Secondary School Students' Construction Processes of Square Root Concept with Realistic Problems: An APOS Perspective

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Abstract: *This study aims to examine the construction process of the square root concept of secondary school students in Realistic Mathematics Education (RME) based environment. The study was carried out in an 8th-grade classroom with 16 students from a secondary school in a city in the Black Sea Region. The research was designed qualitatively as a case study. To measure the preliminary knowledge of students about the square root concept, a test was prepared. The students were grouped heterogeneously according to the results obtained from the test and observations. The students were asked to solve two contextual problems in their groups. The teaching process was supported by in-group and inter-group discussions. Interviews were conducted two times with each of the three participants who were selected from the groups by purposeful sampling and whose readiness levels were advanced, intermediate, and lower-intermediate. The data from the transcripts of the teaching process, in-group discussions, interviews, and also individual and group worksheets and observations, were evaluated with content analysis within the framework of APOS theory. As a result of the study, the concept of the square root of square numbers was constructed as an object by all of the participants. Also, the participants could determine the location of the square roots of non-square positive integers between two natural numbers. It has been determined that having strong coordination is very important, and exponential numbers, area and perimeter measurement, unit, and rational-decimal numbers are used in this coordination.*

INTRODUCTION

Numbers, their properties, and relations cover the main and important part of the mathematics curriculums taught in schools. Natural numbers and their operations on them which are started to be taught in primary school expand with integers and rational numbers in secondary school. At the end of secondary school, irrational numbers are started to construct, and real numbers and complex numbers are taught in high school. During this period, high school students understand the number system and compare its characteristics with others (MNE, 2018; NCTM, 2000).

It is essential to understand irrational numbers for extending and reconstructing rational numbers to real numbers (Sirotic and Zazkis, 2007a). However, the case of this extension is particularly dramatic. The students have difficulties with the strict hierarchy among number sets and the abstract structure of irrational numbers, and also they have misconceptions about them (Arcavi, Bruckheimer, and Ben-Zvi, 1987; Fischbein, Jehiam and Cohen, 1995; Guven, Çekmez, and Karataş, 2011; Sirotic and Zazkis, 2007a). It has been assumed that the concept of irrational number is intuitively difficult because of their incommensurability. After the surprising discovery of early Greek mathematicians about the existence of incommensurable segments, the rigorous theory of irrational numbers has been fully established with the contribution of Dedekind, Cantor, and Weierstrass in the nineteenth century with a long, historical delay (Fischbein, Jehiam, and Cohen, 1995). So, it would not be plausible to expect students to overcome its epistemological obstacle and understand the concept of irrationality easily.

The advantage of using real-world models and pictorial representations can be considered to improve the understanding of natural or rational numbers, but this cannot look easy for irrational numbers due to their more abstractness. On the other hand, considering that the mathematical concepts arise from real-life needs of knowledge and relations, it is thought that teaching these concepts should be realized in a more informal and real-life context (Campbell and Zazkis, 2002; Gravemeijer, 1999). Thus, when students learn by relating to their own lives and performing their mathematization processes, mathematics will become meaningful for them (Gravemeijer and Terwel, 2000). However, attention to informal meanings and familiar contexts in mathematics education should not be considered separately from the development of conceptual foundations (Campbell and Zazkis, 2002). In this sense, realistic problems in the environments in which the student has actively re-invented the mathematical structures by using his/her paths and the models he/she has developed might serve this purpose (Gravemeijer, 1999). The steps which are firstly the development of operations-relations in some ordinary real-life contexts, then the realizing the same structure in other contexts, and finally, the formulation of the common structure by symbolizing it (Treffers; 1991), will constitute generalization and abstraction (Mitchelmore, 2002). If we consider that the concept is a cognitive structure formed as a result of abstraction (Von Glaserfeld, 1991; Yilmaz and Argun, 2018), the environment in which abstraction will take place will be important in the concept formation process.

Although the necessity and importance of understanding irrational numbers in the transition from the set of rational numbers to the set of real numbers are emphasized by the studies conducted, it is seen that the researches on the subject are still quite insufficient. Also, studies examining how these concepts are abstracted are almost non-existent. When the related studies are examined, understanding of irrational numbers generally is conducted and these studies are carried out on teachers, prospective teachers, and undergraduate students. For example, Sirotic and Zazkis examined prospective secondary school teachers' understandings of the representation of irrational numbers as points on a number line (2007a) and their knowledge regarding the relationship between the two sets, rational and irrational (2007b). Also, Zazkis and Sirotic focused on how different representations influenced their responses concerning for to irrationality (2010). Guven, Cekmez, and Karatas (2011) researched prospective elementary school teachers' understandings of defining rational and irrational numbers, placing them on the number line, and operations with them. Patel and Varma's study (2018) is on undergraduate students from different departments and they examined the performance on a magnitude comparison task about irrationals denoted by radical expressions and on a number line estimation task about these irrational numbers. Fischbein, Jehiam, and Cohen (1995) investigated the presence and the effects of obstacles for 9th and 10th-grade students and prospective teachers: the difficulty to accept that two magnitudes (line segments) may be incommensurable and the difficulty to accept that the set of rational numbers does not cover all the points in an interval. Arbour (2012) studied college science students' understanding and concept images of real, rational, and irrational numbers, and Voskoglou and Kosyvas (2013), designed a general plan in terms of the APOS/ACE treatment for teaching real numbers at high school and college level. Kidron (2018) analyzed 10th-12th graders' conceptions of irrational numbers by using their representation as non-repeating infinite decimals and their conceptions of irrational numbers on the number line.

In the below-mentioned theoretical framework, RME and the role of realistic problems, and APOS theory have been explained.

THEORETICAL FRAMEWORK

Realistic Mathematics Education (RME) and Realistic Problems

The idea of RME is based on that mathematics is a human activity. This approach put forwards that the learner re-invents the formal mathematics knowledge by transforming his/her informal knowledge in real life with the help of contextual problems that are meaningful and experientially real to them (De Lange, 1996; Gravemeijer, 1999; Treffers, 1991). Thus, they can experience these mathematization processes similar to the processes of real mathematicians (Gravemeijer and Terwel, 2000) and this provides learners to construct meaningful knowledge. Here, realistic problems are mathematical problems that students can experience or imagine in real life and are based on meaningful contexts. Besides the importance of contexts, the most crucial criterion for a problem within RME is its offering opportunities for modeling and mathematization (van den Heuvel-Panhuizen, 2005).

'*Guided reinvention through progressive mathematization, didactical phenomenology,* and *self-developed models*' are three key principles of RME for implementing and planning the teaching and learning process. During guided reinvention through progressive mathematization, students configure and organize the realistic problem by finding mathematical aspects of it (Fauzan, 2002) and this strong intuitional component prompts the students to reinvent the mathematical concept (Yilmaz, 2020). Progressive mathematization involves a two-stage process: horizontal and vertical. In the first stage, students transform the realistic problem into a mathematical problem by using their informal strategies, and in the second stage, they produce a new algorithm and move the mathematization process to a higher level in the light of informal strategies and abstract the conception (Gravemeijer and Terwel, 2000; van den Panhuizen and Drijvers, 2014). Didactic phenomenology means that the students make sense of the events as a phenomenon and thereby, they can produce their phenomenon like real mathematicians (Freudenthal, 1983). The self-developed models mean that students' models that they develop with the help of the realistic problem have a crucial role to pose as a bridge between informal and formal knowledge (Fauzan, 2002; van den Heuvel-Panhuizen and Drijvers, 2014) and these models (model for) that are dependent on the context transforms into the models (model of) independent from the problem situation (Zandieh and Rasmussen, 2010).

