connected to contents of the computer scenario. In this interview the subject is explicitly asked to express her/his thoughts. In addition (s)he works on another type of decision tasks, because (s)he has not to make a choice like in the scenario but to judge offered alternatives. In the short oral I report on an exploratory study with 20 teacher students which worked on the scenario *and* were interviewed about decision situations. On the one hand findings from analyzing the interviews confirm students' low sensitivity for complexity of POMI (Fritzlar, 2004). On the other hand they suggest that the vector characterizing sensitivity for complexity should be completed by two further important (and measurable) components: the *ability of differentiation* and the *ability of interpretation*.

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BY USING THE OUTCOME-BASED APPROACH TO STRENGTHEN STUDENTS' LEARNING CAPABILITIES

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Students' learning outcomes are a major indicator for teachers to measure students' performance. Mathematics curriculum reform in Hong Kong has set a clear direction for primary mathematics teachers to follow since the implementation of the curriculum guide (CDC, 2000) in 2002. One focus of the reform is to encourage teachers to use the diversified assessment strategies to help them find out the progress of students and give appropriate feedback (Clarke S., 1998). The purpose of this study is to investigate mathematics learning and teaching in the primary classroom with a particular focus on "Assessment for Learning" through two new initiatives in assessment in Hong Kong, the Learning Outcomes Framework (LOF) and the Basic Competency Assessment (BCA). These two instruments are designed to strengthen the relationship between assessments, learning and teaching. The Learning Outcomes Framework depicts a typical progression of learning outcomes identified from the mathematics curriculum and provides a basis for planning learning tasks and assessment by describing what students should know and be able to do. The Basic Competency Assessment consists of a set of assessment items and together with remedial materials based on a set of basic competencies. Recent research reveals that the majority of teachers in Hong Kong largely still use the traditional assessment method in a routine, "paper and pencil" mode as the main way to collect and obtain a comprehensive view of student's achievement and performance (Wong, N.Y., et al, 1999). The study reported here was set up with the intention to encourage more non-paper-pencil assessment in the classroom such as class discussion, oral presentation and investigative activities. In collaboration with more than thirty elementary schools, various assessment materials were developed for selected topics over the last three years. More than 200 school visits with fifty classroom observations have been conducted and over 10000 students' annotated work have been examined. Preliminary analysis confirms that feedback is a crucial component of any learning and teaching cycle. Without proper feedback, assessment becomes meaningless. Effective feedback must be able to help students know what they can and cannot do and where their strengths and weaknesses lie.

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PATTERNS OF STUDENTS' INTERACTIONS WHILE DOING GEOMETRIC PROOFS IN GROUPS

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This paper attempts to discuss the patterns derived from analyzing students' verbal interactions while doing geometric proofs in groups and the extent of participation of the members of the group. It would be interesting to have a closer look at how students tackle geometric proofs by themselves, especially in a situation like in the Philippines where students speak different languages and large class size and limited resources are common. With the documentation procedure used by the Learner's Perspective Study, a rich data were generated to show a picture of the classroom scene, even private talks of students were documented, data that were difficult to gather in the past.

It emerged that few students, attempted to do geometric proofs by themselves. For those who did, they followed the two-column procedure of the teacher. Indeed, much of the learning of geometric concepts has been rote (Clements and Battista, 1992). It was also found that despite the language policy to use English as the medium of instruction for mathematics, it appeared natural for the students to use the mixed language, they either code-switched or code-mixed to Filipino and English to express their ideas, although, mathematical terms, phrases and relationships remained in English. It was evident that the extent of participation of most students was not substantial to make a claim that indeed students at this level were ready to do formal proofs or that cooperative learning worked for this class even in terms of overall participation. Thus, a research on students' readiness to do formal proofs at this level and given situation should be conducted. While result also confirmed that group work is still done superficially in Philippine classrooms (Pascua, 1993), nevertheless, the students' positive responses to the use of group work was hopeful. It could be a matter of conducting more studies that focus on "when" and "how" to use group work in situations where students are bi/multilingual, class size is large and resources are limited.

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DIALOGUE: A TOOL FOR CREATING MATHEMATICAL PROOF

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This research examines students' engagement in a dialogue as a means towards creating proofs. In this study I adopt the communicational approach to cognition, based on the learning-as-participation metaphor, and conceptualisation of thinking as an instance of communication (Sfard, 2001). In this approach learning mathematics is an initiation to a certain type of discourse: literate mathematical discourse (Sfard and Cole, 2002). Sfard considers four dimensions along which literate mathematical discourses can be distinguished from other types of communication: (1) the mathematical vocabulary, (2) their special mediating tools, (3) their discursive routines, and (4) their particular endorsed narratives.

Considering the idea that thinking is a kind of communication that one has with oneself, in this research, I encouraged students to write down the dialogue that they have with themselves while they were thinking to understand or create a proof. Participants in this study were 83 pre-service elementary school teachers. To introduce the idea of writing proof through a dialogue the participants received a sample of a dialogue. The sample dialogue was between two imaginary persona, EXPLORER, the one who tries to prove the proposition, and WHYer, the one who asks all the possible questions related to the process of the proof. The main idea of designing these two personas was to consider two aspects of the character of an individual who is proving a mathematical statement. In this regards, writing down the dialogue may provide students an opportunity to reflect on their thinking process and to organize it in a convincing way. In this perspective the dialogue can be considered as an intermediate stage between having an overview of a proof and writing a formal mathematical proof.

The results show the method of proving through writing a dialogue would be practical heuristic for involving students in the process of creating a mathematical proof. Indeed, the paradigm of dialogues provided students with a flexible environment where they could cultivate their reasoning in the form of a literate mathematical discourse.

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BEGINNING TEACHERS IN MATHEMATICAL INQUIRY: EMERGING COLLECTIVES

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In response to the question, "What is mind?" recent research examining the relation between culture and cognition talks about the *social mind*, the *discursive mind*, *mind as action*, and the *collective mind*. In this paper we draw on complexity theory (Davis, 2004: Davis & Simmt, 2003; Maturana & Varela, 1987) and sociocultural theory (Bakhtin, 1986; Foreman, 2003; Vygotsky, 1986) to investigate how learning collectives evolve and are understood from the perspectives of multiple participants. In particular, we examine the learning collective of beginning teachers as they plan and engage in teaching and its relation to the learning collectives, which arise in their classrooms. “There was a point at which I stopped and realized that the class pretty much taught itself, given the environment we worked to create.” (Beginning teacher, Reflective Journal, Aug. 26, 2005). The findings suggest that as beginning teachers work collectively to explore and plan meaningful mathematical experiences for their learners, they are themselves engaged in an environment similar to the one that they are attempting to create for their classroom.

The data come from a qualitative longitudinal study examining the experiences of 12 beginning teachers working collaboratively to build and facilitate an inquiry-based mathematics learning environment for prospective teachers. An examination of their mathematical activities and discourses helps us understand how these beginning teachers make meaning and develop identities while engaging in sustained exploration of mathematics, and mathematics teaching and learning.

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LEARNING TRAJECTORY OF FRACTION IN ELEMENTARY EDUCATION MATHEMATICS

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The teaching and learning of fractions have been a problematic in mathematics education, particularly in elementary school. Many students are unable to do simple calculation involving fractions. Diagnostic survey conducted by Indonesia Ministry of Education revealed that nearly 30% of junior secondary students (13 years) in adding the fractions 1/4 and 2/5 simply added the numerators and denominators that led them to a wrong answer as 3/9. Most students lacked understanding of decimal number values. Less than one student in six could correctly order the three decimal 0.55 .... 0.8 .... 0.14 from smallest to largest. There were several widespread systematic errors. For example, more than two-third of the students considered 0.8 to be smaller than 0.14, because they evaluated decimal numbers as if they were whole numbers (Somerset, 1997).

This short oral presentation describes the findings of second year research out of three years planning on the development of learning materials about fraction in elementary education based on realistic mathematics education (RME) theory. This research has been conducted by development research approach. The main aspect of this approach is cyclic processes of thought experiment and teaching experiment. The research holds the description of learning trajectory about fraction for elementary education. The trajectory consists of several themes relevant to local contexts following the sequences of: (1) fair sharing; (2) repeated division by two (halving); (3) repeated division by 10; and (4) decimal and percent (Hadi, 2005).

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ELEMENTARY EDUCATION STUDENTS' AFFECT TOWARDS AND ADVANCEMENT IN MATHEMATICS

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The relationship between affective and cognitive development has been of great interest for educational research. Summarising a larger body of research on self-concept and achievement, Chapman, Tunmer and Prochnow (2000) suggested that the causal relationship is dependant on age. In the early school years, the causal direction would be from achievement to self-concept. This would first change into a reciprocal linkage and later the direction would be predominantly from self-concept to achievement. If this hypothesis is correct, we should find self-concept to be more important predictor for teacher students' future achievement than their achievement.

The research project "Elementary teachers' mathematics" (Academy of Finland project #8201695) draws on data collected of 269 trainee teachers at three Finnish universities (see Hannula *et al*. 2005). Questionnaires were administered in the beginning and in the end of a mathematics education course. Students' view of mathematics (including self-concept) and their mathematical skills were measured.

In general the effect of the course was to even out some of the student differences. Those students who had performed in the best or worst quartile in the pre-test, tended to change towards an average performance. The effect of background variables (gender and previous mathematics studies) remained constant or decreased from pre- to post-test. The results do not support the hypotheses that among older students the causality would be mainly from self-concept to achievement. However, the lack of causality from self-concept to achievement may be due to intervention strategies to promote positive affect, which were applied in all institutions involved.

The six clusters that were identified earlier (Hannula et al. 2005) did continue to have differences in their post-test results and there was some difference in the advancement of 'encouraged' and 'diligent' students. A closer analysis revealed that the student encouragement from their family had a different effect on student advancement according to their achievement in the pre-test.

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PRE-SERVICE MATHEMATICS TEACHERS: CHALLENGES CONNECTING THEORY WITH PRACTICE

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The National Council of Teachers of Mathematics (NCTM, 1991) created a vision for secondary mathematics classroom, described as mathematical communities. A longitudinal study was conducted to investigate examine the kinds of challenges that pre-service secondary teachers face as they enter the mathematics classroom.

Theoretical perspective and methods. The apprenticeship model described by Lave and Wenger (1991) portrays learning as an enculturation of novices into a way of thinking and doing that reflects the practices of an expert. In this study, first author assumed the role of an expert who was a researcher/participant as she supervised and mentored pre-service teachers. Four sets of data were collected to investigate the challenges that pre-service teachers face as they move from the theoretical to the practical realm of the secondary mathematics classroom. These data were analyzed using constant comparative analysis to categorize the types of situations that arose during teaching that caused a challenge for pre-service teachers to interpret or make an instructional decision.

Results. Four basic categories emerged from the analysis: the need to "cover material," differences in teaching philosophies, questioning, and student thinking. Perhaps the most difficult challenge to overcome is the challenge of asking appropriate questions to elicit student thinking. Pre-service teachers must become aware of the variety and breadth of intention behind classroom questions. Good questions are those which extend the students' thinking about a problem. Rarely do pre-service teachers pose challenging questions such as "compare and contrast the circumference and the area of a circle." More often, they ask: "What is the circumference of a circle?"

Discussions and implications. Even though pre-service teachers do face challenges as they move from the theoretical realm of professional education courses to the practical realm of the secondary mathematics classroom, they do not often recognize these challenges without discussion with an observer/mentor. In rare instances when these challenges are recognized, pre-service teachers do not often have the support of a learning community in order to gain insight for making the instructional decisions necessary.

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EFFECTIVENESS OF VIDEO-CASE BASED ELEMENTARY MATHEMATICS TEACHER TRAINING: AN EXPLORATORY STUDY IN CHINA

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Video Cases have showed optimistic prospects to the training of teachers. However, there are not enough empirical evidences to convince the positive influences of the promising teacher training approach (Silver, et al., 2005). In China, in order to tackle the in-service mathematics teacher training tensed by the implementation of the new mathematics curriculum in 2001, a new model of teacher professional development, called "action education" has been developed and popularized recently. One product of "action education" is Video Cases to record the critical components of the process of the collaborative learning which includes a case discussion work sheet, video clips, case questions and case assessments (Bao, et al., 2005). This paper reports an exploratory study on the effectiveness of Video-Case based elementary mathematics teacher training program. There are sixteen mathematics teachers at elementary schools participating in a five-day summer course in 2005, in Macau, China. Through questionnaires, interviews, and video-taping, the following three questions were investigated: (a) Can the Video-Case based teacher training program affect the trainees' mathematics concepts? (b)Can it improve the trainees' analysis and reflection abilities of mathematics lessons? (c) What are the feedbacks of the trainees towards Video-Case based training program? It was found that: (a) Although as a whole, the video-case based training program has no significant influence on the change of teachers' attitudes, the participants have significant change in their views about the abstractness of mathematics and the relationship between mathematics and daily life context, (b) the ways of examining a lesson seemed to change from pedagogy orientation to making a balance between mathematics content and pedagogical content, (c) participants felt that video-case based training program has positive functions in improving their understanding of particular mathematics content, mathematical pedagogy, the ability in examination of lessons and in reflecting upon their own practice.

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IMPROVING STUDENTS' LEVEL OF GEOMETRICAL THINKING THROUGH TEACHER'S REGULATING ROLES

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This paper is a part of a two-year long project conducted during August 2003- July 2005. The purposes of the project were to analyze junior high school students' mathematical learning process, specifically, geometrical learning process and to investigate how to improve students' geometrical learning, in particular, to raise students' level of geometrical thinking. In the first year, 12 pairs of students from 6-8 grades were asked to solve open-ended problems by thinking aloud method and were individually interviewed after problem-solving session. Major data for analysis were 36 protocols of three open-ended problems, students' written works, and transcriptions from the interview sessions. According to van Hiele's levels of thinking (Lester, 1988), Students demonstrated that their geometrical thinking were limited to merely level 0 (9 pairs) and level 1 (3 pairs) during their problem-solving session in the first session. In the second session, while the interviewers interviewed each student, they took the role of facilitator (Schoenfeld, 1992) which encourages the students to reflect on their solving processes using the 'what', 'how', and 'why' questions. The role of researcher as an instructional interview initiated a new way of communication between teacher and students. This role improved students' metacognition in using language to explain their geometrical ideas. This language is intimately link with geometrical concepts. It is found that change of their language usage illustrating change of levels of students' geometrical thinking. In the second year, the research was conducted in the actual classrooms of the school where the students in the first year enrolled their study. Open-approach method (Nohda, 1984) was used in the classroom as a method of teaching with teachers taking facilitative roles similarly to researcher's role in the first year. The teachers took the facilitative roles asking 'what' 'how' and 'why' questions during the problem posing, small-group, and whole-class discussion sessions. The teachers had been tried to encourage the students to express their ideas and reason the problem solution the way they did. The results revealed that these roles affected developmental change of level of students' geometrical thinking which also improves students' geometrical learning process.

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WHAT'S THE CONNECTION BETWEEN EARS AND DICE? THEY BOTH PROMOTE PROBABLISTIC UNDERSTANDING

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"You have two ears and I have one ear. So, to count two ears on me and one ear on you-is that the same thing? No! It's two different things, taken from two different places, so it's not the same thing" **.**Thus ended a "mathematical argument" among a group of students whether the sample space of two rolled dice is 21 or 36? We present here one episode from research that examined self-constructed probability concepts among 7th and 8th grade pupils with the aid of game tasks. In this episode the rules of the game were: Role two dice. If the sum of the two is 2, 3, 4, 10, 11 or 12, player A gets a point. If the sum is 5, 6, 7, 8 or 9, player B gets a point. First to get 10 points is the winner. If the game is not "fair", suggest rules for fair games and defend your suggestions (Amit, 1999; Alston & Maher, 2003).

Fairness was understood to be an equal chance for both players to win. Thus, it was necessary to calculate the sample space. Most claimed it was 21 events; a few claimed it was 36. The disagreement arose from the disparity: ***What do you count as an "event"?*** Sample space 21 was based on the number of pairings that comprise each sum. By this logic (5, 6) or (6, 5) comprise the same event. Considering these as two different events (obtaining 36) Aviv argued: *"5+6 and 6+5 are the same numbers, but it's not the same way. I can get 5 first then 6, or 6 first then 5."* Linoi objected*: "5+6 equals 6+5; it's the same situation because of the* ***commutative law."*** She mobilized a heavy arithmetical weapon, the commutative law, and Aviv lost the battle. The argumentation and persuasion continued, Heli, favouring a sample space of 36 hypothesized: *"If the dice were* ***different colours****, it would be easier to tell the difference."* (She exhibited 5, 6 and 6, 5 by switching two white dice). Tal objected: *"We have to* ***find the sum****,* ***not the sources of the numbers****."* Then Tamir came up with a surprising metaphor: "*[Suppose]* ***I have one ear and you have two ears;*** *vice versa is not the same"*. Tamir persuaded his classmates that 5, 6 is different from 6, 5. The conflict was solved, a consensus concerning "events" and "sample space" was reached and rules for fair games were suggested. The students went through a process of ***conflict, argumentation, persuasion and consensus***, and developed new understandings of probability concepts. Examples will be presented in the session.

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TACTILE PERCEPTION IN 3D GEOMETRY

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In our previous research aimed at learning about pupils understanding of geometrical solids using games and non-standard tasks, we found pupils tactile manipulation of solids could be broken down into three levels, global, random and systematic (Littler, Jirotková, 2004), which mirrored the first three Van Hiele's levels of insight into 3D geometry (Van Hiele, 1986). We wish to report a further development of this research in which the pupils had to verbalise their perception of solids whilst undertaking a task which required them to sort solids tactilely aimed at linking manipulation, classification and communication. In describing the tactile perception of a solid, the pupil has to rely on the mental image which is created in his/her mind, find suitable language to describe their thought processes and then use this information to classify the solid. The aim of the research was to see whether using tactile perception only, the link between manipulation and classification held and whether the necessity to communicate their thoughts verbally related to either of these phenomena.

Pupils in both the Czech Republic and the United Kingdom had thirteen solids to sort into two groups, such that in one of the groups at least all the solids had a common attribute. The solids were selected so that some would be very familiar, others less so and some which would be new to them. The pupils were unable to see the solids as they completed the task. Their manipulation was videoed and their commentaries were audio recorded.

The manipulative process of the pupils was linked closely with their language level and their ability to classify at the three levels of manipulation. Pupils who used global manipulation, that is holding the solid and getting a gestalt impression of it used everyday language to describe it, such as 'it is pointed' and often their classification criterion was vague. Those using random manipulation used geometrical language but often 2D rather than 3D, they often used the shapes of the faces for classification. The systematic manipulators more often used correct language and were able to express their criterion of classification clearly. Our presentation in addition to giving examples and details of the results of the research will show videos of the pupils undertaking the task.

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HIGH ACHIEVING STUDENTS' CONCEPTIONS OF LIMITS

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Students learning limits of functions perceive and treat limits differently. A study on students' conceptual development of limits of functions was conducted at a Swedish university (Juter, 2006). The results imply differences in high achieving and low achieving students' work with limits, but also a lack of differences at some points.

The students' developments and abilities were studied in terms of concept images (Tall & Vinner, 1981) in the sense that their actions, such as problem solving and reasoning, were considered traces of their mental representations of concepts. A concept image of a concept comprises all mental representations of that concept and is linked to related concepts in a web.

As could be expected, high achieving students' abstraction abilities were more developed than other students'. The former group were to a much higher degree than the latter able to link theory to problem solving and explain the meaning of, for example, the limit definition. The students were studied during a semester and there were similarities of the high achieving students' developments with the historical development of limits that the other students did not reveal. Several similarities were linked to abstraction and formality.

Students with positive attitudes to mathematics in general were better limit problem solvers. Most of the high achieving students thought that they had control over the concept of limits, but many of the low achieving students also claimed to have control even if that was not the case. An unjustifiably strong self confidence can prevent students from further work on erroneous or incomplete parts of their concept images.

There were no clear patterns of students' mental representations of limits as exact values or approximations, limits as objects or processes, and limits as attainable or unattainable for functions. Of the 15 students interviewed, only two showed a coherent trace of their concept images. Both students were high achievers. The lack of patterns in all students' concept images, particularly in the high achievers', points to the complex nature of limits and the challenge to teach and learn limits.

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READING VISUAL REPRESENTATIONS OF DATA WITH KINDERGARTEN CHILDREN

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Recently, it has been stressed that the inclusion of data handling in the school mathematics curriculum has to be strengthened (NCTM, 1989). However, this recommendation has not yet been accepted from teachers in primary or secondary school, as they underestimate the significance of statistics in relation to the other topics of the mathematical program or they do not have the necessary background in this topic. On the other hand, little research has been done on students' thinking in data handling (Shaugnessy et al., 1998).

The visual representation of data has been mentioned by many researchers as a comprehensible and fruitful context to discuss statistical data (Shaugnessy et al., 1998). Towards this effort the purpose of this study is to investigate the capabilities of kindergarten children to read visual representations of data. More specifically, we tried to find out if some types of visual data representation are more appropriate for this age in order to introduce elements of statistics in preschool education.

The research was realized in five kindergarten schools and 50 children were interviewed. The interview concluded six problems with different types of data representation: a pictogram, a three-dimensional block graph, a two-dimensional block graph, a set- diagram, a bar chart and a cyclic diagram (Haylock and Cockburn, 1997).

The results showed that the kindergarten children can read the different types of diagrams, as they can answer questions about the information that these diagrams give. More difficulties seemed to be presented with the process of counting in the two-dimensional block graph as well as with the reading of the cyclic diagram.

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STUDENTS' USE OF GESTURES TO SUPPORT MATHEMATICAL UNDERSTANDINGS IN GEOMETRY

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An after school Math Club provided the context for a teacher research study on the processes students used to facilitate their talk about mathematics. Gestures supported the mathematics talk as students used them in problem solving to replace the names of shapes, locate shapes within designs and as a tool for supporting mathematical arguments.

This teacher research study examined the processes students used to facilitate the mathematics talk in an after school Math Club in a predominantly Mexican/Mexican American working class neighborhood. Drawing from sociocultural theory and research on mathematics talk (Elbers & de Haan, 2005; Morales, Khisty & Chval, 2003; Moschkovich, 2002), the findings suggest that mathematics talk occurs within a community of practice. Working in small groups on mathematical tasks based on geometry, the students focused on their understandings of the problems and possible solutions. Using ethnographic methods in data collection and analyses, audio tapes and videotapes demonstrated the students' use of gestures in their explanations.

This presentation focuses on how and when students use gestures to communicate with each other while solving problems together. In particular I will discuss a mathematical task in which students had to determine the number of rectangles that could be generated from a drawing of a rectangular array of squares. Some of the findings include how gestures were used to replace the names of shapes and their characteristics, to locate a shape in a geometric design, and to act as a critical tool to support mathematical argument. This important aspect of mathematics talk adds to the previous research of Dominguez_ (2005) whose findings provide insights about students' mathematical knowledge as it develops through gestures and speech.

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MATHEMATICAL ABILITIES FOR DEVELOPING UNDERSTANDING OF FORMAL PROOF

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In this note, which is a part of an on going study, we focus on the ability of the identification of functions in different representations in the frame of Tall's "three worlds theory" (Tall, 2004), and their relation to the ability of understanding formal proofs. The data was collected by a questionnaire administered to 353 12th grade students, who had mathematics as a major subject. The questionnaire consisted of 30 tasks which aimed to investigate the students' ability to operate in the embodied, proceptual and formal word in the context of functions (23 questions) and to understand a formal proof which is given and related to students' curriculum (5 questions). From data analysis we conclude that the vast majority of the students had serious difficulties to identify functions in different representations. In fact they could identify only representations similar to those that had been taught. Their responses also did not indicate any understanding of formal proof even in problems related to their curriculum. Few students could identify functions from their representations. These students could also understand some aspects of formal proof. It seems that the ability to identify functions in different representations is necessary in order to develop some understanding of formal proof. By discussing these results in terms of Tall's "three world theory", we can say that the vast majority of 12th grade students could operate partly in the embodied and the proceptual world in the context of functions but they did not show any understanding of formal proof. Only a small number of students could operate sufficiently in the embodied and proceptual world. These students could also partly understand formal proof. It appears that, in the context of functions, the development of understanding of formal proof requires the ability to operate in the embodied and the proceptual world.

Acknowledgments The present study was funded through the program EPEAEK II in the framework of the project "Pythagoras II-Support of University Research Groups" with 75% from European Funds and 25% from National Funds.

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AN ANALYSIS OF CONNECTIONS BETWEEN ERRORS AND PRIOR KNOWLEDGE IN DECIMAL CALCULATION

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To teach decimals with related prior knowledge is necessary for meaningful and conceptual understanding. As decimals are connected inappropriately to prior knowledge, various errors occur (Drexel, 1997). Although there have been many studies on students' error patterns, little research has been conducted with regard to the connections between errors and prior knowledge, in particular with decimal computation (Bassarear, 2001).

Given this, 68 students in grade 6 were investigated with regard to their common errors. According to the related prior knowledge, decimal calculation error patterns were classified into decimal point error, natural number operation error, and fraction knowledge error. Intensive clinical interviews were conducted with 8 students who showed representative errors as a part of our attempts to probe the nature and cause of such errors.

It was found that students showing errors had difficulty in understanding decimals as another representation of decimal fraction. For example, students decided $\frac{2}{100}$ to be bigger than 0.02, because decimals were the smaller. Students also could not connect the meaning of multiplication and division of natural numbers with that of decimals. Specifically, they posed word problems corresponding to 16÷4 but not for 1.6÷4. Students also made errors due to the confusion of algorithms. For instance, in computing 2.02 x 2.6, they calculated 2x2=4 and 2x6=12 only to produce 4.12.

Against these errors, a series of teaching experiment was designed and implemented. It was emphasized that students had a lot of opportunities to connect decimals with decimal fractions by using such as base-ten blocks. Students were also encouraged to discover algorithms on the basis of the meaning of operations of decimals. Several illustrative classroom episodes will be included in the presentation, which demonstrates the process by which students overcome their errors and develop conceptual understanding.

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INSIGHTS INTO PRIMARY TEACHERS' INTERPRETATIONS OF STUDENTS' WRITTEN ANSWERS IN MATHEMATICS: A STUDY

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In order to judge students' mathematical attainments, teachers read pupils' productions (texts) in an interpretive and contextualized manner, based on "the resources they bring to bear as they 'read' the students' mathematical performance from these texts. These (resources) … arise from the teachers' personal, social and cultural history and from their current positioning within a particular discourse" (Morgan & Watson, 2002).

In the last decade, a small number of studies attempted to identify the resources utilized and the positions adopted by teachers during the process of assessing students' texts. Among such resources were found to be teachers' personal knowledge of and beliefs about mathematics and their expectations of how mathematical knowledge can be communicated (Morgan & Watson, 2002). On the other hand, the positions identified were "teacher-examiner, using either own or externally determined criteria", "teacher- students' advocate" and "teacher-adviser" (Morgan et al., 2002). The results of the above studies show that teachers draw on individual as well as collective resources from different and often contradictory discourses and the way they are positioned within them may lead a teacher to assess differently a student in different times and contexts.

The study reported here is part of a larger research project, which aimed at examining the resources which Greek primary teachers draw on and the positions they adopt within the pedagogical discourse of assessment. The data analyzed here came from 553 primary teachers, who were asked to evaluate in writing the authentic solutions to a word problem of four students, which differ with respect to the features characterizing a mathematical text. The results indicate that teachers' assessment of students' written work is indeed determined by the positions they adopt within the discourse they deploy in practice and the resources available to these positions. More importantly, teachers tend to switch between positions and often use contradictory resources in assessing, due to incompatibilities inherent in the discourse employed or to the teachers' ambivalence or confusion about the resources that may be drawn on.

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THE ROLE OF PROOF

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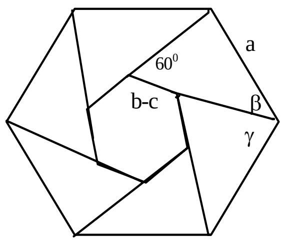
In the three teaching experiments with 17 – 18 years old students the role of proof was evaluated. The personal involvement of the students through the experiment was increased. The analysis of students' work shows some obstacles on the way to expected achievements and raises some additional tasks concerning the role of proof in the learning process.

MATHEMATICAL STARTING POINT OF TEACHING EXPERIMENTS

The starting point of the experiment is the following theorem with the exciting proof.

Theorem: If in a triangle the measure of angle α is 60°, then the area of the triangle is given by the formula: $S = \frac{\sqrt{3}}{4}\left[a^2 - (b-c)^2\right]$ (*).

Proof:



If in the triangle ABC with lengths of sides a, b, c there is given an angle α equal to 60°, then $\beta + \gamma = 120°$. Six such triangles together can be formed in a regular hexagon with side a. In the interior of this hexagon is a regular hexagon with side b-c. Computing the areas of both hexagons we obtain formula (*).

Description of three teaching experiments

Having the experiences from the starting point through the learning process to the end point, which was finding the proof of new theorem (slightly changed starting theorem) as one of the objectives, is the main stream of all three experiments.

The experiment 1 bases on the ready made proof of theorem 1. In the experiment 2 the students were asked to prove the stated theorem by themselves. The experiment 3 started as guided discovery in order to find the theorem and its proof.

The results of increased students' personal involvement give us some additional didactical tasks.

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A COMPARISON OF MATHEMATICALLY GIFTED AND NON-GIFTED STUDENTS IN INTUITIVELY BASED, PROBABILISTIC MISCONCEPTIONS

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This study was based on previous studies on the thinking process of mathematically gifted students and on probabilistic misconceptions, in particular intuition-based probabilistic misconceptions. The researchers presumed that mathematically gifted students differ from their peers in intuition-based probabilistic misconceptions as shown by previous studies. We investigated some differences of mathematically gifted and non-gifted groups of students in probabilistic misconceptions. Intuition-based probabilistic misconceptions were stronger in non-gifted than gifted students. And that the gifted group differed from non-gifted group in the sources of misconceptions.

In the following, we described the findings referring to each question.

Question 1. When tossing a coin, there are two possible outcomes : either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time?

Table 1 The answers to Question 1(in percentages) (* : misconceptions)

Answers	Gifted	General
Greater than the chance of getting tails	***24.2****	25.8
Smaller than the chance of getting tails	6.1	***37.1****
Equal to the chance of getting tails (correct answer)	69.7	37.1

Question 2. In a lotto game, one has to choose 6 numbers from a total of 40. Vered has chosen 1, 2, 3, 4, 5, 6. Ruth has chosen 39, 1, 17, 33, 8, 27. Who has a greater chance of winning?

Table 2 The answers to Question 2(in percentages) (* : misconceptions)

Answers	Gifted	General
Vered has a greater chance of winning	6.1	0.0
Ruth has a greater chance of winning	***48.5****	***89.0****
Vered and Ruth have the same chance to win (correct answer)	45.5	11.0

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EXPLORING TEACHING AND LEARNING OF LETTERS IN ALGEBRA: A REPORT FROM A LEARNING STUDY

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How could a learning situation be designed to promote the learning of letters representing variables in equations? In this paper we present results from a study about algebra in a Swedish school involving 76 pupils (10 and 11 years old) and their teachers. The model used – Learning Study (Marton et al., 2004) - involves teachers and researchers working co-operatively as partners in an iterative process, gathering data about teaching and pupils' learning, analysing the data, planning and revising the lessonplan. The aim was to enhance pupils' learning of the relation problem – equation. Video data from three lessons and results from a post-test were analysed. The variation in learning outcomes on the post-test was related to the variation as regards to how the content was handled during the three lessons.

The analysis demonstrated that the three lessons were different as regards to those aspects of an equation that was brought out; particularly, the arbitrary choice of letters representing the variables. In lesson 1 this particular aspect was never brought out at all. Only after studying the test results and watching the video recording, the teachers realised that this aspect had been taken-for-granted in lesson 1, so they changed the lesson plan in this respect and taught the topic differently. Varying the positions of the symbols in the equation, as happened in lesson 1, was probably not enough for the pupils to learn the arbitrary choice of letters. Our interpretation is that it was likely that the way letters was presented in lesson 1 made it possible for the pupils to see letters as shorthand for objects (c.f. Küchemann, in Booth, 1984). The second and the third lesson, however, were designed to bring out the idea that the letters used are arbitrary chosen. This was done by constituting a particular pattern of variation; in this case by using 'counter examples' and different letters for the same example. This change of the lesson was reflected in the learning outcomes. The pupils in class 2 and 3 performed better on the post test as regards to being able to exchange the letters in the equation. Thus, it is likely that more pupils in class 2 and 3 have learned that the variables could be represented by any letters.

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TEACHING MATHEMATICS TO INDIGENOUS STUDENTS AND PUPILS FROM MULTICULTURAL BACKGROUNDS: LESSONS FROM NEW ZEALAND AND SASKATCHEWAN

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Although the New Zealand Math curriculum guide acknowledges the failure to help all pupils achieve their learning potential, it uses Maori terms and suggests the use of Maori cultural knowledge in explaining mathematical concepts such as computation and estimation. Similarly, Saskatchewan teachers are directed to make necessary changes to suit student needs through a concept known as the Adaptive Dimension. Teachers are encouraged to guide students to design their own problems, demonstrate the applicability of math through integration with areas of study and daily life, and incorporate mathematical ideas associated with the traditional Indian and Métis culture (Saskatchewan Education, 1996). Yet research by writers from both regions suggests that the attitudes of Math teachers and preservice training instructors may contribute to the gap between policies that are meant to privilege learners and the actual occurrences in Math classrooms which comprise of indigenous students and pupils from multicultural backgrounds.

Forty-seven Math teachers from both regions were surveyed in Spring 2004. Despite research that suggests native students are failing behind, as well as an understanding on the part of teachers that culture impacts pedagogy, the findings revealed that teachers did not make adjustments to the curriculum. Indigenous students and pupils of ethnic descent are treated more or less the same as other students in the mathematics classroom as the subject matter is perceived to be divorced from cultural aspects. While there is a disagreement about the extent to which indigenous counting systems and problem solving techniques are present in the curriculum, they are moderately represented at best. Teachers do build on other types of knowledge such as those found in domestic tasks and the workplace in order to link math to other experiences and some structural adjustments are used more often. Perhaps this is done to help students reach curriculum targets, which teachers feel they need in order to succeed in the 'real world'.

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LIMITATIONS OF A PARTITIVE FRACTION SCHEME IN DEVELOPING MULTIPLICATIVE REASONING ABOUT FRACTIONS

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This paper presents the results of a case study that investigated a seventh grader's difficulty in developing multiplicative reasoning in fraction contexts. The study finds that the difficulties are related to a student's view of a unit fraction as a part, not independent of, but belonging to a unit to which the unit fraction refers, preventing him from conceiving fractions as multiplicative operations.

This study investigated how a seventh grader, who established a partitive fraction scheme but has not yet developed the concept of an iterable fractional unit, had difficulty in developing multiplicative reasoning in fraction contexts. A partitive fraction scheme involves embeddness and equi-partitioning scheme but doesn't require a unit fraction to be generally an iterable fractional unit (Steffe, 2002). It is therefore probable that a student may have difficulty in multiplicatively reasoning in a fraction context because he or she may consider a unit fraction as a part, not independent of, but belonging to the unit to which the unit fraction refers. Such a possibility was investigated in the study by conducting a series of teaching experiments with Mike, who was a 7th grader. The study was conducted over a semester long period, totalling 16-videotaped sessions. In each session, 20-30 minutes long, Mike was asked to work with computer software called Tool for Interactive Mathematical Activity (Olive & Steffe, 1994). The TIMA software allows students to make rectangular regions called bars and partition the bars into parts, the parts into subparts, etc. This provided us with opportunities to observe how middle graders develop algebraic and quantitative thinking. The research finds that Mike's lack of a concept of an iterable fractional unit may create difficulty in developing multiplicative reasoning, evidenced by three indications: 1) for the parts produced, Mike referred to the bar from which the parts originated without any concern about a given unit involved; 2) Mike dealt with a common fraction as a unit fraction by initializing a unit fraction in terms of the common fraction that was supposed to refer to the unit fraction; 3) Mike preferred taking off parts or adding parts when asked to make a bar so that a given bar is a fraction of the bar. These findings lead to the conclusion that a partitive fraction scheme lacking a concept of an iterable fractional unit led Mike to rely on the bar from which parts originated when finding referents for the parts, prevented him from coordinating the units considered, and restricted him from dealing with various units.

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TEACHERS' REFLECTION AND SELF-ASSESSMENT THROUGH THE USE OF A VIDEOTAPE OF THEIR OWN MATHEMATICS INSTRUCTION

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Despite the widespread mathematics education reform toward student-centered instruction, many teachers experience difficulties not only in understanding but also implementing it (Pang, 2005). Given this, it is significant for teachers to look closely at their own instruction and to contemplate whether they employ student-centered teaching practice. A powerful tool to do this is to use videotaping (Artzt & Armour-Thomas, 2002). This study investigated how teachers would change via reflection and self-assessment through the use of a videotape of their instruction.

The participants were 9 elementary school teachers who attended an intensive graduate course of 48 hours during a winter vacation. As a prerequisite course assignment, each teacher designed a student-centered mathematics instruction and videotaped it. While taking the course, the participants learned how to examine instructional practice in terms of tasks, learning environment and discourse, and how to probe teacher cognitions in terms of goal, knowledge, and beliefs. They then analysed the videotaped instruction step by step, presented a self-assessment paper, and received various feedbacks from the teacher-educator and the other teachers. The participants also completed two kinds of questionnaires: (a) on the understanding of good mathematics instruction, and (b) on the experience of analysing their own teaching practice.

