MA3403 Algebraic Topology
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Lecture 05

5. Relative homology and long exact sequences

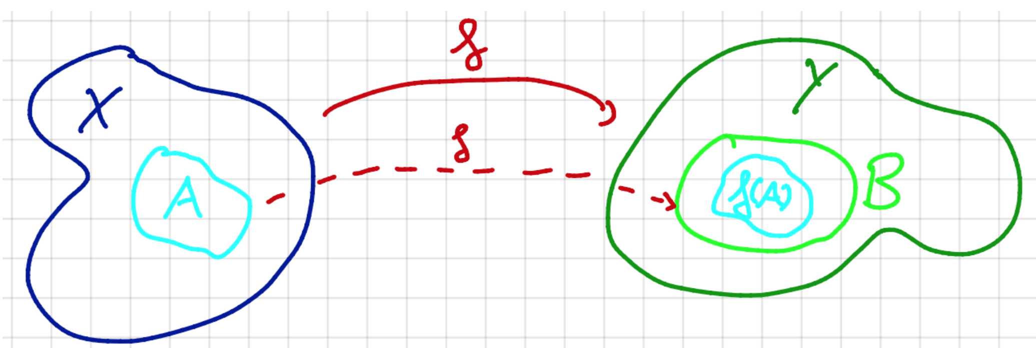
If we want to show that singular homology groups are useful, we need to be able to compute them. For H_0 that was not so difficult. But for $n \geq 1$, we need to develop some techniques.

In general, if you would like to compute something for spaces, it is always a **good idea to think about the relation to subspaces**. Maybe the information on smaller subspaces provides insides on the whole space. That is the idea we are going to exploit now for homology groups.

Let X be a topological space and let $A \subset X$ be a subset. We can consider (X,A) as a **pair of spaces**. If (Y,B) is another such pair, then we denote by

$$C((X,A),(Y,B)) := \{f \in C(X,Y) : f(A) \subset B\}$$

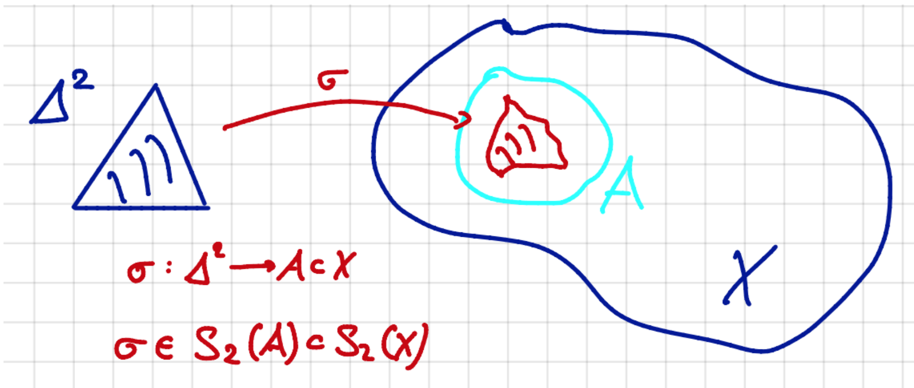
the set of continuous maps which respect the subspaces. In fact, we get a category $\mathbf{Top_2}$ of pairs of topological spaces.



Given a pair of spaces $A \subset X$, any n-simplex of A defines an n-simplex on X:

$$(\Delta^n \xrightarrow{\sigma} A) \mapsto (\Delta^n \xrightarrow{\sigma} A \subset X).$$

The induced map $S_n(A) \to S_n(X)$ is **injective**. Hence we are going to identify $S_n(A)$ with its image in $S_n(X)$ and consider $S_n(A)$ as a **subgroup** of $S_n(X)$.



Definiton: Relative chains

We define the group of **relative n-chains** by

$$S_n(X,A) := S_n(X)/S_n(A).$$

The group $S_n(X,A)$ is free, since the quotient map sends basis elements to basis elements, and is generated by the classes of n-simplices of X whose image is not entirely contained in A.