The design of instruction to be prepared according to the principles of RME requires: starting to learn with a concrete foundation with rich content problems that support mathematical organization and following informal solution processes of the students in context (constructing and concretizing); using models and schemes that may arise during the solution process of the problem to complete the gaps between students' abstraction levels and to facilitate their cognitive transitions (levels and models); giving special assignments to the students to reveal their free products and thoughts (reflection and special assignment); working with groups (social context and interaction) and providing students to learn the concept in a spiral relation with related prior knowledge (structuring and interviewing) (Treffers, 1991; Gravemeijer and Terwel, 2000; van den Heuvel-Panhuizen and Drijvers, 2014).

APOS Theory

Piaget's one major idea is reflective abstraction (1980) which is the main mechanism for the mental constructions and logico-mathematical structures in the development of the thought of individuals. APOS Theory is based on his idea and developed on the terms Action-Process- Object-Schema. As a constructivist theory, it gives a theoretical framework about how mathematical concepts can be learned and examined in the process of their formation.

According to this theory, mental structures called genetic decomposition are built in the mind while learning a concept. This construction can vary from one individual to other and can also be different from the definition/formulation of the concept. With the possible genetic decompositions, suggestions can be made as a model about how to design instruction for students and, what differences and difficulties in acquiring the concept might reveal. The individual makes sense of the concept with certain mental structures and these structures (*Action, Process, Object,* and *Schema*) include mental mechanisms (*interiorization, coordination, (de)encapsulation, reversal, generalization,* and *thematization*) (Asiala et al., 1997; Arnon et al., 2014). During the construction, mental objects that have been previously constructed begin to be transformed in the action stage where external stimulus is needed, and this stage is static (Arnon et al., 2014). As it is controlled and reflected, the action no longer needs external stimulus and is consciously interiorized, the process stage is passed, and then a dynamic structure is formed. These internal processes allow individuals to make sense of perceived phenomena (Dubinsky, 2002). A new process can be achieved by the coordination of different processes or the reversal of the process, as well as the process stage by interiorizing the actions. If the individual perceives this dynamic structure as a whole and realizes that he/she can apply different actions and processes to it, the process is encapsulated as an object and thus a static structure is obtained (Asiala, et al., 1997). Sometimes the object(s) can be de-encapsulated by returning to the process stage from which it was acquired, thus objectifying new processes by coordination between these processes (Arnon et al., 2014). A coherent collection of all these actions, processes, and objects constructs the schema. At the end of the construction process, if the learner is successful, the problem or new object has been assimilated by the schema. And also, the schema includes other schemas related to the concept. When not successful, his/her existing schema might be accommodated to handle the new phenomenon (Dubinsky, 2002).

Remembering that the concepts of irrational and real numbers were first learned in secondary school, it is thought that the examination of the formation of these concepts at this level is very important and necessary due to the lack of studies conducted at this level. Also, it is thought that examining the formation of these concepts by providing a transition environment from informal situations to formal situations through problems based on real-life contexts will give important ideas about how these concepts can be learned as well as how they can be taught.

In this study, 8^{th}-grade learners' construction process of the irrational number concept is examined with APOS theory by trying to define its genetic decomposition. The study is focused on the square root concept and the learning environment is grounded in the theory of RME. Therefore, the research question of the study is considered as *'how is the 8^{th}-grade students' construction process of square root concept in RME based teaching environment?'*.

METHOD

Research Design

In this study, we qualitatively researched the 8^{th}-grade students' construction processes of the square root concept with realistic problems. The research was conducted in a case-study pattern to investigate their constructions in depth (Yin; 2003). The study teaching process has in accordance with the RME approach and focuses to construct with realistic problems. The mental structures and mechanisms of the students were interpreted within the context of the APOS theory.

Participants

This study was conducted with three participants selected from a 16-student class in a state secondary school in the Black Sea region where one of the researchers was a mathematics teacher. A purposeful sampling method was used to reveal the mathematical lattice in students' minds regarding their conceptual understanding of the square root concept. The participants were selected voluntarily according to the following criteria:

Since the learning and teaching environment of the study would take place with heterogeneous groups, a test (explained in data collection tools) including the subjects related to the square root concept was applied to all the students in the class to determine the groups before conducting the activities. After this test was applied, 16 students were divided into heterogeneous groups of four, according to this test results, the observations, and the opinions of the other teachers about their academic achievement and personal characteristics. In the formation of heterogeneous groups, the difference in the readiness of the students in the same group about the concept of square root, their interaction within the group, and the mathematical skills of the students in the same group was considered. Three participants, one from each group and each level group (advanced, intermediate, lower-intermediate), were selected for interviews as in Table 1.

Groups	Students in Groups	Participants	Gender	Levels
Group 1	S1, S7, S8, S9	S1	Male	Advanced
Group 2	S2, S3, S10, S11	S3	Male	Intermediate
Group 3	S4, S6, S12, S13	-	-	-
Group 4	S5, S14, S15, S16	S5	Female	Lower-Intermediate

Table 1: Participants in groups and their levels

Teaching Process and Data Collection

The teaching and learning process was planned according to the principles of RME. It was tried to design a natural environment where students could think freely in groups. In the learning environment, the mathematization processes of the students were tried to be supported by guiding. Contextual problems provided students with a structured beginning that helps the formation of the concept, and students could relate the given problem with their existing knowledge and skills. In addition, informal solution strategies, didactic phenomena, and the models they create could be revealed in the process of solving problems and abstracting the concept. After the problems were solved in the groups, interviews were conducted.

Data collection tools were readiness test, observations, interviews, and worksheets which were used by the participants during activities in groups and during the interviews. The test consisted of a total of 14 questions, one of which was true/false, and the others were short-answered and open-ended, including the concepts that form the basis of the square root and its properties. The test included units of length and area and their transformations, perimeter and area of square and rectangle, rational numbers and decimals, prime numbers, factors and multiples, and exponential expressions and their properties. The test was finalized by taking the opinions of four mathematics educators and applying it to the 8th-grade students of a different secondary school.

From the beginning of the semester, the researcher (who is the teacher) regularly got information about the students from the course teachers and the class mentor. She had the opportunity to observe the students in the lessons, thus getting to know the classroom environment, having information about both the academic and personal situations of the students, and getting an idea about which students to choose as participants. Observations were continued during the teaching and learning process. What the students thought, what kind of conversations they had with their group friends, and how they reacted when faced with contextual problems during the lessons were observed. During the lessons, each group was recorded with video cameras, and the whole class was recorded with an extra video camera. Subsequently, interviews were also recorded.

To reveal what the students thought in the groups, each student used different colored pencils in the group and personal papers, and thus, it tried to follow them. They were asked to write down all their thoughts on their papers and it was collected at the end of each lesson.