Initial teacher cognitions of their own instruction were divided into student-centered, teacher-centered, and intermediate practice. Close lesson analyses showed the differences between teacher cognitions and actual instruction. For instance, a teacher who originally conceived her instruction as student-centered came to realize that she forced students to master a given learning theme and demonstrated her strategy to construct a pattern. After the course, teachers were more enthusiastic to understand and implement student-centered instruction. In the presentation, we will illustrate representative cases of teacher change and discuss the effect of thorough reflection and self-assessment on professional development.

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A CASE STUDY ON THE INTRODUCING METHODS OF THE IRRATIONAL NUMBERS BASED ON THE FREUDENTHAL'S MATHEMATISING INSTRUCTION

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As research on the instruction method of the concept of irrational numbers, this thesis is theoretically based on the Freudenthal's Mathematising Instruction Theory and a conducted case study in order to find an introduction method of the irrational numbers. The purpose of this research is to provide practical information about the instruction method of irrational numbers. For this, research questions have been chosen as follow: 1. What differences are there between the introducing method of irrational numbers based on the Freudenthal's Mathematising Instruction and in the Korea textbook? 2. What are the Characteristics of the learning process shown in class using introducing instruction of irrational numbers based on the Freudenthal's Mathematising Instruction?

METHOD OF RESEARCH

For questions 1 and 2, I conducted literature review and case study respectively. For the case study, I, as a participant observer, videotaped and transcribed the course of classes, collected data such as reports of students' learning activities, information gathered through interviews, and field notes. The result was analyzed from three viewpoints such as the characteristics of problems, the application of mathematical means, and the development level of irrational numbers concepts.

MAJOR FINDINGS OF THIS STUDY

First, during the course of learning new mathematical concept, it was shown that the students tend to link their previously known mathematical knowledge, which is called reflective thinking. They understood the existence of irrational numbers, as another not as rational number, by knowing the fact that the one side of regular square, of which the area is not a square number, cannot be explained with rational number. Second, students began to understand irrational numbers as existing through learning activities of The Wheel of Theodorus. Also, by measuring the length of irrational numbers by themselves, students began to make comparable strategy with rational number. To the extent, it is shown that the activities for measuring the length of irrational numbers help students to improve their conceptional understanding on irrational numbers. Third, using various mathematical tools for stimulating students' reflective thinking is helpful for building the concept of irrational numbers. However, close attention and guidance are needed due to the difference among students on understanding the meaning of mathematical tools.

A CASE STUDY OF AN ELEMENTARY SCHOOL TEACHER'S PROFESSIONAL DEVELOPMENT ON MATHEMATICS TEACHING IN CONTEXT

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There is a popular trend in mathematic curriculum reform worldwide over the last two decades. One consequence of curriculum reform is a change in teaching methods and this time it is from traditional teaching methods (ex. lecturing) to reform teaching methods (ex. group discussion). In the case of Taiwan, the elementary mathematics curriculum reform was launched in 1996. The present paper is a study on an elementary school teacher's professional development in mathematic teaching within this reform timeframe.

From theories proposed by Cooney(2001), Perry(1970) and Baster Magolda(1992), one can conclude that contextuality induces doubt, which may trigger some change(s) in belief and/or teaching behaviors. The researchers also indicate that the last stage for one's development in intellectuality and/or thinking is to perceive and/or think things "in context".

The current case study is Ms. Huang who has a six-year teaching experiences in elementary school. The research methods are interviews and classroom observations. There are a total of 13 interviews and 5 classroom observations.

The results showed that Ms. Huang changed from a sole discussion method to a combination of discussion and lecturing methods. She realized that the discussion method was only applicable under certain context, in schools where students' abilities are better and/or students have extra assistances such as dedicated parental supports and/or high cram-school enrollment. Therefore, it is essential to encompass all teaching methods in the teacher-training program and to educate potential teachers that there are no inherently good or bad teaching methods but they should learn to how choose the most suitable teaching method/strategy under different contexts.

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"BUT AFTER ALL, WE'LL NEED THIS FOR SCHOOL" MAKING SENSE OF MATHEMATICAL REQUIREMENTS IN TEACHER EDUCATION – INTERVENTION AND EVALUATION

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For creating quality in mathematics classrooms, teachers need high mathematical abilities and deep insights into mathematics. Thus, German preservice grammar school teachers in mathematics attend mathematics courses together with prospective mathematicians for about two years. Two problems can regularly be observed:

1. Disappointing achievements of teacher students in the exams.
2. Even higher achieving students being not capable of bridging the gap between the acquired axiomatic knowledge on academic mathematics and school mathematics - with its emphasis on meaning, pre-formal thinking, contexts etc.

These problems interdepend: Low achievement is not only a matter of individual capabilities. Instead, evaluations have shown that one major obstacle for teacher students to engage in academic mathematics is a *lack of individual sense* ("I won't become a full mathematician but *only* a mathematics teacher, why do I have to learn all this?"), mostly combined with a *problematic self-perception* as "second class"-students. This perceived irrelevance of academic mathematics for the future professional life mostly roots in missing connections between school and academic mathematics in the first year mathematics courses.

The presented project tackled these problems by a pragmatic but effective intervention. The central first year lecture 2005/06 in calculus was accompanied by especially designed tutorials for teacher students. Individual work and small group activities were initiated by about 80% of usual tasks and 20% special "teacher tasks". These "teacher tasks" *bridge the gap between school and academic mathematics* by different strategies, e.g. emphasising understanding and meaning, drawing connections by working on textbooks tasks, reflecting the development of mathematical knowledge, giving opportunity to experience exactification as useful for analysing textbooks and pupils' conceptions (e.g. on infinity). In this way, the intervention aims at making sense and developing a "*didactically sensitive understanding of mathematics*" (Hefendehl-Hebeker 2002). First evaluations have shown that the students in the project experienced academic mathematics as directly linked to school mathematics and academic accuracy being necessary.

In the contribution, we will present concrete examples of tasks, the underlying task design principles, their treatment by and effects on students. Empirical evidence can be given that the intervention enhance making sense of the mathematical requirements for prospective teachers.

DEVELOPING PRIMARY STUDENTS' COGNITIVE SKILLS THROUGH INTERACTIVE MATHEMATICS LESSONS

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Mathematics curriculum reform at the primary level in Hong Kong has been a concern of primary mathematics teachers since the implementation of the new curriculum in 2002 which has set out the general directions for curriculum development in the country for the next 10 years (Curriculum Development Council, 2000). Major changes are the role of the teacher as a facilitator in the classroom, helping students develop cognitive skills and fostering students' motivation and interest in mathematics. The reform propels teachers towards a paradigm shift from a largely textbook-based teacher-centred approach to a more interactive and learner-centred approach. For the paradigm shift to succeed, mind set and culture changes are necessary.

The purpose of this study (Leung, K.M., in preparation, 2007) is to examine the teaching and learning of mathematics in the HK primary classroom, particularly in the context of the recent curriculum reform. The majority of teachers in Hong Kong still use the textbook as the main resource for their teaching and their teaching practices remain largely traditional. At the same time, the Hong Kong curriculum reform documentation suggests that a learner-focused approach is likelier to meet the best interests of learners; that diversified learning and teaching strategies are likelier to suit the different needs of learners; and, that part of the teachers' role should be to facilitate students to learn how to learn and to develop higher-order thinking skills in mathematics.

This paper deals with successes and difficulties six teachers go through in two schools as they move on to a student-centred approach by applying mathematical tasks and classroom discussion to develop students' cognitive skills. Evidence collected supports the use of this approach to fulfil three major learning goals in mathematics: promoting students' involvement and engagement in the lesson by allowing students to voice their own ideas; helping them develop better understanding by allowing them to think and verbalise their thinking; and finally, helping students develop communication skills such as confidence to voice their own opinions in public and the ability to do so in a clear and concise way.

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A STUDY ON THE EFFECTS OF MULTIPLE REPRESENTATION CURRICULUM ON FRACTION NUMBER LEARNING SCHEMES FOR FOURTH GRADE CHILDREN

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The purpose of this study is to investigate effects of Multiple Representation Curriculum (MRC, design based on literature and Ministry of Education, 1993) on fraction number learning for fourth grade children. The basis for analyzes on fraction learning rest on theoretical background by Olive and Steffe (2002), and the five fraction schemes by Tzur (1995, 2003). A quasi-experimental design with pretest, posttest, delayed test and interview was adopted. Two fourth grade classes were selected (control group; n=27, experimental group; n=31). Analyzes included quantitative (ANCOVA and MANOVA) and qualitative interviews (9 cases, 3 from each of high, medium, and low group) to capture fractional schemes. The findings were as follows:

Development of MRC. It was feasible to apply cognitive psychology in representation to develop MRC. Only performances of fractional composition and decomposition subscale were significantly higher than the control group on posttest and delayed tests.

Number Concepts. Children who received MRC were performing better in fraction words. Children from nine cases were more elaborative in concepts of fraction words and part-whole relation, when compared to prior experimental teaching. However, there was only limited development in equivalent fraction concepts.

Fraction Schemes. After MRC instruction, equi-partitioning and iterative fraction schemes of nine cases were facilitated. Children also understood principles on “fairness” and “exhaustion”. However, there was virtually no major development in recursive partitioning scheme.

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TEACHERS' KNOWLEDGE ABOUT DEFINITIONS: DIFFERENTIATING BETWEEN MOTIVATION AND PROOF

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Theorems are different from definitions. They convey mathematical truths that have been proved. Definitions are arbitrary, agreed upon conventions. They may be motivated but not proved. Are teachers aware of this distinction? Edwards and Ward (2004) reported on university students' confusion between these two concepts. In his article on the concept of exponentiation, Vinner (1977) claimed that there is often confusion and lack of distinction between definitions and theorems. Borasi (1992) described how extending the definition of exponents beyond positive integers may be used to explore the nature of mathematics. This study uses the context of exponentiation as a springboard to investigate junior high school teachers' knowledge of the nature of definitions, focusing on the differentiation between definitions and theorems, between motivation and proof.

Three teachers were interviewed individually. Each interview began with general and open questions related to the teaching of exponents which led to a discussion on definitions and theorems. Results showed that all three teachers knew that theorems must be proved; yet only one teacher was sure that definitions could not be proved. Asked for the meaning of a definition in mathematics, one teacher answered rather vaguely "it is some kind of rule" while another said "it is the result of a theorem."

It was within the context of geometry, that all three teachers felt most comfortable discussing definitions and theorems. Given that the geometry curriculum emphasizes the specific teaching and learning of definitions, theorems, proofs, and the interrelationships between them, it is not surprising that within this context teachers expressed knowledge of the arbitrariness of definitions and the use of definitions in proofs and theorems. However, when discussing exponential expressions, the teachers displayed only partial knowledge concerning definitions and theorems. Only one teacher spontaneously used the word definition with regard to exponents. When reverting to algebra, one teacher became confused between the acts of defining and proving. That teachers' knowledge of definitions was dependent on the context is an important finding, indicating that continued research on teachers' knowledge of the nature of definitions should be expanded to include various contexts.

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SUPPORTING TEACHERS ON MAINTAINING HIGH-LEVEL INSTRUCTIONAL TASKS IN CLASSROOM BY USING RESEARCH-BASED CASES

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Case method can now be used in teacher education in many countries (Dolk & den Hertog, 2001; Lin, 2002). These studies do not indicate that how cases increases teachers' awareness of different levels required in instructional tasks resulting in students' different thinking. The level of thinking in which students engage determines what they will learn. Stein and her associates (2000) differentiate four levels of cognitive demand of instructional tasks as memorization, procedures without connection, procedures with connection, and doing mathematics. This study was intended to examine how teachers maintained high-level cognitive demand when the tasks were carried out by using research-based cases. Eight teachers, enrolling in a course called "Theory and Practice of Case Method (TPCM)" in summer program at university, participated in this study. The cases presented in a video form were integrated into the TPCM course. After viewing a video case, each case was immediately discussed in small groups and then discussed in a whole class. Each teacher was encouraged to put what (s)he learned from the TPCM course into classroom practice in the following school year. They took turns observing each other during the school semester when the summer course was ended. This study conducted within-case and cross-case analyses to examine how the teachers learned about the cognitive demands from video research-based cases carried out in classrooms. Cross-case analyses were conducted to identify the similarities across cases and the differences among them, and overall patterns. It is concluded that the use of cases supporting teachers on increasing their awareness of the importance of differentiating levels of cognitive demand of tasks determining students thinking. The case discussion created the opportunity of raising the level of discussing among teachers toward a deeper analysis of the relationship. The effect of using cases on teacher's thinking about teaching was affected by the factors including selection of tasks, questions of thought-provoking questionings, and pressure for explanation required in the tasks evolved during a lesson.

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TOWARDS AN ANTI-ESSENTIALIST VIEW OF TECHNOLOGY IN MATHEMATICS EDUCATION: THE CASE OF WEBQUEST

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Both theoretical and empirical support to a view on a secondary mathematics teacher' use of WebQuest is presented in this talk. In particular, we argue that the use of a software package for teaching is not only linked to the school curriculum but also strongly linked to what a teacher sees in it. By treating software packages as texts and secondary mathematics teachers as readers of such texts from an anti-essentialist viewpoint of technology (Lins, 2002; Grint and Woolgar, 1997), this paper discusses the analysis of a case study - The WebQuest of Peter - of Costa's Master research work.

Peter, a secondary mathematics teacher from a state school in São Paulo (Brazil), was interviewed both in front of and away from a computer, talking about and describing *his* WebQuest. He also had two of his lessons within a WebQuest environment observed. Methodological issues on how the research had been designed will be given in the talk.

Costa's research studies aimed to look at *what is actually being said* by a secondary mathematics teacher about WebQuest, and to investigate to what extent this is linked to the teacher's use of WebQuest in the classroom, in his teaching. Here, to look at 'what is actually being said' means to look at what meanings are being produced by the teacher for WebQuest. One of our assumptions is that the software package which reaches the classroom environment is not *the* software that once had been designed but rather *a* software: the one that the teacher has constituted. The WebQuest presented in a classroom is *a* WebQuest: *the WebQuest of the teacher*.

One of the said features of WebQuest is *its methodology* that allows the teacher designs a lesson or some lessons in an environment which envolves evaluating the student's learning process of the mathematical topic raised. From the case study, *seeing* and *treating* WebQuest as such has shown not to be the case. The methodology proposed within WebQuest has nothing to do with *the WebQuest of Peter* at the time he was interviewed. This does not imply that it will never be. New meanings can be or will be produced by Peter for WebQuest, as meaning production is to be viewed and understood as a process rather than something static and fixed. The point is the importance of such *awareness* of *the WebQuest of the teacher* in order to understand *how* and *why* WebQuest is being taken and used in a classroom in a certain way.

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COMPARING TEACHING OF COMMON MATHEMATICAL TASKS IN DIFFERENT COUNTRIES

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This paper presents some of the results of the three year Socrates – Comenius project "Implementation of Innovative Approaches to the Teaching of Mathematics". The project brought together teachers and teacher-trainers from four countries.

In the first year, groups of collaborating teachers from primary and secondary schools working with university teachers, developed and trialled tasks concerning different mathematical topics 3D geometry (CZ); functional thinking (CZ); patterns leading to algebra (UK); early number sense (GR); regular polygons (DE)). In the second year, the tasks were exchanged between the four countries and trialled. This enabled comparisons to be made of the way the teachers in different countries functioned in the classroom (Tzekaki & Littler, 2005).

The role of the teachers is very important in the development of mathematical knowledge (Voigt, 1996, Steinbring, 1997, Sakonidis et al., 2001). The trials of common tasks in different countries gave a special opportunity to compare the communicative patterns the teachers adopted and the ways they handled the same mathematical ideas in diverse educational and cultural environments.

The paper illustrates and attempts to analyse comparatively the teaching approaches for some of the trialled units, presenting similarities and differences in the ways the teachers developed the tasks in the classrooms and the way in which the students worked with them.

The Project involves the following Institutions: Charles University, Prague, Czech Republic, Aristotle University, Thessaloniki, Greece, Kassel University, Germany, University of Derby, United Kingdom.

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NEW APPROACH OF NEUROCOGNITION IN MATHEMATICAL EDUCATION RESEARCH AND FURTHER IMPLICATIONS

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The breakthrough development in the neuroscience recently, the technique of Event-related-potential (ERP) is widely used in the research of brain science. ERP can record the potential change caused by an outside cognition stimulus to and withdrawn from brain and it is a non-intrusive brain image technique. One of the advantages of EPR is high time resolution and it is very useful to study the brain activities in the cognition process (Rockstroh et al, 1982). We design 4 instruments on the computer based on Noetling (1980) of ratio concept which were Graphic Easy to Hard(GEH), Graphic Hard to Easy(GHE), Symbolic Easy to Hard(SEH), and Symbolic Hard to Easy(SHE). There were 47 senior high school students(17 years old) participating in this study. Each test includes 64 trials; the interval of each trial setting as 10 seconds. In the ERP recording session, EEG was recorded from 40 channels Neuroscan EEG/ERP system (Nuamps) Figure1 is the average signal of P4Channel of one subject taking SHE. The latency period of P300 is 229ms. The cognition process starts at 229ms (figure 2) after the tester receives the visual signal and P700 appears at 774ms (figure 3). Figure 4 shows the average of Fz channel of one tester taking SHE. The green line shows the average of easy level, the blue line shows the average of middle difficult level, and the red line shows the most difficult level. We find the vibration of the green line is small and the vibration of the red line is large that means the tester pay less attention to process the easy ratio problem and pay more attention to process the difficult ratio program.

Figure 1

Figure 2

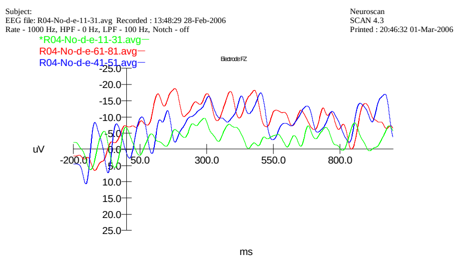

Figure 3

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REASONING AND GENERALIZTING ABOUT FUNCTIONAL RELATIONSHIP IN A GRADE 2 CLASSROOM

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This reports on Grade 2 children's exploration of mathematical generalizations through integration of multiple representations of functional relationship. Three aspects of generalization are discussed: integration of representations, translation and application across representations, and generalization of sets of generalizations.

Children's ability to make mathematical generalizations may rely on the integration of more than one representation of a mathematical idea (e.g. Moss & Case, 1999). This teaching study explored children's reasoning and generalizing about functional relationship in linking geometric growing patterns with ordinal position numbers.

These children had not been taught multiplication conventionally, but "invented" it as needed in order to solve simple teacher-introduced and more challenging student-generated mystery pattern rules. Even children who routinely struggled with arithmetic demonstrated a conceptual grasp of what it meant to multiply, by correctly describing a geometric array for a given pattern position.

In working with composite functions, the children spontaneously introduced the "zero-th" pattern position that they explained was a "big clue" to finding the pattern rule because it isolated the constant. They further discovered the power of zero as a coefficient in generating non-sequential pairs of numbers for the function machine (e.g. Willoughby, 1997), where any input number produced the same output number.

All the children were able to generalize function rules from quantifiable instances (e.g. Kieran). Most were able to transfer this understanding to unfamiliar contexts. Further, some were able to generalize types of pattern rules from sets of rules or generalizations, Piaget's "reflecting abstraction" (Piaget, 2001[1977]).

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THE MATHEMATICAL PERFORMANCES IN SOLVING THE NORMING PROBLEM

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The thought of unitizing and norming can help teachers to analyse students' problem solving. If one can unitize a quantity and view that unit as a reference unit, and reinterpret another quantity (a compared quantity), then, he/she has the ability of norming (Lamon, 1994). This paper, a partial result of Ma's study (1999), investigated students' performances regarding solving the norming problem. The participants in this study were 12 sixth graders from Taichung County, Taiwan. The test, Mathematical Problem-Solving Performance Measurement, was given after the problem solving session from textbook was finished. In addition, students' meta-cognitive thinking was investigated. The original test items were developed by Pan (1993), based on Polya's problem-solving model. The norming problems were in Problems IV and V. This paper would take Problem IV as an example.

Problem IV: John takes the test from 8:20 a.m., and the test duration is 60 minutes. When John finishes the test, he finds out that the time left is 1/5 of the time spent. What time is it when he finishes the exam paper?

The results indicated only three students had the abilities of norming and solved the problem. Two of these three students, used the time spent (TS) as a unit, and the other used the time left (TL) as a unit. They were able to measure the other quality based on the unit they chose. Students were able to reinterpreted TL=(1/5)TS (using TS as a reference unit), or TS=5TL (using TL as a reference unit). Only one student solved the problem with relational understanding, using ratio relation, 1:6=□:60. The results also indicated students might not really understand the relation even if they built. For example, one student wrote "60÷(1/5 + 1)", but he/she could not explain why. Another correctly spoke "some×1/5 + some = 60", but he/she did not know how to write down the equation and calculate "some". He/she solve it using trial and error. Thus, the researcher found that students could use the thought of norming to help figure out what to do when they started on a problem. They had their own viewpoints to select their reference unit. They applied the grouping concept to reinterpret a compared quantity such as "5" TL by TS/TL=5. However, some students had difficulty on understanding the relation or operation related to that procedural.

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THE EDUCATION OF REASONING: INVESTIGATING ARGUMENTATION AND PROOF

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The purpose of my research is to identify differences in children's argumentation that can be attributed to being exposed to a series of lessons on 'mathematical proof'. Students were asked to discuss mathematical problems in groups of four before and after having been taught three lessons of mathematical proof (whatever the teacher perceived that to be). The group sessions were tape recorded and transcribed. This short oral communication is about the methods used to analyse these transcriptions. Having all but disappeared from British mathematics classrooms since the 1960s, mathematical proof is staging a comeback, with its role and value being extensively debated by experts in the field (see for example PME 26, 177-184, 225-232, 230-235, 281-288, 408-415). Amongst many other reasons for reemphasising mathematical proof, one argument is that by proving something mathematically, general reasoning skills will improve (Hanna, 2000). Comparing argumentations (in this context the processes of negotiating mathematical problems and forming an argument) either side of being taught mathematical proof, provides insight into the effect exposure to mathematical proof can have on students' reasoning skills.

The analysis techniques adopted in this research are fourfold. Firstly the data were analysed using a model based on Toulmin (1958) to identify claims, data, justifications, qualifications, and rebuttals. This allowed the *structure* of the argumentations to be compared independently of the particular question. Secondly, the utterances classified as justifications were then compared across questions, and classified into several categories to allow the *types* of justifications students used to be compared, independently of the question. In contrast, the third stage of analysis related to the individual questions with each argumentation being summarised into *arguments* which were then compared. The forth method of analysis used the work of Lakatos (1976) to analyse the utterances classed as rebuttals, comparing when they were made, what they were questioning (claims or reasoning), and whether the frequency and type of rebuttals changed. In one school there were noticeable changes in students' argumentations after the proof lessons, in another, minor ones. One school's students produced significantly different argumentations from the other students in all group sessions.

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THE EFFECT OF REPHRASING WORD PROBLEMS ON THE ACHIEVEMENTS OF ARAB STUDENTS IN MATHEMATICS

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In this paper we present an approach to help the student solve word problems properly, by rephrasing the word problem.

Language plays a significant role in math. education. Language is the learning device and the device which forms the student's knowledge in math, his ability to define concepts, express mathematical ideas and solve mathematical problems; difficulties in the language are seen more in word problems. (Austin, 1979; Radford, 2003)

The connection between language and mathematical achievements has a more distinctive significance regarding the Arab student. This is due to the fact that the language which is used in the schools and in textbooks is Arabic. It is far different than the language used in everyday conversations with family and friends (the spoken Arabic); this delays his comprehension of word problems. (Raiker, 2002)

Our research examines whether or not rephrasing word problems can effect the achievements of the Arab student in Mathematics. In order to do so we selected 538 4th and 5th grade Arab students. Eight word problems were chosen according to their potential, in regard to the language, for disrupting and delaying the students' achievements. The problems were divided among the groups; each group was presented with four problems. The problems where written in a simpler language, preserving the content and mathematical difficulty.

Two different verses of exams were created: verse1 contains the four original problems and four rephrased problems. Verse2 contains the four original problems that appear in verse1, only now they have been rephrased, and the four rephrased problems that appear in verse1. The students received both verses of the exam

The results show that there was a significant improvement in the students' achievements in the second verse of the exam, which contained the rephrased problems. It may be concluded that rephrasing word problems has a significant effect on the Arab student's achievements in mathematics.

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AN APPROACH TO EARLY ALGEBRA USING TECHNOLOGY AS AN ENHANCEMENT

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In recent years, the importance of early algebra learning has been underscored by both the National Council of Teachers of Mathematics (2000) and the RAND Mathematics Study Panel (2003). Although Robert B. Davis (1964) wrote materials to introduce children to early algebra ideas over 40 years ago, only recently has the efficacy of his approach been studied in detail. We report here on a three-month intervention with 13-year olds, using Davis' materials and the CasioClass Pad tool.

In our research, we investigate how students build and use multiple representations of function ideas. Our study takes place in the context of an informal, after-school, mathematics program[1] in an urban economically disadvantaged community. The students met for six, one and one-half hour sessions over a period of three months. All sessions were videotaped. The Casio ClassPad technology tool supplemented tasks from Davis' Early Algebra. The students used an emulator software, ClassPad Manager that had available displays of multiple representations of linear functions. The tasks and tools made available to the students different representations of mathematical objects. In our report, we present a case study of Chris, who moved freely among multiple representations (graphic, tabular, symbolic), using certain features of the CasioClass Pad to verify his work. Chris used certain key ideas of his representations to recognize equivalent problem structures.

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[1] This research is an outgrowth of the National Science Foundation funded project, Research on Informal Mathematical Learning (IML), directed by Carolyn A. Maher, Arthur B. Powell, and Keith Weber (REC-0309062) with support from the MetroMath Center, Rutgers University.

TEACHERS' BELIEFS AND COMPETENCIES OF CREATIVE MATHEMATICAL ACTIVITIES

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This paper presents the result of the preliminary researches carried among the mathematics teachers. These research deals with the skills of taking creative mathematical activities by the teachers. It also deals with the awareness of the need of developing different kinds of these activities among students. Through this creative mathematical activity I understand (after M. Klakla, 2002) its following kinds: putting and verification of hypotheses; transfer of the method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issue); creative receiving, processing and using of the mathematical information; discipline and criticism of thinking; problems generation in the process of the method transfer; problems prolonging; placing the problems in open situations. The aim of this research has been to construct the exploratory tool in the form of the diagnostic questionnaire. This questionnaire can diagnose awareness of the mathematics teachers related to the creative mathematical activities. In order to reach my goal I studied:

- what mathematics teachers understand by the creative mathematical activity of the student,
- whether they are able to give the examples of such activities and tasks, which develop this activities particularly,
- whether they recognize creative mathematical activities, whether they use some didactic endeavors, which have the purpose of developing those activities,
- whether they know how to provoke those mathematical activities, and therefore whether the students have the possibility of developing those activities during mathematics learning process.

The pilot research lets put forward a hypothesis, that *among the mathematics teachers the awareness of that what is the creative mathematical activity, and also the awareness of the necessity of formation of this activity, is insufficient.*

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MANIPULATIVE REPRESENTATION: WHAT DO YOUNG CHILDREN KNOW ABOUT IT?

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The initial teaching process of numbers and operations is commonly accompanied with manipulative representations. Manipulative representations (cubes in this study), like other representations, have both “unique components” (e.g., quantity, colour) and “unique usage models” (e.g., dynamic ways of presenting number sentences) (Fischbein, 1994; Lesh & Doerr, 2000). The present study purports to examine how kindergarten children and first graders produce and identify numbers and addition and subtraction number sentences with cubes.

One hundred and fifty four children (48 kindergarten children and 106 first graders) participated in the study. In the production phase, participants were asked to represent, with cubes, in their own way, four numbers (5, 8, 13, 20), two addition number sentences (5+1=6, 12+3=15) and two subtraction number sentences (5-2=3, 13-2=11). In the identification phase, participants were presented with adequate and inadequate representations of the numbers and the number sentences and asked if each of the representations is an adequate representation of a certain number or number sentence.

Unexpectedly, more kindergarten children than first graders succeeded in producing numbers with cubes. This unexpected finding could result from the differences in the components that the kindergarten children and the first graders depicted to represent numbers. The kindergarten children exploited only “quantity” while the first graders used both “quantity" and “colour" (the latter choice led to inadequate representations). At the identification phase, most children, both kindergarten children and first graders, regarded quantity as an essential characteristic of number representation. Colour was regarded as an essential characteristic by about half of the first graders.

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LANGUAGE, POWER AND MATHEMATICS LEARNING

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What language is appropriate for learning mathematics in a multilingual context? Why is that language appropriate? Who decides? In this presentation we will explore these questions by using data collected through learner interviews with Grade 10 learners in a township school in Limpopo, South Africa. The presentation draws from a wider study investigating the relationship between language and mathematics in multilingual classrooms in South Africa. We use the theoretical construct of cultural models (Gee, 1999) to explore why learners prefer particular languages for learning and teaching. Debates on language(s) of learning and teaching mathematics in black schools in South Africa have dominated discussions in public domain and research since independence (e.g. Adler, 2001). This issue is, however, is not specific South Africa. Multilingualism is an increasingly common feature of urban mathematics classrooms around the world and the underachievement of blacks, minority or immigrant learners is a concern in many countries. Furthermore, mathematical communication is now seen as a central aspect of learning school mathematics. Learners are now expected to use language to communicate mathematical ideas and concepts both orally and in writing (Moschkovich, 2002). This emphasis on mathematical communication raises pedagogic and political questions about which and whose language(s) to use for learning and teaching mathematics. The study reported shows that language choices of learners are largely based on gaining access to English as the language of power rather than gaining access to mathematical ideas and knowledge. Learners displayed this cultural model despite the fact that when they were given a mathematical problem to solve they drew more on their home language (Sepedi) more than they did on English, a language they are still learning. This study highlights the need to address language and learning issues in mathematics education not only at the level of pedagogy but also at the socio-political level.

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CHILDREN LEARNING AS PARTICIPATION IN WEB-BASED COMMUNITIES OF PRACTICE

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Assuming learning as participation in communities of practice and taking a situated perspective on learning (Lave & Wenger, 1991; Wenger, 1998) we looked at children practice within the activities of project WebLabs. Data collection was based on video recording of groups of children in selected sessions and the material published on the webreports by children. Data analysis enabled us to describe children's practice in the project and find evidence of learning in the following categories:

(i) The emergence of a shared repertoire, where we include:

- vocabulary instantiated in ways of approaching problems, questions, demands, challenges (using technology, programming and modelling), ways of representing and sharing ideas (describing their work, ideas and thinking, commenting other people work and ideas, building from others' ideas to go further in their own, e.g. constructing an Encyclopaedia on Randomness)
- an emerging valuing of crossing boundaries (both cultural as well as in specific knowledge domains, e.g. on Number Sequences);

(ii) The co-definition of mutual engagement, which is visible through:

- an emerging acceptance of partiality of knowledge as a positive contribute to the knowledge of the community as a whole group and not as sign of 'not knowing' things (e.g. the exchange of modes of proving that a certain ToonTalk robot produces a certain sequence of numbers);
- an emerging sense of responsibility for the overall achievement, i.e., the joint enterprise where children feel that they have a voice (e.g. their contribution to the improving of the software making children experiencing a strong sense of belonging to a team in a project);
- an emerging sense of ability and pleasure in going deeper in their ideas and products (a kind of localized depth) by way of a set of conditions, namely: interaction with powerful computational tools, interaction with teachers and researchers who help sustaining collaboration (acting as peers in the exchanges within their specific tasks) and possibilities for innovative representations.

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RECOGNIZING MATHEMATICAL COMPETENCES

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Simone – a 45 years old lady quite confident in life despite the fact that she left the school when she was 14 – works at a well known supermarket where she is responsible for a variety of daily tasks. When we talk to her we understand that she is able to deal with quite sophisticated processes. All her colleagues and the supervisor of the supermarket agree that Simone is a quite competent person. She would like to study more but she is in fact facing the need to hold a diploma of the 12th grade in order to get a promotion.

For a number of years, both cultural studies and studies in the tradition of etnomathematics provide accounts of the particular ways people organize, adapt and build up mathematical structures and forms of thought in order to make sense of everyday activities. Now, the educational systems face the challenge of recognizing, validating, and certifying mathematical (and more general) competences in people such as Simone. This is an enormous challenge to education – and in particular to mathematics education – as, for example, millions of people in Europe who didn't follow the regular compulsory schooling (but who want to acquire the certification of the basic or secondary studies, valuing their personal and professional experience) are potential candidates to see their competences valued and recognized by the educational system. In the case of the Portuguese population, the estimated number of potential people to apply to a system of recognition of competences is around four hundred thousand in the next five years.

Key questions to the mathematics education community emerge from this picture: what do we mean today by *being mathematically competent*? what are the dimensions of the field of competences that can be considered relevant? relevant to whom? and relevant to what? How can we specify criteria of evidence that help to identify a mathematical competence? how does a person recognize his or her competence? how does one learn how to recognize mathematical competence in someone who didn't go to school? what is the role of the person who wants to see their competences recognized? what is involved in the process of recognition of mathematical competences? who owns the right to recognize competence in the other?

Adopting a situated learning framework that considers learning as participation in communities of practice (Lave & Wenger, 1991), we address these questions as a way to start discussing the process of recognition of mathematical competences.

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MATHEMATICS TEACHERS' PREPARATION PROGRAM: DETERMINING THE BALANCE BETWEEN MATHEMATICS AND PEDAGOGY

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Teacher educators in the university need to know what types of knowledge and what levels of knowledge acquisition are necessary to become effective mathematics teachers. One theoretical model of teacher knowledge suggests seven domains of teachers' professional knowledge: knowledge of subject matter, pedagogical content knowledge, knowledge of other content, knowledge of the curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical knowledge (Shulman & Grossman, 1988; Wilson et al., 1987). In this study, mathematics teachers' preparation programs in five Malaysian universities and one in Singapore were examined. Data was collected through interviews with two key personnel from each university and questionnaires were administered to 268 final year mathematics education students. The interview focused on the curriculum structure of the training programs, entrance requirement and what an ideal mathematics teachers' preparation program is. It was found that two major beliefs in mathematics teachers' preparation programs are dominant, preparation of mathematicians cum teacher and preparation of mathematics teachers. The questionnaire seek to gain information on five dimensions; students' confidence in teaching, pedagogical content knowledge, level of anxiety, views of mathematics and importance of implementing certain aspects in teaching. Significant differences were established between universities in all five dimensions, however the universities whose programs were built on the foundation of training 'mathematicians cum teacher' or 'mathematics teachers' do not show a definite pattern in their students' perception in the five dimensions measured. Based on the analyses of the interviews, questionnaires and examination of program handbooks, a recommendation is made on the ideal balance between components of mathematics, pedagogy and other related knowledge for a mathematics teacher preparation program.

MATHEMATICS REGISTER ACQUISTION

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This paper examines how a teacher in a high school mathematics classroom facilitates her students into acquiring and using the specialist language of this subject. Using ideas from second language acquisition, the teachers' strategies for language development are grouped into four stages.

ACQUIRING THE MATHEMATICS REGISTER IN CLASSROOMS

Much has been written about the use of language in mathematics classrooms as a cultural artefact or tool. Mathematicians make use of it as an effective means of communication both in presenting their ideas but also in the development of new ones (Meaney, 2005). There is a recognition that in mathematics classrooms, students need to be gain control of the mathematics register as part of their apprenticeship into the community of mathematicians. The mathematics register of a natural language such as English "includes both the terminology and grammatical constructions which occur repeatedly when discussing mathematics" (Meaney, 2002, p 178). Subject registers develop because they are a valuable way to fulfil the function or need for communication of the specialists who work within this subject (Halliday, 1988).

During the course of 2002, a Year 10 class was regularly audio recorded and field notes kept of whiteboard notes. Students' work was also photocopied and some were interviewed thrice during the year. An analysis was then made of this material to see how the teacher was facilitating students' acquisition and use of mathematical language. A resemblance was noted to models of second language acquisition such as those of Gass (1997). Consequently, a model of mathematics register acquisition is proposed. This paper describes the four stages in the model using examples from the classroom interactions. The four stages are: Noticing; Intake; Integration; and Output. It is argued that students need opportunities at all four stages of the model to acquire new aspects of the mathematics register.

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DEVELOPMENT OF SPATIAL ABILITIES

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How does the teaching unit „We Build a Village” further the development of spatial abilities? We report on research results from an investigation in eight primary schools with 376 pupils.

DESIGN

"Begreifen" is a German word with a double meaning. On the one hand "begreifen" means to touch or to feel physically (with your hands), on the other hand it means "understanding" (in your head). Thus "begreifen" is one of the keys to develop spatial thinking, it is a synonym for Piaget's "thinking is internalized action". With regard to this view we developed the teaching unit "We Build a Village" to introduce geometry concepts of plane and three dimensional geometry in primary schools (8 lessons, age 8-10). The children work with about 35 different geometric solids. Many activities form the lessons: Sorting and classifying, folding, drawing, cutting, constructing developments, using plasticine, building solids and houses by their developments.

In the research project we wanted to measure via pre- and post-test how the teaching unit furthers the development of spatial abilities. To cover the three factors from Thurstone (*Spatial Relations, Visualisation, Spatial Orientation*) we choose and/or constructed 9 tasks with altogether 31 items. Control classes got the same tests with a regular teaching.

RESULTS

task	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**all**
pre-test	,527	,739	,548	,652	,476	,601	,503	,261	,307	,513
post-test	,665	,799	,652	,785	,584	,732	,590	,453	,313	,619
signif.	,000	,051	,000	,000	,001	,000	,000	,000	,868	,000

Table 1: Test results experimental group (N = 184)

Comparing the pre-test results the control group showed a higher performance than the experimental group (means 0,548 vs. 0,513, p=0,025). But in the post-test there were no more significant differences. A t-test also indicates that there are gender differences, although small for most of the test items. For more details see "Reference".