Since the boundary operator is defined via composition with the face maps, it satisfies

$$\partial(S_n(A)) \subset S_{n-1}(A) \subset S_{n-1}(X).$$

For, if the image of $\sigma\colon \Delta^n \to X$ lies in A, then so does the image of the compisite $\Delta^{n-1} \hookrightarrow \Delta^n \to X$.

Thus ∂ **induces a homomorphism** $\bar{\partial}$ on $S_n(X,A)$ and we have a commutative diagram

$$\begin{array}{ccc} S_n(X) & \longrightarrow & S_n(X,A) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\bar{\partial}} \\ S_{n-1}(X) & \longrightarrow & S_{n-1}(X,A). \end{array}$$

Since $\partial \circ \partial = 0$ and since $S_n(X) \to S_n(X,A)$ is surjective, we also have

$$\bar{\partial} \circ \bar{\partial} = \mathbf{0}.$$

We define **relative n-cycles** and **relative n-boundaries** by

$$Z_n(X,A) := \operatorname{Ker}(\bar{\partial}\colon S_n(X,A) \to S_{n-1}(X,A)) \text{ and}$$
$$B_n(X,A) := \operatorname{Im}(\bar{\partial}\colon S_{n+1}(X,A) \to S_n(X,A)).$$

Definition: Relative homology

The **nth relative homology group** of the pair (X,A) is defined as

$$H_n(X,A) := Z_n(X,A)/B_n(X,A).$$

Roughly speaking, relative homology groups measure the difference between the homology of X and the homology of A. Let us try to make this more precise. That an n-chain c in $S_n(X)$ represents a **relative** n-cycle means that $\bar{\partial}(\bar{c}) = 0$ in $S_n(X)/S_n(A)$, i.e., $\partial(c) \in S_n(A)$. Hence it just means that the imgage of the boundary of c lies in A.

So let us consider the **preimage** of $Z_n(X,A)$ under the quotient map $S_n(X) \to S_n(X,A)$ and define

$$Z'_n(X,A) := \{c \in S_n(X) : \partial(c) \in S_{n-1}(A)\}.$$

Similarly, that an n-chain c in $S_n(X)$ represents a **relative** n-boundary means that there is an $n+1$-chain b such that

$$c \equiv \partial(b) \mod S_n(A), \text{ i.e., } c - \partial(b) \in S_n(A).$$

Hence the **preimage** of $B_n(X,A)$ under the quotient map is

$$B'_n(X,A) := \{c \in S_n(X) : \exists\, b \in S_{n+1}(X) \text{ such that } c - \partial(b) \in S_n(A)\}.$$

Now we observe that $Z_n(X,A) = Z'_n(X,A)/S_n(A)$ (since $S_n(X,A)$ is $S_n(X)/S_n(A)$) and $B_n(X,A) = B'_n(X,A)/S_n(A)$. Hence we get

$$H_n(X,A) = \frac{Z_n(X,A)}{B_n(X,A)} = \frac{Z'_n(X,A)/S_n(A)}{B'_n(X,A)/S_n(A)} = \frac{Z'_n(X,A)}{B'_n(X,A)}.$$

In other words, we could also have used the latter quotient to define $H_n(X,A)$.

Empty subspaces

As a special case with $A = \emptyset$ we get

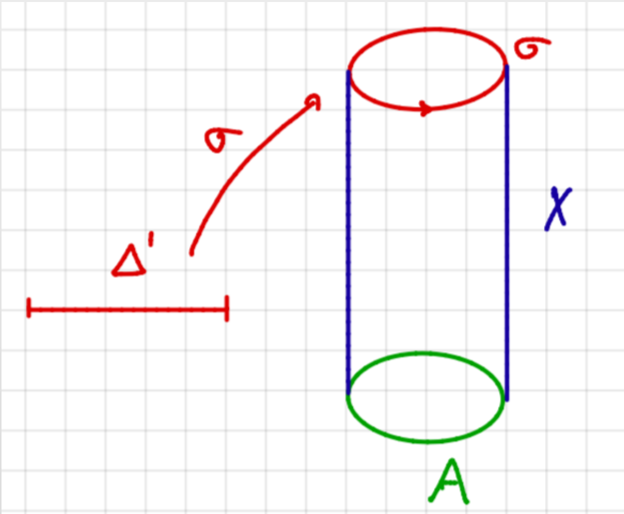
$$Z'_n(X,\emptyset) = Z_n(X),\ B'_n(X,\emptyset) = B_n(X), \text{ and } H_n(X,\emptyset) = H_n(X).$$

Now let us have a look at two **examples** to see how the images of simplices in $H_n(X)$ and $H_n(X,A)$ can differ.

