Derivation of anisotropic matrix for bi-dimensional strain-gradient elasticity behavior

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The different forms of second order elasticity operators, in Mindlin's strain-gradient elasticity, are given for a bi-dimensional physical space. These different forms are obtained according to the different symmetry classes of a material media. Dimensional aspects are discussed together with observations made on the physical behavior of such a media.

1. Introduction

Designing lightweight and innovating materials is nowadays one of the most important challenge for material's engineering. The aim is to reach high mechanical properties with low density materials. To achieve such contradictory objectives the scientific community has, for several years, focused its attention on cellular materials structured at the mesoscale. Designing in optimal way such a material, requires to understand at the same time the relation between architecture and physical properties, and the explicit method to calculate these properties. According to a geometric definition of a RVE (Representative Volume Elementary) a classical way to obtain the overall behavior of our cellular material is to use homogenization theory. It is well known that classical homogenization relied on a clear scale separation between the geometric pattern and mechanical fields. If the scale separation is not broad enough, the classical theory fail to predict the overall behavior. As shown by Boutin (1996) and Forest (1998), if we want to keep a continuum description we have to consider a generalized continuum to model the substitution material resulting from the homogenization process. In particular, if we are designing millimetric microstructural materials to be implemented in centimetric structures (such as, for instance, hollow spheres stacking for acoustical absorber (Gasser, 2003)) we cannot take the strong scale separation for granted. And so, second order elastic effects have to be taken into account in the homogenization approach.

In order to achieve this goal some basic facts about constitutive behavior of strain-gradient elasticity will be recalled in Section 2. In Section 3 a mathematical transformation will introduced allowing us to handle easily the higher order tensors that define our behavior. Using this framework all the operators we need we will obtained in Section 4. This paper will be concluded, in Section 5, by some complementary remarks on the physics of such a behavior.

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2. Mindlin's strain-gradient elasticity

In classical elasticity theory stress at a material point is related to strain through the classical elasticity tensor. This relation, usually known as Hooke's law, is written in tensorial fashion in the following way:

$$\sigma_{(ij)} = E_{\underline{(ij)}\ \underline{(lm)}}\varepsilon_{(lm)} \tag{1}$$

Where $\sigma_{(ij)}$ is the symmetrical-stress tensor, $\varepsilon_{(lm)}$ the strain tensor and $E_{\underline{(ij)}\ \underline{(lm)}}$ the tensor describing our material property. The notation () stands for the minor symmetries whereas $\underline{..}$ stands for the major one.

In the case of Mindlin elasticity the material state at a material point also depends on the strain gradient. We shall note $K_{(lm)n}$ the strain-gradient tensor, which is formally defined as:

$$K_{(lm)n} = \frac{\partial \varepsilon_{(lm)}}{\partial x_n} = \varepsilon_{(lm),n} \tag{2}$$

where the notation, n mean the derivation of the operator along n. This strain-gradient elasticity is also known as the type II Mindlin's elasticity (Mindlin and Eshel, 1968).

Taking into account strain-gradient effect in the mechanical formulation led one to define symmetrically the hyperstress tensor $S_{(ij)k}$. So the knowledge, in each material point, of the stress tensor completed by the hyperstress one allows to compute the effective tensor $\tau_{(ij)}$. This tensor is defined as:

$$\tau_{(ij)} = \sigma_{(ij)} - S_{(ij)k,k} \tag{3}$$

and is the one to consider to calculate the local equilibrium (Forest, 2004). Tensors $\sigma_{(ij)}$ and $S_{(ij)k}$ are related with $\varepsilon_{(lm)}$ and $K_{(lm)n}$ through the following general constitutive relation:

$$\sigma_{(ij)} = E_{\underline{(ij)}\ \underline{(lm)}}\varepsilon_{(lm)} + M_{(ij)(lm)n}K_{(lm)n} \tag{4}$$

$$S_{(ij)k} = M_{(ij)k(lm)}\varepsilon_{(lm)} + A_{\underline{(ij)k}\ \underline{(lm)n}}K_{(lm)n} \tag{5}$$

where the tensor $A_{\underline{(ij)k}\ \underline{(lm)n}}$ is the second order elasticity tensor and $M_{(ij)(lm)n}$ the coupling tensor between first and second order elasticity.