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AN ENCOUNTER BETWEEN QUEER THEORY AND MATHEMATICS EDUCATION

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This presentation arises from a dissatisfaction with the ways that some feminist work on the male dominance of mathematics sets up oppositions between how men and boys vs. women and girls relate to the subject and between their preferred pedagogies. I suggest that such approaches reinforce the ways that mathematics itself is constructed through a series of gendered oppositions: objective vs. subjective, rational vs. emotional etc. and the ways these oppositions set limits on people's identifications with the subject. Instead, I stage an encounter between mathematics education and queer theory. Queer theory is based on a collection of ideas about language, practice, identity and the relationships between them that are very different from commonsense ones (Wilchins, 2004). I will collect it into three strategies and apply these to the idea of mathematical ability. Strategy 1: *troubling binaries* aims to show how the two sides of a binary are not separate, but actually each term in an opposition requires its other, the one 'only becomes intelligible through the difference to its other…This definitional interdependence' (Luhmann, 1998: 144) threatens the distinctiveness of the more powerful term, and so forms of normalization are needed to protect it from the threat of collapsing into its Other. Making visible these forms of normalization is part of this strategy. Strategy 2: *stopping all the binaries lining up* aims to undermine binaries by disrupting 'the desire for the neat arrangement of dichotomous sexual and gendered difference' (Luhmann, 1998: 145), disturbing the way that positionings in binaries line up predictably. As Sedgwick (1994: 6, original emphasis) asks: 'What if instead there were a practice of valuing the ways in which meanings and institutions can be at loose ends with each other? What if the richest junctures weren't the ones where *everything means the same thing?*' Strategy 3: *telling stories* has as its rationale that binaries are kept in place by the fiction of the rational subject. This strategy refuses this identity, shifting from assuming and affirming identities to looking at how a reader becomes part of the text, looking at what identifications are possible and using story-writing as a way of intervening in these processes, re-inscribing people, objects and words, opening up possibilities for new identifications and meanings.

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ESTABLISHING A MATHEMATICS LEARNING COMMUNITY IN THE STUDY OF MATHEMATICS FOR TEACHING

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The presentation offers for a discussion a conceptual framework for research that uses complexity science perspective in the research endeavouring to focus on the quality of mathematics teacher education programs and to bring together all mathematics practitioners in the study of the question of what mathematics teachers need to know for teaching. This framework guides our research that focuses on establishing a learning community among mathematicians, mathematics educators and mathematics teachers and practitioners in the study and development of mathematics for teaching. The research builds on an innovative core mathematics program, Mathematics Integrated Computer Application (MICA) launched by the Department of mathematics at our Institution. The program integrates computers, applications and modelling where students make extensive use of technology in ways that support their growth in mathematics. In the core half-credit project-oriented MICA I course first year students learn to investigate mathematical concepts by use of computer programs they design themselves (VisualBasic.NET). As final project concurrent education students design an innovative, interactive, highly engaging and user-friendly computer environment to teach one or two mathematical concepts of K-12 level, often called a learning object. For example, a 9-task adventure with Herculus about perimeter and area (grade 4) or a journey through MathVille for learning the exponent laws (grade 9). In our research project, these learning objects are “put into work” in Preservice education (Faculty of education) where the issues of pedagogy and didactics are discussed and in schools where the issues of practice surrounding the learning objects are discussed.

Traditionally in mathematics education research, knowledge has been viewed as something that can be possessed or as a commodity to be held. Applied to mathematics teacher education such view assumes that teachers’ knowledge of formal mathematics can be abstracted from their formal mathematics education and applied to their teaching. Drawing from complexity science perspective we challenge this view arguing that knowledge is recursive, dynamic phenomenon that emerges form the interactions of individual and environment. Using some preliminary results from our research we illustrate how we might conceptualize mathematics teacher education as a learning system that emerges from the interaction of university instructors (mathematicians and mathematics educators), student teachers and practicing teachers.

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TALKING MATHEMATICS IN A SECOND LANGUAGE

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Conversation has been for decades viewed as a central means of teaching and learning mathematics. During my teaching years I have been concerned with lack of discussion in my mathematics lessons. It is the norm that students say very little and rarely pose questions. In response to this, I carried out a study for my Masters degree (2004) to investigate how discourse emerges and evolves, as students interact with mathematics, with other students, and with the teacher. Language appeared to have played a central role in the conversations that emerged in this Namibian class of 35 grade 11 students. English, which is a second language to both students and the teacher, is the medium of all instruction. Switching to one's native language is not allowed in Namibian secondary mathematics classrooms. Similarly all discussion amongst students is expected to be in English and the teacher is to discourage the use of other languages. Even though both the teacher and students share a common native language they have to abandon it and speak only English.

One of the problems students worked on in small groups was to "find three consecutive numbers whose sum is 78". The term 'consecutive' presented another problem to all students as they tried to make sense of this exercise. One group constructed an equation $x+x+x = 78$ whereas the second group used the equation $x+y+z=78$. Both groups found their strategies problematic. The former could not work out the other two variables while the latter was faced with three unknown variables. In a third group, students managed to consult an English dictionary and find out the definition of the term "consecutive". However, these students were still unable to translate their finding into an algebraic form. What interests me here is the question of whether language or the inability to translate the notion of "consecutive numbers" into algebra was the constraint in students' meaning making of the problem. Had these students been allowed to switch languages or had the situation been explained to them in their mother tongue, would they have managed to make clear sense of and solve the assigned word problem?

In conclusion I argue that in order for students to explicate their mathematical thought they need competency and fluency in both mathematics and the language used. But given the political measures of rigorously employing a foreign language as the sole medium of instruction, how is a teacher to make learning mathematics meaningful and manageable to her non-English speaking students? This situation calls for serious consideration in all countries with similar circumstances.

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OBJECTS IN MOTION: MAPS OR GRAPHS?

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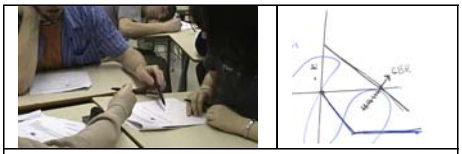
In this paper, we investigate the way in which Grade 10 students interpret and produce graphs related to the motion of objects in a multi-layered artifact environment. The students' mathematical activity around graphs is seen as an instance of the students' social acquisition of historically constituted cultural forms of mathematical-scientific reflections. Drawing on a semiotic approach, our analysis focuses on the manner in which symbols, artifacts, speech and kinaesthetic actions mediate the students' mathematical activity in their process of explaining and predicting motion phenomena.

Historically speaking, reflections about motion in general rest on abstract cultural concepts of space and time. Contemporary forms of mathematical-scientific reflections about motion are carried out through complex artifacts such as Cartesian graphs and algebraic formulas where qualitative and quantitative ideas of co-variations are emphasized. Drawing on a semiotic-cultural theoretical framework (Radford, 2003), we discuss the way in which Grade 10 students interacted with various artifacts in order to explain, predict and make sense of the motion of objects. The classroom activity was based on a problem where a car was moved between two points-A and B-through a remote control and data position (time, space) was collected using two Calculator Based Rangers –CBRs–each one connected to a TI-83+ calculator. In the first part of the activity, the students were required to produce the graph of the relationship space-time that each CBR would generate. Then, with the help of the teacher, two students conducted the experiment in front the class. The students were asked to explain differences and similarities between their graphs and the calculators'. In the next problem they had to imagine the case of an *ideal* CBR placed about the middle of AB. The multi-semiotic data analysis (which includes video, speech, gestures, symbol- and artifact-use analyses) shows how a subtle coordination between gestures, speech and symbol accounts for a first objectification of the abstract meanings conveyed by a Cartesian graph. The data analysis also suggests that the attainment of cultural forms of mathematical-scientific reflection requires a shift from graphs as *maps* to graphs as expressions of *co-variational relationships*.



One group of students discussing the problem of the ideal CBR

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RESEARCHING THE APPEARANCE OF MATHEMATICAL ARGUMENTATION

Christina Misailidou

This communication aims to propose a methodology for researching the pupils' reasoning development in a small group discussion environment. Such a methodology allows the identification of the appearance of mathematical discourse and of the factors that influence such an appearance. This methodology was tested with two different items representing two different contexts of proportion related tasks: a 'paint' context and a 'sharing' context.

A discourse analysis approach was adopted as the pupils discourse throughout the discussions was taken as indication of their reasoning. Thus, the 'development of reasoning' was replaced by the manageable 'development of discourse'. It became possible to track down the development of the pupils discourse by employing the idea of a 'discursive path' (Misailidou and Williams, 2004). A 'discursive path' is defined as the evolution of the pupil's argumentation in the discussion. Each discursive path was comprised by one or more stages. Toulmin's (1958) method for the analysis of arguments was used to represent each stage of the discursive path. Consequently, the pupils' development of discourse was represented in discrete parts with a standardised format that could be compared or combined. This discreteness of the parts allowed the identification of critical factors of the discussion that were hypothesized to have influenced the appearance of a new stage. These factors were significant components of each discursive path.

Characteristic discursive stages will be presented from several discursive paths in order to demonstrate the detection of the change in the quality of the discourse and the identification of the emergence of mathematical argumentation. The discourse analysis approach presented in this communication is proposed as a practical method for assessing the development of individual or collective discourse particularly when this is mediated by cultural models.

Acknowledgement This study was funded by the Economic and Social Research Council (ESRC, No: R42200034284)

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TEACHERS' PEDAGOGICAL CONTENT KNOWLEGE IN THE TEACHING OF QUADRILATERALS

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Inquiry into teachers' pedagogical content knowledge (PCK) has been particularly active since about 1985. Further to Shulman's notion of pedagogical content knowledge, different perspectives of pedagogical content knowledge from different educationists have been reformulated the concept in various way, for example, the work of Marks (1990) and others. Based on the review of the these studies, we build the analytical framework of PCK of this current study upon four components, namely, subject matter knowledge, knowledge of students' understanding, curriculum knowledge, and knowledge of instructional strategies.

This study reported in this paper took place in Hong Kong. The aim was to explore primary school mathematics teachers' PCK in the teaching of quadrilaterals.

The subjects included 8 primary mathematics teachers from a Hong Kong school. The school has a good reputation and good school performance in its local district. In this study, a semi-structured interview was chosen to be the data collection tool. A key feature in the instrument is using scenarios as a tool to explore: (1) The teachers' knowledge about quadrilaterals and the teaching and learning of quadrilaterals in classroom; (2) the teachers' actions in to the situations presented; and (3) the teachers' knowledge supporting their decision and action.

In the investigation of this small sample, we find that the teachers were confident in what they had learnt during their schooling. However, they did not realize that their subject matter knowledge was not sufficient for their teaching. Some of the teachers showed a poor understanding of quadrilaterals, an inadequate understanding of their students and inadequate update of curriculum knowledge.

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OUT-OF-SCHOOL EXPERTS IN MATHEMATICS CLASSES

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I comment on the use of out-of-school experts in secondary mathematics classes. This focus emerged in a project which sought to link school mathematics to out-of-school mathematical activities. Eight teachers were involved with the project and they were free to pursue any activity they wished. All but one consulted an ‘outside expert’ in a field. The literature on outside experts in mathematics classes is virtually non-existent. I comment briefly on three teacher researchers and outside experts. Teacher 1 (T1) made contact with the person in charge of packaging (OE1) for a tea company. T1’s class were given a task to produce *shelf-ready* tea carton packaging, an outer box containing six cartons to enable supermarket staff to remove part of the outer packaging by using a perforated tear strip. Students were given the full design brief. Students, in groups, made many early designs before taking one design further. T2 made contact with a health service data analyst (OE2) who monitors, analyses and represents data from medical clinics in a geographical area. The class, in groups, worked on a series of data sheets and statistical hypotheses based on data from OE2. T3 approached a small business consultant (OE3) and worked on the finances of setting up a plumbing business. Using spreadsheets they used ‘sensitivity analysis’ to prepare ‘optimistic’ and ‘pessimistic’ approaches to setting up a business. I comment on: real scenarios, student motivation and learning and assessment.

Real scenarios. All three outside experts provided real scenarios. OE1’s packaging problem could not have been more authentic – the students worked on the exact same problem he had recently been given by a supermarket chain. OE2’s data was real data even though details were made anonymous. T2 and I knew that we could ‘make up’ health related data but we wanted real data. OE3 presented a method for analysing data rather than data itself. The method had to be simplified but the essence of the real method was not, in T3’s and my opinion, adversely affected.

Student motivation. Although the classroom presence of these OEs was not necessary for task design, their presence clearly motivated the students. One reason for this, I posit, it that there was no student suspicion that the teacher was giving them a pseudo-problem, here was something real to engage with.

Learning and assessment. All three OEs are highly numerate but they are not mathematics teachers. They did not assist the students in learning new mathematical skills or concepts but helped the students to make mathematical sense of the situations they presented. Their contribution to assessment was also not concerned with skills and concepts but in judging the relevance of student solutions to workplace problems. This, however, raises other difficulties as there was strong evidence that T2 judged student presentations with regard to mathematical content whilst OE2 judged them with regard to relevance and these two judgements clashed.

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A SEQUEL TO TRENDS IN INTERNATIONAL MATHEMATICS AND SCIENCE STUDY (TIMSS), 2003, IN BOTSWANA

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This paper highlights actions undertaken by the Ministry of Education (Botswana) to counteract concerns raised by TIMSS 2003. TIMSS's exists to improve the teaching and learning of mathematics and science by availing information on students' achievement as per different types of curricula, instructional practices and school environments around the world. In a quest for quality education envisaged in its policy documents (Vision 2016, Revised National Policy on Education, 1994) Botswana participated in the study for the first time in 2003, using form one (year 8) students in junior secondary schools. A total number of 46 countries participate in the study. The performance of Botswana students was a mean score of 366.3 which is lower than the international benchmark (400). Singapore attained the highest mean score of (605). The scale used has a mean of 500 and a standard deviation of 100. The worst performance by Botswana students was in geometry as opposed to number, measurement and algebra. Variation in performance could probably be due to emphasis on the different content areas in the curriculum, preparedness of the teachers to teach the content areas, the extent to which the content was taught to the students and textbooks suitability. Educationists across the spectrum and stakeholders met for a 3 day workshop (14-16 February 2006) to disseminate TIMSS 2003 report, scrutinizing its frameworks against Junior Certificate Mathematics Syllabus, consider in-serving teachers on specific topics, and strategies that may work best for individual topics and popularize mathematics through various means. The report was sent to teachers in all junior secondary schools to customize it to their environment. The paper's thrust is on strategies the teachers could use to improve students' performance in mathematics. The discussion will encourage participants to contribute cutting edge solutions on issues raised in the paper.

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THE PATTERN AND STRUCTURE MATHEMATICS AWARENESS PROJECT (PASMAP)

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A school-based numeracy initiative was conducted in one NSW metropolitan elementary school, which trialled an innovative approach to teaching low achievers in mathematics. The project involved 683 low-achieving students aged from 5 to 12 years, and 27 teachers, over a 9-month period using a Pattern and Structure Assessment (PASA) instrument and a Pattern and Structure Mathematics Awareness Program (PASMAP). PASMAP was developed through research on the early development of pattern and structure in mathematics (Mulligan et al, 2004; Mulligan et al, 2005; Thomas et al., 2002). Our hypothesis proposed that, if low achievers have a poor awareness of pattern and structure, then their mathematical achievement could be improved if they were explicitly taught to recognise and use pattern and structure across a range of mathematical content domains.

The first researcher worked with the teachers to develop and implement structured learning experiences in Years K-6 that focused on key mathematical structures and patterns. The teachers first administered the Pattern and Structure Assessment (PASA) interview to their students, and the results were used to allocate students to small groups for instruction. Teachers attended regular workshops, and additional numeracy support staff and resources were provided to enable them to implement PASA and PASMAP. Many activities developed students' visual memory as they observed, recalled and represented numerical and spatial structures in processes such as counting, partitioning, subitising, grouping and unitising. Students were reassessed after 9 months after teaching the PASMAP. There was a marked improvement in PASA scores and conceptual understanding, particularly in the early grades. Substantial improvements were also found in school-based and system-wide measures of mathematical achievement. Further analysis and an independent evaluation indicated that PASMAP had addressed a range of difficulties of which teachers were unaware.

It was not possible to conduct this project as a controlled study. Nevertheless, the results suggested that explicit teaching of mathematical pattern and structure, delivered in a manner suitable for low-achieving students, had the potential to radically improve students' mathematics and teacher pedagogical knowledge within a relatively short time frame.

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PRIMARY PUPILS' MATHEMATICS ACHIEVEMENT: DIFFERENCES INDICATORS

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This paper discusses a study in which sex, school and age differences in mathematics achievement of primary-five pupils were investigated in Eastern Uganda. The study sought to investigate indicators of differences in mathematics achievement of primary school going pupils. The sample consisted of 120 primary five (grade 4 or 5th-year of school) pupils (56 boys and 64 girls) in four primary schools in one district of Eastern Uganda. Data were collected using a standard Mathematics Achievement Test administered to the pupils and marked by a trained examiner. The Mathematics Achievement Test was prepared in a standard Uganda National Examinations Board (UNEB) format comprising of two sections: Section A containing 30 short structured questions and; and section B comprising of 12 long structured questions carrying 70 marks. The test-paper covered topics included in the intended primary school mathematics syllabus for the primary-five level: algebra, geometry, operation on numbers, set concepts, numeration system and place-value, measures, graphs and interpretation of information. The data-analysis focussed on comparisons of the mathematics achievement of the pupils by sex, by school and by age as indicators. The results indicate that generally although the boys' means in the schools were higher than those of the girls the differences were not significant. However, the differences were statistically significant in favour of boys in one primary school. Students' performance when compared by schools show highly statistically significant differences in mathematics achievement between schools. The 12-year-old pupils outperformed the other age groups among the pupils studied. It is concluded that it is mainly schools that indicate differences in achievement of pupils, while sex and age only show slight differences. These results have implications for the teaching of mathematics in the different schools. It is argued that the teaching of mathematics be addressed in different primary schools to improve pupils' achievement. There is need for further studies of teacher practices in the teaching of mathematics in the various primary schools.

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IMAGES OF FUNCTIONS DEFINED IN PIECES: THE CASE OF 'NON-INFLECTION'-INFLECTION POINTS

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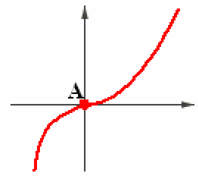
Teachers are continuously encouraged to conduct class discussions where students present their solutions, raise assumptions, and evaluate each others' suggestions (e.g., NCTM, 2000). Clearly, the ability to conduct such lessons is dependent on the teachers' knowledge regarding the validity of students' suggested ideas (SMK), as well as on the teachers' ability to pose challenging follow-up problems (PCK). Research findings, however, indicate that prospective teachers' SMK and PCK related to various mathematical topics are not always satisfactory and hence, there is a call to promote this knowledge. Here we focus on an activity aimed at promoting prospective teachers' SMK and PCK regarding functions-in-pieces. Our questions were: (1) what are the prospective teachers' concept images of functions-in-pieces with reference to the notions "continuity", "differentiability", "extreme points" and "inflection points"? (2) What follow-up tasks may promote prospective teachers' related knowledge? and (3) Which of these tasks do the prospective teachers regard as "good tasks" and why? We investigated 23 prospective secondary school mathematics teachers' conceptions at Tel Aviv University. They were asked to solve the task:



Look at the function $f(x)=\begin{cases} tgx, & for\ x<0 \\ x^2, & for\ x\geq 0 \end{cases}$ and answer the questions:

1. In your opinion, is the function f(x) **continuous** at A? Explain.
2. In your opinion, is the function f(x) **differentiable** at A? Explain.
3. In your opinion, is A an **extreme point**? Explain.
4. In your opinion, is A an **inflection point**? Explain.

and to suggest some follow-up questions. The participants' solutions reflected their views of functions-in-pieces, and their grasp of the notions "continuity", "differentiability", "extreme points" and "inflection points". For example, prospective teachers claimed that "the function has a 'non-inflection', inflection point at x = 0, because on the one hand it changes from concave down to concave up, and on the other hand it's not differentiable at zero". This led to a discussion about different mathematical definitions to a concept and to the examination of the equivalency of these definitions. A number of follow-up tasks were suggested and discussed from a mathematical and from a didactical point of view. This type of activities (e.g., Tsamir & Ovodenko, 2005) seem to be valuable in teacher education.

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A COMPARATIVE ANALYSIS OF ELEMENTARY MATHEMATICS TEXTBOOKS OF KOREA AND SINGAPORE

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This study analysed comparatively Korean elementary mathematics textbooks and Singaporean counterparts against an increasing international concern about Asian mathematics curriculum. Singaporean and Korean students demonstrated their superior mathematical achievement in recent international comparison studies such as TIMSS 1995, 1999 and 2003 (Mullis, Martin, Gonzalez, & Chrostowski, 2004). Given this, it is informative to look closely at Korean and Singaporean mathematics education, specifically their textbooks related directly to teaching and learning. The two kinds of textbooks were compared and contrasted as a part of our attempts to find out what would be the main characteristics, including similarities and differences.

For an in-depth study, the scope of the analysis was restricted to geometry and measurement. Geometry has an old history like numbers and is very important part of human life. Measurement is closely connected with other areas of curriculum as well as daily life. Despite the importance of geometry and measurement, there have been little studies on these areas, in particular in the context of textbook analysis. The analysis was conducted in two stages. First, the textbooks were compared in terms of an overall unit structure, the contents to be covered in each grade, and the periods of introducing and dealing with main learning themes. Second the textbooks were compared and contrasted in terms of the main characteristics of constructing mathematical contents with their demonstrative examples.

Both countries emphasized mathematical thinking, in that there were questions requiring students' reflection and thinking skills in learning process. Whereas Korean textbooks used block learning, Singaporean employed repeated learning. The latter also used the activity of classifying multiple figures as the main method to introduce concepts so that students might develop a deep understanding of mathematical structure (Bassarear, 2001). Whereas Korean textbooks consisted of typical examples of figures, Singaporean included various examples consistent with the principle of mathematical variability. This study will lead to a meaningful discussion on the design and development of main instructional materials with detailed illustrations.

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SUBSTITUTIONS ON ALGEBRAIC STATEMENTS, BASED ON ASSOCIATIONS IN NATURAL REASONING

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In this work we focus on natural language students' reasoning, when they analyse statements written in symbolic language. We observe that when looking for conditions under which algebraic statements are true, they make instantiations with numbers which are familiar to them. By analysing the counterexamples, they condense their typical characters and identify them as being of a *certain type* ("heuristic of representativeness", Tversky and Kahneman, 1974). According to this well-known psychic function, the identified characters will remain as a definition of the type ("typical definition", Duval, 1995). The attitude which characterises the typical definition, could be "*what is valid for a unit is valid for the whole*" (in particular, the property of being a counterexample of the statement). That allows us to interpret how students reformulate the original statement. By removing the identified type (i. e. for them, the set of counterexamples), they believe that the new statement does not get counterexamples any more. Then, they do not feel the need to study the truth or falseness of the new statement.

We will present the production of a college student, Brenda, which is especially illustrative of the procedure of reformulation presented here.

The problem:

"Decide if the following implication is true or false: $\forall x \in \mathbb{R}$: $2x^2 > x(x+1) \Rightarrow x > 1$"

When solving the problem, Brenda considers diverse examples, $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = -1$, $x = -2$, $x = -3$, $x = -4$ analysing the value of truth of the antecedent and the consequent in each case. She concludes, correctly, that the statement is false, because "*it is possible to find values of x smaller than 1 that fulfil* $2x^2 > x(x+1)$"

She is asked to explain how she arrived to the answer.

Brenda says that "-2, -3, -4 *are counterexamples, because for them the antecedent is true and the consequent is false*".

According to the task, Brenda could have finished there, but she adds, immediately:

"*Ah, it was* $|x|$ *that had to be put! What is true is* $\forall x \in \mathbb{R}$: $2x^2 > x(x+1) \Rightarrow |x| > 1$".

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VIRTUAL LEARNING ENVIRONMENTS AND PRIMARY TEACHERS' PROFESSIONAL DEVELOPMENT

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The research question -How does collaborative learning in asynchronous discussion groups result in enhancing academic discourse and knowledge construction in Mathematics Education?- has been examined in a study involving 65 students, working in 3 electronic discussion groups. The transcripts of the discussions were coded and analysed to characterize how primary teachers make sense of professional problems in primary mathematics teaching. The results point out the influence of interactions (way of participation) in virtual learning environments on primary teachers' professional development.

THE MAIN THEMES OF THE STUDY

From a socio-cultural point of view, becoming a mathematics teacher means acquiring an understanding of the teaching of mathematics as a practice. Here, teacher development might be understood as the progressive participation in a community of practice through the use of "conceptual tools" for understanding and handling the teaching task (Lave & Wenger, 1991). From these perspectives a learning environment was designed integrating: a) a case describing the difficulties of a primary pupil with division algorithm, b) theoretical information on the learning of the decimal number system and c) three virtual debates to favour interaction among primary teachers .

The evolution of the way in which the mathematics primary teachers participated in this environment and the characteristics of the generated discourse are indications of the professional development. As the participations in the virtual debates were performed as written texts, we considered the Sfard's (2001) meta-discursive rules as the moulders and the enablers and navigators of communicational activities that regulate the communicative efforts. We identified the following categories of "mode of participation" in the debates that display to have some influence on professional development: response, questions for reflection, questions for clarification, responses for clarification, disagreement, refutation, endorsement and clarification.

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CYPRIOT PRESERVICE PRIMARY SCHOOL TEACHERS' SUBJECT-MATTER KNOWLEDGE OF MATHEMATICS

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From a variety of perspectives, research on teachers' knowledge typically focuses on their subject-matter knowledge (SMK) (e.g. Ball, 1990). Some researchers have investigated teachers' understanding of different topics in mathematics and others have suggested that SMK is related to the way teachers teach (e.g. Rowland, Huckstep and Thwaites, 2004).

In the Cypriot literature this area of research has been neglected. Therefore I set out to investigate Cypriot preservice teachers' SMK, focusing on their procedural and conceptual understanding of different topics in mathematics. The participants (n=38) were university students in the final year of their training programme and represented a variety of mathematical backgrounds. I looked for differences among participants with respect to their understanding and whether these could be explained (a) by their exposure to school mathematics (b) by their experience of mathematics in their university training. The data were collected through semi-structured interviews and written tasks. This structure enhanced triangulation of data and increased the validity of my findings.

Qualitative analysis of the data showed that the participants' knowledge was predominantly procedural rather than conceptual. Those who had more experience of mathematics at school performed better in some tasks, but still lacked conceptual understanding. Moreover, it seemed that experience of mathematics at university did not help the participants to overcome their shortcomings in conceptual understanding. This might suggest first, that university mathematics puts emphasis on procedural understanding and secondly, that their university tutors considered that participants had the SMK they needed (for elementary teaching) from school, and therefore they emphasised pedagogical knowledge. In summary, my study suggests that Cypriot teacher preparation programmes could benefit from rethinking how they approach SMK. The methodology of my study allows only limited generalisation. However, my results should help the design of future research in the field, aspects of which I will address in my presentation.

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PHENOMENOLOGICAL MATHEMATICS TEACHING: A CHALLENGE TO PROSPECTIVE MATHEMATICS TEACHERS

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The teaching of mathematics is slow to change, if indeed it changes at all. It is a challenge in the work of teaching teachers to wonder why this is so (see also Pehkonen 2005). The aim of my research is to define my concept of phenomenological mathematics teaching and to reinforce the share of components related to experiential teaching in the training we offer our prospective teachers. In the course of the research I shall be examining how the views and thoughts of the future teachers of mathematics participating in my research develop during their studies as they become familiar with the phenomenological teaching of mathematics. In the early stages of my research I have linked the following to the phenomenological teaching of mathematics: the interactive, experiential, communal, explorative, observational components and that of mathematics as a language or as a language education.

The approach is through action research, the points of departure being the further development of the researcher's own work, professional growth as a teacher of teachers and a description of the teacher's and students' didactic processes occurring in the course of the action research. The target group of my study comprises students beginning their studies in autumn 2005 on a master's programme at the University of Tampere (n = 6). They are major subject students in our department and in addition to studying education they have a compulsory component of mathematics as a minor subject. The first data collected this autumn include these students' reflective essays, advance tasks and exercises forming part of a lecture series in Basic Studies in education, likewise lecture material planned by myself and deliberations in that connection. These were supplemented in December 2005 by interviews. I shall compile material over three consecutive years of study, when some of the students will have completed the bachelor's degree on the way to the master's degree and some of them will already be in the final phases of their master's degrees.

The data I have collected will probably correspond well to the material produced through narrative research, thus the method of analysis used will be narrative analysis. In my presentation I shall focus on the following aspects:

1. What is the phenomenological mathematics teaching?
2. What are the results of my first data?

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USING THE DEBATE TO EDUCATE FUTURE TEACHERS OF MATHEMATICS

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In a previous report (Proulx, in press), I have highlighted a diverse range of future teachers' perceptions of their mathematics teacher education program. These ideas have prompted me to question the tendency to organize teacher education programs around conformity to 'best practices' or other idealized conceptions of mathematics teaching. Emerging from these critiques, issues of the role that a mathematics teacher education program can play – its goals – brought me to support the idea that teacher education should focus on developing and educating knowledgeable and reflective professionals – that are able to support their claims and teaching actions. As the teacher educator, my intention is to create opportunities for future teachers to learn about and develop a personal sense of "what it means to teach mathematics" This brought me to look for ways to open spaces for prospective teachers in which they could construe rationales and take positions concerning their teaching actions. I have enabled these spaces by creating periodic debates on issues of interest in mathematics teaching (e.g., algorithms, calculators, rote learning). I intend to report on the learning opportunities that two specific 45-minute debates have offered, debates centered on (1) "Should we teach standard algorithms for addition and subtraction?", and (2) "Should we permit the usage of calculators in elementary schools?"

These debates – brought about by asking student teachers to reflect, argue and interact with colleagues – have provided them with opportunities to develop a personal and robust understanding of what teaching children mathematics implies. I intend to report on the type of position that future teachers undertook. For example, in the first debate, some argued for the importance for students to understand what they do and not only "how" to do, while some added that "standard" algorithms were good ways to create a coherence for the methods used throughout the years. Other student teachers highlighted that automatisms had to be acquired and that algorithms played that role, whereas some warned that there was more to addition and subtraction and that instruction should not only focus on algorithms, and several explained that if students can solve the problems with any method than the goal was achieved. In addition, these positions did not appear to be ephemeral, for student teachers re-used and argued for them throughout the whole semester, showing the potential and richness that these educative debates brought – and how it participated in the development and enlargement of their "vision" of mathematics teaching.

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SURPRISE ON THE WAY FROM CHANGE OF LENGTH TO CHANGE OF AREA

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Research in mathematics education reports on many difficulties, which students have, concerning the area concept. One of them, as described by Friedlander & Lappan (1987), deals with identifying the change in the area of a shape caused by a change of its length by a linear scale factor: "The principle of area growth presents many cognitive difficulties…it requires the recognition of the (somewhat counter-intuitive) fact that the enlargement of a figure by a linear scale factor of n will increase its area by a factor of n^2" (p.140).

Other researchers investigate students' cognitive thinking that shapes with same perimeters must have same area and vice versa (for example, Linchevsky, 1985; Hoffer& Hoffer, 1992). Stavy & Tirosh (2000) claim that in a broad perspective this response could be viewed as an example to the general **intuitive rule**: "Same A → Same B". We see in the same approach the intuitive rule mentioned above by Friedlander & Lappan as: "Same change in length → Same change in area". For this aim we designed sequence of activities which includes various assignments dealing with the essence of the area concept and it's measuring, and with the change in the area of a shape caused by a linear change of its sides' length. In these activities the students make use of concrete materials, visual components and Dynamic Geometry software which provides a source for geometrical surprises followed by feedback to students' actions. As reasoning reflects understanding and conviction (Hershkowitz, 1998), our interest is to follow students' reasoning concerning their dilemmas and findings, while working on these activities. Therefore we videotaped and analysed pairs of grade 4 students, working and talking with their partner and answering interviewer questions occasionally.

In the presentation we will briefly describe the activities, report and analyse students' responses. We will especially focus in surprises they have, if and how the surprise, together with the concrete-visual experiences they accumulated in the activities, have an affect on their explanations of the area's change.

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DISCOVERING OF REGULARITY (BY 11-YEARS OLD CHILDREN)

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In polish school, practice of teaching of the mathematics students most often considers tasks requiring using concrete, ready knowledge. Particularly they don't solve problems concerning patterns and regularity. Because of both limited hours of mathematics and overfilled programme of teaching the teachers are not able to attach anything from beyond the programme, what corresponds with the world trends of teaching of mathematics. However, we are expecting that the new programmes, which are being prepared in Poland, will take the following abilities and activities into consideration: solving problems, looking for patterns, representing, explaining, justifying, finding counter examples. However, before we start to prepare some considerations of such problems, we need to recognize the natural children's strategies in this area.

In may 2005 I carried out research in the group of 15-years old students, which consider the noticing of regularity. Taken into account the results I've decided to go further and start new research which included elementary school's pupils. My question was if 11-years old pupils whose math knowledge is less than 15-years old students, were able to notice regularity and discover the same or different relationships as their older friends. The method of research, which were carried out in November 2005 was atomic analysis of work of particular pupils and the analysis of the film which was showing both children's work and the conversation between them and with the teacher. As the research tool I have chosen the task with arithmetic and geometric content which required the noticing and generalization of some relationships.As a result of my analysis I have distinguished a lot of categories and strategies connected with the solution of the task.

Main conclusion was that pupils could discover the regularity but they weren't able to generalize it. Although they were working in definite, consistent way, they weren't able to work successfully on "abstract elements" which couldn't be seen by them. Generalization may be not the spontaneous ability and should be educated in an intentional way by teacher.

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USING MANIPULATIVES TO TEACH STUDENTS IN COLLEGE DEVELOPMENTAL MATH CLASSES ABOUT FRACTIONS

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Kean University   Felician College

Fractions can be problematic for some students in college developmental (remedial) mathematics classes. We discuss efforts to help these students develop a better understanding of fractions through the use of objects such as Cuisenaire rods.

Reynolds (2005) and others have reported that the use of Cuisenaire rods can help elementary school students develop robust understanding of fractions and operations on fractions. We report here on the use of similar activities with college students.

Kamii, Lewis, and Kirkland (2001) say that manipulatives can help students to gain mathematical understanding provided that students use them to reflect on mathematical relationships. Glass and Maher (2002) demonstrated that it is not too late to introduce rich mathematical experiences in a college mathematics class; they note that college students who are encouraged to think about their solutions can develop mathematical understanding. We discuss here how students in developmental mathematics classes developed techniques for understanding fractions by building and utilizing models.

We gave students the opportunity to use Cuisenaire rods to model fractions and relationships between fractions. We found that some students continued to rely on paper-and-pencil methods; some others used models but did not relate them to mathematical operations. But some students were able to use the rods to devise appropriate models for problem situations involving fractions, and they derived algorithms for the basic operations on fractions.

We conclude that manipulatives have a place in developmental college mathematics. Although not all students have benefited from their use, they should be available as one resource that might help students to develop conceptual understanding.

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SIXTH GRADERS' ABILITY TO GENERALIZE PATTERNS IN ALGEBRA: ISSUES AND INSIGHTS[*]

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This study presents findings drawn from pre- and post-interviews of eleven 6th grade students (6 females, 6 males, average age: 11) on five algebra tasks that involve generalization. We address the basic question: What abilities do they have that influence the manner in which they express and justify generalizations in algebra? Results indicate that the students established generality figurally, numerically, and analogically at the proto-representational level, and that they were capable of symbolic generalization at the representational level towards the end of two sequences of classroom teaching experiments (CTEs). The CTEs have been initially drawn from three algebra units of the Mathematics-in-Context curriculum (Operations, Expressions and Formulas, and Building Formulas) that underwent further revision based on classroom and institutional contexts. We present an empirically-driven theory of generalizing types that consists of two levels, namely: proto-representational and representational. At the proto-representational level, the students started to notice pattern attributes or relationships either figurally, numerically, or analogically. At the representational level, variables played an important role in expressing generality. From the data, we note that students with no knowledge of variables expressed generality at the iconic level through verbal and nonverbal forms that imitated what they actually perceived. Students with some knowledge of variables used variable forms that were either indexical or symbol. That is, those students who worked at the indexical level produced partial and situated generalizations, while those who worked at the symbolic level produced full symbolic generalizations. Results of the postinterview also show that the students preferred to set up a general formula for a problem task before dealing with near and far generalization tasks. Further, they assessed the equivalence of two or more generalizations for the same pattern mainly by substitution. Finally, justifying an equivalent symbolic generalization depended on whether it is constructive or deconstructive, and 10 out of 11 students could only justify constructive ones.

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[*] This paper is based upon work supported by the National Science Foundation under Grant No. REC 0448649 awarded to the first author. The opinions expressed are those of the authors and do not represent those of the NSF.

VARIETY OF REPRESENTATIONAL ENVIRONMENTS IN EARLY GEOMETRY

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Representations play an essential role in the process of mathematics teaching and learning because they help us grasp and understand abstract notions. The use of several different forms of representation is not only a characteristic feature of cognition in mathematics but also a necessary prerequisite for the use of mathematics in solving real problems. It transpires that to understand mathematics means, among others, to be able to represent mathematical objects and relationships among them using various semiotic representation systems and to be able to transform and interpret these representations. It is evident that the process of representation and understanding of concepts is purely individual. The need to communicate and make understood led to establishing of social conventions for the use of representation means. We meet at pupils in the geometry classroom with the conventional representation systems which are proposed to them by teacher and also with the pupils' individual representation means. Visual representation, mainly various pictures and three-dimensional models, is the most frequently used tool to show geometric figures but is in no way the only type of representation. The use of word games, stories, and situation sketches is rather unusual in the teaching of geometry but it can motivate pupils and serve as the basis for discussion and problem solving.

