Absolute geometry

1.1 Axiom system

We will follow the axiom system of Hilbert, given in 1899. The base of it are primitive terms and primitive relations. We will divide the axioms into five parts:

1. Incidence
2. Order
3. Congruence
4. Continuity
5. Parallels

1.1.1 Incidence

Primitive terms: point, line, plane
Primitive relation: "lies on"(containment)

Axiom I1 *For every two points there exists exactly one line that contains them both.*

Axiom I2 *For every three points, not lying on the same line, there exists exactly one plane that contains all of them.*

Axiom I3 *If two points of a line lie in a plane, then every point of the line lies in the plane.*

Axiom I4 *If two planes have a point in common, then they have another point in common.*

Axiom I5 *There exists four points not lying in a plane.*

Remark 1.1 *The incidence structure does not imply that we have infinitely many points. We may consider 4 points $\{A, B, C, D\}$. Then, the lines can be the two-element subsets and the planes the three-element subsets.*

1.1.2 Order

Primitive relation: "betweeness"

Notion: If A, B, C are points of a line then $(ABC) := B$ means the "B is between A and C".

Axiom O1 *If (ABC) then (CBA).*

Axiom O2 *If A and B are two points of a line, there exists at least one point C on the line AB such that (ABC).*

Axiom O3 *Of any three points situated on a line, there is no more than one which lies between the other two.*

Axiom O4 (Pasch) *Let A, B, C be three points not lying on the same line and let e be a line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line e passes trough a point D such that (ADB), it will also pass trough a point E such that either (BEC) or (AEC).*

Remark 1.2 *The first three order axioms do not implies that the line has infinitely many points. Let a model of the line be the edge of a regular pentagon. We define the betweeness such that (XYZ) if and only if the triangle of X, Y, and Z is an isosceles triangle with Y as edge vertex.*

Lemma 1.3 *In the Pasch axiom, either (BEC) or (AEC) but not both at once.*

Proof. Indirect: $(AEC) \wedge (BFC) \Rightarrow (DEF)$

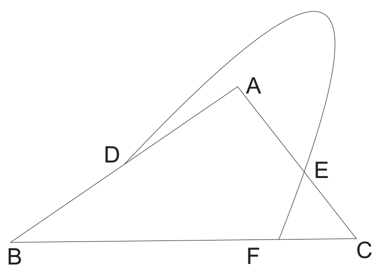

Figure 1.1: Proof: Lemma 1.3

We consider the Pasch axiom for the non-collinear points D, B, F and the line of AC (see Figure 1.1). Therefore either (BAD) or (BCF), but we supposed in the axiom that (ADB) and in the indirect assumption that (BFC). ■

Definition 1.4 *Let A and B be two points. The segment $\overline{AB}$ consists of those points of C, for which (ACB).*

Lemma 1.5 *Any segment has a point.*

Proof.

1. C is not on the line of AB
2. Let D be such that (ACD) (see Axiom O2)
3. Let E be such that (DBE) (see Axiom O2)
4. We use the Pasch axiom for the points A, D, B and the line $CE \Rightarrow \exists F : (AFB)$

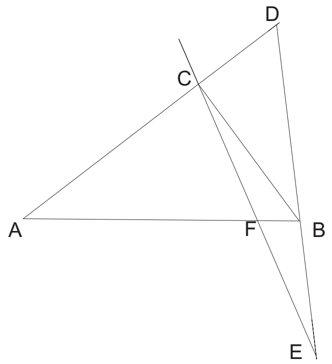

Figure 1.2: Proof: Lemma 1.5

■

Corollary 1.6 *We can define polygonal chain, polygon, half-plane, ray, angle and convexity.*

Theorem 1.7 (Jordan) *Any simple, closed polygon splits the plane into two parts. One of the parts does not contain rays t this is called the inside of the polygon.*

1.1.3 Congruence

Primitive relation: "congruence of segments and angles"

Axiom C1 *The congruence of segments and angles are equivalence relation.*

Axiom C2 *Every segment can be laid off upon a given side of a given point of a given line.*

Axiom C3 *Let $\overline{AB}$ and $\overline{BC}$ be two segments of a line which have no points in common aside from B and let $\overline{A'B'}$ and $\overline{B'C'}$ be two segments of the same or of another line, likwise no point other than B' in common. Then, if $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, we have $\overline{AC} \cong \overline{A'C'}$.*