As explained by Triantafyllidis and Bardenhagen (1996) in a three dimension physical space for a centro-symmetric media, this coupling tensor will vanish. In a bidimensional space this tensor would vanish for any media that is even order rotational invariant (Auffray et al., accepted for publication). For both cases the former constitutive relations could be rewritten:

$$\sigma_{(ij)} = E_{\underline{(ij)}\ \underline{(lm)}}\varepsilon_{(lm)} \tag{6}$$
$$S_{(ij)k} = A_{\underline{(ij)k}\ \underline{(lm)n}}K_{(lm)n} \tag{7}$$

In this study we will focus our attention on operators describing $A_{\underline{(ij)k}\ \underline{(lm)n}}$ for each material's symmetry classes. The different expressions of the operators are necessary for a correct numerical implementation of that kind of behavior in FEM code. First of all, in order to handle the tensor formerly defined, mathematical transformation should be introduced, allowing us to turn our 2-dimensional 6th-order tensor into a 6-dimensional 2nd-order tensor.[1] This will allow us to rewrite the second order constitutive relation as:

$$\widehat{S}_{\alpha} = \widehat{A}_{(\alpha\beta)}\widehat{K}_{\beta} \tag{8}$$

3. Change of space

We aim at obtaining the operators defined in Eqs. (6) and (7) to implement in FEM-code to compute strain-gradient elasticity. We are dealing here especially in 2-D space, nevertheless most of our approach would still be valid in 3-D space. The first order elasticity was studied in depth by Mehrabadi and Cowin (1990). So, our attention will be focused on the 2nd-order elasticity. In a 2-D space the 3rd-order tensor $K_{(lm)n}$ belong to a 6-D vector space, and the fully anisotropic tensor $A_{\underline{(ij)k}\ \underline{(lm)n}}$ would belong to an 21-D vector space.

As a 6th-order tensor is not an easy object to handle, a transformation will now be introduced to turn that object into a 2nd-order tensor. Let's begin with some remarks about matrix representations of a tensor.

3.1. Matrix representation

In the case of Hooke's law the classical Voigt matrix representation of constitutive equation is:

[1] The permutation order-dimension is just a coincidence, in 3-D the same transformation would turn a 3-dimensional 6th-order tensor into a 18-dimensional 2nd-order tensor.

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1113} & E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2213} & E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3313} & E_{3312} \\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2313} & E_{2312} \\ E_{1311} & E_{1322} & E_{1333} & E_{1323} & E_{1313} & E_{1312} \\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1213} & E_{1212} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} \tag{9}$$

But, as explained by Mehrabadi and Cowin (1990), this notation does not define a 2nd-order tensor, it is just a common matricial representation. A rigorous expression of that relation in a tensorial fashion will be:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{pmatrix} = \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & \sqrt{2}E_{1123} & \sqrt{2}E_{1113} & \sqrt{2}E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & \sqrt{2}E_{2223} & \sqrt{2}E_{2213} & \sqrt{2}E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & \sqrt{2}E_{3323} & \sqrt{2}E_{3313} & \sqrt{2}E_{3312} \\ \sqrt{2}E_{2311} & \sqrt{2}E_{2322} & \sqrt{2}E_{2333} & 2E_{2323} & 2E_{2313} & 2E_{2312} \\ \sqrt{2}E_{1311} & \sqrt{2}E_{1322} & \sqrt{2}E_{1333} & 2E_{1323} & 2E_{1313} & 2E_{1312} \\ \sqrt{2}E_{1211} & \sqrt{2}E_{1222} & \sqrt{2}E_{1233} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{pmatrix} \tag{10}$$

In the same way the rigorous way of representing the 6th-order tensor A as a 2nd-order, one according to its symmetries, is:

$$\begin{pmatrix} S_{111} \\ S_{222} \\ S_{221} \\ S_{112} \\ \sqrt{2}S_{122} \\ \sqrt{2}S_{121} \end{pmatrix} = \begin{pmatrix} A_{111111} & A_{111222} & A_{111221} & A_{111112} & \sqrt{2}A_{111122} & \sqrt{2}A_{111121} \\ A_{222111} & A_{222222} & A_{222221} & A_{222112} & \sqrt{2}A_{222122} & \sqrt{2}A_{222121} \\ A_{221111} & A_{221222} & A_{221221} & A_{221112} & \sqrt{2}A_{221122} & \sqrt{2}A_{221121} \\ A_{112111} & A_{112222} & A_{112221} & A_{112112} & \sqrt{2}A_{112122} & \sqrt{2}A_{112121} \\ \sqrt{2}A_{122111} & \sqrt{2}A_{122222} & \sqrt{2}A_{122221} & \sqrt{2}A_{122112} & 2A_{122122} & 2A_{122121} \\ \sqrt{2}A_{121111} & \sqrt{2}A_{121222} & \sqrt{2}A_{121221} & \sqrt{2}A_{121112} & 2A_{121122} & 2A_{121121} \end{pmatrix} \begin{pmatrix} K_{111} \\ K_{222} \\ K_{221} \\ K_{112} \\ \sqrt{2}K_{122} \\ \sqrt{2}K_{121} \end{pmatrix} \tag{11}$$