An experiment was carried out with the aim to observe influences of different forms of representation on pupils' learning. All activities used in the experiment were focused on one issue – parallels and perpendiculars. The geometric problem is introduced through story described a situation on a city map. After listening to the story, pupils drew the street plan – transformed a verbal representation into an iconic representation. The second representational environment was concerned non-verbal communication by means pictures, models or gestures. The activities related to the use of these means developed the pupils' communication skills and their geometry imagery. The third representational environment was a geometrical sketch used verbal and non-verbal means. The dialogues provided clues important for identifying geometric actors in the represented situation. The description was not strictly geometrical but included metaphors and comparisons to real items as well. The geometrical sketch was both a basis for discussion and at the same time motivated students to solve related problems. The observation and the analysis of pupils' activities were conducive to identification of three principal roles of representations: motivational, communicational and cognitive.

The contribution was supported by the grant project GACR 406/03/D052 and by the research plan AV0Z10190503.

FROM RESEARCH ON USING PROBLEMS RELATED TO FUNCTIONAL EQUATIONS AS MULTIFUNCTIONAL TOOLS FOR REVEALING SUBJECT MATTER KNOWLEDGE OF FUNCTIONS IN FUTURE MATHEMATICS TEACHERS

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This paper deals with research on future teachers' *subject matter knowledge*. By this term I understand skills and knowledge of mathematics, its methods and history, which are indispensable for teaching. As a result of substantial modifications of Even's (1990) theoretical framework I have distinguished six *components* of teachers' subject matter knowledge of a mathematical concept (Sajka 2005a, 2005b).

There is a need to find appropriate tools for continuous assessment of that competence during the course of mathematics pre-service teacher training. The hypothesis being verified states that **problems related to functional equations can be used as new multifunctional tools for revealing subject matter knowledge of functions.**

This paper presents an excerpt from the current research on that hypothesis. I put forward one problem and analyse several responses from students of Mathematics at Pedagogical University of Cracow. Their answers show that this task is a multifunctional research tool since it can reveal simultaneously several components of the subject matter knowledge concerning functions. Furthermore, it can reveal both positive and negative aspects of understanding the concept of function (Klakla, Klakla, Nawrocki & Nowecki, 1989; Sajka, 2003, 2005a).

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EXPERT AND NOVICE PRIMARY TEACHERS' INTERVENING IN STUDENTS' MATHEMATICAL ACTIVITY: AN INSIGHT

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A considerable part of the research concerned with teaching mathematics has focused on the teaching skills exhibited by expert versus novice teachers. The relevant studies, viewing teaching as a complex cognitive skill, suggest that experts' cognitive schemata for content and pedagogy in mathematics teaching are more elaborated, interconnected and accessible than those of the novices (e.g., Livingston & Borko, 1990).

Despite its value, the above research fails to adequately address the issue of the quality of mathematics instruction. Koehler & Grows (1992) argue that this can be done by examining specific teaching behaviors and the way classroom events come together to give rise to a meaningful learning situation. To this direction, recent studies, considering the classroom as a social context, where mathematical knowledge is negotiated and constructed, focus on the particular actions taken by teachers in their attempt to help students build their own mathematical meanings.

In an earlier study, placed in the above framework, we examined expert primary teachers' interventions during inquiry-oriented classroom mathematical activity. The results showed that the teachers of the sample tended to intervene in a very directive way, canceling students' initiatives (Kaldrimidou et al., 2003). In the present study, we look at the same research problem, but for a group of perspective primary teachers. In particular, 12 student teachers in the fourth (final) year of their studies were asked to plan and carry out two lessons based on activities promoting the negotiation of the mathematical meaning in the classroom. The analysis of the 24 lessons observed appear to indicate that, although novice teachers' teaching approaches were less rich, flexible and varied than the experts' in the earlier study, their intervention practices tended to be more of a conceptual character and less directive compared to the experts', thus pointing to a challenging area of development for teacher training courses.

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EXAMINING TEACHERS' REFLECTIONS ABOUT MATHEMATICS TEACHING, LEARNING AND ASSESSMENT

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This research reports an investigation of the interplay between teachers' reflective thinking and changes they perceived to make in their approach to mathematics teaching, learning and assessment. Seven high school teachers willing to engage in collaborative research and to learn more about their own pedagogy and professional development are the main participants of this study (Peter-Koop, Santos-Wagner, Breen, & Begg, 2003). Several studies have shown that reflection plays a key role in teachers' learning and professional development (e.g., Wood, 2001). The research design followed Chapman (2001) ideas of conducting research to understand school mathematics teacher growth in a humanistic way and Gates (2001) arguments to search for social roots for the investigation of teacher belief systems. Data source for this study come from memories from positive and negative past experiences as learners and teachers of mathematics; metaphors about mathematics and its pedagogy; open-ended interviews; classroom observations; and sharing, analysis, reflection and discussion of the information collected. In this presentation part of the initial data analysis will be shared as well as discussion of how these research procedures can help teachers develop professionally (Wood, 2001).

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ANALYZING STUDENTS' THOUGHT PROCESS IN REVEALING CORRESPONDENCE BETWEEN FORMULAS AND GEOMETRICAL OBJECTS

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The language of mathematics is a language of formulas. Therefore, to know mathematics is to know how to interpret, use, and construct formulas. Understanding of this conceptual idea is important for mathematics teaching, where "many difficulties emerge because incapability to relate the algebraic code to the semantics of the natural language" (Bazzini, L. 1998, p.112). Our research is aimed to analyze students' thinking in the process of revealing correspondences between formulas and geometrical objects.

In order to eliminate the effect of previous knowledge, and to increase students' cognitive interest and motivation, we have constructed nonstandard tasks in 3D analytical geometry. We start from simple nonlinear equations of linear objects in 3D space, such as straight lines, line segments and planes and switch to more sophisticated equations, which are connected to the cube in 3D space. We analyze how the students try to describe this basic geometric figure by means of words and by simple equations and how the different equations arise from different understandings of what a cube is: a solid body, a surface only, a cube-frame or cube vertexes only. We did these experimental studies with college students in remedial mathematics courses and in teaching calculus. The results of this study indicate some misconceptions and mental obstacles in thought process of students in solving graph-formula problems which need special attention of mathematics teachers to prevent this.

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”NO NEED TO EXPLAIN, WE HAD THE SAME” AN EMPIRICAL STUDY ON OBSTACLES AND INTENTIONS IN THE USE OF MATHEMATICAL LANGUAGE IN UPPER SECONDARY SCHOOL

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The discussion of language factors in mathematical education opens a broad field of research. Especially the important role of language in mathematics classrooms as a means of communication and as a content of learning has drawn the attention on student’s language proficiency. Various empirical evidences have been cited for the mathematical language being an obstacle of understanding and learning and for low achievement in the use of mathematical language by students.

As classroom studies how shown, communication about mathematics often proceeds in colloquial instead of technical language, student’s statements often stay on an imprecise level with rather fragmentary comments and misunderstandings . Why is the acquisition of mathematical language that difficult?

Language use in mathematics classrooms is related to psychological, sociolocial and linguistic aspects. The presented research project bases on a learner-oriented point of view, adopting a pragmalinguistic and constructivist perspective which implies the conceptualisation of language use as an individual activity that aims at accomplishing a personal intention and learning as an individual process of knowledge construction. Rather than collecting student’s difficulties in language proficiency, this perspective suggests to focus the research on the individual perspectives, motivations and intentions while using language in mathematical situations.

The empirical study starts from the following research questions:

Which obstacles do the students meet when they use mathematical language, resp. which aspects influence their decision in favour of or against the use of it?

The study is carried out with upper secondary school students in a two-step empirical design that combines observation and stimulated recall interviews. In a first step, pairs of students were asked to work collaboratively in a diagnostical learning environment, enhancing their active exchange of ideas. In a second step of stimulated recall, the students are invited to reflect their work and to comment metalinguistic aspects of the learning situation.

The collected data is analysed qualitatively with respect to the students’ communicative activities and their formulated intentions. The first results of the still ongoing study give hints to the importance of individual intentions: Even in a learning environment that creates situations of communication about mathematics, the proband’s intentions suggest not automatically the use of mathematical language.

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CHARACTERISTICS OF MALAYSIAN STUDENTS UNDERSTANDING ABOUT FUNCTIONS

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The topic of function is important topic in secondary mathematics curriculum. This paper reports on a recent research conducted on twelve secondary students for the purpose of investigating their ways of thinking about functions. Understanding students thinking can be a useful pedagogical knowledge to facilitate them towards constructing a more sophisticated form of mathematics (Steffe, 1983).

Main data were obtained from four seperate clinical interviews sessions which were video taped and later analysed. Analysis carried out were mainly informed by grounded theory approach (Glaser & Strauss, 1967) where transcriptions were summarised per student per task and validated with a second interpretator.

Results showed that there are three main characteristics of students understanding:

a) Weak understanding of the formal definition of function: Ten students displayed various inconsistentcies and incomplete ideas about formal definitions of functions. Only two students indicated a complete understanding and they were able to apply it in graphical and analytic forms.

b) Confusions about functions, function values and algebraic operations of functions: Persistent conflicts was observed related to quadratic functions, equations and graphs. Many students also thought algebraic operation of functions are only limited to inverse and composition while addition and multiplication were not possible.

c) Strategies used to solve problems: Three strategies involving three different ways of thinking were used by students to solved composite functions fg(x) problems: Images of g(x) becomes object of fg(x); substitute g(x) followed by f(x) and substitute x of f(x) with g(x).

Majority have not reach formal object conceptions and two students were at prefunction conception (Breidenbach. et al, 1992). Only two students indicated flexible ways of thinking and use of strategies when solving all the problems. More concentrated effort need to focus on developing meaningful conceptual orientated activities in Malaysian classrooms.

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MATHEMATICAL INDUCTION VIA CONCEPTUAL REPRESENTATION

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Mathematical knowledge includes fundamental relationships between conceptual and procedural knowledge. When concepts and procedures are not connected, students may generate answers but not understand what they are doing (Hiebert & Lefevre, 1986). Many mathematical educators present several models for understanding (Pierie & Kieren, 1989; Herscovics & Bergoren, 1984). All these models agree that the intuitive knowledge is a necessary condition for understanding; moreover, each model of understanding is divided into levels, where the first level is the intuitive knowledge and the later relates to formalization.

The aim of this study is to characterize the method of teaching the Mathematical Induction Principle (MIP) by a professor of mathematics. We based our methods on observations during our subject's lessons the MIP (the fifth of Peano's axioms). Our analysis is couched in Herscovics and Bergoren model of understanding (1987): Making use of comprehension characterized by means of imagistic representation, our subject constructed the intuitive understanding by relating Peano's axioms to Genesis in the Bible (*At the beginning god created …*). *By translating the stages of the world's creation as it appears in Genesis to mathematical language as it expressed by Peano's axioms* our subject formed the procedural level of understanding. Solving problems such as: "*Let* $a: N \to R$ *be a function defined as* $a(1)=2; a(k+1)=2a(k)$ *, does this function defined for all natural numbers?*"; our professor expressed the MIP as a mathematical invariant; so he emphasized the mathematical abstraction of the concept. Using the MIP in proving problems like: "*prove the inequality* $2^n < 1/n^2 , \forall n < -4, n \in Z$ ", draws the formalization level of understanding. Our main conclusion is deriving a cognitive net for different conceptions of the MIP.

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EXPLORING THE MEANINGS OF EVENTS IN MATHEMATICS CLASSROOM FROM LEARNERS' PERSPECTIVE

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The Learner's Perspective Study (LPS) is an international study of the practices and associated meanings in 'well-taught' eighth-grade mathematics in participating countries (Clarke, 2004). Given the fact that teaching and learning are interdependent activities within a common setting, classroom practices should be studied as such. The LPS has the potential in that it literally focuses on the perspectives of *learners* in mathematics classroom as well as those of teachers.

The methodology employed in the LPS offered the teachers and the students the opportunity in post-lesson video-stimulated interviews to identify for the interviewer those events in the lesson they had just experienced that the participants felt to be significant. This paper reports on the analysis of post-lesson video-stimulated interviews with the students in three eighth-grade grade mathematics classrooms in Tokyo. The interviews protocols with thirty pairs of students were analysed.

The analysis described in this paper shows that nearly half of the students interviewed identified the event related to "understanding/thinking" category as the one that should be happened in a "good" lesson. Also, about quarter of the students identified "whole class discussion" as the component of a "good" lesson. While they share views on "significant" lesson events and on what should be happened in a good lesson, there are differences both within a classroom and among three classrooms in students' perceptions and construction of different meanings associated to the events they have experienced. The strengths of methodology employed in the LPS are highlighted and issues for further research raised by the analysis are discussed

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A STUDY ON THE LAW OF LARGE NUMBERS INSTRUCTION THROUGH COMPUTER SIMULATION

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There is a tendency for students not to recognize the aspect of dispersion of the relative frequencies under the mathematical probability when trials of an event are repeated(Dooren, 2002; Shaughnessy, 2003). To learn the feature adequately, students should be given opportunities to develop their intuition for a likely range of outcomes in actual repeated trials through computer simulation(Freudenthal, 1972). The purpose of this study is the following.; Firstly, it examines how the nature of mathematical knowledge, the law of large numbers, may be changed when being taught by the use of a computer. Secondly, it develops specific teaching material that introduces the law using computer simulation. For the first purpose, we analysed the results of previous research on the teaching method where computers were used. It was found that the didactic transposition method was explaining the law as $\lim_{n\to\infty} \frac{X}{n} = p$ rather than $\lim_{n\to\infty} P(|\frac{X}{n} - p| < \varepsilon) = 1$ where for an event, the number of throws n, the relative frequency $\frac{X}{n}$, the mathematical probability p, and any $\varepsilon > 0$. On the statistical point of view, the law that $\lim_{n\to\infty} P(|\frac{X}{n} - p| < \varepsilon) = 1$ is held seems to be explained as confirming that $P(|\frac{X}{N} - p| < \varepsilon) > P(|\frac{X}{n} - p| < \varepsilon)$ is realized for finitely-carried numbers N and n with $N > n$ (Jung, 1992). For the second purpose, then, we developed specific teaching material such as the following.; For example, consider the situation that the law is realized in a case where spot 3 appears when a die is thrown. Carry 100 simulations that a die is thrown 12 times and 100 simulations that a die is thrown 120 times. Then, for any $0 < \varepsilon < 0.05$, compare the frequencies of the cases of $|\frac{X}{12} - \frac{1}{6}| < \varepsilon$ and $|\frac{X}{120} - \frac{1}{6}| < \varepsilon$. And, finally confirm the law by examining more frequencies in the latter simulation than ones in the former.

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MATHEMATICS LEARNING QUALITY FOR GIFTED JUNIOR HIGH SCHOOL STUDENTS IN TAIWAN

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There is a wide range of indices such as test scores, students' grades, anxiety indices, and self-efficacy scales of mathematical performance in students' learning. It is desirable to investigate the learning quality, with different point of view, of the population who do have high achievement. We adopt and revise the questionnaire (named mathematics learning quality, MLQ), developed by Shy et.al with perspective of the quality of life, to describe the quality of mathematical learning of gifted students. That will yield a new understanding on gifted students' feeling on their mathematics learning.

The data consists of 255 junior high school students in the gifted classes in the central Taiwan. Statistical analysis was performed. And a total of 5 students in different class were selected to have an extensive interview with the researcher to discuss the meaningfulness of all items in the proposed questionnaire and to confirm the results from the questionnaire.

The major findings showed that the score of the MLQ is highly correlated with the teachers who were able to satisfy students' need in providing good quality of the content knowledge. But the correlation with the teaching models and teachers' personal characteristic is not significant. Those findings showed a great difference between gifted and average students as the average students are more affected by the teaching models instead of the quality of the context.

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A PRESERVICE TEACHER'S GROWTH IN SUBJECT MATTER KNOWLEDGE WHILE PLANNING A TRIGONOMETRY LESSON

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The purpose of this study is to investigate the ways preservice teachers grow in their subject matter knowledge of mathematics during lesson planning. Lesson plans have traditionally been used by teacher educators to evaluate their preservice teachers' use of pedagogy (Berenson & Nason, 2003). However, pedagogical knowledge is just one category of teacher knowledge. Another important category of teacher knowledge is subject matter knowledge (Shulman, 1986). An important source of teacher knowledge is the practice of teaching. In teacher education research, the growth of pedagogical knowledge has been a major focus, yet the growth of subject matter knowledge from practice has often been ignored (Ball & McDiarmid, 1990). Berenson & Nason (2003) found that the instructional representations created by preservice teachers during lesson planning reveal the depth of their subject matter knowledge, thereby providing a link between their pedagogical knowledge and their subject matter knowledge.

As part of an ongoing case study about the evolution of learning to teach mathematics, this investigation examines an instance of growth in subject matter knowledge of a preservice teacher. This example of growth occurred while she was planning a trigonometry lesson to teach as part of her student teaching requirements. Data were collected from her student teaching portfolio which included lesson plans and pre- and post-lesson written reflections. While planning a lesson on the unit circle, she researched lesson ideas in relation to her plan to present a visual representation of the unit circle. She asserted her belief in her pre-lesson reflection that visual representations can make concepts accessible to more students. In her post-lesson reflection, she stated that because of her lesson research, she finally understood how the reciprocals of the standard trigonometric functions are derived. These findings imply that teacher educators should attend to the growth of subject matter as well as the use of pedagogy in preservice teachers' lesson plans.

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COMPARING NUMBERS: COUNTING-BASED AND UNIT-BASED APPROACHES

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This study used design research methodology to investigate students' understanding of the concept of a unit and its role in assigning a number to quantities, by probing their methods for comparing numbers. The research questions were: What methods do children use to compare numbers? Do children include fractions in their comparisons? Does an emphasis on the concept of unit impact children's approach to comparing numbers?

Researchers (Sophian et al, 1997; Stafylidou & Vosniadou, 2004) see a relationship between students' errors in solving problems involving comparing fractions and experiences they have with whole numbers. For example, when asked to name a number between 1/5 and 2/5, students often say there are none, which is attributed to over-generalizing a pattern known to work with whole numbers while not recognizing the density of the number. Sophian, however, posits that if students worked with whole number in ways that emphasized an understanding of unit, they might not be as limited in their solutions. Twenty students in grades 4 and 5, who were part of a research and development project based on Davydov's approach to teaching primary-school mathematics (Dougherty, 2003) and 10 students in grade 6 who were not part of the project were given a sequence of tasks: (1) Find a number to complete the statement 513 < __, (2) What is the largest number that would make the statement true?; and (3) What is the smallest number that would make the statement true? (The fourth and fifth grade students also solved $513_{(\text{base six})}$ < __.) Students' approaches were grouped into two categories: a counting-based approach and a unit-based approach. Students who used a counting-based approach looked at the place values independently: for example, as indicated by completing the first statement with $524_{(\text{base six})}$ or 624 in base 10; the second by describing the number as having "a lot" of 5s or 9s (depending on the base); and the third statement with 514 or $514_{(\text{base six})}$. Students who used a unit-based approach demonstrated an understanding that whole units could be divided to create partial units (decimals and fractions) to answer the last question. The largest number (task 2) could not be written; it was infinity. The smallest number (task 3) could not be written; but it had to have at least one partial unit, however small, more than 513.

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MATHEMATICALLY GIFTED 6TH GRADE KOREAN STUDENTS' PROOF LEVEL FOR A GEOMETRIC PROBLEM

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We would like to share some findings of the research on the proof level and understanding of constituents of proving by four mathematically gifted 6th grade Korean students. Subjects of this study are four male students T, S, J, and Y, who are 6th graders(11 or 12-year-old) in elementary schools in Gyeonggi province (with over 1/4 of the total population of the Republic of Korea) and receiving special education for mathematically gifted students with governmental support. The students belong to the upper 1% group in mathematics among the 6th grade students.

We assigned the students with geometric problem and analysed their proof level and difficulty in thinking related to the understanding of constituents of proving. The geometric task used for this study is to find a new square whose is equal to the sum of the areas of two squares, and to justify the method. The study design is qualitative. A case study methodology was employed. Data collected include observation, in-depth interviews, and the activity sheets of the problem-solving process of the task by the four subject students. Analysis of data was made based on the proof level suggested by Waring(2000), the constituents of proving presented by Seo(1999), and methods of approach obtained by our own pilot test.

The result of the study indicated that all were well aware of the necessity of proving, with the exception of student T, and although they needed some assistance, they were capable of proving in a familiar setting(T, S, J, Y belongs to Proof level 2, 3, 4, 4 respectively). Concerning the understanding of constituents of proving, they all understood the usage of basic principles, but showed difficulty in the conjunction of deduction rules, utilization of appropriate figures such as auxiliary line, diversity and completeness of examination, and distinction of assumption and conclusion. In particular, all four students had difficulty in recognizing the necessity of proving the evident-looking facts. Meanwhile, students who mainly used the numerical approach tended to have difficulty in generalizing the facts that they had learned from the figures and in utilizing them in the proving process. Based on such results, we could recognize the possibility and necessity of more systematic, if not formal, instruction on proving for mathematically gifted elementary school students.

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PROBABILITY REASONING LEVEL OF GIFTED STUDENTS IN MATHEMATICS

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This research analysed the probability reasoning level of gifted students in light of the four levels suggested by Jones et al. (1999). The difference in probability reasoning between ordinary students and gifted students was confirmed, and the difference between gifted students attending an elementary school and those attending a middle school was also confirmed. The results were shown in the following table.

Level		**Ordinary elementary school students**	**Gifted students in elementary school(GES)**	**Gifted students in middle school(GMS)**
Level 1	**1.1**	16(42%)		
	1.2	17(45%)	5(26%)	2(12%)
Level 2	**2.1**	2(5%)	1(5%)	1(6%)
	2.2	2(5%)	2(11%)	3(18%)
Level 3	**3.1**	1(3%)	3(16%)	2(12%)
	3.2		8(42%)	9(53%)
	3.3			
Level 4	**4.1**			
	4.2			
Total		**38(100%)**	**19(100%)**	**17(100%)**

Ordinary students and gifted students showed a remarkable difference in the probability reasoning level; the difference between gifted students in elementary school and those in middle school was found to be relatively not so big. It was confirmed that those students who are gifted in mathematics, different from ordinary students, have the tendency to understand a problematic situation structurally. However, as we could not find a student who reached level 4, we confirmed the need to develop lower-level task than the one used in this research and carry out another experiment.

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“This work was supported by Korea Research Foundation Grant funded by Korea Government(MOEHRD, Basic Research Promotion Fund) (KRF-2005-079-BS0123).”

ANALYSIS OF MATHEMATICALLY GIFTED 5TH AND 6TH GRADE STUDENTS' PROCESS OF SOLVING "STRAIGHT LINE PEG PUZZLE"

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Studies on mathematically gifted students have been conducted following Krutetskii(1976). There still exists a necessity for a more detailed research on how these students' mathematical competence is actually displayed during the problem solving process. In this study, it was attempted to analyse the thinking process in the problem solving in which 5 mathematically gifted students, who belong to the upper 0.05% group in the 5th and 6th grade of elementary school in Korea. They solve and generalize the straight line peg puzzle shown below.

<Move the pegs so that their colours will be in reverse order, by either jumping to the next hole or jumping over one peg. (1)Find the least number of movements when there are <u>more numbers of pegs</u>. (2) Explain the method of attaining the least number of movements.>

The results of the research are as follows:

1. When mathematically gifted students deal with a specific case, they tend to use it not merely as a particular case but rather as a generic example. Mathematically gifted students were accustomed to figuring out a generic example (n,n), even in particular examples such as (3,3) and (5,5).

2. Mathematically gifted students have a tendency of trying to find immediately the structure of the generic case of (n,n) by figuring out the key structure of the task.

3. Mathematically gifted students who were 5th and 6th graders in elementary school also could provide generalization with ease using relational expression of two variables such as (n, m). When expanding to (n,m), the methods of solving the puzzle that the students offered were divided into 3 patterns: 1) Generalization by fixing n and varying m. 2) Generalization after confirming a few examples by fixing the difference between n and m. 3) Generalization of (n,m) applying the method of moving and generic mathematical expression used for (n,n).

4. Mathematically gifted students showed a tendency of not depending on concrete materials in the problem solving process. As it was difficult to figure out the moving process by actually moving the pegs, they preferred using simple images or symbols to show the moving process in each stage.

"This work was supported by Korea Research Foundation Grant funded by Korea Government(MOEHRD, Basic Research Promotion Fund) (KRF-2005-079-BS0123)."

STANDARD MATHEMATICS DISCOURSES OF DEVELOPMENTAL ALGEBRA UNDERGRADUATES

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There is a tension, or perhaps merely a complementarity, in what is understood as a student mathematical discourse. A discourse is a way of speaking to "identify oneself as a member of a socially meaningful group" (Gee 1996, p. 131). At times, a student's verbal bid to be recognized as a mathematical thinker goes unrecognized by the instructor (Moshkovich 2003). On the other hand, participation in mathematical discourses, even when a students lacks full understanding of the concepts, may be a necessary phase in learning mathematics (Sfard 2001). Students who are able to acquire a standard discourse can use this as a resource for making themselves understood and taken seriously. This is a particularly important issue for urban undergraduate students of diverse ethnicities who must make their way in a university dominated by the speech forms of Euro-American academics. This discussion describes the form and context of standard mathematics discourses spontaneously spoken by undergraduate developmental algebra students as they solve problems in constructivist, full-class guided discussions.

Transcribed recordings show that students in the study used a variety of standard mathematical discourses such as procedural problem-solving moves, naming methods or formulas, and comparing or evaluating methods. Less common were spontaneous generalizations. The context of these discussions suggests that the standard discourses that were most closely linked to collaborative problem-solving through class discussion were procedural moves, comparing methods, and spontaneous generalizations. However, standard discourses often were not deeply connected to the act of collective problem-solving, in other words, moving the math along. The examples are intended to create discussion on the extent to which we can identify a standard mathematics discourse in spontaneous speech, the degree to which we value the thinking associated with these discourses, and the role of standard discourses in achieving educational equity in undergraduate mathematics programs.

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NOVICE STUDENTS, EXPERIENCED MATHEMATICIANS, AND ADVANCED MATHEMATICAL THINKING PROCESSES

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The comprehensive aim of my research project is to understand how novice students encounter mathematics studies at university. In this presentation, I will focus on the development of mathematical ability and understanding, by examining qualitative differences between novices and experienced mathematicians working with a mathematical task.

Two novice students, one experienced student, one graduate student and two senior lecturers have solved an exercise about inverse functions within a topic dealt within the novice students' current calculus course. The task resembled those from ordinary calculus textbooks. However, solving the task demanded a more thorough analysis of the function to be inverted. In their reasoning and solutions, I have looked for traces of advanced mathematical thinking processes, where representing and translating together with abstraction plays important roles (Dreyfus, 1991). The ability to use these processes increase with mathematical experiences. Thus, it could be expected that the mathematicians had an easier access to theses processes, while solving the task.

Results indicate that the students use processes of advanced mathematical thinking in their work with the task. They discuss possible graphical representations, and make efforts to make a visual picture of the function and its inverse. They also try to work with the formal definition of inverse functions, but without success. While they are working with the task, they tend to refer to a *local mathematical context*, where previous lectures, the textbook, and students' peers play essential roles. This limits the outcome of their advanced mathematical thinking processes. Contrary to the novices, the lecturers draw on a wide range of mathematical experiences. They are well acquainted with formal definition and also discuss inverse functions according to mathematical functions in general. Thus, their advanced mathematical thinking processes in combination with a more *global mathematical context*, gives theme more favourable conditions to solve the task. This leaves us with an additional perspective on advanced mathematical thinking; the local or the global character of the mathematical context of advanced mathematical thinking processes appears to have significant impact on mathematical understanding and ability to work with and solve a mathematical task.

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THE “SOIL” OF EXTENDED PROBLEMS: THE CULTURAL BACKGROUND OF THE CHINESE MATHEMATICS TEACHING PRACTICE

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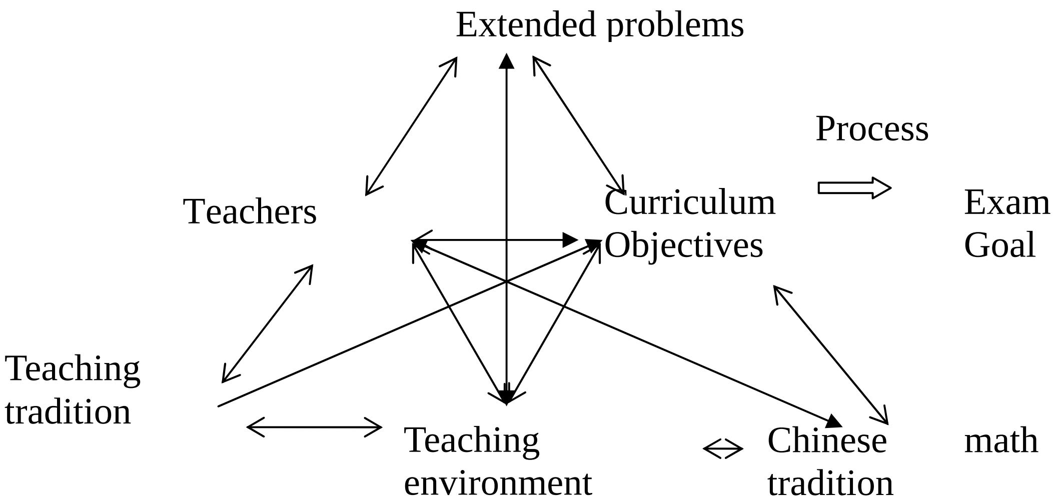
Recently *Extended* (variation) problems, which are Stretched example problems by changing the three components of conditions, conclusions, or deduction process, are consistently identified as an important element in Chinese mathematics education and become a hot topic in China. This paper looks into the origin of *Extended* problems in Chinese mathematics teaching practice through its cultural background. The paper will also discuss the five factors conducive to the development of *Extended* problems in the field of Chinese mathematics teaching, including: (a) the goal of examination; (b) the curriculum objectives — “two bases”; (c) the teaching environment; (d) the teaching tradition; and (e) the Chinese mathematics tradition.

In order to portray the character of Extended problems in the implementation of curriculum, the following model is adopted based on Engeström`s structure of a human activity system In fact, Extended problems are the “double scaffolding instruments ”of Chinese math teachers to link textbook problems to exam problems of “national achievement standards” in the individual level and the classroom level at same time in the centralization of the education system to achieve “Double



Foundation” and entry university goal, under special Chinese teaching environment: rich in population but relatively poor in resources. Extended problems also show the traditional continuity of structure and function of Chinese ancient math tradition of “Suanjing Shishu” (《算经十书》), traditional conception and mode in the modern Chinese classrooms.

IMMERSION IN MATHEMATICAL INQUIRY: THE EXPERIENCES OF BEGINNING TEACHERS

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Research suggests that facilitating effective mathematics inquiry poses substantial challenges for beginning teachers at both the elementary and secondary levels. For elementary teachers, the research suggests that a lack of understanding of mathematics may inhibit effectively implementing inquiry-oriented mathematics programs (Ball, 1990; Ball & Bass, 2002). For secondary teachers, while some of the same concerns exist about the procedural nature of their mathematics knowledge, the real challenge is to help them shift their teaching practices from traditional delivery models to more inquiry-oriented approaches (Stigler & Hiebert, 1999).

This longitudinal study probes such challenges as it investigates the experiences of 150 beginning elementary teachers who participate in a one-week inquiry-based mathematics environment (a.k.a. math camp) that is facilitated by 12 newly graduated secondary mathematics teachers. Through questionnaires and focus groups we examine the experiences of both groups during the math camp and through their first two years of teaching in regular school classrooms.

Theoretically we draw on complexity theory (Davis & Simmt, 2003; Johnson, 2001) to understand the math camp learning environment as a dynamic learning system, and we draw on sociocultural theory to investigate how these beginning teachers construct themselves as learners and teachers during math camp and then in the context of their assignments in school classrooms. This allows us to see that while both elementary and secondary teachers view mathematics teaching and learning in similar ways during the inquiry experience, there are noted differences as they move into classroom settings.

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THE WHOLE IDEA: ENGAGING TEACHERS WITH INQUIRY BASED LEARNING APPROACHES

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This presentation is focused on a teacher professional development initiative that incorporates inquiry-based learning approaches with year 8 mathematics teachers. Contemporary approaches to teaching have encouraged teacher educators to move beyond 'lecture-plus-tutorial' models in light of learners' prior experiences. A long term school-based research experience, entitled the WHOLE (Why and How Our Learning Evolves) initiative, is discussed as the basis for developing teachers' understandings of how children learn mathematics. The collaborative research employs a contextual approach and requires teachers to actively carry out problem-based tasks and subsequently gather students' responses. An outline of the experience including the methodology and results of the initial investigation into middle-school teachers' beliefs about mathematics and engagement including why some students are disengaged or lack resilience is given. Consequently, the long-term classroom research study seeks to improve student engagement and persistence through modeling classroom vignettes. Shriki and Lavy (2005) report that teacher education programs have little, if any, effect on teachers' beliefs about teaching and learning unless innovative approaches are introduced in a gradual and continuous manner. It is also argued by Dweck (2000), that understanding students' views of intelligence is an essential aspect for successfully changing classroom pedagogies. This presentation will report on design research projects developed collaboratively with teachers to investigate the implications for inquiry approaches. Ames (1992) argued that teachers can influence students' *approach to learning* through careful task design. Ames suggested that tasks should include meaningful reasons for students to engage, perhaps through being personally relevant, that it is desirable that students feel a sense of control, and that students experience a variety of task types, including those that foster social awareness. This complements suggestions from Gee (2004) that tasks be customized to match the readiness of the learner both for those who experience difficulty and for those for whom the core task is not challenging.

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THE TEACHING MODES: A CONCEPTUAL FRAMEWORK FOR TEACHER EDUCATION

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In this communication, I will present the conceptual framework that I constructed and used to conduct a teacher development experiment with a group of four Portuguese secondary mathematics student teachers over the course of their year-long student teaching practicum. This framework was built based on existing research and theoretical developments in the field of mathematics education. The major goal of the study was to trace and understand how the teaching modes of the participants evolved throughout their student teaching experience.

The three teaching modes that formed this study's conceptual framework – evaluative teaching, interpretive teaching, and generative teaching modes – were each comprised of teachers' interrelated classroom questioning, listening, and responding approaches (e.g., Davis, 1997; Nicol, 1999; Tomás Ferreira & Presmeg, 2005), which are associated with the following dimensions: teachers' key beliefs about mathematics and its teaching and learning, their dominant patterns of classroom interaction, and their levels of reflective thinking. The teaching dimensions related to the teaching modes have been shown to be critically important to educational research, especially concerning secondary mathematics teacher education and systemic change.

Under the overall research design of this study, the conceptual framework itself was investigated for its adequacy to analyze and interpret classroom teaching with the goal of improving mathematics instruction. The conceptual framework proved to be a useful tool for designing and conducting professional development programs that have a focus on classroom communication issues, and it showed to be a very promising resource for analyzing classroom teaching and stimulating reflective thinking with the goal of increasingly enacting a generative teaching mode. The generative teaching mode is the one that resonates with current reform documents and recommendations for school mathematics in many countries including Portugal.

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DIDACTIC DECISIONS: THE CASE OF REFLEXION

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In this communication, we present an ongoing research concerned with decisions teachers are led to take in their practice, especially those that aim at favoring learning of a given mathematical concept, which we call didactic decisions. The mathematical concept we chose for our research is the reflection (in France, this concept is known since primary school, but it becomes an object of a systematic study in Grade 6). Our purpose is to study the constraints that affect teachers' didactic decisions and the elements that support them.

It is generally agreed that the teacher's didactic decisions will be more suitable if s/he knows the conceptions her/his students have with respect to the mathematical concept to be learnt. For this reason, we started by modeling students' conceptions related to the reflection. Our model has been developed in the context of the cK¢ (knowledge, conception, concept) model (Balacheff 1995). Within this model, a conception C is defined by four elements: P, a set of problems in the solution of which C is involved; R, a set of operators that allow to transform a given problem in another; Σ, a control structure that guarantees a non contradiction of C; and L, a representation system that allows expressing the elements from P, R and Σ. The model views a learning as a process of passing from one conception to another. We suppose that for a given conception C, there are problems that can reinforce C, and the problems that can reveal the limits of C and thus destabilize it. This leads us to hypothesize that passing from an initial conception C1 to a target one Cn consists of several stages, each one being determined by problems aiming at the evolution of C1 towards Cn.

The main theoretical framework within which we are studying teachers' didactic decisions is constituted by a model of teacher's activity (Margolinas & al., 2005). In order to model teachers' didactic decisions, we have chosen a setting where the teacher is not interacting with students. In fact, an ordinary classroom setting would not allow us to access to the decisions referring to the same student. Therefore, we have created an artificial situation in which we have provided teachers with four students' productions and we have asked them to design a teaching sequence for each student. The analysis of the experiment is in progress.