Example: Relative cycles on the cylinder

Let $X = S^1 \times [0,1]$ be a cyclinder over the circle, and let the subspace $A = S^1 \times 0 \subset X$ be the **bottom circle**.
We construct an element in the relative homology $H_1(X,A)$ by taking a **1-simplex**

$$\sigma\colon \Delta^1 \to X,$$
$$(te_1, (1-t)e_0) \mapsto (\cos(2\pi t), \sin(2\pi t), 1).$$



Since σ is a closed curve in X, we have $\sigma(e_0) = \sigma(e_1)$. Hence its boundary vanishes:

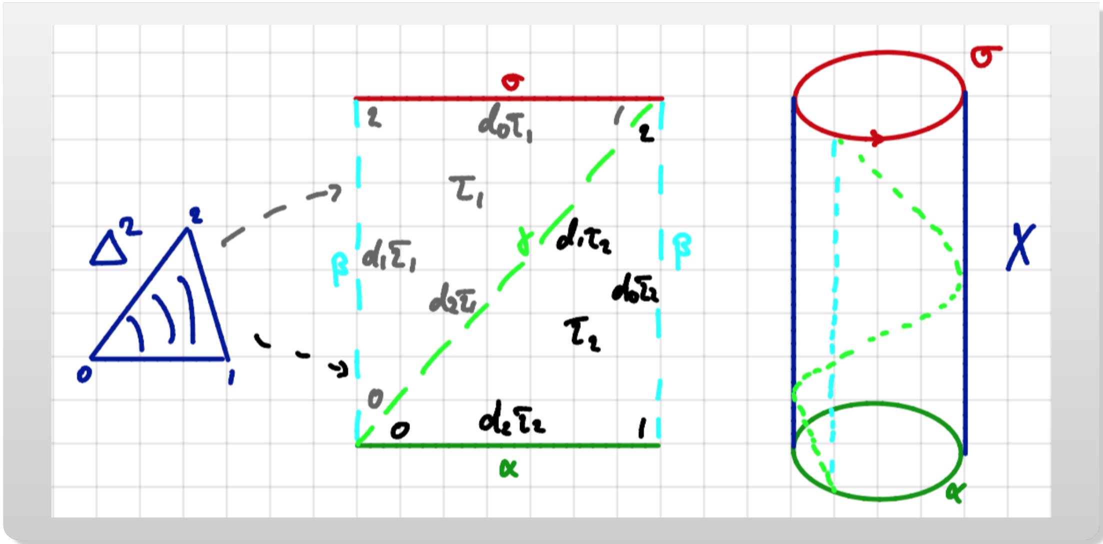
$$\partial(\sigma) = \sigma(e_1) - \sigma(e_0) = 0.$$

Therefore, $\sigma \in Z_1(X) \subset Z_1'(X,A)$. We will see very soon, that σ, in fact, represents a **nontrivial class in** $H_1(X)$.
However, the image of σ in the relative homology group $H_1(X,A)$ vanishes. For, consider the 2-chains τ_1 and τ_2 as indicated in the picture. Then we have

$$\begin{aligned}\partial(\tau_1 + \tau_2) &= d_0(\tau_1) - d_1(\tau_1) + d_2(\tau_1) + d_0(\tau_2) - d_1(\tau_2) + d_2(\tau_2)\\ &= \sigma - \beta + \gamma + \beta - \gamma + \alpha\\ &= \sigma + \alpha \text{ with } \alpha \in S_1(A).\end{aligned}$$