That is a true tensorial way of writing the constitutive relation $S_{(ij)k} = A_{\underline{(ij)k}\ \underline{(lm)n}}K_{(lm)n}$. An example for that representation is the following. According to Mindlin and Eshel (1968) in indicial the isotropic strain-gradient relation could be written as:

$$S_{ijk} = \frac{1}{2}a_1(K_{llj}\delta_{ik} + K_{lli}\delta_{kj} + 2K_{lkl}\delta_{ij}) + a_2(K_{ljl}\delta ik + K_{lil}\delta_{jk}) + 2a_3K_{llk}\delta_{ij} + 2a_4K_{ijk} + a_5(K_{jki} + K_{ikj}) \tag{12}$$

So the tensorial representation of that relation is:

$$\begin{pmatrix} S_{111} \\ S_{222} \\ S_{221} \\ S_{112} \\ \sqrt{2}S_{122} \\ \sqrt{2}S_{121} \end{pmatrix} = \begin{pmatrix} c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_2 & 0 & c_4 & 0 & c_5 & 0 \\ 0 & c_2 & 0 & c_4 & 0 & c_5 \\ c_3 & 0 & c_5 & 0 & c_6 & 0 \\ 0 & c_3 & 0 & c_5 & 0 & c_6 \end{pmatrix} \begin{pmatrix} K_{111} \\ K_{222} \\ K_{221} \\ K_{112} \\ \sqrt{2}K_{122} \\ \sqrt{2}K_{121} \end{pmatrix} \tag{13}$$

with

$$c_1 = 2(a_1 + a_2 + a_3 + a_4 + a_5); \quad c_2 = a_1 + 2a_2; \quad c_3 = \sqrt{2}\left(\frac{1}{2}a_1 + a_3\right);$$
$$c_4 = 2(a_2 + a_4); \quad c_5 = \sqrt{2}\left(\frac{1}{2}a_1 + a_5\right); \quad c_6 = a_3 + 2a_4 + a_5$$

Let's detail the way this transformation works.

3.2. Change of space formalism

The change of space could be expressed by the following diagram (14):

$$\begin{array}{ccccc} \mathbb{E} & \overset{f}{\rightarrow} & \mathbb{E}^n & \overset{g}{\rightarrow} & \mathbb{E}^{2n} \\ & & \downarrow h & & \downarrow h^* \\ & & \widehat{\mathbb{E}} & \rightarrow^{\hat{g}} & \widehat{\mathbb{E}}^2 \end{array} \tag{14}$$

$\mathbb{E}$ is the physical space, a vector space which basis vectors are e_i, i varying from 1 to d for a d-dimensional physical space. Vector space of higher dimensions could be generated by the self tensor product of the initial space. The space obtained by $n-1$ self product of $\mathbb{E}$ will be noted $\mathbb{E}^n$ and its dimension is d^n. So $\mathbb{E}^3$ is a d^3-dimensional vector space which base is $e_i \otimes e_j \otimes e_k$. An endomorphism on this space belong to an 2nd-order vector space of d^{2n} dimensions. In the same time $\mathbb{E}^n$ could be associated with $\widehat{\mathbb{E}}$ a true d^n-dimensional physical space which vectors basis are e_α. And so elements of End($\widehat{\mathbb{E}}$) are 2nd-order tensor belonging to $\widehat{\mathbb{E}}^2$. The basis of that space is $e_\alpha \otimes e_\beta$. For the sake of simplicity the indexes symmetries of the different spaces were neglected, taking them into account don't change the philosophy of our transformation. An orthonormal basis of $\widehat{\mathbb{S}}$ will now be constructed (with the index symmetry now), and the application h will be defined.

3.2.1. Construction of equivalent basis

In strain-gradient elasticity $\mathbb{S}^3$ is the vector space of K and S. This space is symmetric with respect of the first two indices permutation. Let's construct the 6-D space $\widehat{\mathbb{S}}$; its basis vectors $\underline{\hat{e}_\alpha}$ could be expressed as:

$$\underline{\hat{e}_1} = \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_1; \quad \underline{\hat{e}_2} = \underline{e}_2 \otimes \underline{e}_2 \otimes \underline{e}_2; \quad \underline{\hat{e}_3} = \underline{e}_2 \otimes \underline{e}_2 \otimes \underline{e}_1;$$

$$\underline{\hat{e}_4} = \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_2; \quad \underline{\hat{e}_5} = \frac{1}{\sqrt{2}}(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) \otimes \underline{e}_2; \quad \underline{\hat{e}_6} = \frac{1}{\sqrt{2}}(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) \otimes \underline{e}_1$$

The orthonormality of $\underline{e_i}$ implies the one of $\underline{\hat{e}_\alpha}$ and so we got:

$$\underline{\hat{e}_\alpha} \cdot \underline{\hat{e}_\beta} = \delta_{\alpha\beta} \tag{15}$$

for α and β varying from 1 to 6. $\delta_{\alpha\beta}$ stands for the classical Kronecker symbol. This implies the expression of h linking de coefficients of $\mathbb{S}^n$ with those of $\widehat{\mathbb{S}}$. So given $T_{(ij)k}$ in $\mathbb{S}^3$ and $\widehat{T}_\alpha$, its image in $\widehat{\mathbb{S}}$, we got h defined by:

$$\widehat{T}_\alpha = h(T_{ijk}) = \begin{cases} T_{ijk} & i = j \\ \sqrt{2}T_{ijk} & i \neq j \end{cases} \tag{16}$$