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EVALUATING A LARGE-SCALE NATIONAL PROGRAM FOR INCORPORATING COMPUTATIONAL TECHNOLOGIES TO MATHEMATICS CLASSROOMS

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In 1997, the Mexican government launched a national program[1] to incorporate computational technologies (Spreadsheets, Cabri-Géomètre, Graphing Calculators, Logo) to the mathematics curriculum of middle-schools (age 12-15). It has attempted to incorporate results from international mathematics education research to the "real world". In particular, the pedagogical model is based on that of mathematical microworlds (Hoyles & Noss, 1992), with emphasis on student-centred activities, student collaboration, and a mediating role of the teacher (Ursini & Rojano, 2000). The program is now in a phase of massive expansion, having been implemented in 10 out of 32 states, with the ultimate aim of being used in every school in the country. Since 1997, many small associated researches have been carried out. We are now attempting a nation-wide evaluation. From a theoretical point of view, the complexities of evaluating innovative computational environments – especially when they aim to be systemic – are far from resolved; we therefore are facing a difficult task. We have been researching two facets: (i) the use and implementation of the tools, materials and pedagogical model (using a variety of quantitative and qualitative instruments: observations, interviews, questionnaires); (ii) students learning. For the latter, due to the large-scale of the study, we have needed to rely so far on traditional school-mathematics items and quantitative techniques (tests and academic scores). The most outstanding findings related to (i), show the difficulties that teachers have in adapting to the proposed pedagogical model: e.g. difficulties listening to students, in allowing students to discuss their explorations, in connecting the activities to other knowledge and in building conceptual understanding. Many teachers' lack of confidence has also led them to abandon the use of the tools. Thus, the activities and technology seem to have become instruments for learning in very few classrooms. Results related to students learning (ii) have been highly inconsistent, and there is evidence of a correlation to inconsistencies in teachers' implementations. But we also need to determine exactly what it is that students are gaining from the use of the computational tools (perhaps mathematical abilities rather than specific knowledge content) and design instruments that measure that more specifically.

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[1] The program is known as EMAT (Teaching Mathematics with Technology). Parts of its evaluation have been financed by Conacyt Research Grants No. G26338S and No. 44632.

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SYMMETRY: EQUALITY OR A DYNAMIC TRANSFORMATION?

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This study is a part of a research project which focuses on students' geometrical thinking about the concept of reflection as part of symmetry transformations. Rosen (1995) indicates that "symmetry is immunity to a possible change" (p.2). This change is considered as a transformation of a geometrical object that preserves the geometrical properties of the object. This seems to be related to the "concept of invariance" which promotes intuitive reasoning (Otte, 1997, p.48). Leikin, Berman και Zaslavsky (1997), extending the previous definition, assume that symmetry involves three elements: a geometrical object, its properties and a transformation. Considering these assumptions we suggest that reflection includes an initial geometrical object, a process of transformation and the produced object identical to the initial one. The equality of the initial and the produced object is not the sole criterion for reflection since the procedure of transformation demands the existence of one to one correspondence of the objects' elements.

In this research study 11 students of the Department of Primary Education of University of Patras participated in a volunteer base. We distinguished three main phases in the research process. The first concerned students' participations in a classroom teaching experiment. The plan was for the students to work on tasks involving the concept of reflection. In the second phase 11 students participated in clinical interviews. This research phase focused on the investigation of students' conceptions concerning the concept of symmetry. In the third phase students participated in a written task. This phase aimed to the investigation of the development of students' conceptions that was possibly invoked by the previous phases.

One of the main issues that emerged from the analysis of the data was the lack of the one to one correspondence of the objects' elements during reflection in students' responses. Students seemed to base their justification only on the equality of the initial and the produced geometrical object.

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TEACHING CHILDREN TO COUNT: DOES KNOWLEDGE OF THEORY MATTER?

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Three novice elementary school teachers were observed teaching counting to children aged 4-5 years. This was part of a larger longitudinal study that aims to investigate how mathematics teaching might be developed through reflection focusing on mathematical content. The teachers were in their second term of teaching. The feedback, during interviews which followed the observed lessons, were based on a framework termed the 'Knowledge Quartet' (KQ), developed by Rowland, Huckstep and Thwaites (2004). This offers a structure for the observation of mathematics teaching with a focus on the ways in which mathematics content knowledge, both SMK and PCK (Shulman 1986), contributes to classroom teaching practices.

The teachers' knowledge of pedagogical 'theory' and their conceptions of mathematics teaching affected the way they approached their teaching. One teacher made it clear that when planning her lesson she had started from her knowledge of the pre-requisites for counting (Gelman and Gallistel, 1978) and this was apparent in her teaching. This was coded as an instance of Theoretical Underpinning of Pedagogy [TUP] a subcategory of the Foundation dimension of the KQ. Evidence suggests this teacher has a strong *'learner-focused'* conception of mathematics teaching (Kuhs and Ball 1986). She encouraged children to use and articulate their own strategies and used these as starting points to reinforce ideas and enable them to progress. The other two teachers demonstrated more tenuous TUP in their teaching and discussion. They were able to remember some theoretical knowledge that related to their lessons when quizzed, but admitted they had not drawn on it in their planning, or been explicitly aware of its significance. Their lessons were teacher-focused and concentrated on teaching pre-determined procedures.

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STUDENT CONCEPTIONS AND TEXTBOOK MESSAGES: POLYGONS

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According to the theory of figural concepts and concept learning (Fischbein, 1993), the first objective of the study was to investigate the process of interaction between figural and conceptual aspects while defining a concept and using it to explain why figures are classes of objects. Furthermore, the second objective of the study was to assess the potential impact the instructional environment (textbook, in particular) might have on the process of interaction between figural and conceptual aspects.

The first objective was achieved through face-to-face interview that was carried out by six high socio economic eighth-grade 14 years old female students identified by their mathematics teacher as high, average, and low achievers in mathematics. The second objective was achieved through the content analysis of lessons on polygons and quadrangles from the 7th grade mathematics textbook used by the students that participated to this study. The topics covered in this present paper (polygons, paralellogram, rectangles, and square) were studied extensively at grade seven. The textbooks provided a blueprint for content coverage and instructional sequence, as was confirmed by the teacher.

The comparison of the three level students revealed that all students' definitions (except one high level student) related to rectangles, and average and low level students' definitions related to parallelogram were partitional. That is, they do not allow the inclusion of the squares among the rectangles, and squares and rectangles among the parallelograms. Furthermore, difficulty in distinguishing a concept from its name are quite prevalent among average and low achiever students. The investigation of the textbook indicated that these results are not surprising as mostly prototypical figures related to rectangle and parallelogram are used all through the geometric module. The word parallelogram is used with a figure where its auxiliary or subsidiary sides are oblique and the word rectangle is used with a figure which has two long and two short sides. Figures often provide an instantiation of a definition, not a general and rigorous proof. Students, however, do not see this distinction. Students seem to have spontaneously made up prototypes of the rectangle and parallelogram through figures adopted in the textbook. The difficulty in coordinating the figural and conceptual aspects was because they failed to capitalize on conceptual understanding to produce figural understanding but focus on figural understanding to produce conceptual understanding.

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STUDENTS' ERRORS IN TRANSFORMING TERMS AND EQUATIONS

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Teachers reported upper secondary (grade 9, 15 years old) students frequently having difficulties in transforming terms and equations, a task repeatedly covered by the curriculum in grades 5-8. Using the framework of Malle (1993), we wanted to quantify the frequency of eventual difficulties and compare them with the error types proposed by Malle. The results showed quite serious problems with some types of tasks and verified some of the error types. Some suggestions as to a change of the framework will be made.

THEORETICAL BACKGROUND

Malle (1993) developed a model for algebraic transformations and proposed the following error types:

1. Errors in information assimilation and information processing (2 sub-types)

2. Errors in recalling, processing, or using schemes (8 sub-types)

3. Disturbance of concentration

METHODOLOGY

We used written questionnaires consisting of 13 problems (numerical terms, equations, terms with variables) with increasing difficulty. All of the tasks have been taken from a regular textbook and rated "easy" to "medium" by the textbook authors. The tasks only contained algebraic structures that students have been confronted with several times in their regular school classes. The questionnaires have been answered by 180 students of grade 9. There was no time limit (average time was 40 minutes), the use of calculators was not allowed, and students were asked to do all side calculations, notes, etc. on their questionnaire.

RESULTS

32 % error rate in numerical terms, 37 % error rate in equations, 50 % error rates in terms with variables suggest massive problems of students in transforming terms and equations. Malles framework was generally useful, but several error sub-types proposed by Malle did not occur at all, some of the proposed sub-types were ambiguous and overlapping and would require redesign.

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MATHEMATICS WITH TECHNOLOGY: SEX VS. GENDER ATTITUDES

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The results of a previous study (Sanchez et al., 2004) suggested that all 12-15 years old middle-school students in Mexico, tend to have a positive attitude towards mathematics and towards mathematics taught with computers. However, boys tended to be significantly more self-confident than girls in their capability to do maths. The present study aims to investigate if and how attitudes and self-confidence in maths change through schooling. A sample of 870 students (12-13 years old) starting the first year of secondary school and using computers once a week to support their mathematics learning at school, will be evaluated during three years in order to see how these tendencies change. The AMMEC scale (Ursini et al., 2004) is used to measure students' attitudes: towards mathematics (sub-scale 1); towards mathematics taught with computers (subscale 2); self-confidence (subscale 3). The first year data were analyzed considering sex and gender differences. To determine students' gender traits, the Bem Sex Role Inventory (Bem, 1981) was validated for our population and used. The categories describing the gender traits are: masculine, feminine, androgynous (high masculine and feminine traits) and undifferentiated (low masculine and feminine traits). When sex differences were considered, significantly more boys than girls showed a neutral attitude towards both, mathematics taught with and without computers, while significantly more girls than boys showed a slightly negative attitude. Boys' self-confidence to do maths was significantly more positive than girls'. However, when gender was considered we found that the majority of both, boys with masculine, androgynous and undifferentiated gender traits (89.1% of boys) and girls with undifferentiated gender traits (22% of girls) tended to have a significantly more neutral than negative attitude towards maths. For maths taught with computers the majority of boys with masculine and undifferentiated gender traits (59.2% of boys) and girls with undifferentiated gender traits (22% of girls) tended to have a significantly more neutral than negative attitude. Concerning self-confidence to do math the majority of boys, independently of their gender traits and the majority of girls with androgynous gender traits (22.6% of girls) tended to have a significantly more neutral than negative perception of their capabilities.

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Grant from Fondo Sectorial de Investigación para la Educación. Project number: SEPSEByN-2003-CO1-22.

FRANCISCA USES DECIMAL NUMBERS: A CASE STUDY

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This report derives from an extensive qualitative research carried out at a night elementary school in México City. The case we present here is of a nineteen year old girl who was studying fifth grade at that school, and who solved problems in which the use of decimal numbers and their basic arithmetic operations were implied.

Bell, Swan, and Taylor (1981) said that the involvement of the problem solver in selecting the adequate operations is very important. We were guided by those research findings when posing problems for the questionnaire and the interview of our research study so that those problems could have a "familiar" end "real" presentation (according to Centeno, 1988, UNESCO, 1997) for the possibilities of the subjects that would take part in the research.

The girl was selected for this case study on the basis of responses to questionnaire applied to students of fourth, fifth, and sixth grades of that school. The main methodological instrument was the semi-structured interview (based on Valdememoros, 1998). In general, all the resources displayed while carrying out the case study were centered in answering this question: How does an adult who solves problems in which the use of decimal numbers and their basic arithmetic operations are implied assign meaning to this type of numbers? In the initial exploratory questionnaire Francisca made three mistakes. She based some answers on an inadequate interpretation of the decimal expression as if it were a positive integer. In the interviews Francisca showed evidence of assigning meaning to decimal numbers involved in the problems she solved by two means. First, by the concrete referents cited in the tasks, that is, the implied situations, experiences, actions, and relations that made them "familiar". The other means of assigning meaning to decimal numbers, the decimal point, and their place value was constituted by the mathematical referents she used to formulate processes of equivalence between decimal and natural expressions (preferably) or between decimal and fractional expressions (to a lesser extent).

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DEVELOPMENT OF NUMERICAL ESTIMATION IN GRADE 1 TO 3

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Although estimation skills are very useful in everyday life, generally not much attention is devoted to this subject in the primary school mathematics curriculum. This is unfortunate: not only are children usually very bad at estimating, recent studies suggest that there may be a link between estimation skill and general mathematical ability (Siegler & Booth, 2005; Dowker, 2003).

In the present study children of Grade 1, 2, and 3 (age 7.2, 8.3, and 9.2 respectively) were tested on four domains of estimation: estimation of the position of a number on an empty 0-60 number line (Siegler & Booth, 2004), estimation of numerosity, and estimation of area and length using a given reference (Pike & Forrester, 1997). Research questions were: 1) Are these four domains of estimation interrelated? 2) How does estimation skill develop over grades? 3) Is there a relation between estimation skill and mathematical ability?

Factor analysis showed that two different factors could be discerned: 1) estimation of area and length, and 2) number line estimation and numerosity estimation. This distinction also appeared in the development over grades: area and length estimation scores showed no improvement from G1 to G2 but did improve from G2 to G3, whereas number line estimation and numerosity estimation improved from G1 to G2 but not from G2 to G3. Math achievement scores correlated with 3 of the 4 estimation domains in Grade 1, but the relation between estimation scores and general math scores disappeared in higher grades. Results will be presented and discussed.

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MATHEMATICS EDUCATION AND NEUROSCIENCES (MENS)

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The Mathematics Education and Neurosciences (MENS) project is a four-year PhD study on how young children (aged four to seven years) solve mathematical problems. This project is unique in that it integrates neuropsychological with cognitive psychological results in order to better understand the development of mathematical abilities in young children.

THE PROJECT

The main research question of the MENS project is whether and how children who are prone to experiencing problems in the development of mathematical abilities can be identified soon enough for teachers to guide the learning process and minimize the ongoing problems. In order for this question to be answered, the cognitive psychological study will focus on the specific use and development of strategies for solving mathematical problems, as well as on how a child's profile of strategy usage compares to the child's results on the mathematical tasks. At a later stage, during the neuropsychological study, EEG signals of children performing mathematical tasks will be monitored. These signals will then be compared for children of varying mathematical abilities and varying strategy preferences. The present Short Oral Communication will focus on the strategy study and its implications for the rest of the project.

THE STRATEGY STUDY

The main purposes of the strategy study are to classify strategies and to cluster them into counting or geometrical methodologies. The strategies will then be related to number sense and spatial tasks, and to mathematical performance. The outcomes will contribute to developing tasks that are appropriate for five-year olds, and to understanding and refining definitions for number sense and spatial reasoning. The study is of a qualitative nature: the cognitive psychologist interviews the children as they complete the number sense and spatial reasoning tasks. The children are repeatedly asked how and why they solve a problem in a certain way. This invites discussions that lead to a comprehensive picture of the children's thinking processes.

After completing the strategy study, the children will take part in an EEG study that will compare neuropsychological results with cognitive results on strategy usage. The longitudinal component of the study will involve repeating this procedure in one and two years in order to be able to remark on the significance of strategy usage and its relation to possible problems in the development of mathematical thinking.

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SYMBOLIZING AND MODELING TO PROMOTE A FLEXIBLE USE OF THE MINUS SIGN IN ALGEBRAIC OPERATIONS

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Until now, many researches which addressed negative numbers focused on the idea of concept. They searched in the history of these numbers the explanations of the current students' difficulties. If these difficulties in using negative numbers cannot be challenged, some questions still come up. Therefore, we think that other theoretical points of view can be useful to enlarge our understanding of that issue. The social cultural approaches seem helpful because of their original point of view stemming from Vygotsky's principles. The primary focus of these approaches is signs and no more concept. It implies the idea that signs and meaning co-emerge. If we examine the negative numbers from this point of view, we can say that, on the one hand, the main characteristic of negative numbers is the presence of the minus sign, and on the other hand, the ways this sign will be used by the learners will be interrelated with the meaning they will attribute to it.

Previous works (Vlassis, 2004) showed that the students developed a lot of erroneous strategies in reducing polynomials when minus signs were present in the expressions. The students couldn't use these signs in a flexible manner: they were unable to consider the minus as a unary sign that is attached to the term, and, at the same time, as a binary sign used to make an operation.

In that context, we experimented with situations aimed at 8th grade students. The objective was to develop a flexibility in using mathematical signs in polynomial reductions. The sequence was based on the modeling perspective principles (Gravemeijer, 2002) and draws on the students' informal modeling to facilitate their adequate use of the signs. The analysis of the development of the sequence shows students' evident progress in the use of the signs because of the evolution of the oral and written discourse based on the goings and comings between the intuitive models and the formal symbols.

In the presentation, some extracts of the teaching experiment will be presented.

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AN ANALYSIS OF PRESERVICE TEACHERS' ESTIMATION STRATEGIES WITHIN THE CONTEXT OF WHOLE NUMBERS, FRACTIONS, DECIMALS, AND PERCENTS

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The goal of the reported research was to test and refine the Estimation Thinking Framework (Volkova, 2005), which was developed based on the analysis of the existing research on computational estimation, as well as on the data collected from student interviews in the 2005 study, and was grounded in Case's (1996) theory of cognitive development with regard to quantitative thought. According to Case's (1996) theory, children's cognitive growth proceeds through four stages: sensorimotor (ages 1-18 months), interrelational (ages 1.5-5 years), dimensional (ages 5-11), and vectorial (ages 11-18). Middle school students' cognitive growth occurs during the vectorial stage. The data collected for the Volkova (2005) study showed that middle school students, chosen from the top level of their mathematics classes, were, in fact, on different levels of thinking with regard to computational estimation. This finding is more in line with the van Hiele (1959/1984) model than with the Case (1996) model. Thus, even though the Volkova (2005) framework was developed to characterize middle school students' thinking in estimation, further research was necessary to explore the framework's applicability to estimators in other age groups. For this purpose, preservice teachers were selected as participants in the study, since their ages are beyond the range of Case's vectorial stage. Collective case study methodology was used to explore the issue. Interview data revealed that the descriptors of the framework are adequate in characterizing preservice teachers' thinking in estimation. The combined findings of the current study and the original Volkova (2005) study confirm that the levels of the framework are not age-specific and can be successfully used to characterize levels of thinking with regard to estimation in participants of not only middle school age, but also of ages beyond Case's vectorial stage (11-18 years of age) – particularly, preservice K-8 teachers. However, further research may be warranted to explore applicability of the framework to estimators of other groups (e.g., age, study major, profession, etc.).

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WHAT DOES IT MEAN TO INTERPRET STUDENTS' TALK AND ACTIONS?

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This study aims to examine the structural nature of teachers' interpretation of students' talk and action, while engaged in mathematical problem solving. The presentation concentrates on empirical analysis of the interpretation of 12 elementary school teachers who participated in an in-service workshop. The data was collected from individual interviews with each teacher that centered on episodes from her students' problem-solving sessions. The interviews were transcribed and analysed qualitatively.Analysis of the individual interviews centers on two dimensions: types of interpretations and focus of interpretations. The latter examined the focus in the teachers' interviews on the following three aspects in the students' work: cognitive, socio-cultural and affective. Data analysis indicates that all 12 teachers referred in their interview to all three aspects, but to different extents. Averages of the relative frequencies of the aspects, which the teacher refers to, are: 70% - cognitive aspect (range 49% to 87%); 19% - socio-cultural aspect (range 3% to 44%); and 11% - affective aspect (range 3% to 20%). Four types of teachers' interpretations of students' talk and actions were identified: (a) reporting – the teacher reconstructs what the students were saying/doing, (b) meaning – the teacher explains or justifies the students' talk or actions, (c) associating – the teacher connects the event to the students' educational/social/cultural history, and (d) inferring – the teacher connects the event to herself, to her role as a teacher, or to potential future actions of hers. The first two types – reporting and meaning – relate to the event itself, to the students' work, and are defined as internal interpretations, whereas the other two types reach out of the event to other contexts in order to interpret it, and are defined as external interpretations. The averages of the relative frequencies of the interpretations' types are: 11% - reporting (range 1% to 20%); 63% - meaning (range 41% to 83%); 17% - associating (range 4% to 28%); and 9% - inferring (range 0% to 20%).

A two dimensional analysis of the teachers' interviews (types and focus of interpretations) reveals four structural profiles of interpretation, defined as follows: (1) internal-cognitive interpretation – focus on the event (the students' work) with little connections to external contexts, together with an emphasis on cognitive aspects of the students' work, (2) internal-varied – focus on the event with reference to different aspects of the students' work, (3) external-cognitive – using external contexts to explain the event or inferring new insights, together with an emphasis on cognitive aspects of the students' work, and (4) external-varied – using external contexts with reference to the different aspects.

THE RESEARCH OF CO-TEACHING MATH BETWEEN EXPERIENCED AND PRE-SERVICE TEACHERS IN ELEMENTARY SCHOOL

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This article describes a qualitative study of the experienced and pre-service teacher co-teaching models in math class. In this paper, the researcher explores some experience of student teachers and classroom teachers in math curriculum. Generally speaking, co-teaching means teachers planned, taught and evaluated lessons together, and were encouraged to implement different teaching approaches. This research focuses on the teaching and learning interaction between 'experienced teacher' and 'pre-service teacher'.

Shulman (1987) pointed the teacher knowledge is included 'content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts and knowledge of educational ends, purposes, and values and their philosophical and historical ground**'**. This research focuses on content knowledge(CK), general pedagogical knowledge(PK), and pedagogical content knowledge(PCK).

There are four kinds of roles in this study—1. PCK experienced teacher: was graduated from Math Education College and had rich teaching experience. 2. PK experienced teacher: was graduated from General Education College and had rich teaching experience. 3. CK pre-service teacher: was studying in Math College and Teacher Education Program. 4. PK pre-service teacher: was studying in Management College and Teacher Education Program. This study used the narrative approach to analysis two directions: First, the PK experienced teacher and CK pre-service teacher co-teaching interaction; secondly, the PCK experienced teacher and PK pre-service teacher co-teaching interaction. Narrative analysis makes it clear that the process of co-teaching is inevitably.

The purpose of this study considers the effect of co-teaching as two kinds of models to improve the math teaching practicum in elementary school. Pre-service teacher and experienced teachers both appreciated the opportunity to reflect their math teaching. In this article, co-teaching arose from the different knowledge background of pre-service teacher and experienced teachers. The findings will supply math educator and teacher education for reference.

THE INFLUENCE OF TEACHING ON TRANSFORMAING MATH THINKING – THE DIVISION THOERY OF POLYNOMIALS

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The purpose of this study was to investigate the discrepancy of teaching ways the teachers considered to be effective and those which the students could have actually learned. The topic was division theory of polynomials and its related theorems in the 10^{th} grade. We employed a qualitative method with inductive analysis to do the research. Samples were 83 students of 2 classes and their math teachers (A and B) in a municipal high school in Taipei. Parts of the findings are the following:

Teaching emphasizing not on procedure or examples but on mathematics structures disadvantages students' thinking transformation: when A taught the division theory, he explained, with formal expressions such as $a = bq + r$ and $0 \le r < |b|$, the concepts of $f(x) = g(x)q(x) + r(x)$ and $\deg r(x) < \deg g(x)$. He emphasized on the difference between the limitations of remainders in integer division and in polynomial division, but gave no numerical example. Afterwards, students' activated thinking were "remainder polynomial (RP) < divisor polynomial (DP)", "$0 \le$RP< DP", "$0 \le$RP<$|$DP$|$". B did not emphasize on the mathematics structure, but used the division theory to express the numerical outcome of a long-division, and her students could better transform the limitation in integer division to "deg RP < deg DP".

Language can not successfully pass down thinking: when both teachers taught synthetic division, e.g. when they divide a polynomial by $4x - 3$, they used oral language to emphasize many times the divisor in the synthetic division was x - 3/4 but did not emphasize on using 3/4 in the operation. However, most students could activate the thinking of the latter, but only half of them knew the former.

Contradicting types of thinking coexist: Both A and B tried very hard to transform "remainder < divisor" to "deg RP < deg DP", but the transformed thinking of some students was still "RP < DP". Besides, though some students' thinking was successfully transformed to "deg RP < deg DP", many of them still had the thinking of "RP < DP". These 2 types of thinking coexisted in students' minds.

If teaching does not emphasize on the connection to the theory behind concepts, then students' thinking are less flexible: teachers used conclusions of the remainder theorem to teach the factor theorem. We took the example of dividing $f(x)$ by $x - 3$. The teachers substituted 3 into $f(x)$, and if $f(3) = 0$, then $x - 3$ is a factor of $f(x)$. They did not go back to $f(x) = g(x)q(x) + r(x)$, which resulted in lacking the connection to the theory of division. When students were asked to judge if $2(x-3)$ was a factor of $f(x)$, almost none of them connected to the division theory, and hence misjudged it.

SEARCHING FOR COMMON GROUND: MATHEMATICAL MEANING IN MULTILINGUAL CLASSROOMS

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In rural Eastern Cape classrooms where English is officially the language of instruction, the teachers and learners face the task of teaching and learning mathematics in their second language, which exponentially increases possibilities for ambiguity and the development of misconceptions

This paper describes a language and mathematics intervention with seven Grade 10 Xhosa main language teachers in the Keiskamma Hoek region. When teachers use mainly English for explanation, and simply re-iterate the concepts in Xhosa, opportunities for meaningful learner-centered discourse are limited, usually resulting in rote learning of procedures in English and little opportunity to develop formal mathematics talk and writing in English (Setati and Adler, 2001, Heugh, 2005).

The in-service teachers taking part in the intervention strategy are all studying part-time for a BEd (Further Education and Training) degree at the Nelson Mandela Metropolitan University (NMMU) in Port Elizabeth. The teachers also meet as a cluster once a month to workshop non-formal strategies to promote language acquisition and use in a mathematical context. They are videotaped in their classroom and they are interviewed twice a term. Standardized tests, assignments and examinations are used to track learner performance throughout the year.

The data generated to date by this work in progress suggest that emphasis should be placed on code switching which enables the learners to use their main language in order to aid cognition as has been noted by Setati and Adler (2001). These findings have implications for second-language learners in that, if they are not to become marginalized in mathematics because of limited language acquisition, appropriate, meaningful and integrated language and mathematics interventions need to be developed, implemented and researched in order to assist them to reach their potential in mathematics.

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ARE BELIEFS AND PRACTICES CONGRUENT OR DISJOINT? A PRE-SERVICE VIEW

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The worldwide trend of curriculum change in schools requires teachers to be effective in engaging learners in problems in context, but do teachers believe this approach will enhance learning, and do they endeavour to use a problem-solving approach in the classroom?

Research shows that there is not always a correlation between the beliefs teachers verbalize and their practice in the classroom (Lerman, 2002). This study investigates the attitudes and beliefs of five pre-service Post-Graduate Certificate of Education (PGCE) teachers at Nelson Mandela Metropolitan University (NMMU) towards the outcomes-based teaching style required for the Further Education and Training (FET) band for mathematics in schools in South Africa.

The students' beliefs were elicited using individual interviews, questionnaires and self-evaluations. They completed an instrument to investigate teachers' beliefs developed by Pehkonen and Törner (2004) and they were given an opportunity to showcase their creative teaching styles in videotaped lessons In this paper the views of one particular student, Sarah, are interrogated in order to gauge whether her elicited beliefs were constant or contradictory, and whether her expressed beliefs were mirrored in her classroom practice.

During the course of a single lesson, Sarah displayed various, and at times opposing, teaching styles. It appears that the motives of the teacher's activity do not necessarily depend on espoused beliefs, but emerge in the course of complex classroom interactions (Skott, 2004).

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WORKING MEMORY AND CHILDREN'S MATHEMATICS: IMPLICATIONS FOR THE CLASSROOM

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Children of 9 and 10 years of age were given a range of working memory and mathematical measures in a bid to unravel the complicated relationship between these two. The results are discussed in terms of an increased understanding of the role of working memory in children's acquisition of mathematical skills and possible educational implications.

A large body of evidence now exists highlighting the importance of working memory in children's mathematics. The precise nature of this relationship is not clear, with different components of the working memory model (Baddeley and Hitch, 1974; Baddeley, 1986) being implicated.

The large numbers involved and the potential for associative interference suggest that multiplication facts may be encoded as a learned phonological sequence (Dehaene, 1998). The nature of children's early mathematical experiences suggests that the ability to visualise and mentally manipulate sets of objects may be a good predictor of success with addition, and therefore help with linking addends and their sum in memory (Klein and Bisanz, 2004).

As part of a larger study attempting to unravel this complex relationship, primary school children were given measures of phonological and visual working memory, inhibition, addition and multiplication. The children's predominant strategy for the solution of the addition and multiplication problems was also noted. The results suggest that different operations are linked to different working memory components.

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MATHEMATICS EDUCATION REFORM IN THE UNITED STATES: THE CASE OF EIGHTH-GRADE

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For nearly two decades, there has been a concerted effort to reform mathematics education in the U.S. to improve the quality of students' mathematical knowledge. These changes are delineated in several reform documents (e.g., National Council of Teachers of Mathematics 1989, 2000). Yet, recent research indicates that teaching observed did not reflect the pedagogical practices believed to be essential to quality mathematics learning (Jacobs, et al., 2006). In this study, 3 U.S. schools drawn from data from the Learner's Perspective Study were analysed, for the purpose of examining how these classrooms are realizing the goals of the reform in school mathematics instruction. The analysis reported in this presentation attempts to provide descriptions of specific teacher practices but also insight into issues for the learners. Analysis of the data revealed that the difficulties faced in an attempt to reform mathematics education in the U.S. consists of more than just a lack of instruction that focuses on conceptual learning. As Nathan and Knuth (2003) state, the instructional practices "tend to focus on the mechanics of symbol manipulation, rarely addressing the conceptual underpinnings of those symbols and procedures" (p.180). One does not truly fathom how procedural the U.S. instructional practice is. In the analysis of these lessons the presentations are divested not only of reasons, but are also completely devoid of any richness of thought that allows the learner to reason and gain insight into what one is doing mathematically when using the procedure. Consequently, the computational or procedural fluency advocated by NCTM (2000) is not found in these lessons. The implications of these results raise issues for the learner and the development of their mathematical learning practices.

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A MODELING PERSPECTIVE ON PROBLEM SOLVING IN STUDENTS' MATHEMATICS PROJECT

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This case study explored modeling of problem solving in students' mathematics project by three 8th-grade students as they participated in model-eliciting activity, model-exploration activity, model-adaptation activity and journal writing in problem solving. This article describes the development of student's ability of description, modeling in their mathematics project through team cooperative discussion, verification, presentation and comment in an inquiry-oriented teaching environment, and using discourse analysis to team cooperative verifying process, evaluation and reflection that are excerpted from students' discourses and journal writings.

The form of students' journal writing used in this study included log, journal and expository writing (Strackbein and Tillman, 1987). Theoretically, "model development sequences" (Lesh and Cramer, 2003) , "three modes of inference making employed in sense-making activities" and "a past instance of semiosis can become the object of new semiosis" (Kehle and Lester, 2003) were used in this case study as foundation of writing text, discourse analysis and modeling activities.

METHODOLOGY

Essentially, researcher as teacher adopted inquiry-oriented teaching strategy. In practice, there is an open-ended problem task called Diagonal-Sandwich-Taking of 8×8 chessboard in this study, and the students of this case study developed a project from this task. We audio taped teaching and interview sessions, and adopted discourse analysis (Gee, 1999) to organize, analyse, classify, and consolidate the data which included discourse and writing text, then determined themes.

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DEVELOPMENT OF A QUESTIONNAIRE TO MEASURE TEACHERS' MATHEMATICS-RELATED BELIEFS

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This article reports the development of an instrument, the Teachers' Mathematics-related Beliefs Questionnaire (TMrBQ). Teachers' beliefs affect their teaching behaviors and decisions (Richardson, 1996; Thompson, 1992; Hart, 2002). Many questionnaires were developed for measuring teachers' beliefs. These questionnaires were used to measure teachers' beliefs more about the mathematics education (e.g., about the nature of mathematics, mathematics teaching and learning mathematics) and less about the other beliefs. The research efforts of using these questionnaires always showed that some of them have described consistencies between teachers' beliefs and their practice, whereas others have identified inconsistencies (Raymond, 1997). Some researchers indicated that the beliefs of a person are like belief systems (Green, 1971; Richardson, 1996) including not only the beliefs about the mathematics nature, teaching and learning, but also other beliefs such as the beliefs about the social context and the self. In order to know teachers' beliefs better, we tried to develop an instrument to measure teachers' beliefs based on belief systems perspective rather than belief only. A framework of teachers' mathematics-related beliefs used in this study to develop a questionnaire, the TMrBQ, was borrowed from Eynde, Corte, & Verschaffel's (2002) framework of students' mathematics-related beliefs. The questionnaire includes three subscales. Each subscale includes several items representing different beliefs positions. Participants rank each item on a six-point scale. The items were empirically based and described from the teachers' perspective, but the issues and subcategories covered were validated by panel experts. The latest version was administered to 172 in-service mathematics teachers. Combined conceptions and conflicting thoughts about the nature of mathematics education, social context and self were detected. The Cronbach α for the entire questionnaire is 0.96, for the three subscales, ranged from 0.87 to 0.96. Factor analysis proved the constructs of the TMrBQ. The results of using the TMrBQ for assessing teachers' beliefs will be discussed also.

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POSTER PRESENTATIONS

A COMPARATIVE ANALYSIS OF MATHEMATICS ACHIEVEMENT AND ATTITUDES OF MALE AND FEMALE STUDENTS IN BOTSWANA SECONDARY SCHOOLS

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This study investigated into the differences between male and female students' attitudes toward mathematics and their mathematics achievement in Botswana. The sample consisted of 156 senior secondary school students in Gaborone. Five dimensions of attitudes, namely; motivation, confidence, value, enjoyment and gender related beliefs were considered. It was noted that gender disparity in education is a global issue (UNMDG, n.d.). Aiken (1976) showed that mathematics achievement is influenced by attitudes. In Botswana, achievement, attitudes and choice of career among secondary school pupils were found to be different between male and female students and these differences have been attributed to factors such as gender stereotyping, class interaction, societal expectation, family expectation, socio-economic environments, among others (Kaino, 2002; Taiwo & Molobe 1994 & Marope 1995). The outcome of this research however showed that male and female students in Botswana secondary schools differed only in their gender related beliefs. The study recommended that parents and teachers should foster positive attitudes towards mathematics among their children and students respectively.

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LOGICAL-MATHEMATICAL LEARNING FOR STUDENT WITH DOWN'S SYNDROME

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University of La Laguna

In this poster we give the results of a study undertaken with 13 students with Down's Syndrome regarding their acquisition of logical-mathematical concepts.

Various research works on mathematics learning for people with Down's Syndrome show that when these students are given the opportunity to learn, integrate in schools and follow the same curriculum (having, of course, adapted concepts) as students without disabilities, they can achieve greater success than has been assumed until the present (Nye, et al., 2001; Monari, 2002).

The logical concepts involved are *classification*, *ordering*, *one to one mapping* and *quantifier*. Analysis was made of activities undertaken in Intelligent Tutorial designed to reinforce numerical concepts in infant education (Aguilar and others, 2005). This tutorial action was adapted to the characteristics of each student, there being activities at various levels of difficulty.

The results show that students with Down's Syndrome can acquire a certain degree of understanding of logical-mathematical concepts, as they had already achieved more successes than failures in non-routine tasks on these concepts. Our results ratify those results given by Nye et al. (2001) which show that Down's Syndrome students not only learn *processes* of memory but also can get to understand mathematical *concepts*.

Ordering was the most complex activity for all the students, demonstrating that this concept is in need of curricular adaptation. It was also difficult for the students to identify logical relationships of a higher order, that is, those relationships in which the connections between elements are not direct (in other words, the objects are not the same), but where the students need to abstract the relationships that links them.

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THE MATH FAIR AS A BRIDGE BETWEEN MATHEMATICS AND MATHEMATICS EDUCATION, THE UNIVERSITY AND ELEMENTARY OR JUNIOR HIGH SCHOOL

Mary Beisiegel

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As the mathematical counterpart to a traditional Science Fair, the Math Fair evolved in the Department of Mathematical and Statistical Sciences at the University of Alberta through collaboration between mathematics professors Jim Timourian, Andy Liu, and Ted Lewis and educators in Edmonton Public Schools. The purpose of the Math Fair was developed with recognition of problem solving as “common to most of mathematics, it is frequently an explicit part of the mathematics curriculum and it encourages skills in students that can be applied in all areas of their lives” (Lewis, 2002). The Math Fair represents a bridge between the university and the greater education community in Edmonton, as school children and their teachers have the opportunity to experience a day of mathematical problem solving at the university.

Students in Elementary Education at the University of Alberta are required to take the course Math 160: Higher Arithmetic. Along with traditional assessment measures, such as homework assignments and exams, the course is structured such that one component of the students’ final mark is based on the development and presentation of a mathematical problem or puzzle to elementary school students who come to the University of Alberta for the Math Fair. As a sessional lecturer in the Department of Mathematical and Statistical Sciences assigned to teach Math 160, I have had the opportunity to work with pre-service teachers in their Math Fair project development, planning, and presentation.

The purpose of the poster will be to present to a broad audience the planning of and my own experience with the Math Fair. The poster presentation will elaborate on the planning and logistics of the Math Fair, such as the coordination with elementary and middle schools and university services. It will also expand on the directions given to pre-service teachers for their project selection, development, and planning. The poster presentation will also incorporate pictures of past Math Fairs and include some puzzles that may be presented. Lastly, the poster presentation will include some recommendations for planning a Math Fair based on previous experiences.

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ONE TEACHING EPISODE FROM A LEARNER'S, AN OBSERVER'S AND A TEACHER'S POINT OF VIEW A CASE STUDY

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One of the central issues currently discussed is that of analyzing culture of the mathematics classroom (e.g. Seeger at al., 1998). In this contribution we will focus on verbal communication means as an integral part of communication in mathematics classroom. They represent key sources of cognition but at the same time they may also be the source of misunderstanding and mistakes. The presented case study is a part of a longitudinal research focusing on communication, its course, modifications of the language of mathematics under the influence of the teacher's instructional intentions, as well as specific communication means used in mathematics. It uses LPS methodology (Clark, 2001) for analysing the influence of teacher's behaviour on students' performances. In the Learner's Perspective Study (http://www.edfac.unimelb.edu.au/DSME/lps/) sequences of ten lessons are analysed, documented using three video cameras and supplemented by reconstructive accounts of classroom participants obtained in post-lesson video-stimulated interviews.