Axiom C4 *An angle congruent to a given angle can be added to a given ray of a given line of a given plane in both half-planes determined by the given line.*

Axiom C5 *If, in two triangles ABC and $A'B'C'$ the congruences $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$ and $BAC\angle \cong B'A'C'\angle$ hold, then the congruence $\overline{BC} \cong \overline{B'C'}$ also holds.*

Remark 1.8 *With congruence we can*

- *compare segments and angles.*
- *introduce "length" like an additive non-negative, real function.*
- *introduce right angle like an angle divides the plane into four congruent parts.*
- *use the Descartes coordinate system with analytic geometry.*

1.1.4 Continuity

Axiom 1.9 (Archimedes) *If $\overline{AB}$ and $\overline{CD}$ are any segments then there exists an integer number n such that n segments $\overline{C_nD_n}$ constructed contiguously from A, along the ray from A through B, will pass beyond the point B.*

Axiom 1.10 (Cantor) *If $\overline{P_iQ_i}$ $i = 1 \dots \infty$ are closed segments of a line such that $\overline{P_iQ_i} \supseteq \overline{P_jQ_j}$ if $j > i$, then $\bigcap_{i=1}^{\infty} \overline{P_iQ_i} \neq \emptyset$.*

Axiom 1.11 (Dedekind) *For every partition of all the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.*

Theorem 1.12 *The Axioms 1.9 and 1.10 are equivalent with the Axiom 1.11.*

Proof. ONLY $(D) \Rightarrow (C)$
Let $A := \{S \in e | \exists i(SP_iQ_i)\}$ and $B := \{T \in e | \exists i(P_iQ_iT)\}$. Using the axiom, we get a point C such that $\forall S \in A$ and $\forall T \in B : (SCT) \Rightarrow (\forall i : C \in \overline{P_iQ_i})$. ∎

Remark 1.13 *Rational number line satisfies (A) but not (C).*

Question: Exists such model of the line that satisfies incidence, order and congruence but not (A)?
Answer: YES (Veronese)
We introduce ordering as follows:
$(x_1, y_1) > (x_2, y_2) \iff y_1 > y_2 \vee ((y_1 = y_2) \wedge x_1 > x_2)$
A point (x', y') is between (x_1, y_1) and (x_2, y_2) if and only if either $(x_1, y_1) < (x', y') < (x_2, y_2)$ or $(x_1, y_1) > (x', y') > (x_2, y_2)$.

Remark 1.14 *This model do not satisfies (C), e.g. $P_i := (n, 0)$, $Q_i := (-n, 1)$.*

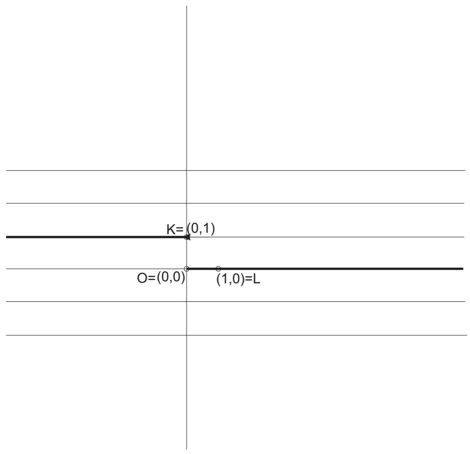

Figure 1.3: Non-Archimedean line model

1.1.5 Parallels

Axiom 1.15 (EUC) *Let a be any line and A a point not on it. Then, there is exactly one line in the plane, determined by a and A, that passes through A and does not intersect a.*

Axiom 1.16 (HYP) *Let a be any line and A a point not on it. Then, there are at least two lines in the plane, determined by a and A, that pass through A and do not intersect a.*

1.2 Absolute theorems

Lemma 1.17 *If we connect the points A and B which lie on different rays of an angle, then inside rays emanating from its vertex O intersect the segment $\overline{AB}$.*

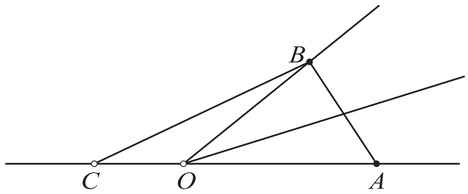

Figure 1.4: Proof: Lemma 1.17

Proof. Let C be such that (AOC) /Axiom O2/. We apply the Pasch axiom for the points A, B, C and the ray. ■