Hence, **modulo** $S_1(A)$, we have $\sigma \in B_1(X,A)$ and

$$[\sigma] = 0 \text{ in } H_1(X,A).$$



And the second example:

Example: Relative cycles on Δ^n

Let us look the standard n-simplex $X = \Delta^n$ as a space on its own. We would like to study it **relative to its boundary**

$$\partial\Delta^n := \bigcup_i \operatorname{Im}\phi_i^n \approx S^{n-1}$$

which is homeomorphic to the $n-1$-dimensional sphere.
There is a special n-simplex in $\operatorname{Sing}_n(\Delta^n) \subset S_n(\Delta^n)$, called the **universal n-simplex**, given by the **identity** map $\iota_n\colon \Delta^n \to \Delta^n$. It is **not a cycle**, since its boundary $\partial(\iota_n) \in S_{n-1}(\Delta^n)$ is the alternating sum of the faces of the n-simplex each of which is a **generator** in $S_{n-1}(\Delta^n)$:

$$\partial(\iota_n) = \sum_i (-1)^i \phi_i^n(\Delta^{n-1}) \neq 0.$$

However, each of these singular simplices lies in $\partial\Delta^n$, and hence $\partial(\iota_n) \in S_{n-1}(\partial\Delta^n)$.
Thus the class $\bar{\iota}_n \in S_n(\Delta^n, \partial\Delta^n)$ is a **relative cycle**. We will see later that the relative homology group $H_n(\Delta^n, \partial\Delta^n)$ is an infinite cyclic group generated by $[\bar{\iota}_n]$.

- **Long exact sequences**

Back to the general case. So let (X,A) be a pair of spaces. We know that the inclusion map $i\colon A \hookrightarrow X$ induces a homomorphism $H_n(i)\colon H_n(A) \to H_n(X)$. Moreover, the map of pairs $j\colon (X,\emptyset) \to (X,A)$ induces a homomorphism

$$H_n(j)\colon H_n(X) \cong H_n(X,\emptyset) \to H_n(X,A).$$

We claim that there is yet another interesting map.

Connectng homomorphism

For all n, there is a **connecting homomorphism**, which is often also called **boundary operator** and therefore usually also denoted by ∂,

$$\partial\colon H_n(X,A) \to H_{n-1}(A),\ [c] \mapsto [\partial(c)]$$

with $c \in Z'_n(X,A)$.

Let us try to make sense of this definition: We just learned that we can represent an element in $H_n(X,A)$ by an element $c \in Z'_n(X,A)$. Then $\partial(c)$ is an element in $S_{n-1}(A)$. In fact, $\partial(c)$ **is a cycle**, since it is a boundary and therefore

$$\partial(\partial(c)) = 0.$$

In particualar, $\partial(c)$ **represents a class** in the homology $H_{n-1}(A)$. Hence we can send $[c]$ under the connecting homomorphism to be the class $[\partial c] \in H_{n-1}(A)$.

It remains to **check that this is well-defined**, i.e., if we choose **another representative** for the class $[c]$ we need to show that we obtain the same class $[\partial(c)]$.

Another representative of $[c]$ in $Z'_n(X,A)$ has the form $c + \partial(b) + a$ with $b \in S_{n+1}(X)$ and $a \in S_n(A)$. Then we get

$$\partial(c + \partial(b) + a) = \partial(c) + \partial(a).$$

But, since $\partial(a) \in B_{n-1}(A)$, we get

$$[\partial(c)] = [\partial(c) + \partial(a)] \text{ in } H_{n-1}(A).$$

Thus, the connecting map is well-defined. And it is a **homomorphism**, since ∂ is a homomorphism.