In this contribution, a specimen of analysis based on the LPS methodology is presented. The observation took place in the eighth year of compulsory education in the town České Budějovice. There were 28 participating students in the episode. The topic dealt with was "Using non-equivalent processes when solving equations". An analysis of the three sources of data (video recordings of the teaching episode, post-lesson interviews with learners and with the teacher) will be used to illustrate the differences in perception of communication from the standpoint of the various participants in the episode. A more detailed analysis of the whole of 10 consecutive lessons can be found in (Binterová, Hošpesová, Novotná, 2006).

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A FRAMEWORK FOR STUDYING CURRICULAR EFFECTS ON STUDENTS' LEARNING: CONCEPTUALIZATION AND DESIGN IN THE LIECAL PROJECT

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University of Delaware

John C. Moyer
Marquette University

This poster describes the framework we use in the LieCal Project, a Longitudinal Investigation of the Effect of Curricula on Algebra Learning in the United States. This USA National Science Foundation-funded project (ESI-0454739) investigates whether the Connected Mathematics Program (CMP) can effectively enhance student learning in algebra. We are conducting the LieCal project in 16 middle schools of an urban school district in the United States. We will follow the students and their teachers for four years as the students move from 6th to 7th, 8th, and 9th grades.

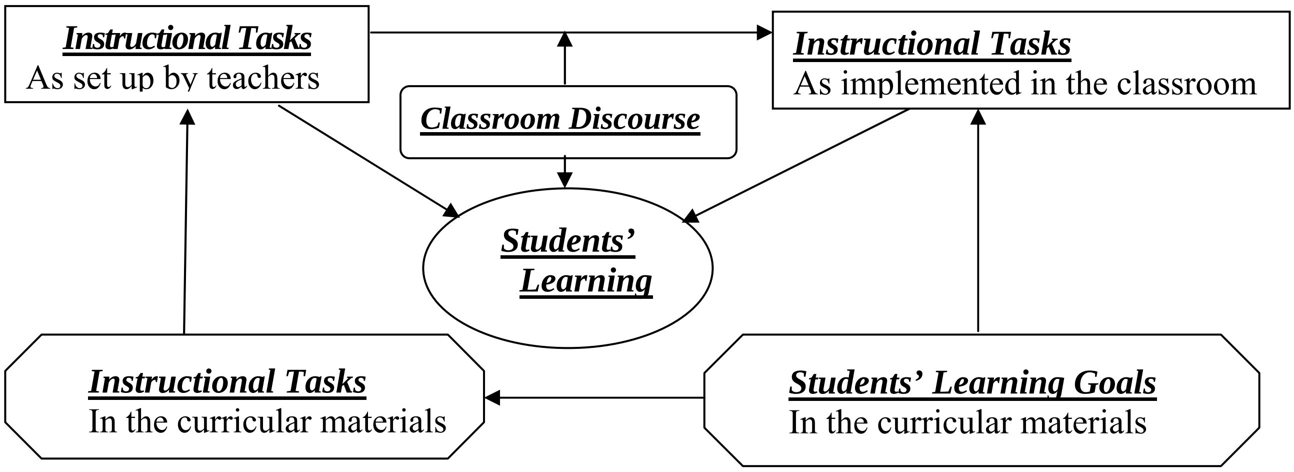

Figure 1: LieCal Research Framework

The framework for the project (see Figure 1) features the assessment of students' learning in a multi-dimensional manner (beyond both symbol manipulation and correctness), the examination of the fidelity of curricular implementation, and the identification of the important features of the curricula being studied.

Students' Learning. We use both state and researcher-administered tests to measure the learning of students using the CMP and Non-CMP curricula. These tests assess a spectrum of algebraic thinking, including computation, thinking, and reasoning.

Curricular Implementation**.** To track the fidelity of curricular implementation we collect four types of data: teacher logs, classroom observations, assigned homework, and pre- and post-instruction interviews.

Features of the CMP and Non-CMP Curricula. We analyse the algebra strands in both the CMP and non-CMP curricula from three inter-related perspectives: (1) goal specification, (2) content coverage, and (3) process coverage.

PRESERVICE ELEMENTARY TEACHERS' CONCEPTUAL UNDERSTANDING OF WORD PROBLEMS

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Word problems [WP] are an integral aspect of school mathematics. They can be used as a basis for understanding mathematics, applying mathematics, and integrating the real world in mathematics learning. However, whether WP are used to their potential depends on the teacher who plays a central role in shaping how they unfold in the classroom. This paper reports on a study of preservice elementary teachers that investigated their conceptual understanding of arithmetic WP and the nature and effect of engaging them in unpacking WP conceptually on this understanding.

The study was conducted with 20 preservice elementary teachers. They worked on arithmetic WP tasks that included: (1) constructing and comparing similar WP; (2) constructing and interpreting WP to demonstrate their sense-making of the meanings of four arithmetic operations (addition, subtraction, division and multiplication) and the WP situations; (3) unpacking a comprehensive set of 37 WP based on the research literature, which provides an exhaustive set of different WP situations, modeled by addition, subtraction, multiplication and division. Unpacking, as used here, involves identifying, interpreting mathematically, and representing in different modes all relevant aspects of the WP, in particular, the meanings of the four arithmetic operations and the WP situations. For example, participants modeled WP situations concretely, drew pictures of arithmetic processes, determined the meanings of the operations in the context of the WP situations, and represented the processes mathematically. Data, drawn from all of the participants' written work for all of the WP tasks, field notes of their in-class discussions, and their journals and post-WP tasks assignments, were used to obtain evidence of the participants' understanding of WP and how their understanding was influenced by the unpacking activity.

The findings showed that the participants initially had a limited way of understanding WP, e.g., focusing on the numbers, isolated words and related operations. However, engaging them in unpacking WP helped them to develop a deeper understanding of, e.g., structures of WP, WP situations, and alternative, real-world interpretations of the four arithmetic operations. The findings suggest that in order to deepen preservice teachers' conceptual knowledge of WP, learning opportunities should include, not only how to solve WP, but also how to analyze and represent WP situations mathematically, how to compare WP, and how to use WP to show interpretations of operations. An approach using a comprehensive set of simple structured arithmetic WP situations and a collaborative learning context driven by concrete representations of the WP and prompts to facilitate mathematical thinking can provide a meaningful basis to help preservice teachers to learn about, and how to unpack, WP conceptually.

MATHEMATICS EDUCATION AND SCHOOL FAILURE: A METHOD TO STUDY THIS RELATION

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The wider sociocultural environment in which a student is educated constitutes a main interpretative context for the development of his/her relation with the knowledge and the learning process(Lerman,1998). The study of students' school failure phenomenon in a well defined region with specific sociocultural and territorial characteristics allows the development of specific criteria for the designing of reparative actions in mathematics education. Our project[1] concerns the relation between mathematics education and school failure in Dodecanese area in Greece which is constituted of twelve islands with a wide population diversity as well with different educational structures in every island.

In this poster we present a research program aimed to investigate variables that they define the relation between mathematics education and school failure. The main phenomenon on which our research focused was the interruption of students' attendance during their compulsory studies and its relation to the students' education in mathematics. The project was realized according to the following phases:

a) Registration of the educational conditions in Dodecanese Islands (the educational institutions, students' leakage, students' achievement in mathematics, students' personal school-routes, teachers' age and training).

b) Investigation of the students' beliefs about mathematics education. We chose to address in students who attended in the 'afternoon schools' in which the attendance interruption was a frequent phenomenon.

c) Investigation of the relation among the students', their parents' and their teachers' beliefs about the school failure and mathematics education in different case studies (attendance interruption, gender, age, achievement in mathematics) (Jensen & Rodgers, 2001- Yin, 2002).

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[1] "*The interaction between mathematics education and school failure in island areas: interpretation and confrontation in the case of Dodecanese Islands*"(Pythagoras I, Aegean University, co-financed by 3rd CSF of EEC).

ENHANCING THE SEVENTH GRADERS' LEARNING ON EQUALITY AXIOM AND LINEAR EQUATION THROUGH INQUIRY-ORIENTED TEACHING AND INTEGRATED MATHEMATICS AND SCIENCE CURRICULUM

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This study aims to enhance the seventh graders' learning on the topics of equality axiom and linear equation while they were engaged in the integrated mathematics and science curriculum through inquiry-oriented teaching. The theoretical framework of the instruction design in this study is the "4E+A" learning cycle which is modified from the 5E learning cycle of Lawson (1989, 1995) and Trowbridge & Bybee (1990). The main consideration is that the "Evaluation" stage of the "5E" learning cycle only evaluates the outcome of student's learning without developing deep understanding of the process of student's learning. Therefore it seems reasonable to amend "Evaluation" into "Assessment", which can be applied to the students' reflection. The research method is based on the case study method, and the research subjects were thirty-five students selected from the seventh grade of a junior high school in Taiwan. They experienced three activities and worked on manipulating situations. The collected data included student's learning sheets, mathematics logs, transformation of class recording into protocol, classroom observation records, and teaching logs. There are three phases in the designed integrated curriculum. The first is guidance activity: to understand the elements of equilibrium phenomenon and further more link to mathematical concepts. The second, synthetic activity: to express an "equilibrium phenomenon" with a mathematical formula and further more construct the concept of equality axiom. The third, study activity: to solve a linear equation with the concept of equality axiom. After analyzing the collected data by descriptive statistics and qualitative method, the results reveal that over 85% of students could understand the equality axiom, and over 70% of students could solve a linear equation with the concept of equality axiom. It seems to show that the integrative mathematics and science curriculum through inquiry-oriented teaching is effective for enhancing seventh grade students to learn the concepts of equality axiom and linear equation.

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TO CONJECTURE THE STAFF DEVELOPMENT MODEL OF MATHEMATICAL TEACHER ACCORDING TO SPARK'S THEORY

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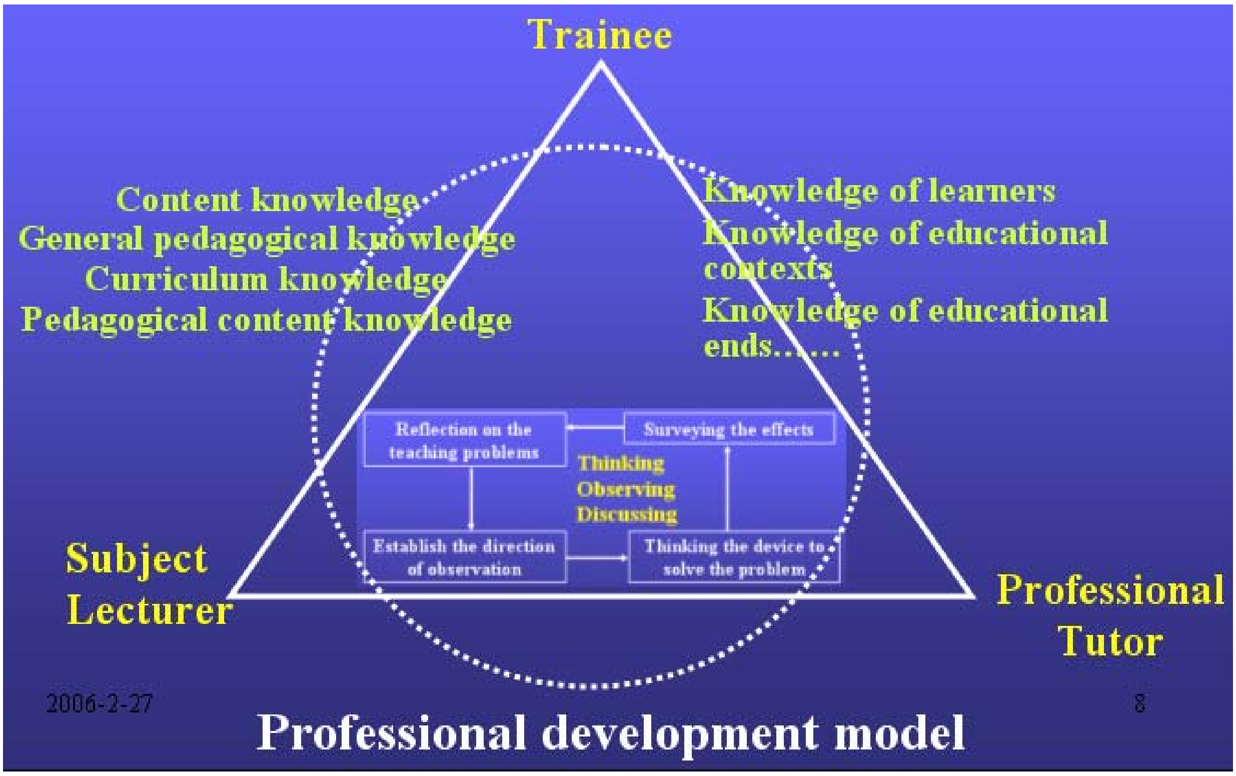
The one who lives in the society which advocate learning lifelong, he will face a learning process which is no end. Properly, there is no exception for mathematic (scientific) teachers. This study is to develop a faultless staff development model. First, to introduce the importance of staff development. Second, to discuss the meaning and intention of staff development. Third, to introduce the staff development models, and to attempt to expound a practicable staff development model. Fourth, to check the appropriate of the staff development model from mathematic (scientific) practical research. Finally, to bring up the oneself critically. The result of the staff development model of mathematical teacher was the following figure. This study hope to provide to the mechanism of teacher and related researchers.



Key words: Staff development. Teacher thinking. Observation. Discussion

DISCOVERY OF IMPLEMENTING TEACHING BY DISCUSSION IN MATHEMATICS CLASSROOMS

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The National Council of Teachers of Mathematics (NCTM,2000) has recommended that teachers need to convert the current setting of the classroom to a mathematics learning community. Interactions and discussions are ways for students to express their understanding, perceptions, reasoning during the learning processes, and help students to embrace their learning into their own knowledge (NCTM,1991). When teachers give students an opportunity to express their understanding, they are finding that students are learning mathematics more naturally (Cobb, Wood &Yackel, 1991). Many teachers reported using reform-oriented teaching practices and actively plan learning experiences that incorporate a range of processes including reasoning and communicating (Anderson & Bobis,2005). Therefore, to discover how to properly and effectively implement teaching by discussion in the mathematics classroom is worthwhile. In Taiwan, teaching by discussion is designed, introduced and implemented based on social constructivism. The researchers spent one year through observations and interviews with three fifth grade teachers who are using the method in their new classrooms. Two of the teachers are first timers, the other teacher has five years of such experience.

Research results shown from the first time teachers who use teaching by discussion method are: 1. The principle of teaching by discussion is easy to understand but difficult to implement; 2. One can only conduct teaching by discussion in a superficial way; 3. It is rather challenging for teachers to flexibly apply teaching by discussion techniques. For the experienced teacher the results are different: 1. They are able to conduct profound teaching by discussion; 2. They are able to alter students' discussion from merely presenting opinions to questioning, debating and proving; 3. It is still very important to follow the sequence of focus on a psychological, sociological, and scientific perspectives (Chung & Chu, 2003) to tune the students to the atmosphere of teaching by discussion. In summary, teachers need to fully grasp the sequel of the mathematical content as well as students' understanding and abilities so they can effectively implement teaching by discussion in mathematics classrooms.

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MODELING TEACHERS' QUESTIONS IN HIGH SCHOOL MATHEMATICS CLASSES

Sara Dalton, Gary Davis & Stephen Hegedus

University of Massachusetts Dartmouth

We examine teachers' practice of asking questions in a mathematics classroom and how it relates to student response, engagement, and how questioning can set the norm for classroom flow. Our video data consists of wirelessly connected classrooms as well as non-connected classrooms. It is not uncommon for teachers to ask a lot of questions in a mathematics classroom. We have looked at classroom videos of several teachers and when we plot the number of questions asked versus time, the plot is uniformly linear (r^2= 0.98). Based on our data, the constant rate of questioning often extends over an entire class period. When a teacher is asking, on average, 1 question per 10 seconds, what sort of time are students given to answer?

Mary Budd Rowe (1972, 1987) introduced the idea of "wait-time": the time from a teacher's question until the teacher speaks again. Her research showed that increasing wait-times to 3 or more seconds has a strong positive effect on student answers; student responses were longer and more accurate, the number of "I don't know" answers, and no answers decreased, more students volunteered appropriate answers, and test scores increased. We look at the descriptive statistics of wait times, and their distribution, for one of the video clips we obtained: these statistics are typical.

Our data collected relates directly to this work done by Rowe; the teachers we've looked at have a mean wait time of 3.1 seconds. The interesting part is that the data is a highly skewed distribution with a high percentage of wait-times that are 3 or more seconds. We have also found that the nature of the questions affects the wait time given by the teacher. In the data we examined, longer wait times were given for questions that related to the teacher's aims for the class whereas shorter wait times were given for questions which did not directly pertain to the objective of the class according to the teacher.

Two of the classes we observed were connected classrooms in which student work was shared in a mathematically meaningful way among the class. This shared student work can lead to complex student peer to peer discussions, student question generation, and teacher questioning drops dramatically (Hegedus, et al., 2006).

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MY ASSISTANT. A DIDACTIC TOOL OF MATHEMATICS FOR PRIMARY SCHOOL TEACHERS

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My assistant is a free access web page that helps primary school teachers in their use and comprehension of the official textbooks and other materials for the mathematics class.

"*My assistant. A didactic tool of mathematics for primary school teachers*" is a piece of software designed and made by faculty of the National Pedagogical University, with the collaboration of the Mexican Mathematical Society, that aims at helping primary school teachers to prepare their mathematics classes.

The construction of this software is a consequence of the results of the research project on "Goals and current contents of mathematics teaching in Mexico". This study was conducted in 1996 - 1999 with the support of the Mexican Mathematical Society, and was funded by the National Council of Science and Technology.

My assistant presents an analysis of the mathematics materials provided by the Ministry of Education (syllabus, official textbooks and cut-out books for children, books and cards of didactic activities for teachers); this analysis is directed to guide primary school teachers in their comprehension and use of the materials.

In seven different options for each school grade, *My assistant* presents: the mathematical content and the skills developed by performing the activities of each lesson and activity; suggestions for associating lessons and activity cards; segments from the teacher's manual about the didactic treatment of specific contents; activities, games, tools, documents, and links to relevant sites on the Internet.

Teachers may access freely *My assistant* through the web sites of the National Pedagogical University (www.upn.mx) and the Mexican Mathematical Society (www.smm.org.mx/miayudante) or directly in http://miayudante.upn.mx/.

The web site was developed using the Linux Operating System. It works on a database in MySQL and HTML, PHP and JavaScript programs. To permanently update this software, the site has an on-line updating system.

During 2005 *My assistant* received more than 95,000 visits, most of them from Mexico and USA, to more than 1,100,000 pages comprehended in the software, consisting both of the mathematics classroom materials for teaching mathematics in public education in Mexico and our contributions to them.

Examples will be provided in a pictorial format in the poster and a laptop of the different pages that are options of the software.

A MODEL TO INTERPRET TEACHER'S PRACTICES IN TECHNOLOGY-BASED ENVIRONMENT

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Equipe DIDIREM, Université Paris 7, France

Existing research on the use of technology in the teaching and learning of mathematics mainly concerns the potentialities for improving teaching/learning. There is a wide gap between these potentialities and the actual situation of classroom use of technology (Jones & Lagrange, 2003). The teachers' practices are to be considered as complex and consistent systems in which the use of technology introduces new factors. Our aim is to study the impact of these new factors on systems of practices, looking at perturbations that technology brings about and at possible re-equilibrations. Ruthven and Hennessy (2002) took into account teachers' view of successful use of technology in order to elaborate a model of their practices. This model helps to understand the consistency between teachers' view of the use of technology and pedagogical concerns, contrasting with the emphasis generally put on changes that technology should bring into epistemology and learning processes.

This poster provides results from a case study about teaching in dynamic geometry (DG) environment. We observed lessons of a teacher in its 7th grade class. In the rationales that this teacher gave for the use of DG we see clearly the way a teacher connects aspirations for improved classroom atmosphere and activity, with potentialities of technology through intermediate themes like in Ruthven's and Hennessy's model. In the actual classroom situation, the connection did not really work because of an underestimation of the need for students' understanding of DG operation. We see teacher's individual technical assistance to students as a way to re-establish the connection by 'scaffolding' students' use of DG.

Combined with classroom observations, the model can help to make sense of phenomena. It helps to understand how a teacher can connect potentialities of a technology to her pedagogical needs, overlooking mathematically meaningful capabilities. The observation shows what happens when the connection does not work: the teacher tries to re-establish the connection by becoming a technical assistant.

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THE GNOMON: AN INSTRUMENT. USING THE GNOMON THE HANDS DRAWS SYMMETRICALLY, BUT THE BRAIN'S ASYMMETRY IS IN CONTROL

Panagiotis Delikanlis

National & Kapodistrian University of Athens, Greece

Left- and right-handed high school children were asked to drop a perpendicular to a line through a point not on the line, using the gnomon.

The gnomon is a figure in its geometrical meaning, or a mechanical instrument, for drawing right angles. (Heath 1956).

Vision for perception and vision for action are subserved by two different "streams of visual processing", the ventral stream and the dorsal stream. The perception of the shape of the gnomon, the perception of the orientation of the line on which a perpendicular is dropping through a point not on the line, and the proper adaptation of the gnomon are actions like Perceptual Orientation Matching and Visuomotor "Posting" (Goodale and Humphrey 2001).

1	2	3	4	5	6	7	8

Table 1: Eight expected positions of left-handed pupils of posting the gnomon

1	2	3	4	5	6	7	8

Table 2: Eight expected positions of right-handed pupils of posting the gnomon

In the study is appeared the relative position of the hands during the performance of drawing. The expected percentage of left-handed pupils was 79% and the right-handed pupils was 69%. The inevitability of brain laterality imposes behavioral differences which must be considered in teaching of mathematics.

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THE TEACHER'S PROACTIVE ROLE IN THE CONTEXT OF WORD PROBLEM SOLVING BY YOUNG BEGINNERS IN ALGEBRA

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University of Liège, Belgium

The difficulties that students meet during the transition from arithmetic to algebra in word problem solving situations were observed in several studies that have notably pointed out the variety of the solving approaches observed as much in the arithmetical reasoning as in the algebraic ones (Bednarz, Radford, Janvier & Lepage, 1992). Other authors have focused on the teacher's role in the sociocultural approach. Gravemeijer, Cobb, Bowers and Whiternack (2000) consider that the teacher has a real proactive role when exploiting situations in the classroom: to fully participate in the debate by suggesting possible solutions, strategies, concepts, questions, to emphasize more some students contributions, to help students to re-formulate an unclear idea or process are roles a teacher can play.

In this study we investigate the issue of two students solving together a problem, focusing on the role played by the teacher in the process. If the students experience considerable difficulties in developing a productive mathematical interaction when they are alone, what if the teacher plays an active part in the process? Through the analysis of guided-by-teacher's interviews of pairs of students solving together a problem none of them had succeeded in solving on his/her own, we envisage types of attitudes of the teacher, organized into a hierarchy according to his growing level of implication in the solving process.

The poster will present the theoretical background of the study. Through interviews extracts, it will also envisage three types of teacher's attitudes organized into a hierarchy according to his growing level of implication in the solving process.

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STUDENTS' GEOMETRICAL THINKING DEVELOPMENT AT GRADE 8 IN SHANGHAI

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The van Hiele model suggests that in geometry education, students' thinking levels are sequential and hierarchical. In this model, the development of students' thinking is not dependent upon age or biological maturation, but on the instruction received (van Hiele, 1984). In the field of research in school geometry, one of the current major concerns is about how to improve pedagogical models and instructional strategies in order to help students to successfully progress from practical geometry to deductive geometry (Royal Society, 2001).

The main aim of this study is to investigate geometry teaching at the lower secondary school level in Shanghai, with particular attention to the relationship of the teaching/learning phases organized by teachers with students' thinking levels demonstrated in classrooms and examination papers at Grade 8 (students age 14). The study focuses on characterizing teaching materials and the interaction between teachers and students in classrooms. It also contributes to identifying effective instructional models and approaches used especially for teaching new geometric theorems in deductive geometry. In the study, two ordinary middle schools in two school districts in Shanghai are sampled. Classroom observation is used, together with data from teacher interviews, student interviews, and students' attainment in mathematics tests and homework.

Analysis of data from the pilot study suggests that an instructional model is consistently used by Chinese teachers in teaching new geometric theorems. An essential teaching strategy used by the Chinese teachers was mutually reinforcing visual and deductive approaches in order to develop students' geometric intuition in the learning of deductive geometry. Based on judgments of students' responses to, and explanations of, questions set by the teacher, students' thinking levels mostly appeared to be between van Hiele levels 2 and 4. Students' thinking in transition from level 1 to 2 was also identified by examining their learning outcomes in test papers. Further analysis of data from the study is continuing to focus on the relation between the instructional structure used by Chinese teachers and their students' geometrical thinking development.

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AN INTERDISCIPLINARY PERSPECTIVE ON LEARNING TO TEACH MATHEMATICAL WRITING

Helen M. Doerr   Kelly Chandler-Olcott   Joanna O. Masingila

Syracuse University

Over the past 15 years, numerous research studies have been published on the importance of mathematical communication for students' learning. Curriculum standards documents (e.g., NCTM, 1989, 2000) have included the processes of communication as part of the framework for the teaching and learning of school mathematics. This focus, in turn, has been reflected in many recent curriculum development projects (Lappan et al., 1998). These standards-based curricular materials provide students with mathematical situations that need to be interpreted through talk, texts, stories, pictures, charts and diagrams. This interdisciplinary study examines how middle grade teachers learned to address the literacy demands of mathematical writing when using such contextually complex curricula. Our methodological approach is the multi-tiered teaching experiment (Lesh & Kelly, 1999) which allows us to collect and interpret data at the researcher level, the teacher level, and the student level while generating and refining principles and products that are useful to researchers and teachers.

We describe two important shifts in the teachers' practices. The first shift occurred as they moved from seeing the curricular materials as barriers to seeing them as providing an opportunity for student learning. The teachers found that the curricular materials provided little useful guidance for making instructional decisions to support students in generating appropriate written responses to the problems in the texts. The teachers began to provide opportunities for writing and to develop "literacy scaffolding tools" that were built on our analyses of students' work. The second shift in the teachers' practices occurred as they saw that opportunities for writing needed to be addressed systematically, in ways that supported the students' development as mathematical writers over time and across grade levels. The teachers responded by creating unit-level writing plans to focus their instructional efforts. They developed a framework that could be used by all the teachers and that would enable them to make visible and able to share their rationales and specific instructional approaches. This study has implications for curriculum design and theorizing about teacher learning.

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MULTIPLICATION MODELS: AN UNEXPECTED ADVENTURE

Dmitri Droujkov — Natural Math®, LLC

Maria Droujkova — North Carolina State University

Creating a poster on twelve multiplication models started as a tiny side project, a visual toy for general audiences – an outgrowth of studies in multiplicative and algebraic reasoning of young children (Droujkov & Droujkova, 2005; Droujkova, 2005; Droujkova & Droujkov, 2005). However, it became clear that creating appropriate visual representations for models led to intricate theoretical questions, while offering these questions to other researchers and education practitioners turned into delicious debates and investigations producing more questions and discussions.

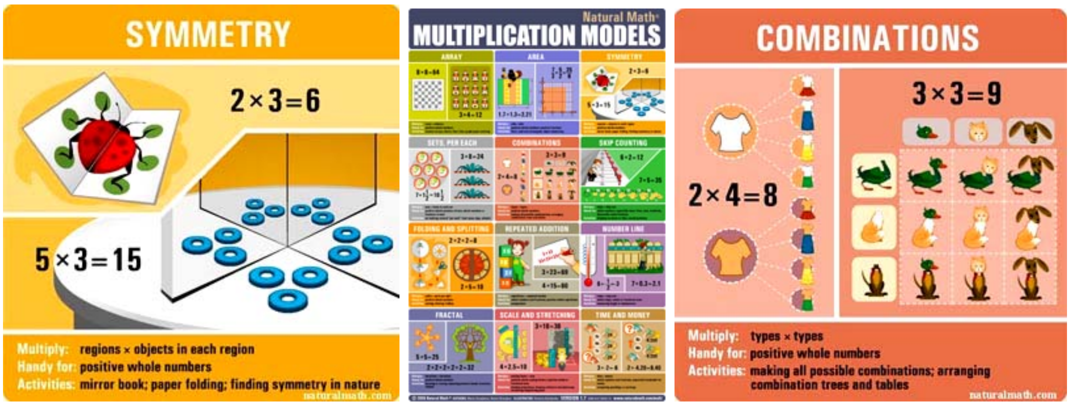

Figure 1: Multiplication models poster and close-ups of two models.

In our poster presentation, we are tracing this visual project through its tens of versions, and discussing the multiplicative reasoning problems that came up. How does the order of multiplicands in the formula correspond to each model? The answer depends on the researchers' country of origin. Do you start skip counting by fives from zero or from five (0-?, 5, 10, 15…)? Can you recommend the "sets" model for fractional numbers? Will children in Southern areas using Fahrenheit understand below-zero temperature well? This project is an example of deep research questions and interesting dilemmas generated by a "simple" elementary topic.

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QUALITATIVE GRIDS AND CYCLIC PATTERNS

Dmitri Droujkov — Natural Math®, LLC

Maria Droujkova — North Carolina State University

In the course of grid teaching experiments with children ages three to ten, we have developed a conceptual framework based on metaphor, three reasoning worlds, and a grid reasoning model (Droujkova, 2005; Droujkova & Droujkov, 2005). *Metaphor*, as a dynamic system where the target (new structures) emerges from the source (objects and actions), explicates teaching and learning mechanisms. The idea of *reasoning worlds*, qualitative, additive and multiplicative, explains relationships between the development of a particular domain, here grid, reasoning, and the development of operation thinking. The *grid reasoning model*'s dualities emerged as categories of children's actions and informed task design in further cycles of teaching experiments.

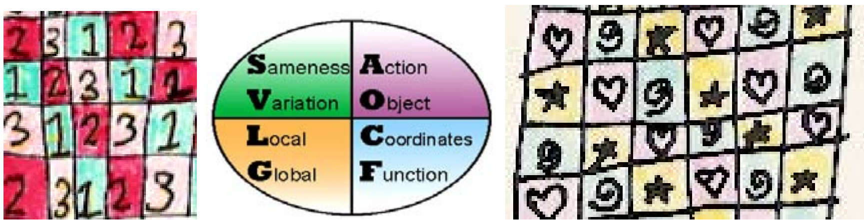

Figure 1: Grid reasoning model and two examples of cyclic pattern grids.

In this paper, we describe a curious grid type: 2D structures based on 1D cycles. Additional information afforded by the second dimension gives students a new point of view on the pattern, making apparent any iteration or reversing mistakes. When the 1D pattern size or structure is changed, there is a structural change in the grid that is so striking it commands children's attention, unlike the 1D pattern change by itself. While paper and pencil activities only promote making single grids and grid pair comparison, software we developed allows dynamic observation of changes as animated sequences of pattern evolution. The grid itself is a multiplicative structure, and cyclic grids additionally support multiplicative reasoning through play with iterable units (1D patterns) and with reversibility (filling the grid in the direction opposite to the original pattern) (Steffe, 1994). A research question pending further investigation is why children love playing with cyclic pattern grids so much.

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LEARNERS' INFLUENCE IN COMPUTER ENVIRONMENTS

Maria Droujkova — North Carolina State University
Dmitri Droujkov — Natural Math®, LLC

In the course of assessment of software at a large Southeastern US university, we developed taxonomy for analysing computer environments based on user influence in the environment. This table summarizes the categories that emerged. Note that most computer learning environments we found belong to the first category, where the role of learners is to take in the information and then to measure themselves next to the unchanging computer world. The poster will provide discussion and examples.

Representative examples: "like a…"	**Learner role and actions**	**Interactions among learners**	**World influence**	Interact with pre-made representations	Collaborate	Create representations
1a. …book or a video: PDF files, e-books, movies, pictures	Take in the information	Outside of the world	-	N	N	N
1b. …quiz: online quizzes, flashcards, multiple choice trees	Answer closed-ended questions	Compare success in answering	Measure yourself in the world	Y	N	N
2a. …dressing room: molecule constructors	Combine pre-made parts or change parameters to make new entities and cause world events based on entities	Share, compare, collect entities; collaborate on causing pre-determined world events	Create combinations, cause pre-determined world events	Y	Y	N
2b. …parametric equation: graph by parameters; linked representations software						
3a. Like paper and crayons: Cabri	Create your own representations and entities, cause emergent world events	Outside of the world	Emergent content, new world events	Y	N	Y
3b. Like a graffiti wall: blog communities, multi-user modeling software		Collaborate on creating emergent content		Y	Y	Y

Figure 1: User's influence in computer environments.

 2006. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, p. 386. Prague: PME.

ALLEVIATING OBSTRUCTIONS TO LEARNING

David Easdown

University of Sydney

Useful general principles are identified which help to alleviate obstructions to comprehension. The author will provide a poster containing novel illustrations and strategies of practical value to all teachers and students of mathematics.

What is Comprehension and How Do We Measure It?

If one imagines the transfer of information, through the medium of language, as an ocean, then on the surface lies the syntax (form) of language, and deep on the seabed lies the semantics (meaning). This leads to a measure of *depth* of understanding. The mathematics teacher needs strategies to enable the student to bring him or herself down from the surface to reside comfortably on the seabed of understanding. Illustrations of several principles, including those mentioned below are taken from the author's own experiences (Easdown (2006)), and also from Pinker (1994).

The Principle of Reflected Blindness.

If one is in a lit room at night without curtains, then one cannot see beyond the windows because of one's own reflection. In order to see what is outside the windows, one must first turn off the light. Seasoned mathematicians must employ strategies for *turning off the light* to see from the novice's point of view.

The Principle of Trivial Complexity

Choosing the quality and volume of detail are essential in effective mathematics teaching. One recalls poor Alice being flummoxed by the White Queen's question: "What's one and one and one and one and one and one and one and one and one?" Carroll (1978). The author will introduce the strategy of *spotlighting*.

The Halmos Principle: Every Good Theorem Has an Accessible Example

A carefully chosen example provides a direct link from the surface to the anchor on the seabed. To be able to relate to the most profound mathematics one must search for context and connections using the simplest elements (Halmos (1980-81)).

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KNOWLEDGE AND INTERPRETATION OF TEACHERS TO THE SCHOOL CONTENT OF PROPORTIONALITY

Homero Enríquez Ramírez Edda Jiménez de la Rosa Barrios

Universidad Pedagógica Nacional

This research analyses the knowledge of ratio and proportion among 65 elementary education teachers for the resolution of lessons on these topics contained in student textbooks. Mexico is one of those few countries where textbooks are both free and compulsory throughout the country.

We share the views of Lo (2004), who states that teacher difficulties may have detrimental effects on the student understanding of these topics. Preliminary results from our research agree with Post, Harel, Behr and Lesh (1991), Simon (1993), and Ma (1999) cited in Lo (2004), who reveal the limitations of teachers' level of understanding of the mathematics they have taught. This research offers specific information about what knowledge a group of teachers posses on the elementary school contents of ratio and proportion, how they interpret these, as well as the strategies they use to solve problems found in their textbooks.

Sixty-five 5th and 6th grade teachers (students aged 10 to 12) took part in this research, in a rural area of Oaxaca. The analysis of their strategies and procedures are based upon three data sources: a) paper and pencil reports of the solutions, b) video recording of teachers explaining their procedure, and c) interviews of teachers giving full details of their strategies and working procedures.

Our results document that the teachers turn to arguments and procedures similar to those reported in investigations of children and teens (Hart 1988, Karplus, Pulos and Stage, 1980), such as comprehension of proportion or dependency on a single strategy. We conclude that the teachers should diversify their didactic explanations and how they answer student questions.

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MATHEMATICAL FLEXIBILITY IN THE DOMAIN OF SCHOOL TRIGONOMETRY: COFUNCTIONS

Cos Fi

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This paper reports results of a study of advanced pre-service secondary school teachers' knowledge of trigonometry. The results show that the preservice teachers' knowledge of cofunction and other trigonometric ideas were not sufficiently robust.

Theoretical Framework: The study took careful account of the accumulated data and theories of teacher knowledge that point to the complexity of knowing (for example, Ball, Bass, & Hill, 2004)

The Idea of Co-Functions: Consider $\sin(x)$ and $\sin(\pi/2 - x) = \cos(x)$. Notice that $x + (\pi/2 - x) = \pi/2$ radians. Hence, the *sine* function and *cosine* function form a complementary function pair.

Methods, Data, and Analysis: In phase 1 of the study, 14 advanced pre-service secondary school mathematics teachers at a large university in the Midwest of the Unites States of America completed two concept maps from emic (CM1) and etic (CM2) perspectives. In phase 2, five of the 14 participants partook in two semi-structured interviews. Participants' relationship clusters of trigonometric functions were analyzed for accuracy and patterns of association.

Results and Discussion: Neither the concept maps nor the interview showed evidence of participants' cognizance of the connections and the use of the prefix co for the complementary nature of the co-function pairs. Participants confused co-functions with reciprocal functions, and inverse functions. See figure 1 and 2.

