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A Theorem by Giusto Bellavitis on a Class of Quadrilaterals

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Abstract. In this note we prove a theorem on quadrilaterals first published by the Italian mathematician Giusto Bellavitis in the 1850s, but that seems to have been overlooked since that time. Where Bellavitis used the functional equivalent of complex numbers to prove the result, we mostly rely on trigonometry. We also prove a converse of the theorem.

1. Introduction

Since antiquity, the properties of various special classes of quadrilaterals have been extensively studied. A class of quadrilaterals that appears to have been little studied is that of those quadrilaterals for which the products of the two pairs of opposite sides are equal. In case a quadrilateral ABCD is cyclic as well, $ABCD$ is usually referred to as a *harmonic* quadrilateral (see [2, pp.90–92], [3, pp.159–160]). Clearly, however, the class of all quadrilaterals $ABCD$ for which $AB \cdot CD = AD \cdot BC$ includes non-cyclic quadrilaterals as well. In particular, all kites are included. As far as we have able to ascertain, no name for this more general class of quadrilaterals has ever been proposed. For the sake of brevity, we will refer to the elements in this class as *balanced* quadrilaterals. In his *Sposizione del metodo delle equipollenze* of 1854, the Italian mathematician Giusto Bellavitis (1803-1880) proved a curious theorem on such balanced quadrilaterals that seems to have been forgotten.[1] In this note, we will give an elementary proof of the theorem. In addition, we will show how the converse of Bellavitis' theorem is (almost) true as well. Our proof of the first theorem is different from that of Bellavitis. The converse is not discussed by Bellavitis at all.

2. Bellavitis' Theorem

Let the lengths of the sides AB, BC, CD and DA of a (convex) quadrilateral $ABCD$ be denoted by a, b, c and d respectively. Similarly, the lengths of the quadrilateral's diagonals AC and BD will be denoted by e and f. Let E be the point of intersection of the two diagonals. The magnitude of $\angle DAB$ will be

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[1] Bellavitis' book is very hard to locate. We actually used Charles-Ange Laisant's 1874 translation into French [1], which is available on-line from the Bibliothèque Nationale. In this translation, the theorem is on p.26 as Corollary III of Bellavitis' derivation of Ptolemy's theorem.

referred to as α, with similar notations for the other angles of the quadrilateral. The magnitudes of $\angle DAC$, $\angle ADB$ etc will be denoted by α_B, δ_C and so on (see Figure 1). Finally, the magnitude of $\angle CED$ will be referred to as ϵ.

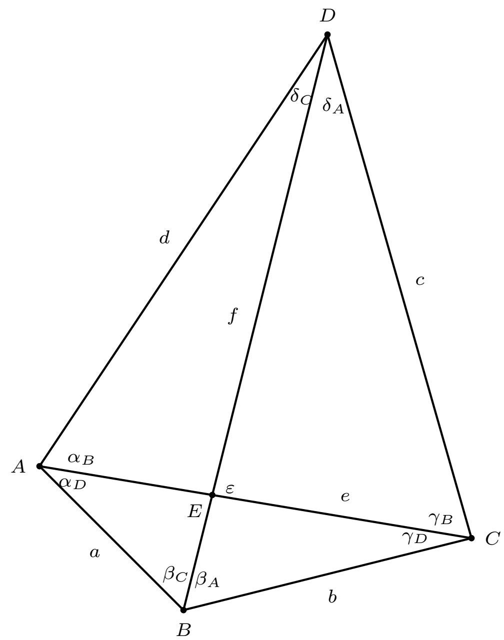

Figure 1. Quadrilateral Notations

With these notations, the following result can be proved.

Theorem 1 (Bellavitis, 1854)**.** *If a* (*convex*) *quadrilateral $ABCD$ is balanced, then*

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ.$$

Note that the convexity condition is a necessary one. The second equality sign does not hold for non-convex quadrilaterals. A trigonometric proof of Bellavitis' Theorem follows from the observation that by the law of sines for any balanced quadrilateral we have

$$\sin\gamma_B \cdot \sin\alpha_D = \sin\alpha_B \cdot \sin\gamma_D,$$

or

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha_D) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma_D).$$

That is,

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha + \alpha_B) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma + \gamma_B),$$

or

$$\cos(\gamma_B + \alpha_D) + \cos(\delta + \alpha) = \cos(\alpha_B + \gamma_D) + \cos(\delta + \gamma).$$

By cycling through, we also have

$$\cos(\delta_C + \beta_A) + \cos(\alpha + \beta) = \cos(\beta_C + \delta_A) + \cos(\alpha + \delta).$$

Since $\cos(\alpha + \beta) = \cos(\delta + \gamma)$, adding these two equations gives

$$\cos(\gamma_B + \alpha_D) + \cos(\delta_C + \beta_A) = \cos(\alpha_B + \gamma_D) + \cos(\beta_C + \delta_A),$$

or

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos\frac{1}{2}(\gamma_B + \alpha_D - \delta_C - \beta_A) \\ =&\cos\frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) \cdot \cos\frac{1}{2}(\alpha_B + \gamma_D - \beta_C - \delta_A). \end{aligned}$$

Now, note that

$$\gamma_B + \alpha_D - \delta_c - \beta_A = 360 - 2\epsilon - \delta - \beta$$

and, likewise

$$\alpha_B + \gamma_D - \beta_C - \delta_A = 2\epsilon - \beta - \delta.$$

Finally,

$$\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) + \frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) = 180^\circ.$$

It follows that

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon + \frac{1}{2}(\beta + \delta)) \\ =&-\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon - \frac{1}{2}(\beta + \delta)), \end{aligned}$$

or

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon) \cos\frac{1}{2}(\delta + \beta) = 0.$$