Two interviews were conducted with each participant at the end of each lesson and before moving on to the next lesson. The interviews were held in an environment where the participants feel comfortable and the students were asked to think aloud. During the interviews, as in the classroom environment, materials (such as chess, rulers, squared objects and calculators) were kept ready, and semi-structured interview questions which were finalized after expert opinions were used. In the first interview, reflections of the participants in *'determining the relationship between the square numbers and the square roots of these numbers'* were examined. Participants were asked to describe how they developed a solution for the relevant problem (Chess Problem 1 in Appendix), what prior information they used while solving it, and how they used it. Also, another problem (Parkour Problem in Appendix) was presented to reveal the formation of the process. In the second interview, reflections of the participants in *'determining the location of the square root of a positive integer between two natural numbers even if this positive integer is not a square number'* were similarly examined after the relevant problem (Chess Problem 2 in Appendix) and other problems (I know! Problem 1 & 2 in Appendix) was presented to reveal the formation of the process. The participants were asked to write down all they thought during the interviews and the solutions to the questions and problems they reached on their papers.

Data Analysis

The data obtained from the transcription of the video recordings of the teaching process and interviews were analyzed via content analysis. After the transcribed data were coded according to APOS theoretical framework, categories were identified by linking different codes (Creswell, 1998). A full consensus has been reached about the formation of data, codes, and categories by researchers to provide the reliability of the study. Also, to ensure that the results can be conveyed in similar media, the obtained findings have been supported with quotations and detailed descriptions have been made. In addition to the transcripts, the written documents collected from the students and the observations were also included in the analysis process. In the study, a genetic decomposition (in Figure 1) was prepared and then, finalized by taking the opinions of two experts. This decomposition gave an idea before the teaching process about how the participants could construct the concept and was useful in designing the teaching process.

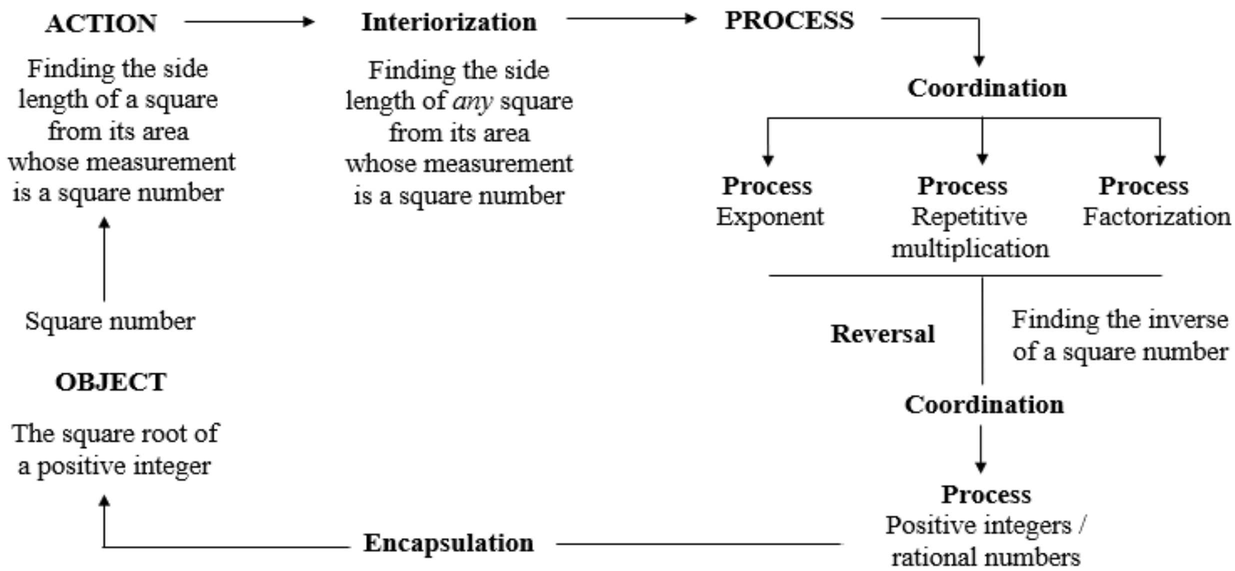


Figure 1. Genetic decomposition of the square root concept

According to genetic decomposition in Figure 1, students start their constructions with a square number in the context of Problem 1. In action, they try to find the side length of the square from the area of it whose measurement is a square number. After interiorizing the process, they could systematically find the side length of any square from the area of it whose measurement is a square number. Also, in the process, students may coordinate the way(s) of finding these numbers with processes of concepts such as exponent, repetitive multiplication, or factorization. Thus, they could systematically find the square roots of square numbers by making sense. The students are asked about the sides of the table in the context of Problem 2, they find the inverse of the square numbers and then they can coordinate the process of the square root of square numbers with the process of positive rational numbers/integers. After determining the location of the square root of a positive integer between two natural numbers even if this number is not a square number and finding the square roots of positive integers in the context of I know! Problems 1 & 2, they can encapsulate the process by using the square root symbol out of the context and can reach the square root of positive integers as an object. The genetic decomposition of the square root concept is given in Figure 1.

RESULTS

The findings related to square root were organized within the APOS theoretical framework. Differences in the construction processes constitute the main criteria of this research. The findings of the study are tried to be given under two sub-titles related to the square root of square numbers and the square root of positive integers by synthesizing constructions of the participants in their group studies and interviews.

Construction Process of the Square Root of a Square Number

S1 and his group mates had hesitations about the solution to the problem because the numerical values were not given. However, after in-group discussions, S1 expressed his thought, *"There are 64 squares in its field. Its side is 8. Then its perimeter would be 32"* and represented it as in the first part of Figure 2. When the researcher asked the reason for his thought about the length of squares, he explained *"It can be 16 meters too, teacher. One of them is 2 meters. Then it will be 16… or we can decrease. It's half a meter. Then it will be 4 meters"*. But he was limited by the thought that there should be only one answer to this problem. S3 said that he and his group mates could not solve the problem through numbers and that they would find it by using x and y variables. He thought about the relation between the number of the squares and the length and explained *"If its lengths were x, then its area would be x times x of* x^2*, and its perimeter would be 4x"* and visualized as in the second part of Figure 2. Similarly, S5 and her group mates thought the number of squares, expressed the area measurement as 64 (square units), and found the perimeter as 32 (units). However, they did not think that the area could take different values and this is considered that there was not enough sign that would make them feel their internalization.

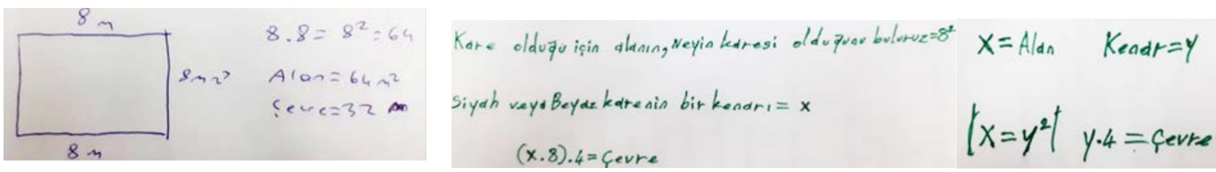


Figure 2. Representations of S1 and S3 in their Group Works on Chess Problem 1

The interviews started with discussions about the problems and the participants' thoughts on them by remembering the activity, and they were asked to explain their thoughts. Then Parkour Problem asked to search for their constructions.