And so do for the strain gradient and the hyperstress tensors:

$$\widehat{K}_\alpha \underline{\hat{e}_\alpha} = h(K_{ijk})\underline{\hat{e}_\alpha}; \quad \widehat{S}_\alpha \underline{\hat{e}_\alpha} = h(S_{ijk})\underline{\hat{e}_\alpha} \tag{17}$$

We could now construct the basis of $\widehat{\mathbb{S}}^2$ by the tensor product of the basis $\widehat{\mathbb{S}}$. So given a tensor $T_{\underline{(ij)k}\ \underline{(lm)n}}$ in $\mathbb{S}^6$ its image $\widehat{T}_{\alpha\beta}$ in $\widehat{\mathbb{S}}^2$ is obtained by the application $h^\star$:

$$\widehat{T}_{\alpha\beta} = h^\star(T_{ijklmn}) = \begin{cases} T_{ijklmn} & i = j \text{ and } j = k \\ \sqrt{2}T_{ijklmn} & i \neq j \text{ and } l = m \text{ or } i = j \text{ and } l \neq m \\ 2T_{ijklmn} & i \neq j \text{ and } l \neq m \end{cases} \tag{18}$$

3.2.2. Derivation of transformation matrix

As the space transformation in now introduced, let's focus on the way an O(2)-orthogonal operator could be transformed into a O(6)-orthogonal operator. Let's $\underline{e'_i}$ be the image of the vectors $\underline{e_i}$ under the action of Q, Q $\in$ O(2). We got:

$$\underline{e'_i} = Q_{ij}\underline{e_j} \tag{19}$$

Let's, in the same way, $\underline{\hat{e}'_\alpha}$ be the image of $\underline{\hat{e}_\alpha}$ under the action of $\widehat{Q}$. $\widehat{Q}$ is defined as the image of Q in O(6) and we got:

$$\underline{\hat{e}'_\alpha} = \widehat{Q}_{\alpha\beta}\underline{\hat{e}_\beta} \tag{20}$$

$\widehat{Q}$ will be expressed, now, as a function of Q. The action of Q on a $\mathbb{E}^3$ could be expressed as:

$$\underline{e'_i} \otimes \underline{e'_j} \otimes \underline{e'_k} = Q_{il}Q_{jm}Q_{kn}\underline{e_l} \otimes \underline{e_m} \otimes \underline{e_n} \tag{21}$$

The same action on a element of the symmetrized space $\mathbb{S}^3$ lead to:

$$\frac{1}{2}(\underline{e'_i} \otimes \underline{e'_j} + \underline{e'_j} \otimes \underline{e'_i}) \otimes \underline{e'_k} = \frac{1}{2}(Q_{il}Q_{jm} + Q_{im}Q_{jl})Q_{kn}\underline{e_l} \otimes \underline{e_m} \otimes \underline{e_n} \tag{22}$$

The operator we just written is the following 6th-order tensor:

$$Q^{\mathbb{S}^3} = \frac{1}{2}(Q_{il}Q_{jm} + Q_{im}Q_{jl})Q_{kn}\underline{e_i} \otimes \underline{e_j} \otimes \underline{e_k} \otimes \underline{e_l} \otimes \underline{e_m} \otimes \underline{e_n} \tag{23}$$

By the $h^\star$ application introduced in the previous section we could turn this 6th-order tensor into a 2nd-order one in $\widehat{\mathbb{S}}^2$. And so:

$$\widehat{Q}_{\alpha\beta}\underline{\hat{e}_\alpha} \otimes \underline{\hat{e}_\beta} = h^\star\left(\frac{1}{2}(Q_{il}Q_{jm} + Q_{im}Q_{jl})Q_{kn}\right)\underline{\hat{e}_\alpha} \otimes \underline{\hat{e}_\beta} \tag{24}$$

The following table sum-up the information about the change of system

α	1	2	3	4	5	6
(i,j,k)	$(1,1,1)$	$(2,2,2)$	$(2,2,1)$	$(1,1,2)$	$(1,2,2)$	$(1,2,1)$

(25)