This almost concludes our proof. Clearly, if neither of the last two factors are equal to zero, the first factor has to be zero and we are done. The last factor, however, will be zero if and only if $ABCD$ is cyclic. It is easy to see that any such quadrilateral has the angle property of Bellavitis' theorem. Therefore, in the case that $ABCD$ is cyclic, Bellavitis' theorem is true. Consequently, we may assume that $ABCD$ is not cyclic and that the third term does not vanish. Likewise, the second factor only vanishes in case $ABCD$ is orthogonal. For such quadrilaterals, we know that $a^2 + c^2 = b^2 + d^2$. In combination with the initial condition $ac = bd$, this implies that each side has to be congruent to an adjacent side. In other words, $ABCD$ has to be a kite. Again, it is easy to see that in that case Bellavitis' theorem is true. We can safely assume that $ABCD$ is not a kite and that the second term does not vanish either. This proves Bellavitis' theorem.

3. The Converse to Bellavitis' Theorem

Now that we have proved Bellavitis' theorem, it is only natural to wonder for exactly which kinds of (convex) quadrilaterals the angle sums $\delta_C + \gamma_B + \beta_A + \alpha_D$ and $\alpha_A + \beta_C + \gamma_D + \delta_A$ are equal. Assuming that the two angle sums are equal and working our way backward from the preceding proof, we find that

$$\sin\gamma_B \cdot \sin\alpha_D + K = \sin\alpha_B \cdot \sin\gamma_D$$

for some K. Likewise,

$$\sin\delta_C \cdot \sin\beta_A = \sin\beta_C \cdot \sin\delta_A + K.$$

So,

$$\frac{\sin\gamma_B}{\sin\alpha_B} - \frac{\sin\gamma_D}{\sin\alpha_D} = -\frac{K}{\sin\alpha_B \cdot \sin\alpha_D}$$

and

$$\frac{\sin\delta_C}{\sin\beta_C} - \frac{\sin\delta_A}{\sin\beta_A} = \frac{K}{\sin\beta_A \cdot \sin\beta_C}$$

or

$$\frac{d}{c} - \frac{a}{b} = -\frac{K}{\sin\alpha_B \cdot \sin\alpha_D}, \qquad \frac{a}{d} - \frac{b}{c} = \frac{K}{\sin\beta_A \cdot \sin\beta_C}.$$

If $K = 0$, we have $bd = ac$ and $ABCD$ is balanced. If $K \neq 0$, it follows that

$$\frac{d}{b} = \frac{\sin\beta_A \cdot \sin\beta_C}{\sin\alpha_B \cdot \sin\alpha_D}.$$

Cycling through twice also gives us

$$\frac{b}{d} = \frac{\sin\delta_C \cdot \sin\delta_A}{\sin\gamma_D \cdot \sin\gamma_B}.$$

We find

$$\sin\beta_A \cdot \sin\beta_C \cdot \sin\delta_C \cdot \sin\delta_A = \sin\alpha_B \cdot \sin\alpha_D \cdot \sin\gamma_D \cdot \sin\gamma_B.$$

Division of each side by $abcd$ and grouping the factors in the numerators and denominators appropriately shows that this equation is equivalent to the equation

$$R_{ABC} \cdot R_{ADC} = R_{BAC} \cdot R_{BCD},$$

where R_{ABC} denotes the radius of the circumcircle to the triangle ABC etc. Now, the area of ABC is equal to both $abe/4R_{ABC}$ and $\frac{1}{2}e \cdot EB \cdot \sin\epsilon$ with similar expressions for ADC, BAC, and BCD. Consequently, the relation between the four circumradii can be rewritten to the form $EB \cdot EC = EA \cdot EC$. But this means that $ABCD$ has to be cyclic. We have the following result:

Theorem 2. *Any* (*convex*) *quadrilateral* $ABCD$ *for which*

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ$$

is either cyclic or balanced.

4. Conclusion

We have not been able to find any references to Bellavitis' theorem other than in the *Sposizione*. Bellavitis was clearly mostly interested in the theorem because it allowed him to showcase the power of his method of equipollences.[2] Indeed, the *Sposizione* features a fair number of (minor) results on quadrilaterals that are proved using the method of equipollences. Most of these were definitely well-known at the time. This suggests that perhaps our particular result was reasonably well-known at the time as well. Alternatively, Bellavitis may have derived the theorem in one of the many papers that he published between 1833, when he first published on the method, and 1854. These earlier publications, however, are extremely hard to locate and we have not been able to consult any.[3] Whether the theorem originated with Bellavitis or not, it is not entirely surprising that this result seems to have been forgotten. The sums $\alpha_B+\beta_C+\gamma_D+\delta_A$ and $\beta_A+\gamma_B+\delta_C+\alpha_D$ do not usually show up in plane geometry. We do hope to finish up a paper shortly, however, in which these sums play a role as part of a generalization of Ptolemy's theorem to arbitrary (convex) quadrilaterals.

References

[1] G. Bellavitis, *Exposition de la méthode des equipollences* (traduit par C-A. Laisant) (Paris: Gauthier-Villars, 1874)

[2] W. Gallatly, *The Modern Geometry of the Triangle* (2nd ed.) (London: Hodgson, 1913)

[3] M. Simon, *Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, Teubner, Leipzig, 1906 (= *Jahresberichte der Deutschen Mathematiker-Vereinigung. Ergänzungsbände*, B. I).

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[2] This method essentially amounted to a sometimes awkward mix of vector methods and the use of complex numbers in a purely geometrical disguise. In fact, for those interested in the use of complex numbers in plane geometry, it might be a worthwhile exercise to rework Bellavitis' equipollences proof of his theorem to one that uses complex numbers only. This should not be too hard.

[3] See the introduction of [1] for a list of references.