During the interview, when the researcher asked S1 whether the area of the chessboard could take another value, he gave examples for the areas like '16 times 16'. He explained these values as "*It could have been a multiple of 8"* because of the 8 tiles. So, it is considered that his interiorization was limited to multiples of 8. Then when it is asked *"What can the area be if it was not 64 …, but something else?"* S1 replied that *"x could happen*" and explained x as *"anything could happen… for example 50"*. Then, when the researcher asked what the edge length can be when the area is 50, his thought was *"…the multiplication of two same numbers does not equal 50... if it is 25, the edge can be 5"*. Here, it is considered that the participant realized which (square) numbers can be as the area value. He explained the strategy he used while trying to find the edge length of the square as *"searching for numbers that are the product of two same numbers"*. Here, it is thought that the participant started to interiorize that there may be different area values, progressed to the process stage, and coordinated with repeated multiplication while finding the square numbers. When he was asked how he could symbolize it, he was able to use the information he learned in the classroom and wrote the notation correctly. S1 answered the researcher's question about the relationship between square root and square concepts *"In the square, we multiply the number by itself, ..., in the square root, ...Hmmm… it says the number that which two equal numbers we can find by multiplying them"*. Thus it is considered that he did a generalization in the process stage.

S3 said that he started solving the problem by thinking about the number of square tiles. Since no length was given in the problem, he said that he thought to set up an equation. He mentioned finding the 'square of what' for the area of the chessboard and gave 8^2 as an example. When the researcher asked what values the area of the square could take, firstly S3 said 12 and when asked to explain, he added that *"the multiplication of the same two numbers cannot be equal to 12"*. He explained his thoughts through reflection upon the variable-algebraic expression he had constructed. He then exemplified as 25 and 100. Thus, while finding the square numbers, he researched which was the multiplication of two same numbers, and coordinated square numbers with repeated multiplication. When he was asked how they could symbolize it, he wrote the notation correctly as in Figure 3.

$\sqrt{25}=5$ $\sqrt{16}=4$

Figure 3. Representation of S3 on Square Root Symbol

Then, when asked about the relationship between the square of a number and its square root, S3 said, *"Square multiplies, that is,* (gives example) *5 times 5. The square root divides...* (thinks) *to the same number..."* and related the concepts of square and square root. When S3 was asked what the square root was for, his answer was *"finding the edge from the area*". When asked how he could represent the number 25, which is his example, he answered, *"square of 5"* and wrote it as $\sqrt{5^2}$. He also explained, *"First, we write the exponential number as a natural number, it becomes 25, then we take its square root, so it will be the same again"*. Thus, it is considered that S3 interiorized the action and progressed to the process stage.

S5 had the idea that by considering the square tiles on the floor as a unit square and its area would be equal to the total number of squares in it. So, the researcher (R) continued to interview giving samples from the square objects around them to make her aware of the units she could think of. Using the square tiles on the floor, she was asked to think about the side lengths of these tiles and was expected to interpret the distance between the door and the trash can in the interview environment through these tiles.

R: *…What can you say about the distance between the door and the trash can?*
S5: *It is about 1 meter.*
R: *What about the square tiles?*
S5: *… 4.5 squares... but.*
R: *So the length of one side of a square is 1 (m)?*
S5: *No.*
R: *What can it be?*
S5: *We divide 100 cm by 4. It is about 20…*

When the researcher asked her whether the area in the problem had values different from 64, S5 thought about the numbers 32 and 128 as related to 64 in terms of taking different values. But when finding the side length, she thought of factoring it and said *"We will find the factors of 32, but I can't find it right now. what and what… If I say 32 is divided by 2, it doesn't work either, I guess it would be wrong. 32 must-have factors…"*. She thought the same for 128 and 56 and could not reach the side length from these numbers. When the researcher continued on the squares whose side length she knew, she said that the value of the area would be 36 by taking the side length 6, 49 by taking the side length 7, and 81 by taking the side length 9, and showed that she interiorized the examples in different situations. When asked about the property that did not change when the area changed, she said *"it becomes its square"*. If the area was found this way when the side was known, it was asked how to find the side when the area was known, and she said *"The number of the square itself… how can I say… the square root is already this operation, we find the number 49 itself, that is, we find 7. Its square root"*. She also correctly symbolized it by remembering the notation in the lesson. Thus, from her expressions that "*the area is always equal to the square of the side length",* it is considered that she interiorized the action and was able to coordinate with the exponential numbers in the process stage.

Finally, the researcher asked the Parkour Problem to the participants and they discussed their solution. S1 paid attention to the segments given as examples in the problem, found the square numbers in these segments, and then correctly determined the other square numbers for parkour 1-3. In the 4th part, he found the number of flags taken by the three athletes competing in both teams separately and calculated the points he collected from these flags. He correctly determined 106 total points for team A and 119 total points for team B (Figure 4).

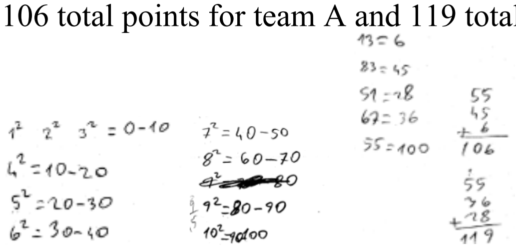


Figure 4. Solution of S1 on Parkour Problem

Since S3 first understood the problem as a flag being planted for every 10 m, he commented that *"10 has no square, 20 has no square… none of them has a square"* and when asked the reason, he said, *"4 times 4 is 16, 5 times 5 is 25... No"*. But he realized that he had misunderstood after he re-read the problem. He said, "*The square of 1 is 1. Then there is the square of… There is a square of 4…"*. S3 then correctly thought about the intervals between the other square numbers and found the places where all the flags could be planted, and then correctly answered the problem for parkour 1-3. When he read the last parkour (4), he could not understand the problem at first, and then said *"There is no square of 1"* in parkour 1, and also added *"There is no square root of 1"* in this parkour. S3 ended his hesitation about number 1 while finding the places where the flags would be planted (Figure 5).

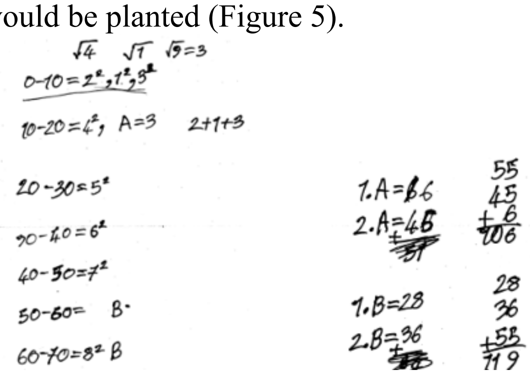


Figure 5. Solutions of S3 on Parkour Problem

On the other hand, while calculating the scores to be taken from the places where the flags are planted, he noticed that the obtained scores were always equal to the base, and questioned whether this was a coincidence.

> **S3:** *...these are (1^2,2^2,3^2,4^2…) we always have the base (1,2,3,4…). Is it a coincidence?... First, we take the square of this, then its square root, and then we find this again. It's not a coincidence…*
>
> **R:** *Then, what is the relation between square and square root?*
>
> **S3:** *It's like inverse operation. For example, it's like adding and subtracting 1. These are inverse operations. So here (he's talking about the square root of 4^2) is 4, here's 5, here's 6, here's 7, here's 8, here's 9.*

It was thought that S3 progressed in the process stage by discovering the relationship between square numbers and the square roots of these numbers as inverses of each other by doing a reversal.