Hence we get a sequence of homomorphisms

$$H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A).$$

It is an exercise to check that the connecting homomorphism is natural:

The connecting homomorphism is functorial

For any $f \in C((X,A),(Y,B))$, the following diagram commutes

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{n-1}(A) & \xrightarrow[H_{n-1}(f_{|A})]{} & H_{n-1}(B). \end{array}$$

In fact, the existence of the connecting map, the above sequence and its properties can be deduced by a purely algebraic process, that we will recall below. For, the relative chain complex fits into the **short exact sequence** of chain complexes

$$0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0.$$

Such a sequence induces a **long exact sequence** in homology of the form

$$\cdots \xrightarrow{H_{n+1}(i)} H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow[H_n(i)]{} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow[H_{n-1}(i)]{} \cdots$$

A digression to homological algebra

Maps of chain complexes

Let A_* and B_* be two chain complexes. A morphism of chain complexes $f_*\colon A_* \to B_*$ is a sequence of homomorphisms $\{f_n\}_{n\in\mathbb{Z}}$

$$f_n\colon A_n \to B_n \text{ such that } f_{n-1} \circ \partial_n^A = \partial_n^B \circ f_n \text{ for all } n \in \mathbb{Z}.$$

A homomorphism of chain complexes induces a homomorphism on homology

$$H_n(f)\colon H_n(A_*) \to H_n(B_*),\ [a] \mapsto [f_n(a)].$$

Check, as an exercise, that this is well-defined.

Consider a short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0, \tag{1}$$

i.e., for every n, the corresponding sequence of abelian groups is exact.

Since f_* and g_* are homomorphisms of chain complexes, they **induce maps on homology** groups

$$H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*). \tag{2}$$

Since $g_n \circ f_n = 0$, we know $H_n(g) \circ H_n(f) = 0$.

But is the sequence **exact** at $H_n(B_*)$, i.e., is $\mathrm{Ker}\,(H_n(g)) = \mathrm{Im}\,(H_n(f))$?

Let us look at an extended picture of the short exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_A} & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_C} & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_A} & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_C} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0
\end{array} \tag{3}$$

Let $[b] \in H_n(B_*)$ such that $H_n(g)([b]) = 0$. In fact, $[b]$ is represented by a **cycle**, i.e., some $b \in B_n$ with $\partial_B(b) = 0$. Since $H_n(g)([b]) = 0$, there is some $\bar{c} \in C_{n+1}$ such that $\partial_C(\bar{c}) = g_n(b)$. **By exactness** of (1), g_{n+1} is **surjective** and there is some $\bar{b} \in B_{n+1}$ with $g_{n+1}(\bar{b}) = \bar{c}$.

Now we can consider $\partial_B(\bar{b}) \in B_n$, and have $g_n(\partial_B(\bar{b})) = \partial_C(\bar{c})$ in C_n. What is the difference $b - \partial_B(\bar{b})$?

Well, it maps to 0 in C_n. **By exactness** of (1), there is some $a \in A_n$ such that $f_n(a) = b - \partial_B(\bar{b})$. Is a a **cycle**, and hence does it represent a homology class?

We know

$$f_{n-1}(\partial_A(a)) = \partial_A(f_n(a)) = \partial_B(b - \partial_B(\bar{b})) = \partial_B(b) - \partial_B(\partial_B(\bar{b})) = \partial_B(b).$$

But we assumed $\partial_B(b) = 0$. Thus $f_{n-1}(\partial_A(a)) = 0$. But since f_{n-1} is **injective**, this implies $\partial_A(a) = 0$. Hence a is indeed a cycle, and therefore represents a homology class $[a] \in H_n(A_*)$. It also follows

$$H_n(f)([a]) = [b - \partial_B(\bar{b}] = [b].$$

Thus sequence (2) is **exact**.

However, the map $H_n(A_*) \xrightarrow{H_n(f_*)} H_n(B_*)$ **may fail to be injective** and the map $H_n(B_*) \xrightarrow{H_n(g_*)} H_n(C_*)$ **may fail to be surjective**. That means sequence (2) does not fit into a short exact sequence, in general.

But we can connect all these sequences together for varying n and obtain a long exact sequence:

The homology long exact sequence

Let $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ be a short exact sequence of chain complexes. Then, for ach n, there is a functorial homomorphism

$$\partial \colon H_n(C_*) \to H_{n-1}(A_*)$$

such that the sequence

$$\cdots \xrightarrow{H_{n+1}(f_*)} H_{n+1}(C_*) \xrightarrow{\partial} H_n(A_*) \xrightarrow{H_n(f_*)} H_n(B_*) \xrightarrow{H_n(g_*)} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f_*)} \cdots$$

is **exact**.