If we consider now $Q \in \mathbb{E}^2$ we got:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{26}$$

we could construct $\widehat{Q} \in \widehat{\mathbb{S}}^2$ as:

$$\widehat{Q} = \begin{pmatrix} Q_{11}^3 & Q_{12}^3 & Q_{12}^2Q_{11} & Q_{11}^2Q_{12} & \sqrt{2}Q_{12}^2Q_{11} & \sqrt{2}Q_{11}^2Q_{12} \\ Q_{21}^3 & Q_{22}^3 & Q_{22}^2Q_{21} & Q_{21}^2Q_{22} & \sqrt{2}Q_{22}^2Q_{21} & \sqrt{2}Q_{21}^2Q_{22} \\ Q_{21}^2Q_{11} & Q_{22}^2Q_{12} & Q_{22}^2Q_{11} & Q_{21}^2Q_{12} & \sqrt{2}Q_{12}Q_{22}Q_{21} & \sqrt{2}Q_{11}Q_{22}Q_{21} \\ Q_{11}^2Q_{21} & Q_{12}^2Q_{22} & Q_{12}^2Q_{21} & Q_{11}^2Q_{22} & \sqrt{2}Q_{11}Q_{12}Q_{22} & \sqrt{2}Q_{11}Q_{12}Q_{21} \\ \sqrt{2}Q_{21}^2Q_{11} & \sqrt{2}Q_{22}^2Q_{12} & \sqrt{2}Q_{12}Q_{22}Q_{21} & \sqrt{2}Q_{11}Q_{22}Q_{21} & (Q_{11}Q_{22}+Q_{12}Q_{21})Q_{22} & (Q_{11}Q_{22}+Q_{12}Q_{21})Q_{21} \\ \sqrt{2}Q_{11}^2Q_{21} & \sqrt{2}Q_{12}^2Q_{22} & \sqrt{2}Q_{12}Q_{22}Q_{11} & \sqrt{2}Q_{11}Q_{21}Q_{12} & (Q_{11}Q_{22}+Q_{12}Q_{21})Q_{12} & (Q_{11}Q_{22}+Q_{12}Q_{21})Q_{11} \end{pmatrix} \tag{27}$$

In the case of O(2) we consider the two following operators: Q_{rot} the rotation operator, Q_{mir} the mirror operator.

$$Q_{rot} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}; \quad Q_{mir} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

Theirs images in O(6) are:

$$\widehat{Q}_{rot} = \begin{pmatrix} c(\theta)^3 & -s(\theta)^3 & c(\theta)s(\theta)^2 & -c(\theta)^2s(\theta) & \sqrt{2}c(\theta)s(\theta)^2 & -\sqrt{2}c(\theta)^2s(\theta) \\ s(\theta)^3 & c(\theta)^3 & c(\theta)^2s(\theta) & c(\theta)s(\theta)^2 & \sqrt{2}c(\theta)^2s(\theta) & \sqrt{2}c(\theta)s(\theta)^2 \\ c(\theta)s(\theta)^2 & -c(\theta)^2s(\theta) & c(\theta)^3 & -s(\theta)^3 & -\sqrt{2}c(\theta)s(\theta)^2 & \sqrt{2}c(\theta)^2s(\theta) \\ c(\theta)^2s(\theta) & c(\theta)s(\theta)^2 & s(\theta)^3 & c(\theta)^3 & -\sqrt{2}c(\theta)^2s(\theta) & -\sqrt{2}c(\theta)s(\theta)^2 \\ \sqrt{2}c(\theta)s(\theta)^2 & -\sqrt{2}c(\theta)^2s(\theta) & -\sqrt{2}c(\theta)s(\theta)^2 & \sqrt{2}c(\theta)^2s(\theta) & c(2\theta)c(\theta) & c(2\theta)s(\theta) \\ \sqrt{2}c(\theta)^2s(\theta) & \sqrt{2}c(\theta)s(\theta)^2 & -\sqrt{2}c(\theta)^2s(\theta) & -\sqrt{2}c(\theta)s(\theta)^2 & -c(2\theta)s(\theta) & c(2\theta)c(\theta) \end{pmatrix} \tag{29}$$

$$\widehat{Q}_{mir} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{30}$$

Where, for the sake of simplicity, $\cos(\theta)$ and $\sin(\theta)$ have been noted $c(\theta)$, and $s(\theta)$. We could check that for $Q \in O(2)$ we got:

$$\widehat{Q}^T\widehat{Q} = Id_{\widehat{S^2}} \tag{31}$$

so the orthogonality of Q implies the one of $\widehat{Q}$ and so $\widehat{Q} \in O(6)$.

With that transformation, completed by the expression of $\widehat{Q}$, we have the tools we need to study the different expressions of $A_{\underline{(ij)k}\ \underline{(lm)n}}$ for different material's symmetry classes.