S5, like the other participants, focused on every 10 meters distance and thought about the solution over the numbers 10, 20, and 30… and could not find an answer. Then the researcher asked, *"What… if it is 5?"* and she answered as *"25" and* added *"between 20 and 30"* to the question of where to plant it and stated that she understood the problem. S5, like the other participants, did not consider 1 as a square number because the square of the number 1 is equal to itself, but assimilated it to an *"identity element"*. Later in the process, she accepted the number 1 as a square number, found the square numbers up to 100, and answered correctly. She also thought that she would get the square root of the sum of the distances where the flags were planted, and therefore added these values. Later, when she was asked to read the problem again, she understood it correctly and solved it (Figure 6).

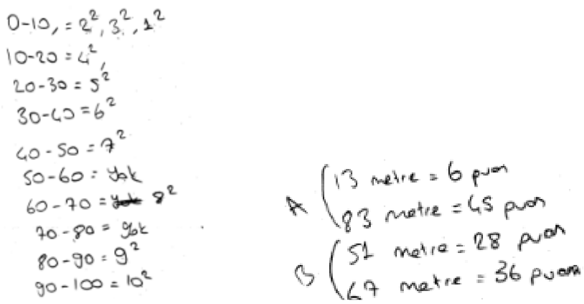


Figure 6. Solution of S5 on Parkour Problem

She realized that the square root of the square of a number (as an exponent) would be equal to this number in the base by saying *"...the square of 2 is 4, square root of 4 is 2. The square of 3 is 9, the square root of 9 is 3 …. The square of 4 is 16, and the square root of 16 is 4* (that's how it gets to 9)*. As directly, we take the bases…"*.

It was observed that S5 had no difficulty in finding what the values of the square roots of square numbers were equal to. Although she couldn't define it exactly, she interiorized that the number in the square root was the square of a number and that the base number was equal to the square root expression. It is thought that the reason why the participant has difficulty in the chess problem is the lack of knowledge in concepts such as area and square unit. In this sense, it is considered that S5 progressed to the process of the square root of square numbers.

Construction Process of the Square Root of a Positive Integer

To encourage the students to conceptualize the square root of positive integers, Chess Problem 2 was presented to them and the solution was discussed by the groups. The participants and their group mates generally started to think that the area of the sticker was 64 (square units) and the area of the table was 81 (square units). Their reasoning was based on the number of squares on the chessboard. Also, they thought of the units as cm. It is realized that their constructions on the unit concept were limited and they had a lack of knowledge about it. For example, S1 told his group about the table: *"…then this will also be a square since the margin will be equal here. If it's square, it will be 9. One side is 9 cm, then according to the previous problem,".* His reasoning was, *"There are 8 squares. 8 times 8 is 64. Which is the smallest number greater than 8, it is 9. Then 9 times 9 equals 81, the area of the table"* (Figure 7). He also expressed the lengths in units *"It must be centimeter. If it is meter, it will be very big…".*

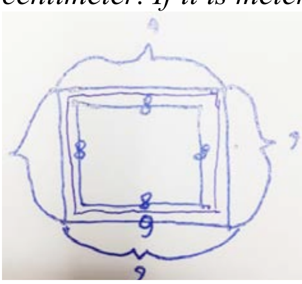


Figure 7. Visual from Group Paper of S1 on Chess Problem 2

The researcher asked if there were other values. The groups of S1 and S3 offered different values related to the area, but S5 and her group mates could not. For example, S1 said *"it must be greater than 64"* and then his group mate S7 added, *"it can be 65, … or 68, then it can be 72".* When the researcher asked what the edge of the table would be if the area was 72 (square unit), S8 stated that it would be between the sticker and the edge of the table. Then S1, stating that it could not be an integer in this case, attributed his thought to 72 being between 64 and 81. When the researcher asked what the side length of the table would be if the area of the sticker was 50 (square unit), S1 said, *"Let's find one side if its area is 50. 7 times 7 is 49. Then it could be 7,5 units. 7,5 times 7,5. Let's find this result"* (Figure 8).

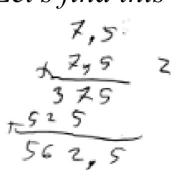


Figure 8. S1'calculation of the square of 7,5

S1 calculated the multiplication wrongly in the decimal notation and the value obtained according to this result caused the participant to hesitate. When the researcher asked them to explain the value of 7.5, S1 said, *"…you asked the side length of a square whose area is 50 square units…"* and for the side length, he said *"Something between 7 and 8…".* When the researcher asked which one was closer, S1 replied, *"It is closer to 7"* and explained the reason as *"7 times 7 is 49. 50,* *because it is closer to 49"*. He also stated, *"If it is closer to 7, then it is less than 7.5"*. Similarly, S3 expressed his thoughts *"There are numbers with commas between 8 and 9"*. The students in his group first thought that a table could be a square if its side lengths were integers. The reason was that when they wanted to divide this square table into unit squares, they thought that the remaining part would be a decimal number and it couldn't be written as a square number. Then, S3 stated that the closest number to 8 was 8.1 which was between 8 and 9. He said *"At first, I thought 81. Then its side was 9… For example, if it was 8.1, its area was 8.1 times 8.1."* When the researcher asked him to answer approximately, he said, *"It is greater than 64 and it is less than 81"* and commented on the side *"Then I can say it is between 8 and 9"*. He continued, *"…there can be a lot of tables… All of them have areas between 64 and 81, and side lengths between 8 and 9"* (Figure 9).

Figure 9. Visual of S3's comment on Table of Chess

It is thought that these participants tended to coordinate the process of the square root of square numbers with the process of positive rational numbers/integers. To see their construction, second interviews were conducted.

In the interview, S1 expressed his solution similar to what he thought in the classroom. When he was asked whether the length of the table to be built could be smaller than 9, he replied, *"Yes, it will be a decimal number"* and gave an example *"It will be 8.5"*. To calculate the measure of the area of the table, he multiplied 8,50 and 8,50 and incorrectly calculated it as 7225,00. S1, who is deficient in operations of decimals, said, *"It is very long. But the side is short… 8.5 is a small number…It should be between 64 and 81"*. He also added to the researcher's question *"Is there only 8.5 between 8 and 9?"* by saying *"No,… 8.4; 8.3; 8.2; 8.1… their areas are also between 64 and 81"*. It is thought that S1 coordinated the square root of square numbers with rational numbers in decimals.

S3 explained his thoughts similarly to S1. When the researcher asked whether the area could have other values, he answered *"It can be 25. If it was 25, one side will be 5. It can be 36… Its side length will be 6 cm."* When the researcher asked, *"Could it be something other than 6?"*, he explained, *"Yes, it can be... Then there can be numbers between 5 and 6 on one side, and the area will be between 25 and 36 at that time"*. When he was asked *"What can you think if you have your area between 25 and 36?"*, he said *"30"* and thought *"the side length of the square will be between 5 and 5,5"* and he explained by saying *"30 is closer to 25"*. He also determined approximately the location for 20 which is between 16 and 25 on the number line he represented by saying *"Since this is (square root of)16, it is between 4 and 5, between 4 and 4,5… it's closer to 4 because it's closer to 16… it can be 4,4."* (Figure 10).