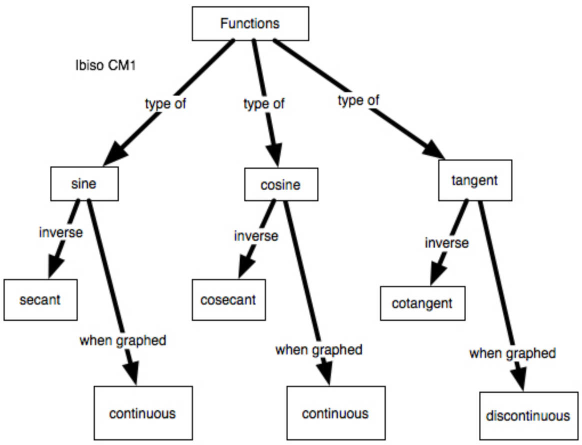

Fig 1: A participant's emic view

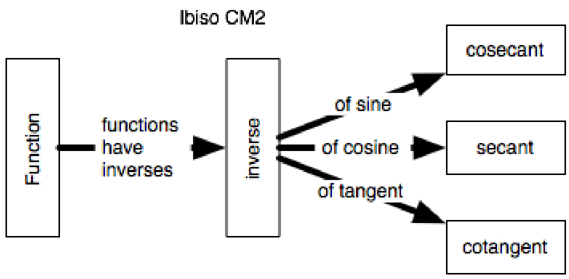

Fig 2: Same participant's etic view

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"MOVING FLUIDLY AMONG WORLDS": MULTISENSORY MATH SOFTWARE

Susan Gerofsky

University of British Columbia

The author, in collaboration with computer scientist Karon Maclean, is embarking on a project to develop software using multisensory human-computer interfaces for secondary school math learning. A key innovation is the use of *haptic* (tangible) interfaces along with audible/ musical and 2D and 3D manipulable visual interfaces. An important principle in the software design is the availability of fluid translation of mathematical representations from one sensory mode and from one computer application to another.

Forms of embodied math that allow students to move easily among virtually-embodied, physically present and abstract worlds of experience, and to work with a sense of purpose and aesthetic satisfaction are especially apropos for new generations of students growing up in a world of multitasking, convergent technologies, bricolage and increasingly available means of design and production.

• There will be built-in interactivity between sensory and algebraic representations, so that learners can change one representation and thus change all of them, highlighting their structural equivalence.

• Fluid movement between virtual, abstract and physically present representations will be facilitated by using CAD to make physical models.

• The software will encourage the importation of mathematical virtual objects into other applications: music authoring, animation, web authoring, desktop publishing and games authoring software.

• We will explore other kinesthetic interfaces (for example, riding a programmed exercise bike up and down the graph of a function).

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MAKING PRACTICE STUDYABLE

Hala Ghousseini and Laurie Sleep
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INTRODUCTION

In the poster presentation, we will articulate the need for professional development opportunities for mathematics teacher educators and argue that such professional development opportunities, similar to teacher professional development, should be situated in practice. However, we will also argue that simply situating professional development in practice does not automatically lead to learning; Professional developers must mediate the learning opportunities available in particular artifacts of practice and help learners become deliberate users of practice in other contexts, i.e., practice must be made *studyable*.

RESEARCH QUESTION

- How can practice be made studyable?
- What work can professional developers do to scaffold the study of practice beyond the mere use of artifacts of teaching?

METHOD

To investigate our research questions, we examined a particular case of practice-based professional development for mathematics teacher educators. Sixty eight teacher educators participated in a week-long summer institute during which they observed daily a laboratory class of 18 prospective elementary teachers taught by Deborah Ball, a professor at the University of Michigan.The teacher educators participated in discussions and analyses related to the laboratory class. One of the main goals of the institute was to enhance participants' understandings of Mathematical Knowledge for Teaching by providing them with opportunities to think about mathematics in new ways and to consider how elementary teachers need to know and use mathematics in their teaching. Another aim was to develop the participants' ability to observe and discuss teaching, as well as to analyze and use records of practice. Data was collected throughout the week in the form of videotapes, fieldnotes, and other artifacts.

FINDINGS

Examining data throughout the week enabled us to capture 5 categories of work that helped make practice studyable: (1) engaging the context, (2) navigating the terrain, (3) developing a disposition of inquiry, (4) providing lenses for viewing, and (5) providing insight into student thinking. Using our poster, we will articulate these categories and their implications for teacher education. Using our poster, we will present the information using a mixture of graphical, pictorial, and textual format.

COGNITIVE ROOTS FOR THE CONCEPT OF ASYMPTOTE

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In this research, we present an alternative approaches for the concept of asymptote. In elementary calculus teaching and textbooks, a (non-vertical) asymptote to a real function f is defined as a straight line r such the difference $|f(x) - r(x)|$ vanishes as x tends to infinity. As many authors have pointed out (e.g. Cornu, 1991), the concept of limit is deeply unfamiliar to students in the early stages of calculus learning and, therefore, not suitable as a starting point for the pedagogical sequence. We propose an approach based on the process of *global magnification* of rational functions: when a rational function is displayed on progressive larger windows, it gradually acquires the aspect of a polynomial function. The process is assisted by simple graphic software. The idea of asymptote appears as a straight line that mingles with the curve when it is zoomed out. Furthermore, this approach leads to a more general idea: the study of real functions asymptotical behaviour.

The research design is based on a qualitative study with a small group of first year undergraduate students in Brazil. We have found evidence that the notion of global magnification is suitable as cognitive root for the concept of non-vertical asymptote (in the sense of Tall, 1989, 2000). In fact, it is based on knowledge that is familiar for the students, and open ways for further theoretical developments. In this presentation, we will outline and illustrate the design of the approach, as well as its theoretical foundation, and report empirical results.

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FLEMISH AND SPANISH HIGH SCHOOL STUDENT'S MATHEMATICS-RELATED BELIEFS SYSTEMS: A COMPARATIVE STUDY

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Over the years there has been a growing body of research on students' beliefs and mathematical learning. Typically, however, scholars have been focussing on one or the other categorie od students'beliefs, e.g., motivational beliefs or beliefs about mathematics. Very few have analyzed the diferent kinds of beliefs in relation to each other, i.e. students' mathematics related beliefs systems. In this poster we will report a comparative study of Flemish and Spanish high school student's mathematics-related belief systems. Two research questions directed our investigation:

- Are Flemish and Spanish sudents' mathematicas-related systems constitudes along the same dimensions or do they have a different structure?
- What are the mathematics-related beliefs of Flemish and Spanish junior high students and how do they relate to gender, achievement level and track level?

We did a survey in wich 379 Flemish and 279 Spanish students were administered the Mathematics-Related Beliefs Questionnaire (MRBQ) that measures four major components of students' mathematics-related belief systems: beliefs about the role and the functioning of the teacher, beliefs about the significance and competence in mathematics, beliefs about mathematics as a social activity, beliefs on mathematics as a domain of excellence (Op 't Eynde, De Corte &Verschaffel, in press and Gómez-Chacón, De Corte, Op 't Eynde, in press). Principal component analyses and variance analyses were performed to identify, respectively, the internal structure of students' mathematics-related belief systems in Flanders and Spain, and their relations with gender, achievement level, and track level. The results of the principal component analyses indicate tha students' mathematics related beliefs sytems in Flanders and Spain are characterized by similar dimensions, but also indicate tha not all of them are strutured identical. Variance analyses on the Flemish data pointed to an overall effect of the three independent variables. In Spain the relation betwee gender, achievement level, and track level appears to be more complicated.

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CUBE NETS – COMPARISON OF WORK IN DIFFERENT COUNTRIES, THE RESULT OF THE EU SOCRATES PROJECT

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The world of arithmetic is firmly structured. Each number has its precise position in the set of all numbers which are strongly fixed by the relation > and operations +, -, *, /. The world of geometry is a community of individuals or small families and there is a large diversity in the linkages between them. From the didactical point of view, arithmetic is suitable for developing abilities systematically, and geometry is more suitable for abilities such as experimenting, discovering, concept creation, hypothesizing and creating mini-structures. Some 3D relationships of a cube's attributes are closely linked to the 2D set of cube nets and the whole topic is suitable for developing students' ability to construct geometrical mini-structures.

We present several results from the Socrates project 'Implementation of Innovative Approaches to the Teaching of Mathematics'. The project is built on the findings of our long term research aimed at spatial intelligence. The theoretical framework of the research is based on the Theory of Generic Model (Hejný, Kratochvílová, 2005) enriched by two ideas of procept (Gray, Tall, 1994; Meissner, 2001).

We proved that 7 year old pupils were able to discover all 11 cube nets if the metaphor "Dressing Mr. Cube" was used. Pupils from the age of 8 were given a sequence of increasing demanding problems and their solving processes yielded deeper understanding of the concept of a cube net and the structure vertex-edge-face of a cube. The results of the work by pupils, student-teachers and teachers from different countries involved in the project will be presented.

Coordinating institution is Charles University in Prague, CZ (D. Jirotková, M. Kubínová, M. Hejný, N. Stehlíková), and the project partners are: Aristotle University of Thessaloniki, GR (M. Tzekaki, G. Barbas), Kassel University, DE, (B. Spindeler, B. Wollring), University of Derby, UK (D. Benson, G. Littler).

Acknowledgment: The project IIATM was supported by Socrates-Comenius 2.1. programme. It does not necessarily reflect the position nor involve the responsibility of EC.

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FROM WORD NOTATION OF RELATIONS BETWEEN CONSTANTS AND UNKNOWN TO ALGEBRAIC NOTATION (PRETEST)

Jan Herman

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In 2004 the pretest proceeding from the Broin's (2002) research was carried out in Prague, with the aim to explore which phenomena attend formation of equational algebraic notation. In contradistinction to the Broin's research the pretest was closely focused on transition from the word notation of relationships between constants and unknown to formal algebraic notation. It explored the sample of 33 13-14 year old pupils before and after being taught algebra.

Phenomena which showed up brought the following answers the pretest questions and ask for further exploration:

1) *Do solving abilities precede abilities to formalize relations?*
   Differences between solving abilities and abilities to formalize relations were not expressive. It is possible to say that those abilities are connected.

2) *Does the ability to write down the algebraic notation of relationships between constants and unknown change after studying algebra?*
   The ability to write down the algebraic notation of relationships between constants and unknown amplifies after studying algebra.

3) *Does the quality of solution and formalization differ in case that the assignment of the problem contains words instead of numerals (two instead of 2)?*
   Pupils who were assigned a test which does not contain numerals were more successful in formalizing the relationships of types $a = x + b$ and $a \,.\, x + b = c$ than pupils assigned a test which contains numerals. This phenomenon occurs before and also after being taught algebra.

4) *Are pupils in the age 13-14 without algebraic experience able to write down verbally set relations between the unknown and constants in an algebraic way?*
   More than one half of the pupils without algebraic experience were able to write down relations corresponding to equations $a \,.\, x = b$ and $a = x + b$ by the equations. More than one third of the pupils were able to write down relationships corresponding to the more complicated equations $a \,.\, x + b = c$.

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MATHEMATICS AND COMMUNITY CAPACITY BUILDING: LISTENING TO AUSTRALIAN ABORIGINAL PEOPLE

Peter Howard — Australian Catholic University

Bob Perry — University of Western Sydney

A key element in enhancing the mathematics learning of Australian Aboriginal students is building community capacity, described as the bringing together of the community's knowledge, skills, commitment and resourcefulness to build on community strengths and address community challenges.

The *Mathematics in Indigenous Contexts* (1999-2006) project identified key aspects of meaningful engagement between schools and Aboriginal communities in the development and implementation of contextualised, relevant and connected mathematics curriculum through culturally appropriate teaching and learning strategies designed to enhance Aboriginal students' mathematics outcomes. The explicit involvement and engagement of Aboriginal parents and community in mathematics curriculum development was the significant factor in the project. Such engagement enhanced the school-community capacity through parent and community involvement in mathematics development and in the strengthening of mutual trust between Aboriginal and non-Aboriginal people. The project participants included: Aboriginal educators; Aboriginal parents and community people; primary and secondary teachers; teacher mentors; Aboriginal and non-Aboriginal students; New South Wales Board of Studies personnel and university mentors.

Data collected during the project included comments from Aboriginal people in relation to their engagement in collaborative mathematics curriculum development in a rural community in western New South Wales, Australia. Overcoming fear, valuing knowledge, ownership and responsibility, non-Aboriginal ignorance, awareness leading to changes, relevant learning places, and explaining were identified as issues relevant to reducing the sociocultural conflicts that many Australian Aboriginal and other Indigenous students encounter when learning mathematics.

Attending to the social and cultural meanings of mathematics learning requires learners and mathematics educators to participate in a discourse of exploration and engagement with cultures. The *Mathematics in Indigenous Contexts* project highlighted the importance of community-school engagement in bringing about the effective reform of mathematics learning and teaching for Aboriginal students, through enhancing community capacity.

The poster will provide a context for the *Mathematics in Indigenous Contexts* project, highlight the voices of Aboriginal people through presentation of their comments and offer examples of successful practice.

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A STUDY ON THE MATHEMTICAL THINKING IN LEARNING PROCESS

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The main aim of this study was to investigate the transformation of mathematical thinking in learning process of junior high school students. The topic of "solving quadratic equations by completing the square" (SQCS) was chosen. We employed naturalistic inquiry method while the observed classrooms were not at all manipulated by us. We used classroom observation, questionnaires, field notes, interview to collect data. Research samples included all the students of 4 classes and their mathematics teachers in the Taipei metropolis area.

After a review of literature, we proposed a thinking process model of students when learning new concepts. The model includes the components: "new stimuli → sensation → perception or figurativeness → judgement or reasoning → output". When students engage in any of the components, they connect to the old concepts, methods, or thinking. We found that when learning occurs, the essence of the components, as well as that of the connection, would change. These changes transform math thinking. For example: before teaching SQCS, when students faced "$x^2+2x=5$", they would have the figurativeness of "both items on the left-hand side have x", and perceived that "we can factor x out", so they would connect to the thinking of solving "$x^2+2x=0$" and came to the judgement of "factorization by common factors", then got "$x(x+2)=5$". However, after learning SQCS, for the same equation students had the figurativeness and perception of "the left-hand side of the equation is a polynomial lacking a constant", connected to the concept of "the formula of perfect square" and "finding the constant to complete the square", then finally reached the output of "$x^2+2x+1=5+1$". We also found that they could not develop the transformation by themselves without learning.

Another important discovery was that teachers' thinking process of SQCS was condensed and the pass of the process was very quick. In teaching, teachers presented some part of the thinking process as automatic actions, which made students produce "packaged procedural thinking" (PPT). The characteristic of PPT is that a student would pack all operational steps as a whole package, and once the package is formed, he/she cannot start thinking from the steps in the middle of the package. For example: the teaching of SQCS includes 7 steps, and we found that when a student made these 7 steps into a PPT, and when asked to solve $x^2-4x-5=0$, he/she would activate this PPT. While he/she encountered $(x+2)^2=9$ in the process of operating this PPT, he/she would have the output of $x+2=\pm 3$. However, if we directly gave students $(x+2)^2=9$ to solve, they did not activate the same thinking but, instead, multiply out expression into $x^2-4x-5=0$ before they tried to find the solutions.

A CASE STUDY ON PRE-SERVICE TEACHERS MAKING MATHEMATICAL MODEL OF VORONOI-DIAGRAM

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INTRODUCTION

Mathematical modeling is one of the important issues in the mathematics education. In particular, but the research about pre-service teachers modeling is not sufficient. Therefore, the purpose of the study reported in this research was to investigate the process of pre-service teachers making mathematical modeling and the potential problems in the process. Generally a model is a representation of reality. A mathematical model is a mathematical structure that can be used to describe and study a real situation. In this research, we accepted Lesh & Doerr's (2003) opinion that models are (consisting of elements, relations, operations, and rule governing interactions).

RESEARCH DESIGN

The research subjects were 26 pre-service teachers, who were separated into 5 groups. We designed the learning activities according to the form of mathematical modeling in our theoretical framework, and investigated the process of making mathematical modeling in the activities.

FINDINGS

For a completely real situation in the real world, we found that students development can be categorized into 2 types.

Type I: introducing mathematics concepts (G2 and G4)

Type II: not introducing mathematics concepts (G1, G3, and G5)

CONCLUSION

This research shows that it is not a easy process to make mathematical modeling form the real situation. 3 of the 5 groups did not introduce mathematics concepts or tools but dealt with daily knowledge instead. Therefore, we have to design proper guiding activities when we want to apply mathematical modeling in our teaching.

The other two groups used mathematics concepts. During their process of thinking they moved around between the mathematical world and the real world.

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A FAST-TRACK APPROACH TO ALGEBRA FOR ADULTS

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Many recent studies on approaches to algebra have focussed on introducing algebraic thinking to very young children, for example Lins and Kaput (2004), Carpenter, Levi, Berman & Pligge (2005), because introducing algebra in the secondary school has often not been successful. The approach described here, incorporates many of the theoretical aspects of introducing algebraic thinking but is also concerned with pragmatic considerations such as speed and efficiency because adults today are in a hurry. These adults are mainly the failures from a school mathematics which has concentrated on manipulative skills in a context-free environment. Thus it is important to begin with a context, as in the Realistic Mathematics Education program, Gravemeijer (1994), and the context familiar to and of interest to most adults is money, so that is our starting point.

Lins and Kaput (2004) identified two key characteristics of algebraic thinking:

> First it involves acts of deliberate generalisation and expressions of generality. Second, it involves, usually as a separate endeavour, reasoning based on the *forms* of syntactically structured generalisations …

It is these two characteristics that I have sought to unify. We begin with sentences involving money which lead to arithmetic expressions. Each sentence is followed by a similar one in which the numbers have been replaced with letters. This is a very basic approach to generalisation but very quickly the scaffolding is removed, the number sentences are dispensed with and the sentences become ever more complicated. Some sentences lead to expressions that can be expressed either as sums or as products and this leads to a context-generated statement of the distributive rule. This becomes our first identity and and students then recognise other identities from the rules of arithmetic. Throughout the course students generalise from numerical examples to create formulas. As well as practicing generalisation, this gives students *ownership* of the formulas.

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THE VALIDITY OF ON-SCREEN ASSESSMENT OF MATHEMATICS

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It has been argued that the introduction of onscreen National Curriculum tests (NCTs) in England is inevitable (Ripley 2004). Indeed, in 2004 QCA announced that by 2009 NCTs would be available on-screen and more recently (QCA 2005) Ken Boston of QCA predicted that technology will transform models of assessment and reporting, rather than simply be applied to existing models.

The Test Development Team at Edexcel, who develop the mathematics NCTs for England have develop test items that could be suitable for on-screen national assessments. The project aims to answers these research questions:

1. What are pupils doing when they tackle mathematics questions on screen?
2. How is their behaviour influenced by (a) the mathematics in the test, (b) the technology used by pupils to access and respond to the test and (c) the language of the test, including the conventions used in NCTs.
3. How are these three elements – mathematics, technology and language – related?

The poster will show examples of onscreen mathematics questions which use colour, animations and interactions. When there are opportunities I will be at the poster with examples of the questions for attendees to try. Pupils' attempts at two particular questions will be shown and analysed in light of the three research questions.

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AFTER USING COMPUTER ALGEBRA SYSTEM, CHANGE OF STUDENTS' RATIONALES AND WRITING

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Adopting the perspective of activity theory, Mellin-Olsen(1981) argues the case for the significance of a student's rationale for engaging in classroom activity. He identified two rationales for learning. These are the S-rationale (Socially significant) and the I-rationale (instrumental). And then, Goodchild(2002) adds the P-rationale (practice) and the N-rationale (no rationale). First, this study surveys the students who have any one of four rationales. After students using Computer Algebra System in class, this paper investigates the changes in students' rationale. Ball & Stacey(2003) say about what students should record when solving problems with CAS. They say that record need reasons, information, the plan, and some answers. Based on this theory, I taught students how to use Computer Algebra System and to write their answers in class. After given classes to students, this paper investigated changes in students' writing by assessment. The subject of this investigation was one class of an urban high school. First, I had an interview with their classroom teacher and mathematics teacher briefly. And then, I taught that class using Computer Algebra System. Those classes were carried out four times. It took one hour each time. I did assessment on students and investigation on students' rationales using a questionnaire before and after whole class. The result showed significant change about students' rationale and writing. Significant numbers of students changed to S-rationale. After those classes, several students had good attitude though they didn't have it before that class. And several students' writing also changed for the better.

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USING A SOCRATES' METHOD IN A COURSE OF MATHEMATICS EDUCATION FOR FUTURE MATHEMATICS TEACHERS

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Future mathematics teachers should be encouraged to consider what the famous dialogue between Socrates and Meno's slave implicates for teaching of mathematics(Fernandez, 1994, p.46) . This research is a case study on investigating the effects of mathematics education in using Socrates' method in a course of mathematics education for future mathematics teachers.

This study was conducted in 2005 to 44 university students(third grade) who entered the department of mathematics education. We took a course in curriculum that was required for the future mathematics teachers in my department. In this course, we began with reading the famous dialogue between Socrates and Meno's slave. And we've analyzed Socrates' questioning of Meno's slave and tried to understand what this dialogue is implicating in mathematics education.

All the participants in this course were divided into 11 groups. Each group designed a secondary mathematics class plan using Socrates' method and practiced their teaching plan by a performance(students' announcements). In the process of these activities, the teacher(researcher) laid an emphasis on enabling the future mathematics teachers to deeply understand Socrates' method and design to a good teaching plan. They tried to apply Socrates' method to today's school mathematics classroom situation.

Figure 1: Scenes of future mathematics teachers' announcements(videotaped)

The researcher observed the students' activities continuously, recording the results of the observation. Every content of students' announcements was videotaped. Individual records including students' thoughts about their teaching practice were written down. We collected them to use as research data. Through the analysis of research data, five effects of mathematics education in using Socrates' method for future mathematics teachers were induced. In the poster, we will summarize this research process visually by diagram and chart. And we will show the videotaped example of future mathematics teachers' announcement using Socrates' method. Also, our research results will be presented with research data.

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BEYOND VISUAL LEVEL: ASSESSMENT OF VAN HIELE LEVEL 2 ACQUISITION

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Transition to Van Hiele level 2 is characterized by a gradual primacy of geometrical structures upon the gestalt unanalysed visual forms and application of geometrical properties of shapes. Until now little research has been made to analyse more systematically levels 1 and 2. One of the questions posited in our research was:

- To what degree have secondary education students substantially progressed from the “visual” level 1 to the “descriptive-analytic” level 2? Special emphasis has been laid on the “application” of the properties of shapes by the students in novel situations, rather than mere “recollection”.

The main research instruments were specially designed paper and pencil tests consisting of six geometrical tasks (two for each topic of congruence, similarity and area, respectively). The problems were presented in a visual context as different as possible from that of a usual geometry textbook. The correct answer could be found either by some geometrical reasoning pertaining to level 2 or by a visual estimate, prone to perceptual misjudgement and easily leading to error. The test questionnaire was administered to 487 students (ages 15 to 23). This sample included subgroups of varying formal geometrical education (from lower secondary up to university mathematics). To examine more securely the strategies adopted by the students we conducted a number of interviews some time after the administration of the written test.

The analysis of the results showed significant differences between subgroups depending on their educational level and an overall tendency to cling to visual strategies despite extensive school geometrical training. For example 40% of the upper secondary twelfth grade students insisted on visual methods in *all* tasks, even after probing during the interview. It is worth noting that the acquisition of level 2 is not guaranteed uniformly even for the Mathematics Department university students: the percentage of them using visual estimates varies from 10% up to 65% depending on the geometrical difficulty of the task, according to their written responses.

These results seem to suggest that the geometry teaching methods in secondary education are not efficient helping the majority of students to surpass visual level 1 and appropriately formulate essential geometrical concepts.

VIDEOPAPERS AND PROFESSIONAL DEVELOPMENT

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The purpose of this poster is to present partial results from an ongoing research on the role of video papers into the professional development of mathematics teachers. We will analyse a collective work of ten school teachers while elaborating a video paper of their use and reflections about Cartesian graphs representing functions of linear movements, generated by paper and pen and by a sensor linked to graphic calculator[1].

From previous PME work (e.g. Ferrara 2004, Nemirovsky 2003) we assumed that video papers are a new gender of communicating and producing research into the field of education and Cartesian graphs of linear movements can be seen as conceptual metaphors. Building on these works we also took into account papers, discussions from ICMI Study 15 on teacher education, mainly strand 2, and focused our investigation in two questions: A) What do teachers learn from different opportunities to work on practice — their own, or others'? In what ways are teachers learning more about mathematics and about the teaching of mathematics, as they work on records or experiences in practice? B) What seems to support the learning of content? Teachers volunteered to participate in this study; they knew they will be videotaping their working with graphing calculators and a movement sensor, their involvement in a seminar as well as their reflections on watching those videos, choosing pieces for a video paper that will be produced as a CD and on the web.Analysis is based on embodiment theory (Nunez 2000, Edwards 2003) and Argumentative Strategy Model (Bolite Frant et all 2004). Partial results addressing both questions will be displayed for discussion during poster presentation.

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[1] Texas Instruments supports this research.

CONCURRENT CALIBRATION DESIGN FOR MATHEMATICS LEARNING PROGRESS INVESTIGATION

Chuan-Ju Lin, Pi-Hsia Hung, National University of Tainan, Taiwan

Suwei Lin, National Hualien University of Education, Taiwan

Developing a common metric is essential to successful applications of item response theory to the issue of Mathematics Learning Progress. In the context of evaluating the learning progress through the grades, calibration run is usually implemented for each grade and linked by using linking coefficients. However, numerous calibration runs would be required while evaluating long-term progress (e.g., grade 2 through grade 7) and linking errors would be accumulated through the linking process. Concurrent calibration could be an alternative to achieve the goal of efficiency and prevent the linking-error problem, but it may perish calibration results from the problem of group heterogeneity. The purpose of this study is to investigate the feasibility of concurrent calibration using the combined data from grade 2 to grade 7.

The data used in this study is derived from 2005 STASA Mathematics testing for the second- through seventh-grade students. We compare, for each grade, b parameters from separate calibration with those from two types of concurrent calibration under Three-Parameter-Logistic Item Response Theory: concurrent calibration over (a) 6 grades (grade 2 through 7), and (b) 3 grades (i.e., grade 2 to grade 4, 3 to 5, 4 to 6, and 5 to 7). Consequently, the correlations between separate calibration runs and concurrent calibrations over 3 grades appear to be high for each grade (0.946 to 0.992). Lower correlations are observed for concurrent calibration over 6 grades (0.577 to 0.880), and the lowest correlation happens for the grade-4 items. The inconsistent calibration results for the forth grade are mainly caused by the relatively difficult common items (i.e., b>1.5) in the grade-4 assessment. However, according to the content specialists in mathematics, concurrent calibration over 6 grades yields surprisingly reasonable proficiency progress of over grades. The results suggested that concurrent calibration over 6 grades may be feasible.

FURTHER INSIGHTS INTO THE PROPORTION REASONING AND THE RATIO CONCEPT

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Noelting 1980a used the ratio of orange ju ice to study students' concepts of ratio and he divided the difficulties of ratio concepts into three stages: intuitive stage, concrete operational stage, and formal operational stage. Streefland (1985) suggested that learning the concepts of ratio is a long process and the formula and calculations should not be taught in the early learning stage. Lybeck (1985) indicated that students tend to find the relationships between extensive quantities. In some other studies, the relationships between internal quantities are also important. But the relationships between internal quantities were used to solve the ratin problems by the students at the age of 12 to 20 (Noelting, 1980b). In the previous studies (Lin, 1984, Streefland, 1985, Feigenson, 2004), there was no research on the impact of representations and different approaches on students' learning the concepts of ratio. This study is to look at the difference of students' learning outcomes in two different kinds of representation (symbolic and graphic) of ratio problems and two different approaches of increasing/decreasing the difficulty of the sequence (easy to hard, hard to easy). We developed 4 instruments based on Noetling (1980) which were Graphic Easy to Hard(GEH), Graphic Hard to Easy(GHE), Symbolic Easy to Hard(SEH), and Symbolic Hard to Easy(SHE). There were 47 senior high school students (17 years old) participating in this study. The results show that different kinds of representation influenced the learning outcome. When testing the symbolic representation of the ratio, there is no significant difference in the order of difficulty and when testing the graphic representation of the ratio, the order makes a significant difference. But opposite results were found in the higher formal operational level.

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COURSEWORK PATTERNS BETWEEN MATHEMATICS AND SCIENCE AMONG SECONDARY STUDENTS

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This research focuses on the following research questions pertaining to the relationship between course sequences in mathematics and science.

1. How many distinct course sequences are there in mathematics and science among secondary students?
2. To what extent are students' mathematics course sequences related to their science course sequences?
3. What student-level and school-level variables influence the relationship between mathematics and science course sequences?

This research uses data from the 2000 (United States) High School Transcript Study (HSTS 2000). HSTS 2000 contains a nationally representative sample of 20,931 high school graduates from 277 public and non-public schools. HSTS collected authentic transcripts of high school graduates as well as basic information on students and schools. The primary statistical technique to examine the relationship between mathematics and science coursework is (advanced) multilevel analysis (see Raudenbush & Bryk, 2002).

This research documents a strong relationship between mathematics and science coursework patterns—patterns of mathematics coursework are significantly related to patterns of science coursework. Taking more advanced mathematics courses is related to taking more advanced science courses. Although this relationship remains strong even after adjustment for student-level and school-level variables, the more academic that students are in mathematics coursework, the more likely that student and school characteristics join in to discriminate students in science coursework. Results highlight a serious concern about mathematics and science coursework among graduates and call for state governments to prescribe not only the number but also the content of mathematics and science courses required for high school graduation. Results also indicate that mathematics coursework is necessary but insufficient to promote advanced science coursework and call for school career counselors to help promote better coursework of students in mathematics and science.

BEGINNING THE LESSON: THE INSTRUCTIONAL PRACTICE OF "REVIEW" IN EIGHT MATHEMATICS CLASSROOMS

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Our poster communicates an instructional practice we call "Review" as it is enacted in the beginning of a lesson over sequences of ten lessons (Mesiti & Clarke, 2006).

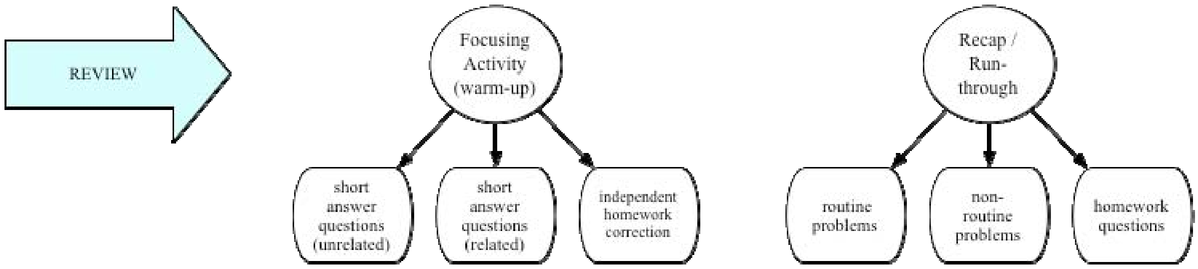

Figure 1. The instructional component "Review"

By examining the practices of eight competent mathematics teachers in Australia, the USA, Sweden and Japan, it has been possible to identify particular dominant components from which these teachers crafted the effective commencement of their lessons. However, it is in the crafting of "Iconic Sequences", alternative ways to begin a mathematics lesson, that we feel the expertise of the competent mathematics teachers is most visible and most readily related to the practices of other classrooms. We identify and illustrate two Iconic Sequences, *Familiarity Breeds Understanding*, and *Student-led Corrected Review* with the use of flowcharts, classroom transcripts excerpts and post-lesson interview excerpts. Both of these teacher-selected strategies were demonstrably associated with effective student learning.

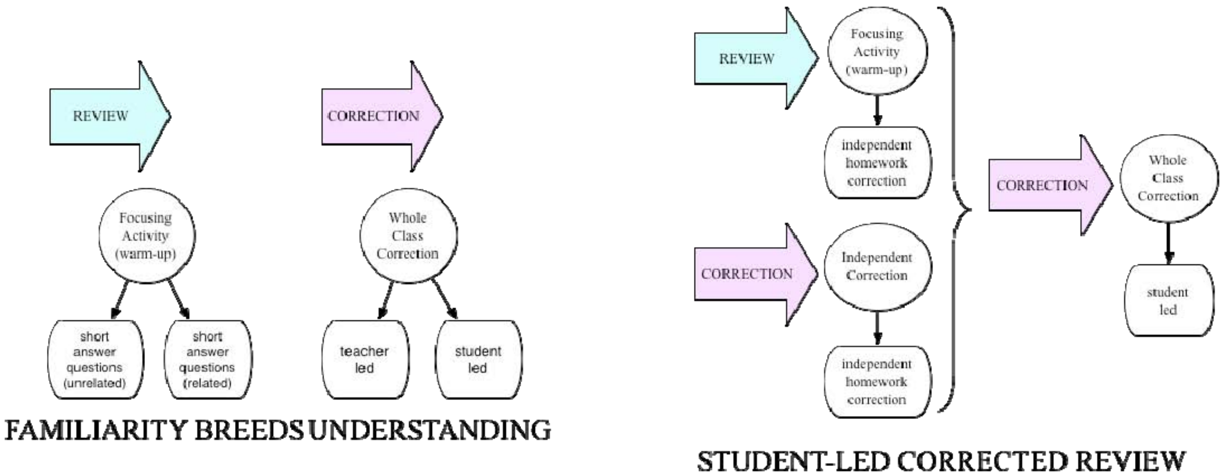

Figure 2. Iconic Sequences

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DEVELOPMENT OF WEB ENVIRONMENT FOR LOWER SECONDARY SCHOOL MATHEMATICS TEACHERS WITH 3D DYNAMIC GEOMETRY SOFTWARE

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This research sets a framework to develop teaching materials using 3D Dynamic Geometry Software (3D DGS henceforth) in lower secondary school geometry. It has also developed digital contents and a web environment for teachers to conduct classes of geometry using 3D DGS. (We have used "Cabri 3D" as the 3D DGS).

Promote "epistemological impacts" in classes: By using DGS, students will deliberate on the process of understanding figures, and also understand a proof (Hoyles & Noss, 2003, p.335). Especially, we can expect considerable improvement with "epistemological impacts" (Balacheff & Kaput, 1996, p.469) in the learning of space geometry by using 3D DGS, but use of DGS among teachers has still not propagated.

Framework of developing materials using 3D DGS: The scope of geometry in lower-secondary schools mainly includes understanding the concept of figures, constructing, and proving their properties (Viewpoint I). 3D DGS can open the possibility of transition of dimensions (Viewpoint II) By combining these viewpoints, we will set the framework of development of teaching materials using 3D DGS.

I / II	Understand	Construct	Prove
No transition of dimension	[A]	[C]	[E]
Transition of dimension	[B]	[D]	[F]

Web environment for teachers using 3D DGS: Focusing on [A] and [B] of the framework, we have developed entire plan of unit "Space geometry" in 7th Grade, teaching plans for classes, 3D DGS files for plans, flash movies that show how to use the files, and worksheets for students. Then, we have linked all of them and delivered them on the web. As a result, teachers can use it to plan and conduct their classes, and the practical expectations of the teacher as a user will be met. (http://www.schoolmath3d.org/index.htm)

Acknowledgement: This research is supported by Grant-in-Aid for Scientific Research), Ministry of Education, Culture, Sports, Science, and Technology (No. 17011031, 17530654, 1753065).

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THE EFFECT OF THE TEACHER'S MODE OF INSTRUCTION INSIDE MATH CLASSROOMS WITH A COMPUTER

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We describe some of the results of a project, sponsored by the Ministry of Education of Mexico, which has as its main objective to find out the changes that are propitiated in the learning and teaching of math, by the use of a computer and a projector inside the classrooms of elementary schools. For this purpose we designed 120 activities with the programming language Java. We based our analysis on several similar frameworks. In an article about cognitively guided instruction, Carpenter et al (2000) stress the importance of the teacher's knowledge about the mathematical thinking of children. The authors identify four levels of teachers' beliefs that correlate with their mode of instruction. In another study by Jacobs and Ambrose (2003) about how interviews applied by teachers to their students can improve their instruction by developing their questioning skills, the authors proposed a classification of the different modes of teachers' interaction during the interview (Directive, Observational, Explorative and Responsive). These define a profile of the teacher. The study consisted of three steps: i) An initial interview with each of the teachers to find out their beliefs about using computers in the classroom. ii) A didactical experiment in classrooms, consisting of eight sessions, using some of the activities designed. iii) A final interview with each of the four teachers to find out the changes in their beliefs and their impressions of the activities and the project as a whole.

The students' cognitive progress and the quality of their interaction greatly depended on the teacher's mode of working. For "Directive-Observational" teachers, we observed a low advance on the conceptual interpretations of students. However, for "Explorative-Responsive" teachers, we observed a significant advance, getting the students to form the concepts, to notice properties and to make some generalizations.

The "Directive-Observational" teacher used the activities mainly to practice or to verify the answers. This matches very well the first level of Carpenter et al. On the other hand, the "Explorative-Responsive" teacher used the activities to introduce and develop concepts, which corresponds with the two higher levels of Carpenter et al.

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MATHEMATICS EDUCATION IN RURAL SCHOOLS

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There is concern in Australia about outcomes of school mathematics in rural areas. Diminishing numbers of students undertake post-compulsory mathematics study; relatively low numbers of rural students participate in higher education mathematics courses; rural students have lower results on the PISA test than the national average; and it is hard to attract qualified teachers to rural areas. This is clearly an equity issue. An Australia wide research and development project is supporting mathematics in rural areas across Australia. This poster addresses issues noted during 15 focus group meetings of teachers, students, and parents in four Victorian rural schools.

While teachers, parents and students all spoke of lifestyle advantages of regional living, they are frustrated at the level of resources and support. The lack of a professional community of mathematics teachers was noted, as well as their relatively poor qualifications. Most students expect to undertake tertiary education, but in fact relatively few students from rural areas progress to university. Although not having completed secondary education themselves, many of the parents saw their children as ideally studying at university. Some were able to relate the needs of the farming community to a good grounding in mathematics. In contrast, the pull of a football career was noted to be great for boys in rural areas. It was noted by teachers that if parents had the financial resources to send their children to boarding schools in urban areas, then they were more likely to do well in mathematics, and it has been shown that these children do actually perform at a higher level on statewide testing than the mean scores for students in rural locations who attend their local schools.