Lemma 1.18 (External angle) *Any external angle of a triangle is larger than its any internal angle not beside it.*

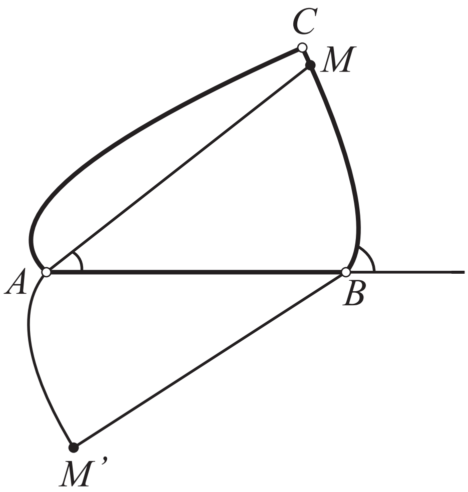

Figure 1.5: Proof: Lemma 1.18

Proof. INDIRECT $CAB\angle > CBD\angle$ where D is such that (ABD).
We lay off the $CBD\angle$ upon the $CAB\angle \Rightarrow M$ by using 1.17 lemma. Let M' be such that (CBM') and $\overline{BM} \cong \overline{M'B}$. Then $ABM'\angle = MBD\angle = MAB\angle \Rightarrow ABM'\triangle \cong BAM\triangle$. Therefore $M'AB\angle \cong ABM\angle \Rightarrow M$, A, M' are collinear and A=B. ■

Remark 1.19 *If two lines intersect another line at the same angle, then they cannot intersect each other. $\Rightarrow$ Exist non-intersecting lines in absolute geometry.*

Lemma 1.20 *If, in two triangles $ABC\triangle$ and $A'B'C'\triangle$ the congruences $\overline{AB} \cong \overline{A'B'}$, $ACB\angle \cong A'C'B'\angle$ and either $ABC\angle \cong A'B'C'\angle$ or $BAC\angle \cong B'A'C'\angle$ hold, then the triangles are congruent.*

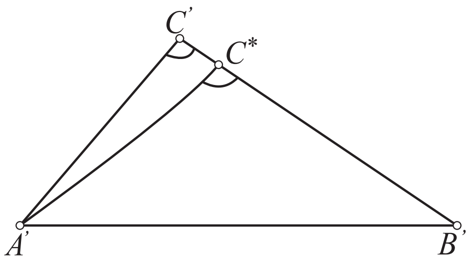

Figure 1.6: Proof: Lemma 1.20

Proof. Let *e.g.* $ABC\angle \cong A'B'C'\angle$ be true. Let C^* be on $B'C'$ such that $BC^* \cong B'C'$. Then $ABC^*\triangle \cong A'B'C'\triangle$ /Axiom C5/, therefore $AC^*B\angle \cong A'C'B'\angle$ and the previous remark implies that $C' = C^*$. ■

Lemma 1.21 *In any triangle, the greater angle is subtended by the greater side and the greater side subtends the greater angle.*

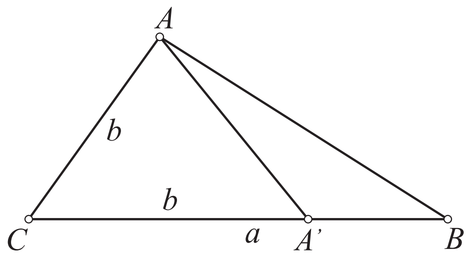

Figure 1.7: Proof: Lemma 1.22

Proof. Suppose that $\overline{AC} < \overline{BC}$. Let A' be such that $\overline{AC} \cong \overline{A'C}$ and $(CA'B)$. C lies on the bisector of $\overline{AA'}$. Therefore $CAA'\angle \cong CA'A\angle$. Using the external angle lemma, we obtain that $CA'A\angle > CBA\angle$. ■

Lemma 1.22 *For any triangle, the sum of the length of any two sides must be greater than the length of the remaining side.*

Proof. Let D be such that (BCD) and $\overline{CD} \cong \overline{AC}$. Then $ACD\triangle$ is isosceles, therefore $CDA\angle \cong CAD\angle$. But $BAD\angle > CAD\angle$ and using the previous lemma, we obtain that $|\overline{BD}| > |\overline{BA}|$. ■