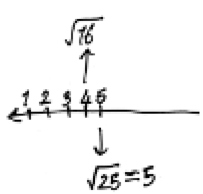


Figure 10. S3' Representation of square roots of numbers on line

S5 also spoke of the areas as 64 and 81 and the lengths as 8 and 9. When the researcher asked whether the table could be smaller, she said, *"Then, it must be between 64 and 81"* and first added, *"I'll take half of 8^2 and 9^2, because I want to get a number between them"*. Then she thought of a value for the area by saying *"64 and 81… can it be 75"* and tried to explain the length of the table by saying *"We need to find the square root of 75… a decimal number"*. But since she could not comment on this side length, she chose 81 as the area of the table again. When the researcher asked for numbers between 8 and 9, she answered *"8.1; 8.2; 8.3;…"* and then the researcher wanted her to draw a square with a side length of 8,2 units. She drew the square and expressed its area as 64,4 by calculating incorrectly. She also compared this area with other areas (64 and 81) and said *"It must be somewhere around"* by showing any point on the edges of the table which she accepted its measurement as 9 units. When asked to visualize the table with a length of 8.2 units, she added rectangles with lengths of 0.2×1 square units at the end of each row. (Figure 11).

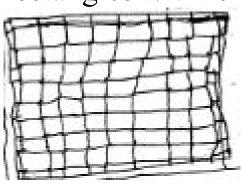


Figure 11. Representation of S5' Table of Chess.

When the researcher asked about the side length of a square with an area of 30 cm^2, she said, *"We find the square root of 30"* and explained as *"5 times 5 is 25. 6 times 6 is 36. Then we need to find something between them… It may be 5,5"*. When asked about 43, she said, *"6 times 6 is 36. Since the next one is 7 times 7, it is 49… It can be 6,5... Because it will be greater than 6 and less than 7"*. When asked whether there were any other numbers between them or not, she said, *"It can be 6,6… whichever is closer to 43…43 is closer to 49"*.

It is considered that participants S1 and S3 could coordinate the square root of square numbers with rational numbers in decimal and interiorized the location of the square root of a positive integer between two positive integers. But it is thought that the coordination of S5 was not strong enough like others.

With the evidence for encapsulation and progress to the object stage, the researcher asked S1 for the value of $2+\sqrt{14}$. S1 said, *"14 doesn't have things,… teacher, in other words, the same multiplier"* and when the researcher asked how he could calculate it, he said *"It must be a decimal number"* and added, *"This number is between 9 and 16"*. He explained the reason as *"Because the least square root of this is 9, not the square root, teacher. The previous 9, that is,… teacher. The next one is 16"*. When the researcher asked what the answer would be if it was 9 instead of 14, he said *"5"* and explained as *"the square root of 9 is... 3"*, if it was 16, he said *"6"* and explained it as *"the square root of 16 is... 4"*, and if it was 14, he said *"a number between 3 and 4"*. When asked about his estimated location, he said, *"It should be closer to 4"* and explained *"because 14 is closer to 16, …it can be 3,8"* and got a result as *"it is 5,8"*.

When the researcher asked S3 for the value of $2+\sqrt{17}$, S3 said, *"First we have to find the square root. The square root is a decimal number, so let's find out which two numbers it is between.*

Between 4^2 and 5^2" and thought its approximate value was *"it is closer to 4 to the power of 2,… it can be 4,2, approximately 4,2"*. Then she had the result as *"approximately 6.2"* (Figure 12).

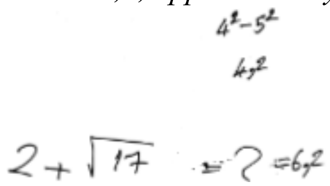


Figure 12. Answer of S3 for $2+\sqrt{17}$

S5 thought the result of $2+\sqrt{14}$ as *"First, we find the square root of 14. 4 times 4 is 16. 3 times 3 is 9. It is a decimal number again… In other words, it is between 3 and 4"*. When she was asked about the closer one, she said, *"3 times 3 is 9. The difference is 5. 4 times 4 is 16. The difference is 2. It is closer to 4… it can be 3.8… the result will be 5.8"*.

The researcher continued the interview by asking I know! Problem. S1 evaluated the answer 'between 6 and 7' of Merve's question 'square root of 46' as *"correct"*. He first said, *"the square root of 36 is 6, the square root of 49 is 7"* and then he tried to correct his expressions, *"the square of 6 is 36… 4 and 9… between 2 and 3"*. Then, he examined the answers given in the problem one by one and found which one gave the correct or incorrect answer. He hesitated about the square root of 2 in the problem, but later when the researcher questioned him, *"No, teacher. It is a decimal number so… a number between 1 and 2"* and explained as *"1 time 1 equals 1. 2 times 2 is 4. It is a number between 1 and 2"*. He thought of the square root of 195 and stated its location between two positive integers and said the closer one. His explanations were *"square of 12 is 144. Square of 13 is 169. 195 is bigger than these numbers"*, and he accepted Merve's answer in the problem as correct and said, *"Square of 10 is 100, square of 20 is 400. 195 is between these numbers"*. When the researcher asked the location of the square root of 195, she answered *"between 14 and 13"* and when asked the closer one, he replied *"14"*. Thus, it has been considered that S1 could estimate the location of the square root of a number between square numbers, state the closer one, and also calculate its value approximately.

S3 thought of the problem as *"…which two integers are the square root of 46? 46… which two numbers I multiply make 46…, 6 times 6 is 36, 7 times 7 is 49,… between 6 and 7. True. …between which two positive numbers is the square root of 87,… the square root of 87,… 9 times 9 is 81, 10 times 10 is 100, this is wrong"*. He said *"1 time 1 is 1. 2 times 2 is 4. Then it will be between 1 and 2. This is wrong…"* and completed the solution correctly. Regarding the square root of 195, he said *"I don't know the square root of 13"* and then calculated the square of 13 as 169, and quickly said that the answer had to be between the squares of 13 and 14.

S5 also examined the answers given in the problem and then explained which one was right or wrong. When asked about the square root of 2, she interpreted it correctly with an uncertain tone and explained as *"it is between 1 and 2,… square of 1 is 1, square of 2 is 4. 2 is between them, therefore… it is something decimal"*. She was also able to find the square root of 195.

It is considered that the participants used the square root symbol outside the original context of geometry and square numbers, extended it to non-square numbers, estimated decimals between integers, and then used it in a new situation with the addition of integers. So, they could construct the square root of natural numbers as an object.

DISCUSSION AND CONCLUSIONS

In the study, it was determined that the genetic decomposition proposed before the RME activities and the schemas of the students seen as a result of these activities were consistent with each other. It has been seen that the environment designed with the problem situations within the framework of this context by associating the concepts with real-life situations generally supports the concept formation. The reason for the difficulties encountered by the participants in constructing the concept of the square root is that they encountered this concept for the first time in both formal and informal life, its irrationality, and therefore it is thought to be more abstract. At the same time, it has been seen that the deficiencies and misconceptions in the preliminary information, which are thought to be necessary for the formation of the square root concept, make the construction of the square root concept difficult and therefore affect it. However, it was observed that the participants were generally able to conceptualize the square root of square numbers and to determine the location of the square root of positive integers between which two positive integers. In this respect, it can be said that the results of this study support the studies (e.g., Bray, and Tangney (2016), Juandi, Kusumah, and Tamur, (2022), Ozdemir and Uzel (2011), Sitorus and Masrayati (2016)) that advocate the positive effect of RME on the teaching of the concepts. In addition, it was determined that the participants knew the square root of a square number and the location of the square root of a (non-square) positive integer between two natural numbers. But sometimes they had difficulties in estimating their locations or calculating approximate values of these numbers when performing operations on square root values or displaying them on the number line. It can be said that this situation is partially similar to the results of studies (e.g. Arbour (2012), Güven, Çekmez ve Karataş, (2011), Ercire, Narlı, and Aksoy, (2016)) which state that students learn irrational and rational numbers at the knowledge level, but have problems in understanding whether a number is irrational or rational.