Proof: This is a typical example of a **diagram chase**. We will illustrate it by constructing the connecting homomorphism ∂ and leave the rest as an exercise. It is more fun and, in fact, easier to do it yourself than to read it. All we need is to look again at the extended picture (3) of the short exact sequence above.

To construct $\partial \colon H_n(C_*) \to H_{n-1}(A_*)$, let $c \in C_n$ be a cycle. Since g_n is **surjective**, there is a $b \in B_n$ with $g_n(b) = c$. Since $\partial_C(c) = 0$ and the diagram **commutes**, we get $g_{n-1}(\partial_B(b)) = \partial_C(g_n(b)) = \partial_C(c) = 0$. Since the horizontal sequences are **exact**, this implies there is an $a \in A_{n-1}$ with $f_{n-1}(a) = \partial_B(b)$. In fact, there is a **unique** such a because f_{n-1} is **injective**.

Moreover, we claim that this a is a cycle. For, since the diagram **commutes**, we have $f_{n-2}(\partial_A(a)) = \partial_B(f_{n-1}(a)) = \partial_B(\partial_B(b)) = 0$. Since f_{n-2} is **injectve**, this implies $\partial_A(a) = 0$.

This means a represents a homology class and we define ∂ by sending the class of c to the class of a.

But we need to check that this does not depend on the choices we have made. So let $b' \in B_n$ be another element with $g_n(b') = c$, and let $a' \in A_{n-1}$ be the element that we find as above. Then we need to show $[a'] = [a]$ in $H_{n-1}(A_*)$, i.e., that $a' - a$ is a boundary. So we need an $\bar{a} \in A_n$ such that $\partial_A(\bar{a}) = a' - a$. We know $g_n(b'-b) = c-c = 0$. By **exactness**, there is an $\bar{a} \in A_n$ with $f_n(\bar{a}) = b'-b$. Since the diagram commutes, we have $f_{n-1}(\partial_A(\bar{a})) = \partial_B(b') - \partial_B(b)$. But we also have $f_{n-1}(a'-a) = \partial_B(b') - \partial_B(b)$ by definition of a' and a. Hence, since f_{n-1} is **injective**, we must have $\partial_A(\bar{a}) = a' - a$.

Finally, it is also clear from the construction that if c is a boundary, then a is zero.

This proves the existence of ∂. Moreover, we know already that the induced homology sequence is exact at $H_n(B_*)$. It remains to check exactness at $H_n(A_*)$ and $H_n(C_*)$. This is left as an exercise. **QED**

Why do we care about long exact sequences?

Well, they are extemely useful. For example, for a pair of space (X,A), if we can show $H_{n+1}(X,A) = 0$ and $H_n(X,A) = 0$, then $H_n(A) \cong H_n(X)$. Long exact sequences will be one of the **main computational tools** for studying interesting homology groups.

Furthermore, there is the famous Five Lemma (here in one of its variations):

Five Lemma

Suppose we have a commutative diagram

$$
\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}
$$

with exact rows. Then

- If f_2 and f_4 are surjective and f_5 injective, then f_3 is surjective.
- If f_2 and f_4 are injective and f_1 surjective, then f_3 is injective.

In particular, if f_1, f_2, f_4, and f_5 are isomorphisms, then f_3 is an isomorphism.

The proof is another diagram chase and left as an exercise. You should definitely do it, it's fun!

Here we rather state two **consequences**:

- Given a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

  in which f' and f'' are isomorphisms. Then f is an isomorphism.
- Back in topology, let $f\colon (X,A) \to (Y,B)$ be a map of pairs of spaces. If any two of $A \to B$, $X \to Y$ and $(X,A) \to (Y,B)$ induce isomorphisms, then so does the third. This observation will simplify our life a lot.