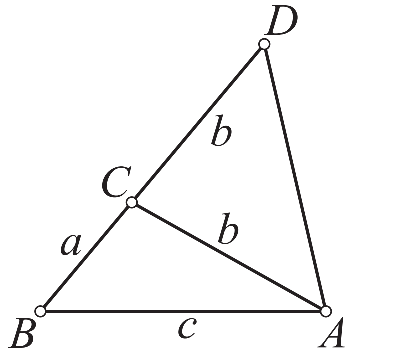

Figure 1.8: Proof: Lemma 1.22

Corollary 1.23 (Polygonal chain inequality) *Let $A_1, A_2, \ldots, A_n$ be a polygonal chain. Then $\sum_{i=1}^{n-1} |\overline{A_iA_{i+1}}| \geq |\overline{A_1A_n}|$.*

Lemma 1.24 (Arm-lemma) *If, in two triangle $ABC\triangle$ and $A'B'C'\triangle$ the congruences $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$ hold and $ABC\angle < A'B'C'\angle$, then $|\overline{AC}| < |\overline{A'C'}|$.*

Proof. We may assume that $A = A'$ and $B = B'$ Due to the lemma 1.17, the line BC intersects AC' and we obtain D.

1. case: (BDC): $AC'C\angle < BC'C\angle \cong BCC'\angle < ACC'\angle$. Therefore using the triangle inequality for the triangle $ACC'\triangle$ we obtain that $|\overline{AC}| < |\overline{A'C'}|$.

2. case: (BCD): In any isosceles triangle, the base angle is smaller then $\frac{\pi}{2}$, since the sum of the two base angle is smaller then the straight angle /External angle lemma/. In our case $BCC'\angle < \frac{\pi}{2} \Rightarrow C'CD\angle > \frac{\pi}{2} \Rightarrow |\overline{C'D}| > |\overline{CD}|$. Using the triangle inequality and, we obtain that: $|\overline{AC}| < |\overline{AD}| + |\overline{DC}| < |\overline{AD}| + |\overline{DC'}| = |\overline{AC'}|$. ■

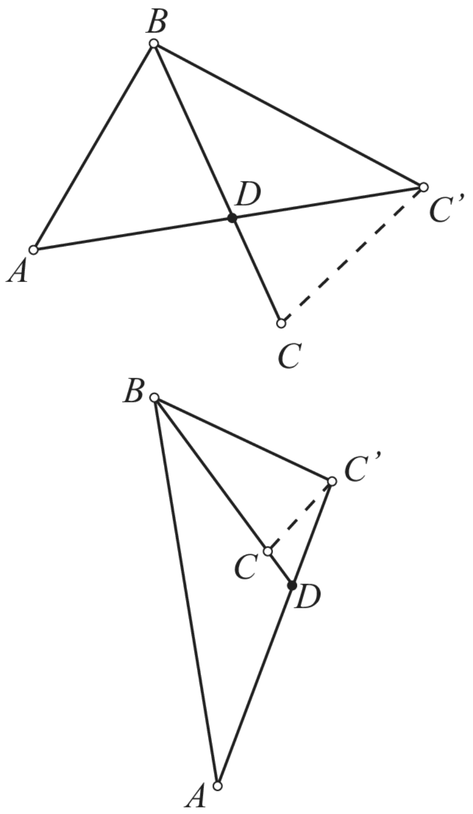

Figure 1.9: Proof: Lemma 1.24

Theorem 1.25 (Legendre I.) *For any triangle, the sum of its internal angles must be less than or equal to π.*

Proof. Indirect The internal angle sum in $ABC\triangle$ is greater than π.

Let $|\overline{A_iA_{i+1}}| = |\overline{AB}|$ be contiguous segments on the line of AB and C_i be such that $A_iC_iA_{i+1}\triangle \cong ACB\triangle$. Due to our indirect assumption $\forall i = 0, 1, \ldots, n-1 : A_iC_iA_{i+1}\angle > C_iA_{i+1}C_{i+1}\angle$. Applying the Arm-lemma: $|\overline{A_0A_1}| = |\overline{A_iA_{i+1}}| > |\overline{C_iC_{i+1}}| = |\overline{C_0C_1}|$. Now, ordering the polygonal chain inequality $|\overline{A_0C_0}| + |\overline{C_0C_1}| + \ldots |\overline{C_{n-1}A_n}| > n|\overline{A_0A_1}|$, we