While constructing the concept of the square root of square numbers, all three participants adhered to the visual of the contextual problem given in the action stage. In this stage, the advanced participant set out with the idea that the area measurement can take value in the square unit and thought that the area of the square can be found by dividing it into unit squares. However, his idea remained attached to the visual in the context and he obtained a single result by considering only one of the different values that can be considered in this modeling problem, where no numerical data was given and the result can take different values. Likewise, the intermediate participant had similar thoughts, and the fact that there was no numerical data in the problem and that he had taken a value for the solution bothered him, and he thought that using variables would be more meaningful. The lower-intermediate participant, who had similar thoughts to other participants, had difficulties due to the inability to construct the area measurement units exactly. It was seen that the main reasons underlying the three participants' adherence to the visual in solving the contextual problem and difficulty in transitioning from the action phase to the process phase were that they could not consider that there could be different numbers of unit squares due to the area-edge relationship while calculating the area of the square, that they could not interiorize the unit concept, and that they have insufficient modeling skills in solving the contextual problem used in the teaching environment.

It was observed that the participants had different thoughts while trying to find the side length of the square given as the chessboard in the contextual problem. In the action stage, the lower-intermediate participant, who has incomplete/incorrect information on the area and perimeter measurement, exponential expression, performed operations such as dividing the area measure of the square by 2 while trying to find the side length of the square whose area is known. It was determined that this participant had thought that the values of the exponential numbers would be found by multiplying the base and the exponent so that the reverse operation would be achieved by dividing by 2. Therefore, he had difficulty in establishing a relationship between the number of unit squares he deals with and the edge area and in making this situation independent from the problem. The advanced participant, who first thought that the area of the given square could be equal to only a single value, later interiorized that the multiples of the side length obtained from this value could also be considered as different values and passed to the process stage. The lower-intermediate participant thought about the area measure instead of the side length and thought that there could be values that are multiples of this area measure value. However, since the numbers he received were not square numbers, he could not find the side length from the area measurement. The intermediate participant established an equation on the area measurement that changes depending on the side length of the given square. In addition, although the participants did not express that the number they should receive has to be a square number, they realized that they had to choose the numbers that are the product of two same numbers or the square of a number as the area value, and they found the square roots of numbers by coordinating this information with factoring, exponential expressions or repeated multiplication.

In the process stage, it was seen that the participants could generally relate the number of unit squares in the given area of the square with the side length of the square, recognize that the meanings of these concepts were different, state that the side lengths could change according to the size of the unit squares taken, but they could not use this information actively in the context of the given problem. It has been observed that the participant who can interiorize this information was the intermediate participant who could say different values for the side length of the given area of the square and could also calculate the area measurement that changes according to these values.

Interestingly, it was observed that the advanced participant had difficulty in interpreting the square root of the square number in exponential form and first thought that the square root of this exponential expression could not be found. On the other hand, the intermediate participant found the square root value of the square number in exponent form without any difficulty and interiorized this knowledge. Also, this participant stated that there is no need to perform operations in finding the square root of the square of a number and that these operations indicate opposite situations. It was observed that the lower-intermediate participant did not make any comments. In addition, it was determined that all three participants stated that 1 cannot be thought of as a square number because it is equal to itself, and therefore, it will not have a square root. Afterwards, it was seen that the participants interiorized the concept of the square root as a situation of finding a number whose square is the given number, and they were able to generalize the knowledge that the square roots of numbers with a square exponent are equal to the number in their base, thanks to the coordination they established. As a result of the interviews, all three participants with different readiness levels conceptualized the square root of square numbers as an object, but the coordination with the unit, area-side relationship, perimeter-side relationship, exponential expressions, and factorization were very important in conceptualization and that the participants expressed 'deleting the square, finding the side from the area, the opposite of squaring' for the concept of the square root as phenomena.

While thinking about the location of the square root of (non-square) positive integers between two natural numbers, the advanced participant, who constructed the decimal representations of rational numbers stronger than the others, carried out coordination by considering other rational numbers between these numbers. During the coordination, this participant made mistakes in multiplying the rational numbers in decimal forms, but these errors did not complicate his thoughts so much. Similarly, it was observed that the intermediate participant established coordination between square numbers and the square roots of these numbers and decimal notations. It can be said that these participants were able to find the location of the square root of (non-square) positive integers between two natural numbers, they were able to determine the closer natural number, and also they could estimate their approximate values. In this sense, it is thought that they could conceptualize the square root of positive integers.

It was observed that the lower-intermediate participant had difficulty in performing operations on non-integer numbers and made mistakes while expressing decimal notations and performing operations in this notation. However, he was able to recognize the value of the square root of (non-square) positive integers between two natural numbers with the guidance of the researcher.

While determining the location of the square root of positive integers between two natural numbers, it was observed that the participants were able to encapsulate that there are infinitely many numbers between two natural numbers and that the square roots of (non-square) positive integers will always locate between these two natural numbers. Also, it has been observed that they could correctly represent the approximate values on the number line. As a result, it can be said that they were able to generalize and construct as an object, and coordination of rational numbers, their decimal notations, and their operations were very important. It has been seen that the concepts that form the basis of the square root concept greatly affect the students' formation of this concept. For this reason, concepts such as area and perimeter measurement, exponential numbers, unit, rational numbers, and decimal representations and their operations should be formed strongly before the teaching process of the concept.

In the study, it was observed that the students were not familiar with the RME-based environment, modeling situations, and had difficulty in reasoning and making comments. For this reason, it will be beneficial to design contextual situations, to associate them with real life that students be familiar with, to present them in an environment where students are allowed to make sense of them, and to use similar teaching designs where each student takes responsibility for their learning. In addition, the students' in-group discussions helped them realize and correct each other's mistakes, and support their learning by sharing their knowledge.

References

[1] Arbour, D. (2012). *Students' Understanding of Real, Rational, and Irrational Numbers* (Doctoral dissertation, Concordia University). Retrieved from https://spectrum.library.concordia.ca/id/eprint/973825/

[2] Arcavi, A., Bruckheimer, M., & Ben-Zvi, R. (1987). History of mathematics for teachers: The case of irrational numbers. *For the learning of mathematics*, *7*(2), 18-23.

[3] Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In *Research in Collegiate mathematics education II.* CBMS issues in mathematics education (Vol. 6, pp. 1–32). Providence, RI: American Mathematical Society.

[4] Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York: Springer.

[5] Bray, A. & Tangney, B. (2016). Enhancing student engagement through the affordances of mobile technology: a 21st-century learning perspective on Realistic Mathematics Education. *Mathematics Education Research Journal 28,* 173–197.

[6] Campbell, S. R., & Zazkis, R. (Eds.). (2002). *Learning and teaching number theory: Research in cognition and instruction* (Vol. 2). Greenwood Publishing Group.

[7] Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions.* Thousand Oaks, CA: Sage.

[8] De Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop et al. (Eds.). *International handbook of mathematics education* (pp. 49-97). The Netherlands: Kluwer Academic Publishers.

[9] Dubinsky, E. (2002). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall, (Ed.) *Advanced Mathematical Thinking.* Dordrecht: Springer.

[10] Ercire, Y. E., Narlı, S., & Aksoy, E. (2016). Learning difficulties about the relationship between irrational number set with rational or real number sets. *Turkish Journal of Computer and Mathematics Education*, *7*(2), 417.