When asked to talk about mathematics, students responded positively. Two factors seemed to feature in students' attitudes: satisfaction at success or feelings of competence; and appreciation of mathematics that was done in relevant contexts. The building of individual confidence was regarded as significant, as was catering for individual needs. There was, however, concern expressed by both parents and teachers that more rural teachers needed to have strong maths discipline knowledge. They commented that it was difficult for rural staff to attend special events such as excursions and mathematics competitions and that teachers are reluctant to attend professional development activities, as qualified relief teachers cannot be found.

Class sizes in secondary schools were regarded as both a benefit of rural education, and problematic. While providing increased opportunity for individualised instruction, some schools found the need to combine different year levels in the same mathematics classrooms disadvantaged the more advanced students. There is difficulty in maintaining a critical mass of academically committed students.

MATHEMAMATICALLY GIFTED STUDENTS' CONCEPTION OF INFINITY

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In this study, the conception of infinity by 19 mathematically gifted students, all of whom were 7th graders in middle school (aged 12), was examined. Gifted students answered three problems that asked them to compare the numbers of the elements of two infinite sets and to explain the reason. Their responses were analysed in five categories; "utilization of part-whole relationship," "all infinite numbers are equal," "infinite numbers cannot be compared," "utilization of 1-1 correspondence" and "utilization of properties of figure". As the students solved the problems without being given formal education on infinity, their conception of infinity as reported in this study can be seen to be spontaneous.

The result of the study shows that many mathematically gifted students did not apply the properties of the finite to infinity in the numerical context, and that some of them demonstrated a rather advanced conception of infinity by informally utilizing 1-1 correspondence. It is notable that 4 among 19 gifted students compared infinite sets utilizing 1-1 correspondence without having learned about the notion, although at an informal level. Utilization of 1-1 correspondence in comparison of infinite sets means the beginning of transition from "potential infinity" to "actual infinity", and from "infinity as a process" to "infinity as an object". Many studies (Fischbein, 2002; Monaghan, 2002; Moreno & Waldegg, 1991) asserted that the great difficulties that students had in dealing with infinity are resulted from that their actual life and thinking are mainly limited to finiteness. However, this study confirmed that mathematically gifted students tend not to apply the properties of finiteness when they deal with infinity. While, in the geometric context, many gifted students showed an ordinary conception of infinity by applying the properties of a figure in the comparison of infinite sets. There needs to be a more in-depth study on where such characteristics of the mathematically gifted students' conception of infinity come from.

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A GOOD MOMENT IN TIME TO STOP ‘SHYING AWAY FROM THE NATURE OF OUR SUBJECT’? TOWARDS AN OVERWRITING OF MATHEMATICAL STEREOTYPES IN POPULAR CULTURE[1]

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The relationship between mathematics and students is often tantalised by perceptions of tedium, difficulty, lack of creativity, elitism and unsociability. *Outside* schooling one influence on young people’s attitudes (and choice of field of study) originates in representations of mathematics and mathematicians in popular culture. While obviously our first priority needs to be with improving the students’ experience of mathematics *within* school, we also need to develop systematic ways of working against stereotyping and towards engineering more favourable, and accurate, representations. Within school we need to openly address these representations: question the inaccurate, undesirable ones and make the most of the rest. Outside school we need to work more closely and systematically with the often well-intended, but not always best-equipped, ‘outsiders’ who create those popular images. The main point I am putting forward here is that the timing for considering questions such as ‘if we were to work towards overwriting stereotypical images, what images would we replace them with?’ and ‘how receptive would lay consciousness would be to those?’ might be rather good. Regarding the former my argument draws upon accounts of mathematical experience by learners from across the educational spectrum. In these accounts mathematics emerges as a powerful way of reasoning, often expressed in highly technical-yet-effective language, and as a hugely rewarding intellectual pursuit and preoccupation. Regarding the latter I first introduce a tool which I call *Visibility Spectrum*. This ranges from Invisibility to Exotic Presence, Political Correctness and Normalisation/Acceptance and is borrowed from works on how other social groups (e.g. black, gay) that used to be/are under-(or inappropriately) represented have gradually gained a more acceptable type of visibility. I then use the Spectrum to examine examples of certain images of mathematics that have been painstakingly reinforced by popular culture (e.g. ‘madness/strangeness’, ‘intelligence as devious artifice’, ‘ivory-towerism’) – thus establishing that the representation is still far from desirable. However, I then conclude, recent signs (e.g. from press, film, TV, theatre, literature and popular music) seem to suggest a shift towards a more intelligent and subtle representation – signs which I exemplify and which, I propose, our community needs to make the most of. Systematically, relentlessly and with gusto!

[1] Based on a Public Lecture for the Norfolk branch of UK’s *Mathematical Association* (02/02/2006) and Nardi, E. (in press, 2006) A good moment in time to stop ‘shying away from the nature of our subject’? Towards an overwriting of mathematical stereotypes in popular culture, *Mathematics Teaching 198.* Also available from late 2006 at: http://www.atm.org.uk/mt/archive.html.

TOWARD REAL CHANGE THROUGH VIRTUAL COMMUNITIES: IMPLICATIONS FOR MATHEMATICS TEACHER EDUCATION

Kathleen T. Nolan

University of Regina

This presentation discusses recent research on creating and sustaining a 'virtual' community of practice with secondary mathematics interns. In addition to exploring how virtual communities can address real theory-practice transitions, the study also discusses implications for a new internship field supervision model

OVERVIEW

In a recent case study with secondary mathematics pre-service interns (Nolan, forthcoming), attempts were made to mentor the interns as they negotiated theory-practice transitions from university courses to school classrooms. The mentoring fell short, however, in that it focused on the pre-service teachers' individual experiences rather than recognizing the benefits of participating in reform-oriented mathematics communities (Van Zoest & Bohl, 2001). In response to this realization, a 'virtual' community of practice was created. Through pictorial and video representation, this poster presents the challenges and implications of creating and sustaining an online reform-focused mathematics community between a faculty advisor and her interns. By introducing 'virtual' (desktop video conferencing) visits with the interns, the faculty-intern supervision process took on a whole new meaning as the students became part of a community, discussing and grappling with the many theory-practice transitions facing them at their various schools, and with their various cooperating teachers and students. The virtual community illustrated the possibilities of expanding conversations simultaneously into several office and classroom spaces, erasing physical boundaries that might normally function to marginalize new theory and perpetuate old practice. The poster also discusses implications for a new internship field supervision model that incorporates communities of theory-practice transitions in mathematics teacher development.

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POWERFUL IDEAS, LEARNING STORIES AND EARLY CHILDHOOD MATHEMATICS

Bob Perry — University of Western Sydney
Elspeth Harley — South Australian Department of Education and Children's Services
Sue Dockett — University of Western Sydney

The Southern Numeracy Initiative (SNI) was established in 2003 in high schools, primary schools and preschools situated in the southern suburbs of Adelaide, South Australia. Its general aim was to improve mathematics and numeracy outcomes through a sustained, collaborative program of professional development and action research, particularly in the areas of pedagogy and assessment. This paper reports work done with preschool educators as part of SNI. It traces how 'powerful ideas' in mathematics were identified in current preschool practice, how they were linked to the Developmental Learning Outcomes in the mandatory curriculum documents and how the technique of learning stories, (narrative assessment) was established as a valid assessment regime compatible with key principles of preschool education.

The South Australian Curriculum, Standards and Accountability (SACSA) Framework organises its preschool offerings according to the Developmental Learning Outcomes (DLOs) including 'Children develop trust and confidence' and 'Children are intellectually inquisitive'. In the SNI preschool project, these DLOs are linked to powerful mathematical ideas through a numeracy matrix consisting of groups of pedagogical questions to provide a basis for teaching, learning and assessment of mathematics in preschools.

The process of assessment being pioneered by the SNI preschool project links a learning story or narrative assessment methodology with the numeracy matrix to help develop a coherent and holistic picture of young children's mathematical development.

The SNI has resulted in early childhood educators actively considering how they can improve their children's mathematics learning—and having the confidence and capability to do so—while maintaining the very important principles on which early childhood education is based. The mathematical power of young children is being recognised, celebrated and assessed in ways that are valid and reliable while, at the same time, are relevant to the children's contexts.

In this poster, examples from the numeracy matrix with related learning stories will be presented so the progress made by the early childhood educators participating in the SNI can be celebrated and shared.

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TRACING THE DEVELOPMENT OF KNOWLEDGE ABOUT MATHEMATICS TEACHING: PARTICIPATION IN VIRTUAL AND PRESENCIAL DISCUSSION GROUPS

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The aim of this research was to analyse how the participation in virtual al discussion groups influenced the elementary teachers' learning in a professional development programme with focus on mathematics education. The elementary teachers had to analyse and reflect on cases from mathematics teaching in presencial sessions and online asynchronous discussion groups. We traced the development of elementary teacher knowledge describing how they participated in these discussion groups. The results indicate how reflecting on mathematics teaching may help primary teachers to develop a more complex view of teaching and learning mathematics.

THE MAIN THEMES OF THE STUDY

Wenger (1998) argues that through the participation in communities of practice and by negotiation of meaning people gain experience about the world, and addition Wenger situates the meaning in a process that involves participation and reification. However, generating productive interactions and establishing communities of practice as a means for enabling mathematics teachers to develop new practices as teachers is a difficult task.

We designed asynchronous discussion groups and presencial discussion groups to establish communities of practice whose focus was the reflection on mathematics teaching in a professional development programme centring on mathematics education (van Huizen et al, 2005). We considered in the discussion groups (a) the focus of reflection (e.g. the mathematics learning), and (b) the way in which elementary teachers participated.

The focus of analysis was on the type of interactions and the conversational chains generated (sequential graphics which show information of the order of participation and the type of interaction) as an aspect of the reification process of knowledge and beliefs. The results indicate how reflecting on mathematics teaching in virtual and real discussion groups may help primary teachers to develop a complex view of teaching and learning mathematics .

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PRESCHOOL CHILDREN´S NUMBER SENSE

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Number sense refers to a general understanding of numbers and operations, including the ability to make judgments and inferences about quantities; and knowledge of the effect of using a number as an operator on other numbers (Sowder, 1995; Sowder & Schappelle, 1989, Yang, 2003). Considering the relevance of number sense in mathematics education and the importance of preschool years for the learning of mathematics at elementary school, this study investigates preschool children's number sense when solving addition and subtraction tasks. Sixty Brazilian preschool children from different social backgrounds took part in this study which was comprised of two tasks. Task 1 investigated whether the child understood the effect of operations on numbers, i.e., that adding increases and subtracting decreases a quantity. Task 2 investigated whether the child understood the effect of inverse operations on numbers; that is, whether the effect would lead to an increase or reduction in the initial quantity, or whether the operations, being the inverse of each other, would not alter the initial quantity. The data were analyzed according to correct responses and justifications given. In Task 1, the number sense based explanations expressed a general idea according to which when the initial quantity increases, the operation involved is that of adding, and that when the initial quantity decreases, the operation used is that of subtracting. In Task 2, the number sense based explanations involved the ability to understand the inverse effect of one operation over the other. The results revealed that besides differences between the two tasks, preschoolers show number sense strategies with regards to addition and subtraction. Differences between social classes are discussed (CNPq).

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A FIRST APPROACH TO STUDENTS' LEARNING OF MATHEMATICAL CONTENTS[1]

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The global aim of our research project is to explore, into a community of inquiry constituted by Mathematics Teacher Educators/researchers and Secondary Mathematics Teachers/researchers, the learning processes of mathematical contents at 'Bachillerato' level (students from age 16 to age 18).

In particular, in this poster we shall report some preliminary results related to the identification of metaconcepts in mathematical contents in the Spanish Curricular Orientations and textbooks related to that level. We contemplate these textbooks as a reflection of the contents that different authors consider adequate for the mentioned level. We focus on Definition, Proof and Modelling, among other. We consider that these metaconcepts are very relevant in students' mathematical learning processes.

Two theoretical ideas underlie this research project: the community of inquiry, as an adequate context for developing a project of these characteristics, and the situated perspective, which give us a framework for considering together learning and teaching. The project is situated in the basic research on Didactic of Mathematics (since it aims to deep on the characteristics of students' learning in a determinate level) but it can be considered as the basis of future applied researches, allowing to use the results in improving the mathematics education of the students of the considered level, and providing a scientific knowledge useful for Secondary Mathematics Teacher Education programs.

[1] This research has been supported by the Ministerio de Educación y Ciencia (Spain) through grant SEJ2005-01283/EDUC (partially financed with FEDER funds).

YOUNG CHILDREN'S MATHEMATICS EDUCATION WITHIN A PHILOSOPHICAL COMMUNITY OF INQUIRY

Abigail Sawyer

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Philosophical communities of inquiry are a specific type of problem-centred and discussion-intensive pedagogy. Such communities have been identified as supportive of mathematical learning. This presentation describes the impact of participation in a philosophical community of inquiry on the mathematics education of one group of young children.

Reform-oriented curricula emphasise the importance of students learning to both understand and use mathematics. Such curricula foreground the solution of 'real-life' problems as a fundamental purpose of mathematics education. Many also promote the use of classroom discussion to advance mathematical understanding.

Some research has suggested that problem-centred and discussion-intensive mathematics programs can privilege middle class students. Boaler (2002), however, has demonstrated that such programs do not disadvantage students from less advantaged social groups, provided teachers make certain features of mathematical discourse explicit to students.

Groves and Doig (2004) have suggested that philosophical communities of inquiry can enable students' participation in mathematical discourse and thereby facilitate mathematical learning. This presentation will use data collected during an explanatory case study to describe the practices within one First and Second Grade class that functions as a philosophical community of inquiry, and explain how these practices impact young students' mathematics learning.

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LEARNING MATHEMATICS IN AUSTRIA: A STUDENT PERSPECTIVE

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In this poster we present the selected results from a student questionnaire administrated to about 1500 Austrian secondary school students (65 classes). The teachers of those students were engaged in (partly year-long) projects to foster good practices in teaching and learning. These projects, which had passed a carefully designed review process, were funded by a specific fund for teacher projects in mathematics, science and technology (MNI-Fund, part of the IMST3 initiative of the Austrian ministry of education, Krainer 2005). In the questionnaire we assessed student interest with respect to their regular classroom teaching as well as their interest while the teacher engaged in their class-room projects, subject-directed anxiety, and subject-related self-concept. In addition, we asked them to report on some of their or their teachers' activities in class (e.g. "How often do you have to listen to a presentation of your teacher?"). All items were taken from either the PISA (2000) questionnaire, or the national PISA supplement of Germany or Austria ("PISA Plus").

In our first analysis, we found that during the projects, teaching was less teacher-centred (students reported more individual and group work). Self-directed student work was positively linked with student interest (correlational analysis). Multi-level analysis further revealed a whole-class impact of about 10%.

In our poster we will briefly present the context of the study, i.e. the teachers' (self-)selection, and teachers' activities within the MNI-Fund. We will reanalyze and contrast the data with the mathematics classes only (27 classes). Finally, we will use our findings to present a picture of how Austrian mathematics students of engaged teachers perceive their mathematical school life.

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THE INFLUENCE OF A MATHEMATICIAN ON HIS STUDENTS' PERCEPTIONS – THE CASE OF MATHEMATICAL INDUCTION

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In this study, we report on a professor of mathematics who also teaches mathematics in a secondary school. We shall concentrate on the "mathematical induction" principle (MIP).The aim of the study is to answer the following questions:

1. How do students, who have learned the MIP in the set language, e.g. "If a subset S of N contains 1 and contains the successor of each of its members, then it contains (and equals) all of N", perceive the principle?
2. How do the students' perceptions relate to the teacher's perception of the principle?

Method: Extensive data was collected on the teacher; he was observed, videotaped the fulltime he taught the subject MIP (for approximately 385 minutes). The aim of this data, was to discover the teacher's conception of the MIP. A 3-question questionnaire, relating to the subject of MIP, was compiled, and administered to two classes of our subject. One class (33 students) learned the subject approximately one year before its students answered the questionnaire; the students in the other class (41) answered it during the period in which they studied the subject.

Conclusion: It can be concluded that the students' perceptions of a mathematical term were influenced by the several representations of this term. Moreover, it was concluded that the students' main perception of MIP was: mathematical induction identifies the natural number set, including formulation; following four representations of the MIP that were exposed by the teacher.

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ANALYSIS ON THE ALGEBRAIC GENERALIZATION OF SOME KOREAN MATHEMATICALLY PROMISING ELEMENTARY STUDENTS

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*Gyeongin National University of Education
**Seongra Elementary School

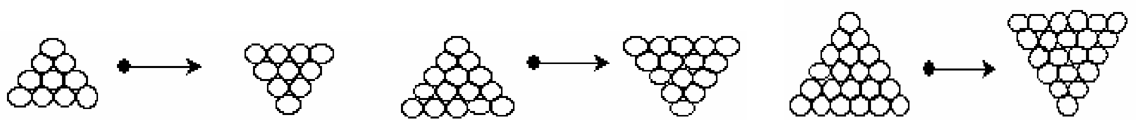
This research is designed to analyze the thinking patterns that some Korean mathematically promising in different 4 level groups(top 0.05%, 0.01%, 1%, 10% by their capability) show in the process of solving generalization problems. There have been many educational researches on generalization(Lannin, 2005; Radford, 1996; Stacey, 1989; Swzfford & Langrall, 2000). And Krutetskii(1976) reported that the mathematically able students were known to be flexible in thinking and have the ability to generalize various patterns. While they doing the given task for 3 hours, we observed with videotape and analyzed characteristics in their generalization processes.

The task(figure 1) is to identify “the least number of pebbles moved” and “how to move the pebbles” to realign pebbles forming a triangle to make a form of an inverted triangle. The question contains geometric patterns for moving the least number of pebbles and numerical patterns for the least number of pebbles moved.

1. Realign pebbles forming a triangle to create a form of an inverted triangle, identifying the following rules.

(1) How to move the least number of pebbles (2) The least number of pebbles moved

[Phase 4] [Phase 5] [Phase 6]



2. Prove that your rules or general formulas suggested are always true.

Figure 1. Pebble Task

The research results can be summarized as follows:

First, the strategies which students applied for problem-solving can be classified into four different types such as (1)relationships among dependent variables, (2)a single relationship, (2)composite relationships, and (4)contextual structures.

Second, key generalization strategies differed by level group.

Third, students differed in terms of the capability to properly use the recognition of contextual structures to identify general numerical relationships. The recognition of contextual structures influences but can not guarantee the identification and expression of general numerical relationships.

“This work was supported by Korea Research Foundation Grant funded by Korea Government (MOEHRD, Basic Research Promotion Fund)(KRF-2005-079-BS0123).”

STUDENTS' LINGUISTIC STRATEGIES FOR SHARED AUTHORITY IN UNDERGRADUATE ALGEBRA DISCUSSIONS

Susan Staats

University of Minnesota

This poster compares the linguistic strategies used by five undergraduate developmental algebra students in collaborative, constructivist problem-solving discussions. It focuses on students' use of pronouns and deictic terms in claiming a position on responsibility (Hill & Irvine 1993) and collaboration in problem solving. Because a student's strategy for interjecting an idea may respond to the linguistic forms used by the instructor or other students just prior to their speech, a particular student's pattern is viewed not only as an intrinsic habit but as a stance on shared authority in a discussion-based constructivist classroom. Analysing linguistic markers of responsibility and negotiated authority offers a counterpoint to student relationships to authority in more traditional classes (Amrit & Fried 2002). The most common pronoun, "you," is ambiguous because it can be understood as either a narrative convention of the mathematics register or as a denial of authority, a marker of procedural rather than active thought. The pronoun "I" and the consistent and varied use of deictic terms both express a strong degree of personal responsibility for the mathematical outcome of a problem, but were common strategies for only a few students. Most students shifted their preferred strategy during units that they understood incompletely, positioning their speech as momentary, emergent improvisations (Martin & Towors 2003) on mathematical content and shared classroom authority.

The major visual elements of the poster will be five horizontal bands that describe each students' strategy, with a brief (2-3 line) example of their predominant strategy, along with examples of notable shifts away from the preferred strategy for each student. The left endpoint of each band introduces the student with a quote indicating a characteristic presentation of self in the class. The right endpoint of each band terminates with student achievement data and their reflections on the class. An mp3 player with headphones will provide excerpts of the students' voices.

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CONTRASTING DECIMAL CONCEPTIONS OF ADULT AND SCHOOL STUDENTS

Kaye Stacey and Vicki Steinle

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Over three hundred adult nursing students completed a test to identify those with incomplete conceptions of decimal numbers. On this test students are presented with a carefully selected set of pairs of decimals and asked to select the larger or write = between them. The test is based on extensive research and contains multiple items grouped into types, based on student behaviour. Whereas this test has revealed many misconceptions amongst school-aged students, the adult students show a different behaviour. Almost all nursing students can compare "typical" pairs of decimals; i.e. Types 1, 2, 3, 5, 9 and 11 in Figure 1. Yet over 30% chose incorrectly on some of the Type 4 items, which are "unusual" as the numbers have the same initial decimal digits and one is a truncation of the other (e.g. 17.353 / 17.35 and 4.666 / 4.66). Other items with higher error rates involved comparisons with zero (Types 8 and 10).

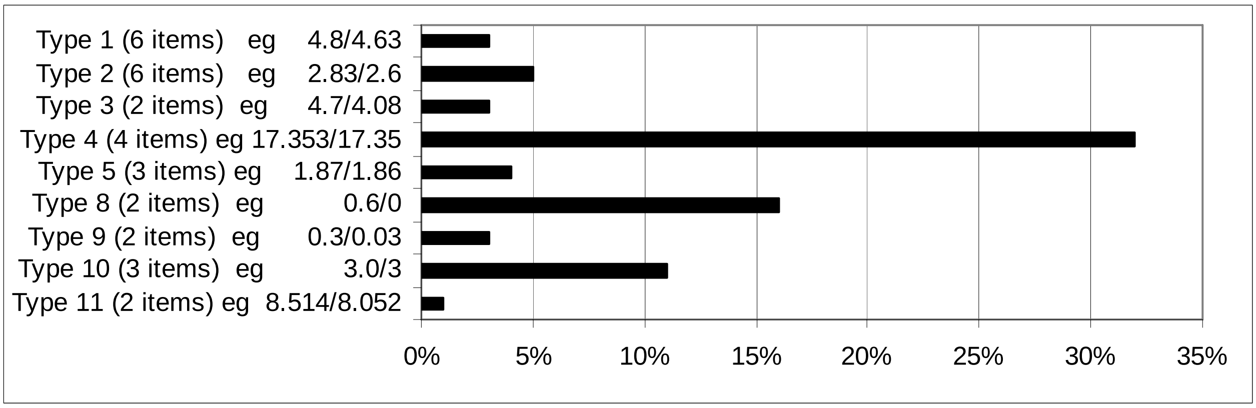

Figure 1: Percentage of Nursing Students with at least one error in each item type.

These results are consistent with the hypothesis that some of these nursing students have limited conceptions of decimals and consequently they compare decimals using various incomplete algorithms; for example compare only 1 or 2 decimal places, or compare from left to right but are unable to proceed when one number stops. These incomplete algorithms are most likely the result of patchy recall of procedures that are no longer (if ever they were) supported by understanding. The task of comparing decimals is simple for those who know a "complete" algorithm and the algorithms themselves are not apparently complicated. However, if they are seen as arbitrary rules, they are hard to remember.

Our previous research shows that many school children demonstrate misconceptions. In contrast, the adult students try to recall any procedure that gives them an answer. At some stage in their learning, they have stopped investing time and effort in "sense-making" (which shows as a misconception) and become content with "procedure-using". This is reinforced when the procedures work most of the time.

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MATHEMATICAL WRITING AND THE DEVELOPMENT OF UNDERSTANDING

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There is a great body of literature focusing on student writing in mathematics (e.g., Morgan, 1998). Its majority report a positive influence of writing on a student's learning. Here, we will focus on a case study of writing at the university level.

The poster will present the mathematical background – a finite arithmetic structure called restricted arithmetic (or RA). RA is congruence modulo 99 in 'disguise'; it means that this fact is not immediately apparent to students. RA is based on an analogy with 'ordinary' arithmetic, which enables students to use their existing knowledge for developing solving strategies and posing problems (Stehlikova, 2004).

The focus of the case study is Molly, a future mathematics teacher, and her mathematical description of knowledge of RA. For the text analysis, we were inspired by Dormolen (1986) and Morgan (1998). The data to be analysed consisted of subsequent versions of the same text. We followed the following attributes: *Structure of Writing, Style of Writing, Presentation of New Concepts, Vocabulary and Symbols, Mathematical Validity.* The categories were subdivided into more specific aspects whose development was followed in the subsequent versions of writing.

Molly's first intention when writing the text was apparently to present what she had discovered in a way it was done and in a way accessible to a student (her didactic intentions were prominent). Later, she was more constrained by institutional demands represented by the experimenter and an imaginary reviewer of her diploma thesis. We contend that Molly's writing reflects her global conception of RA and sheds light on her understanding of some RA concepts.

The existence of several subsequent versions of writing makes it possible to follow the development of her understanding and hypothesise about its possible sources. In the poster, some results of our analysis will be presented showing Molly's development as a learner of mathematics as well as one particular example of changes between individual versions of writing.

Acknowledgment The paper was supported by grant GACR 406/05/2444.

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ENHANCING TEACHERS PROFESSIONAL DEVELOPMENT THROUGH DEVELOPING TEACHING NORMS BASED ON DEVELOPING CLASSROOM LEARNING NORMS

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National Hsinchu University of Education, Taiwan

This study was designed to support teachers' professional community on developing teaching norms based on developing learning norms in classroom communities in which students were willing to engage in discourse. A collaborative team consisted of the researcher and four first-grade teachers. The professional community intended to generate normative aspects of acceptable and appropriate teaching based on discussing teachers' observations about their students' learning mathematic. Classroom observations and routine meetings were used to collect data for the study. The process of generating two teaching norms including students' social autonomy and students' questioning in the professional community and its effect on learning norms in classroom communities were the foci of this paper.

Fostering students' development of intellectual and social autonomy oriented the goal of mathematics teaching involving in this study. Teachers with intellectual autonomy promoted their students becoming as self-directed learners who were used to question, inquire, and figure out the answer in their classroom communities. The teachers' autonomy referring to the study was identified as teachers' willing to participate in the professional community and students' autonomy was clarified as students' willing to participate in the classroom community. It is found that the process of fostering students' intellectual and social autonomy was consistent with that of enhancing teachers' teaching autonomy. The teaching norms promoted the teachers' autonomy in their teaching practice through the dialogues of the professional community and developed the learning norms that promoted students' autonomy in the classroom communities.

The teachers performed the development of classroom learning norms with different paces in different classroom communities. The norm of developing students' learning norms that were evolving and renegotiating within the dialogues of professional community was developed with some acceptable criteria. Then, each teacher taken and shared the norms of students learning, implemented them into her own classroom community, and then improved her teaching autonomy and her students' social and intelligent autonomy in teaching practices. The evidence of two kinds of autonomy affecting mutually was shown in this results.

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THE FEATURES IN THE PROCESS OF MATHEMATICAL MODELING WITH DYNAMIC GEOMETRIC SOFTWARE

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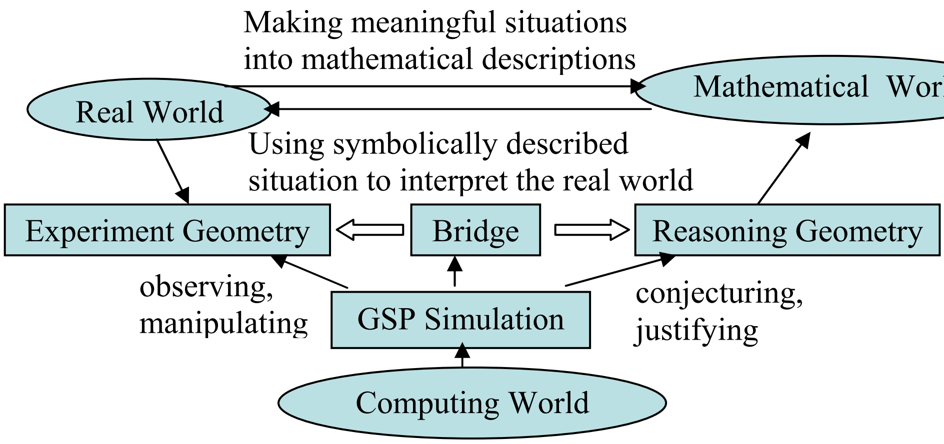
Department of Mathematics, National Taiwan Normal University

The present paper investigates the process of potential teachers making mathematical modeling and the role of computer in this process. Due to the fact that mathematical modeling is important in mathematics education, and integrating computer in mathematics learning has also become a trend, mathematicians have used computers as a tool to simulate the mathematical models of the real world situations. However, students seldom have opportunities to use such devices for making mathematical modeling. As a result, the role of computer tools in mathematical modeling is still not clear, while it's potential in mathematics learning has already been widely recognized.

This presentation describes the findings of a case study on potential teachers, focusing on the process of how they make mathematical modeling with dynamic geometry software. The study shows the following three findings. First, potential teachers explore in between three worlds, which are the real world, the mathematical world, and the computer world. Second, the transition between the three worlds is based on reflective acting. Third, the dynamic geometry software not only simulates the mathematical model, but also connects two aspects of mathematics, which are the experimental mathematics and the reasoning mathematics.



Von Glasersfeld (1992) indicated that knowledge is a result of an individual's construct activity and not a commodity which can be conveyed or instilled by another. Mathematical modeling activities can help some of our potential teachers to recognize learning as a process of knowledge construction. This case study shows that mathematical modeling activities can encourage potential teachers to exploit their own thinking and provide them a means to use mathematics to solve real world tasks.

FINDING INSTRUCTIVE CHARACTERISTICS OF PICTURE BOOKS THAT SUPPORT THE LEARNING OF MATHEMATICS

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This poster addresses the first stage of the PICO-ma project which is the mathematics chapter of a multi-disciplinary and multi-method study into the use of picture books to support kindergartners' learning within the domains of mathematics, literary, and social-emotional development. The research project starts with (stage 1) domain-specific analyses of mathematics-related picture books, followed by (stage 2) design research to develop guidelines for how to use these picture books, and the project concludes with (stage 3) an intervention study to find evidence for the power of picture books.

The first stage of the PICO-ma project is aimed at identifying instructive qualities of picture books that are supposed to contribute to the development of mathematical concepts. This stage started with a review of literature on requirements for picture books to be useful in education (see references). Armed with this knowledge and our own expertise we carried out mathematics-didactical analyses of mathematics-related pictures books in order to detect potentially efficacious characteristics of picture books. These document analyses were combined with the analyses of responses of children when they are read out these books. Both research activities resulted in a first version of an analysis tool for the identification of mathematics-related instructive qualities of picture books. Next, a Delphi procedure was applied to consult a group of experts on the use of picture books in education. This consultation was meant to get a more refined version of the tool that eventually will be used to select the books that will be used in the stages 2 and 3 of the PICO-ma project.

The poster shows the first results of stage 1 of the project and discusses the believed instructive qualities of picture books. In addition, some video clips from reading picture book to children will be shown on a laptop.

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HIGH SCHOOL COURSE PATHWAYS OF HIGH ACHIEVING GIRLS

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Course grades and standardized test scores are factors that influence mathematics course selection (Parsons, Adler, Futterman, Goff, Kaczala, Meece, & Midgley, 1985). As part of a longitudinal study of high achieving female students, we consider the pathways of students' mathematics courses from middle school through high school. We conjecture there is a relationship between these factors and pathways. The variables of interest are: standardized state test scores for Algebra I, Geometry, and Algebra II; Preliminary SAT (PSAT) mathematics scores; and Algebra II course grades. Groups are assigned based on mathematics course selections. Group A consists of students who did not choose a Pre-Calculus or Calculus path. Group B is comprised of students who completed a Pre-Calculus course, and Group C consists of students who completed at least one Calculus course. Only subjects who have measures in each area are included in the analysis. A one-way ANOVA test between the groups and each variable found there are significant differences between all mean scores with a .05 significance level. Bonferroni procedures for multiple comparisons were also conducted (see Table 1).

	A vs. B	A vs. C	B vs. C
Alg I	$t(97) = 2.026, p = .05$	$t(97) = 3.330, p <. 01$	$t(97) = 1.356, p = .18$
Geo	$t(97) = 3.260, p < .01$	$t(97) = 4.292, p < .01$	$t(97) = 1.191, p = .24$
Alg II	$t(97) = 2.638, p < .01$	$t(97) = 3.839, p < .01$	$t(97) = 1.304, p = .20$
PSAT-M	$t(97) = 3.297, p <.01$	$t(97) = 4.632, p < .01$	$t(97) = 1.476, p = .14$
AlgII Grade	$t(97) = 3.346, p <.01$	$t(97) = 5.244, p < .01$	$t(97) = 2.003, p = .05$

Table 1: Bonferroni Results for Multiple Comparisons.

Significant differences between groups A and B were found for all variables except for Algebra I scores and between groups A and C for all measures. There were no significant differences between groups B and C. Groups A and B differ on each of these academic factors except their Algebra I standardized test score. Groups A and C differ on all of the academic factors examined. Groups B and C do not differ on any of the factors. These results imply the decisions to take courses beyond Algebra II significantly correlates with all measures of school mathematics success.

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THE DEVELOPMENTAL STAGES OF REPRESENTATIONS OF SIMPLE REGULAR SPACE FIGURES OF ELEMENTARY SCHOOL STUDENTS

Der-bang Wu[1)], Hsiu-Lan Ma[2)], Dong-Cun Chen[3)]

[1)]National Taichung University, [2)]Ling Tung University,
[3)]Kong Long Elementary School

This study presents partial results from the project "A study of the developmental stages of elementary school children's representation of simple regular space figure", funded by National Science Council of Taiwan (NSCTW, Grant No. NSC 93-2521-S-142-003). It was undertaken to explore students' abilities and developmental stages, developed by Mitchelmore (e.g., 1980), of the representations of the three-dimensional figures drew by elementary school students. The participants were 194 elementary school students, randomly selected from 4 counties/cities in central Taiwan. There were five basic three-dimensional geometric figures, including the cuboid, the cylinder, the cone, the triangular awl, and the triangular cylinder. Meanwhile, the differences between two genders and divergences among six grades were explored. The new developmental stages of elementary school students' representation was as well expanded or created. The relationship between Van Hiele levels of geometric thinking and the developmental stages of the representation of the simple regular space figures was discussed.

The results indicated that: (1) There were no significant differences between boys and girls at the picture representation of five simple regular space figures. (2) Students from higher grades tended to have the higher stage of representation at the picture representation of five simple regular space figures.. (3) The new developing stages of representation were expanded for the cuboid and the cylinder, and developed for the cone, the triangular awl, and the triangular cylinder of 3 simple regular space figures. (4) A positive correlation was found in the relationship between Van Hiele levels of geometric thinking and the developmental stages of the representation of five simple regular space figures. Elementary students with higher van Hiele levels of geometric thinking were classified into higher stages of the representation of five simple regular space figures.

Acknowledgements The research reported in this paper was supported by the National Science Council under Grand No. NSC93-2521-S-142-003. Any opinions and viewpoints expressed are the authors and do not necessarily reflect the views of the National Science Council, Taiwan.

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HOW TO ASSESS MATHEMATICAL THINKING?

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It's important to use real life problems which are related to students' daily life, which have solution more than one and encourage students to solve. In this study, it is aimed to give examples of real life problems that reveal student's thinking way. It is also aimed to highlight how real life problems are beneficial to see different solutions' of students.

MATHEMATICAL THINKING AND REAL LIFE PROBLEMS

Recent calls for reform in mathematics education suggest that students must learn to recognize mathematical elements in contexts, apply appropriate mathematical tools, and engage in mathematical reasoning (Putnam & Reineke, 1993). All these goals put forward the importance of supporting students' mathematical thinking and in order to improve students' mathematical abilities, it is necessary to understand their thinking and reasoning. Teachers obtain more information about what students know and think, the more opportunities they create for student success (Darling-Hammond, 1994). Real life problems are an important part of student engagement in mathematical thinking. The aim of the study is to give examples of real life problems that reveal students' thinking.

METHOD

15 problems are developed which aim to understand students' mathematical thinking while solving problems. The pilot study is conducted with two pupils of 13 years old who are selected purposefully according to their academic success. 7 problems are omitted after pilot study. The rest of them are asked to 300 students who are requested to solve problems by explaining their reasoning. Document analysis is used.

RESULTS

Real life problems which require more than a numerical response are useful to understand their mathematical thinking. These problems demand from students to carefully consider a problem, understand what is needed to solve the problem, choose a plan, carry out the plan, and interpret the solution, which means realizing the steps of mathematical thinking.

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LASTING EFFECTS OF A PROFESSIONAL DEVELOPMENT INITIATIVE

Stefan Zehetmeier

University of Klagenfurt

The project IMST ("Innovations in Mathematics, Science and Technology Teaching", 2000-2004) was an Austrian nation-wide development initiative with the aim to achieve lasting improvement of the teaching of mathematics as well as of science and technology. The main objectives of this initiative were

- to initiate, promote and showcase innovations in the teaching;
- to carry out a scientific analysis and to disseminate such innovations, with the emphasis on generating good practice concepts and to professionalize teachers;
- to encourage practice-oriented, scientifically grounded subject didactics.

One year after the end of the project, data from the participating teachers as well as from principals of involved schools, and the project's teacher educators (who supported the teachers' activities during the project) were collected to answer the following questions:

- Is there any lasting impact resulting from the teachers' participation in the project (that is still effective after the project's termination)? Which types of impact did emerge?
- Which are facilitating (or prejudicial) factors that promoted (or inhibited) teachers' lasting professional development?

The poster presents two exemplary cases of participating teachers' lasting professional development.

The poster's information is presented in terms of text modules (supplemented by some graphics) providing the teachers' journeys of professional development.