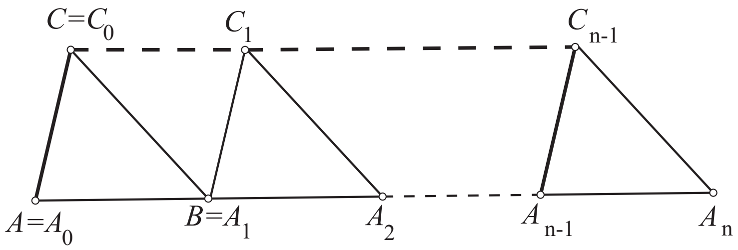

Figure 1.10: Proof: Theorem 1.25

obtain that $n(|\overline{A_0A_1}| - |\overline{C_0C_1}|) < |\overline{A_0C_0}| + |\overline{C_{n-1}A_n}|$ which contradicts the Archimedean axiom. ■

Definition 1.26 *The defect of a triangle is the difference between π and the internal angle sum of the given triangle (additive, non-negative).*

Theorem 1.27 (External angle theorem) *Any external angle of a triangle is larger than or equal to the sum of the other two internal angles.*

Proof. Let D be such that (ABD). In the triangle $ABC\triangle$: $\pi \geq CAB\angle + ABC\angle + ACB\angle$. Therefore, $DBC\angle = \pi - ABC\angle \geq CAB\angle + ACB\angle$. ■

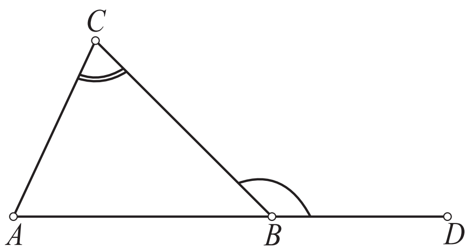

Figure 1.11: Proof: Theorem 1.27

Theorem 1.28 (Legendre II.) *If, the defect of a triangle is* 0*, then the defect of any triangle is* 0*.*

Proof. Assuming, that the defect of $ABC\triangle$ is 0, we can divide it into two right angled

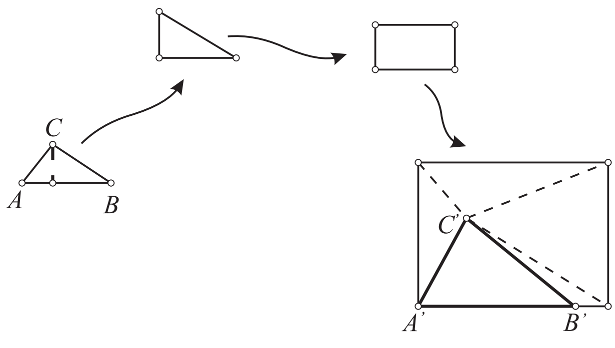

Figure 1.12: Proof: Theorem 1.28

triangle with defect 0. Joining two, we get a rectangle with defect 0. For any triangle $A'B'C'\triangle$, there exists a rectangle, which contains the triangle /Archimedean axiom/. We may divide the rectangle into five triangles. Since the defect is additive and non-negative, the defect of $A'B'C'\triangle$ must be 0. ■

Remark 1.29 *For any non self intersecting polygon, we can define the defect, by dividing it into $n-2$ triangles.*

Theorem 1.30 *The defect of a triangle is* $0 \iff (EUC)$ *is true.*

Proof. Let a be a line and P be a point not on it. Let PT be perpendicular to a (T is on a) and PK be perpendicular to PT. Then, for the Remark 1.19, $PK \cap a = \emptyset$.

$\Rightarrow$ INDIRECT: Let f be such a ray in $TPK\angle$ that $F \cap a = \emptyset$.

We will find a point D, such that $DPK\angle < fPK\angle$. If, we define the points D_i such that $D_1 = B$ and $|\overline{PD_i}| = |\overline{D_i D_{i+1}}|$, then we get for the angle $DD_iT\angle = \frac{1}{2^{i-1}} DD_1T\angle$. Now, f is a ray in $BPD\triangle$ triangle, therefore Lemma 1.17 guarantees the intersection.