[11] Fauzan, A. (2002). *Applying realistic mathematics education (RME) in teaching geometry in Indonesian primary schools*. (Doctoral Dissertation), University of Twente. Retrieved from https://ris.utwente.nl/ws/files/6073228/thesis_Fauzan.pdf

[12] Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics*, *29*(1), 29-44.

[13] Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: Riedel Publishing.

[14] Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2)*,* 155-177.

[15] Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of curriculum studies*, *32*(6), 777-796.

[16] Guven, B., Cekmez, E., & Karatas, I. (2011). Examining preservice elementary mathematics teachers' understandings about irrational numbers. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 21*(5), 401-416.

[17] Juandi, D., Kusumah, Y. S., & Tamur, M. (2022). A meta-analysis of the last two decades of realistic mathematics education approaches. *International Journal of Instruction, 15*(1), 381-400.

[18] Kidron, I. (2018). Students' conceptions of irrational numbers. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 94-118.

[19] Ministry of National Education [MNE]. (2018). *Elementary and middle school mathematics program: Grades 1, 2, 3, 4, 5, 6, 7, and 8.* Turkish Republic Ministry of National Education.

[20] Mitchelmore, M. C. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. In D. Edge and Y. B. Ha (Eds.), *Mathematics education for a knowledge-based era, Proceedings of the Second East Asia Regional Conference on Mathematics Education and the Ninth Southeast Asian Conference on Mathematics Education*, (pp.157–167). Singapore: Association of Mathematics Educators.

[21] National Council of Teachers of Mathematics [NCTM], (2000). *Principles and standards for school mathematics*. Reston, Va. NCTM.

[22] Ozdemir, E. & Uzel, D. (2011). The effect of realistic mathematics education on student achievement and student opinions towards instruction. *Hacettepe University Journal of Education*, *40*, 332-343.

[23] Patel, P., & Varma, S. (2018). How the abstract becomes concrete: Irrational numbers are understood relative to natural numbers and perfect squares. *Cognitive Science*, *42*(5), 1642-1676.

[24] Piaget, J. (1980). Adaptation and intelligence (S. Eames, Trans.). Chicago: University of Chicago Press. (Original work published 1974).

[25] Sitorus, J. & Masrayati. (2016). Students' creative thinking process stages: Implementation of realistic mathematics education. *Thinking Skills and Creativity*, *22*, 111-120.

[26] Sirotic, N., & R. Zazkis. (2007a). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics*. 65(1):49–76.

[27] Sirotic, N., & R. Zazkis. (2007b). Irrational numbers on the number line—Where are they? *International Journal of Mathematical Education in Science and Technology*. *38*(4), 477–488.

[28] Treffers, A. (1991). Realistic mathematics education in the Netherlands 1980-1990. In L. Streefland (Ed.), *Realistic Mathematics Education in Primary Schools.* (pp. 11–20). Utrecht: Freudenthal Institute.

[29] Van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the learning of mathematics*, *25*(2), 2-23.

[30] Van den Heuvel-Panhuizen, M. & Drijvers, P. (2014). "Realistic mathematics education", In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 521-525), London: Springer.

[31] Von Glaserfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45–67). New York: Springer.

[32] Voskoglou, M., & Kosyvas, G. (2012). Analyzing students' difficulties in understanding real numbers. *REDIMAT*, *1*(3), 301-226.

[33] Yilmaz, R. & Argun, Z. (2018). Role of visualization in mathematical abstraction: The case of congruence concept. *International Journal of Education in Mathematics, Science and Technology (IJEMST), 6*(1), 41-57.

[34] Yilmaz, R. (2020). Prospective Mathematics Teachers' Cognitive Competencies on Realistic Mathematics Education. *Journal on Mathematics Education*, *11*(1), 17-44.

[35] Yin R. K. (2003). *Case study research, designs, and methods.* (3rd Ed.). California: Sage Publications.

[36] Zandieh, M. & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *The Journal of Mathematical Behavior*, *29*(2), 57-75.

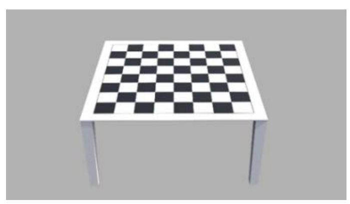
[37] Zazkis, R., & Sirotic, N. (2010). Representing and defining irrational numbers: Exposing the missing link. *CBMS Issues in Mathematics Education, 16,* 1-27.

APPENDIX

Chess Problem

1. The head of the school is considering preparing a place in the school garden where the students can play chess to increase their interest in the game of chess. To do this, he tiled a chess board with black and white tiles in a suitable place in the garden. He wants to surround this area with a security strip so that it will not be stepped on until it dries. Since he knows the measurement of the area from the laid tiles, what minimum length of the strip should he use for this?

2. The head of the school was pleased with the increasing interest in chess and decided to build a chess table for each classroom so that the students could continue playing in cold weather. First, he bought sticky chessboard pictures to apply on the surfaces of the tables. Then he asked a carpenter to make tables that would have equal margins on the edges when these pictures were pasted. If the carpenter wants to reduce the cost, what minimum edge length can the table have?



Parkour Problem

For the parkour race, flags will be planted on the 100-meter linear line at the points where the squares of the positive integers coincide. Let's consider 10-meter segments such as 0-10, 10-20, 20-30, 30-40... starting from the beginning.

Parkour 1: *In how many of the 10-meter segments will multiple flags be planted?*
Parkour 2: *In how many of the 10-meter segments will the flag not be planted?*
Parkour 3: *How many flags will someone pass if leaves the race at 57 meters?*
Parkour 4: *Teams A and B each of them have three athletes racing. Two athletes from team A leave the race at the 13^{th} and 83^{rd} meters. Two athletes from team B leave the race at the 51^{st} and 67^{th} meters. The other two athletes complete the race. For each flag passed, points are scored as the square root of the number on which it is planted.*

Which team wins the race?

I know! Problem

1. *Merve (M) and Elif (E) compete in the TV play "I Know!". The questions come out separately from the envelopes of the two competitors. The moderator announces the scores of the competitors when the questions in the envelopes are finished. The questions (Q) and the answers (A) of the competitors given are as follows:*

Q of M: *Between which two positive integers is the square root of 46?*
A of M: *6 and 7*
Q of E: *Between which two positive integers is the square root of 87?*
A of E: *10 and 11*
Q of M: *Between which two positive integers is the square root of 2?*
A of M: *2 and 4.*
Q of E: *Between which two positive integers is the square root of 101?*
A of E: *10 and 11.*
Q of M: *Between which two positive integers is the square root of 91?*
A of M: *8 and 9.*
Q of E: *Between which two positive integers is the square root of 75?*
A of E: *8 and 9.*

Which competitor will the moderator announces as the winner of the competition?

2. *In the competition, the moderator explains the new rule as "Both competitors will answer the question by pressing the button first. The correct answer will be evaluated and the competitor who gives the correct answer will score points". The competition continues:*

Moderator: *Between which two positive integers is the square root of 195? (Elif presses the button at first.)*
Elif: *12 and 13 (Merve presses the button)*
Merve: *10 and 20. (Elif presses the button)*
Elif: *11 and 12.*

When the moderator confirms that the answer of Elif was correct, Merve objects and claims that her answer is correct. Do you think Merve is right? Please explain.