4. Derivation of anisotropic operators

4.1. Expression of invariance

Let G be a group of operation,a material $\mathscr{M}$ will be said G-invariant if the action of all the element of G transform the material into itself. This set of operation will be noted $G_{\mathscr{M}}$, namely the material symmetry's group, and defined by:

$$G_{\mathscr{M}} = \{Q \in O(2), \quad Q \star \mathscr{M} = \mathscr{M}\} \tag{32}$$

Where $\star$ represents the action of Q upon $\mathscr{M}$. As we are dealing with 2-D materials, our attention will be restricted to the 2-D orthogonal group: O(2). Moreover we know that $G_{\mathscr{M}}$ must be conjugate to a subgroup of O(2) (Zheng and Boehler, 1994). The collection of those subgroups is, according to Armstrong (1988):

$$\Sigma := \{I, Z_n, D_n, SO(2), O(2)\} \tag{33}$$

Where I is the identity group. Z_n is the cyclic group of order n, it is the group of rotations of a chiral figure that possesses an n-fold invariance (cf. Fig. 2 for an example of an Z_3-invariant figure). D_n is the dihedral group of order $2n$, it is the group of operations that leave a regular n-gone invariant (cf. Fig. 1 for the example of an D_3-invariant figure). SO(2) is the continuous group of rotations. The generator of the Z_n-invariance is the matrix Q_{rot}, and for the D_n-invariance the set of generator have to completed with the matrix Q_{mir}.

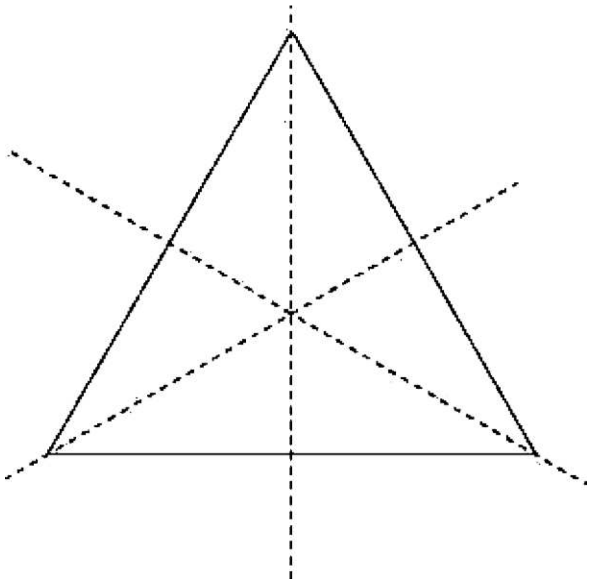


Fig. 1. D_3-invariant figure.

Consider now a physical property $\mathscr{P}$ defined on our material $\mathscr{M}$. The physical group of symmetry of that property could be defined as the set of operations that leave the behavior invariant. This set of operations will be noted $\mathrm{G}_{\mathscr{P}}$, namely the physical's symmetry group and defined as:

$$\mathrm{G}_{\mathscr{P}} = \{\mathrm{Q} \in \mathrm{O}(2), \quad \mathrm{Q} \star \mathscr{P} = \mathscr{P}\} \tag{34}$$

In our case the action of Q upon the tensor A could be rewritten as:

$$\mathrm{G}_{\mathrm{A}} = \{\mathrm{Q} \in \mathrm{O}(2), \quad Q_{io}Q_{jp}Q_{kq}Q_{lr}Q_{ms}Q_{nt}A_{opqrst} = A_{ijklmn}\} \tag{35}$$

By the mean of Neumann's principle (Zheng and Boehler, 1994), we got the inclusion:

$$\mathrm{G}_{\mathscr{M}} \subseteq \mathrm{G}_{\mathscr{P}} \tag{36}$$

this just mean that every operation that leaves our material invariant will let our physical properties invariant. Nevertheless the physical property could appear to be more symmetrical than the material (Auffray, 2008).

In the following subsections, consequences of material symmetries on tensorial components will be studied. This work will be simplified by the use of transformation introduced in Section 3, and so, the group of symmetry of tensor A could be rewritten as:

$$\mathrm{G}_{\mathrm{A}} = \{\mathrm{Q} \in \mathrm{O}(2), \quad \widehat{Q}_{\alpha\gamma}\widehat{Q}_{\beta\delta}\widehat{A}_{\gamma\delta} = \widehat{A}_{\alpha\beta}\} \tag{37}$$

The restriction on tensorial coefficients will be the different solutions of the following matricial system:

$$\widehat{\mathrm{Q}}^{T}\widehat{\mathrm{A}}\widehat{\mathrm{Q}} = \widehat{\mathrm{A}} \tag{38}$$

for Q belonging to the generators of all O(2)-subgroups.

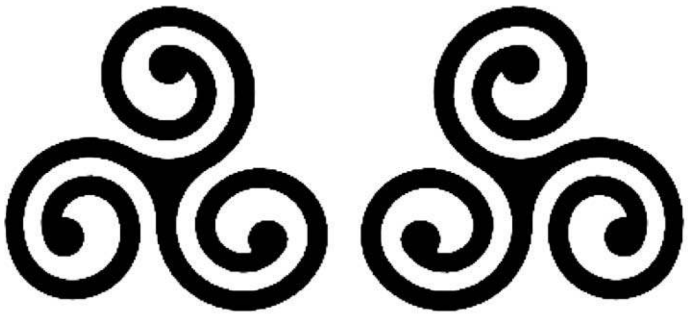
Let's begin by studying the consequence of a Z_n-material invariance on $A_{\underline{(ij)k}\ \underline{(lm)n}}$.