$\Leftarrow$ Let a be the line of AB and $C = P$. Furthermore, let $KCB\angle = CBA\angle$ and $LCA\angle = CAB\angle$. Then $\overrightarrow{CL}$ and $\overrightarrow{CK}$ are such rays, that they do not intersect a $\Rightarrow$ they belong to the same line and the defect of the triangle is 0. ■

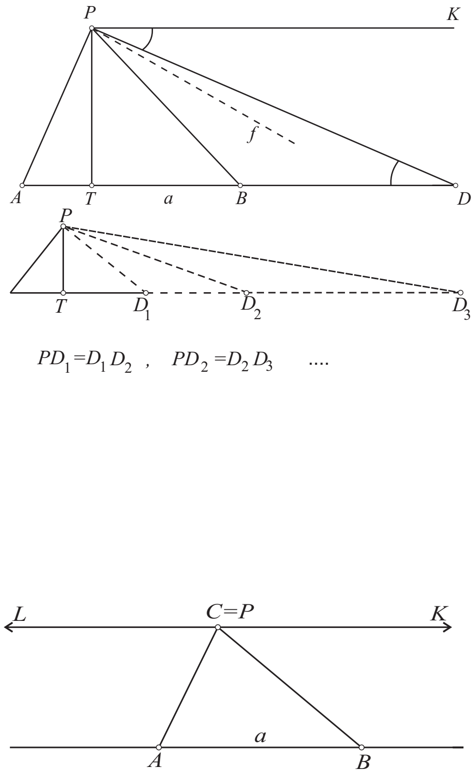

Figure 1.13: Proof: Theorem 1.30

1.3 Orthogonality

Definition 1.31 *Two lines are orthogonal to each other if they divide the plane into four congruent parts.*

Definition 1.32 *A line intersecting a plane is orthogonal to the plane, if it is orthogonal to all the lines, passing through the intersection point.*

Theorem 1.33 *A line intersecting a plane is orthogonal to the plane, if it is orthogonal to two lines, passing through the intersection point.*

Proof. Let a and b be two intersecting lines on a plane and D be their intersection point. Let n be orthogonal to both a and b such that D lies on n. Furthermore, let c be a line in the plane, passing through the point D. There exist always points $A \in a$ and $B \in b$ such that c intersects the segment $\overline{AB} \Rightarrow C$. Let P' be a point on n such that $|\overline{PD}| = |\overline{P'D}|$. Then $PDA\triangle \cong P'DA\triangle$ and $PDB\triangle \cong P'DB\triangle$, since $|\overline{PD}| = |\overline{P'D}|$, they have a side in common and the angles between them is $\frac{\pi}{2}$.

Therefore $|\overline{PA}| = |\overline{P'A}|$ and $|\overline{PB}| = |\overline{P'B}|$. But then $PAB\triangle \cong P'AB\triangle \Rightarrow PAC\angle \cong P'AC\angle \Rightarrow PAC\triangle \cong P'AC\triangle \Rightarrow |\overline{PC}| = |\overline{P'C}| \Rightarrow PDC\triangle \cong P'DC\triangle \Rightarrow PDC\angle \cong P'DC\angle = \frac{\pi}{2}$. ■

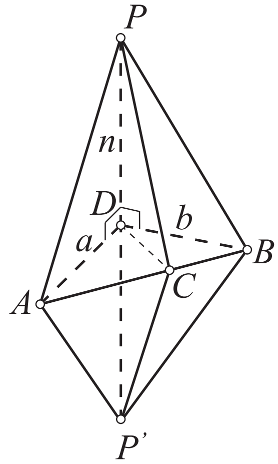

Figure 1.14: Proof: Theorem 1.33

Corollary 1.34 *In the space, the equidistant surface of two point is the orthogonal plane at the midpoint of them.*

Corollary 1.35 *The locus of lines, orthogonal to a given line at a given point of it is a plane, orthogonal to the line at that point.*

Theorem 1.36 *Through a given point A exactly one line can be drawn perpendicularly to a given plane α.*

Proof. *1. case:* A does not lie on α

Let a be an arbitrary line on α and T be a point on a such that $AT \perp a$. First, we draw an orthogonal line to a through T in α, then an orthogonal line to this line through A with foot point D. Let A' be a point on the line of AD such that $|\overline{AD}| = |\overline{A'D}|$ and S be an arbitrary point on a. $A'T \perp a$, since a is perpendicular to the plane of ADT. Then $ADT\triangle \cong A'DT\triangle \Rightarrow |\overline{AT}| = |\overline{A'T}| \Rightarrow ATS\triangle \cong A'TS\triangle \Rightarrow |\overline{AS}| = |\overline{A'S}| \Rightarrow ADS\triangle \cong A'DS\triangle \Rightarrow AA' \perp DS$