4.2. Z_n-material invariance

In the following subsections the following notation will adopted for the matricial coefficients:

- a_{ij} will stand for not final coefficients, some more transformation are needed to reach the minimal expression;
- b_{ij} will be the coefficients of the minimal expression, b_{ij} are independent;
- c_i will be used to make comparison between different forms, the c_i are not independent.

In the same way a non-minimal matrix representation will be noted by a $*$ exponent.



(a) Levorotary triskelion (b) Dextrorotary triskelion

Fig. 2. Z_3-invariant figures.

4.2.1. Z_2-invariance

For Z_2 no restriction will be imposed on A.

4.2.2. Z_3/Z_6-invariance

The Z_3 and the Z_6-invariance will lead to the same following operator:

$$\widehat{\mathrm{A}}^*_{Z_3} = \widehat{\mathrm{A}}^*_{Z_6} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & -2a_{12}-a_{23} & a_{15} & \frac{-a_{12}+a_{23}}{\sqrt{2}} \\ & a_{22} & a_{23} & a_{11}-a_{22}+a_{13} & -\frac{3a_{12}+a_{23}}{\sqrt{2}} & \sqrt{2}(a_{11}-a_{22})+a_{15} \\ & & a_{11}-a_{22}+a_{44} & a_{12} & -a_{15}+\frac{a_{22}-a_{44}}{\sqrt{2}} & \frac{3a_{12}+a_{23}}{\sqrt{2}} \\ & & & a_{44} & \frac{a_{12}-a_{23}}{\sqrt{2}} & -\sqrt{2}a_{11}-a_{15}+\frac{3a_{22}-a_{44}}{\sqrt{2}} \\ & & & & a_{11}-a_{13}-\frac{a_{22}-a_{44}}{2} & 2a_{12} \\ & & & & & -a_{11}-a_{13}+\frac{3a_{22}+a_{44}}{2} \end{pmatrix} \tag{39}$$

This operator seems to depend on 7 different coefficients. But writing of the system:

$$\mathrm{QAQ}^{\mathrm{T}} = 0 \tag{40}$$

shows that there exists a rotation:

$$\frac{\sin(6\theta)}{\cos(6\theta)} = \frac{2a_{12}}{a_{22}-a_{11}} \tag{41}$$

allowing us to reduce the number of parameters from 7 to 6. The operator will have, in an appropriate basis, the following expression:

$$\widehat{\mathrm{A}}_{Z_6} = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} & \frac{b_{11}-b_{33}}{\sqrt{2}}-b_{35} & -\frac{b_{14}}{\sqrt{2}} \\ & b_{22} & -b_{14} & b_{11}-b_{22}+b_{13} & \frac{b_{14}}{\sqrt{2}} & \frac{3b_{11}-b_{33}}{\sqrt{2}}-b_{35}-\sqrt{2}b_{22} \\ & & b_{33} & 0 & b_{35} & -\frac{b_{14}}{\sqrt{2}} \\ & & & b_{33}+b_{22}-b_{11} & \frac{b_{14}}{\sqrt{2}} & \sqrt{2}(b_{22}-b_{11})+b_{35} \\ & & & & \frac{b_{11}+b_{33}}{2}-b_{13} & 0 \\ & & & & & \frac{-3b_{11}+b_{33}}{2}-b_{13}+2b_{22} \end{pmatrix} \tag{42}$$

where b_{ij} coefficients are functions of the former a_{ij} coefficients of $\widehat{\mathrm{A}}^*_{Z_6}$. So, finally, the tensor A is defined by 6 coefficients in its hexatropic chiral class.

An example of a material with such a geometry could be found in Prall and Lakes (1996). The geometry of the honeycombed studied by the authors is described Fig. 3.

4.2.3. Z_4-invariance

For the Z_4-invariance we got:

$$\widehat{\mathrm{A}}^*_{Z_4} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{11} & -a_{14} & a_{13} & -a_{16} & a_{15} \\ & & a_{33} & 0 & a_{35} & a_{36} \\ & & & a_{33} & -a_{36} & a_{35} \\ & & & & a_{55} & 0 \\ & & & & & a_{55} \end{pmatrix} \tag{43}$$

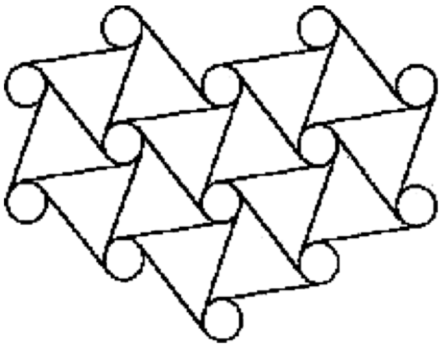


Fig. 3. *Z*6-invariant honeycomb.