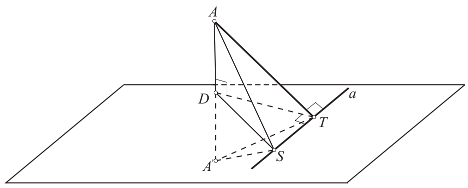

Figure 1.15: Proof: Theorem 1.36

2. case: A lies on α

Our only solution will be the intersection line of the planes, perpendicular to two arbitrary lines, passing through the point A. ■

Theorem 1.37 (of three perpendiculars) *Let a be a line on a given plane and A be a point not lying in the plane. The the foot points T and D of the perpendiculars from A to a and α respectively form a perpendicular line to a.*

Proof. Let S and R be equidistant points on a from T. Then $TAS\triangle \cong TAR\triangle$, therefore $|\overline{AS}| = |\overline{AR}|$. But this implies that $ADS\triangle \cong ADR\triangle \Rightarrow |\overline{DS}| = |\overline{DR}|$. Now $DSR\triangle$ is an isosceles triangle and $DT \perp SR$. ■

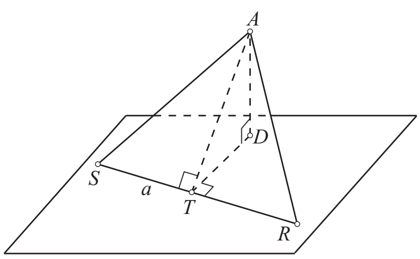

Figure 1.16: Proof: Theorem 1.37

Definition 1.38 *The orthogonal projection of a point of the space for a given plane is the common point of the plane and the line, perpendicular to the plane through the point. The orthogonal projection of a point set is the set of the projected points.*

Definition 1.39 *A mapping is a collineation if the image of collinear points are collinear themselves.*

Theorem 1.40 *The orthogonal projection is a collineation and it preserves incidence.*

Proof. Let A and B be two points on a line and their projection be A' and B' respectively. Then A, A', B and B' are coplanar.
INDIRECT: Let the line e be in the plane, perpendicular to $A'B'$ through A' and F be the midpoint of the segment $\overline{A'B'}$. Reflecting the lines AA', $A'B'$ and e respected to F, we obtain the lines $B'C$, $A'B'$ and f respectively. Since the orthogonal line to the plane through a given point is unique, B is on the line $B'C$.
Now, let A, B and C be collinear points. Then the points A, A', B, B' and A, A', C, C' are coplanar $\Rightarrow$ the two plane are the same $\Rightarrow$ A', B' and C' are collinear points. ■

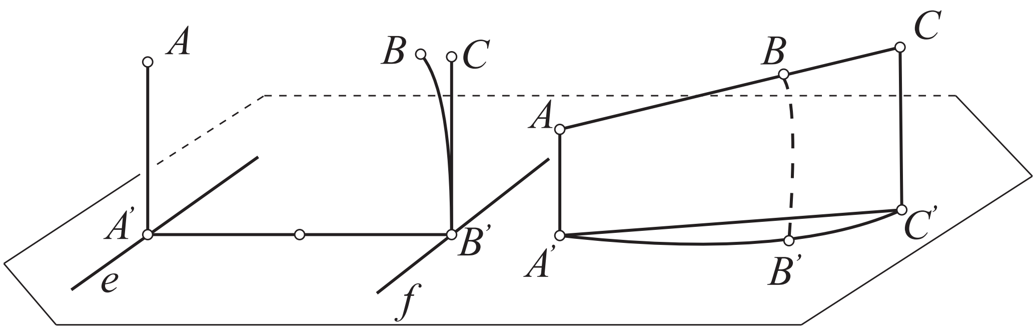

Figure 1.17: Proof: Theorem 1.40

Definition 1.41 *Two planes are orthogonal to each other if one of them contains a line, perpendicular to the other.*

Theorem 1.42 *For every plane and line, there always exists a plane, which is orthogonal to the given plane and contains the given line.*

Proof. If the orthogonal projection of the line is a single point, then it is orthogonal to the plane, and any plane which contains the line will be appropriate.
If the orthogonal projection of the line is a line does not coincide with the original line, then they determine this orthogonal plane.
If the orthogonal projection of the line is a line coincides with the original line, then construct a perpendicular line to the given line in the given plane at any point if it. The orthogonal plane to the constructed line through this point will satisfy the conditions. ■