As in the case of the Z_3/Z_6-invariance a rotation decreasing the operator's number of parameters could be found. The action of the following rotation:

$$\frac{\sin(4\theta)}{\cos(4\theta)} = \frac{2\sqrt{2}(a_{16} - a_{36})}{(a_{11} + a_{33} - 2(a_{13} + a_{55}))} \tag{44}$$

reduces our former operator to the new one:

$$\widehat{\mathrm{A}}_{Z_4} = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} & b_{15} & b_{16} \\ & b_{11} & -b_{14} & b_{13} & -b_{16} & b_{15} \\ & & b_{33} & 0 & b_{35} & b_{16} \\ & & & b_{33} & -b_{16} & b_{35} \\ & & & & b_{55} & 0 \\ & & & & & b_{55} \end{pmatrix} \tag{45}$$

In this new basis the former coefficients a_{16} and a_{36} are now equal and are denoted by the new coefficient b_{16}. And so the number of independent coefficients in the orthotropic chiral class decrease from 9 to 8.

4.2.4. $Z_5/Z_n, n \geqslant 7$-invariance

For a Z_5-invariance, and for any Z_n-invariance in which $n \geqslant 7$, we got the following operator:

$$\widehat{\mathrm{A}}_{\mathrm{SO}(2)} = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} & \frac{b_{11}-b_{33}}{\sqrt{2}} - b_{35} & -\frac{b_{14}}{\sqrt{2}} \\ & b_{11} & -b_{14} & b_{13} & \frac{b_{14}}{\sqrt{2}} & \frac{b_{11}-b_{33}}{\sqrt{2}} - b_{35} \\ & & b_{33} & 0 & b_{35} & -\frac{b_{14}}{\sqrt{2}} \\ & & & b_{33} & \frac{b_{14}}{\sqrt{2}} & b_{35} \\ & & & & \frac{b_{11}+b_{33}}{2} - b_{13} & 0 \\ & & & & & \frac{b_{11}+b_{33}}{2} - b_{13} \end{pmatrix} \tag{46}$$

This symmetry class depends on 5 parameters.

4.2.5. Analysis of the hemitropic class

It can be noticed that for $n \geqslant 7$ the order of the symmetry exceed the order of the tensor A. According to Hermann's theorem (Auffray, 2008), the symmetry group of A must be, in that case, conjugate to a continuous group. As we are dealing with subgroups of O(2) this continuous group must be either SO(2) or O(2). In other words, for a Z_5-invariance, and for any Z_n-invariance in which $n \geqslant 7$, the tensor A must be either hemitropic (SO(2)-invariant) or isotropic (O(2)-invariant). So in our case A is at least hemitropic. Let's

$$b_{11} = c_1; \quad b_{13} = c_2; \quad b_{33} = c_4; \quad b_{35} = c_5$$

where c_i for i varying from 1 to 6 are the Mindlin's coefficients. The following relations are verified:

$$c_3 = \frac{b_{11} - b_{33}}{\sqrt{2}} - b_{35}; \quad c_6 = \frac{b_{11} + b_{33}}{2} - b_{13}$$

and if we note $c_7 = b_{14}$, we finally obtain:

$$
\widehat{A}_{SO(2)} = \begin{pmatrix} c_1 & 0 & c_2 & c_7 & c_3 & -\frac{c_7}{\sqrt{2}} \\ & c_1 & -c_7 & c_2 & \frac{c_7}{\sqrt{2}} & c_3 \\ & & c_4 & 0 & c_5 & -\frac{c_7}{\sqrt{2}} \\ & & & c_4 & \frac{c_7}{\sqrt{2}} & c_5 \\ & & & & c_6 & 0 \\ & & & & & c_6 \end{pmatrix} \tag{47}
$$

This expression differs from the isotropic one (13) by the presence of the c_7 coefficient. But, as A is at least hemitropic, it does not exist any rotation that could make c_7 disappear. So the $Z_5/Z_n, n \geqslant 7$-invariance, lead to a non-isotropic invariance, namely the hemitropic invariance.

4.3. *D_n-material invariance*

For the dihedral-invariance, former results have to be combined with mirror-invariance. This means the invariance of our former operators under the action of $\widehat{Q}_{\text{mir}}$.

4.3.1. *D_2-invariance*

For a D_2-invariance, we obtain the operator:

$$
\widehat{A}_{D_{(2)}} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 & b_{15} & 0 \\ & b_{22} & 0 & b_{24} & 0 & b_{26} \\ & & b_{33} & 0 & b_{35} & 0 \\ & & & b_{44} & 0 & b_{46} \\ & & & & b_{55} & 0 \\ & & & & & b_{66} \end{pmatrix} \tag{48}
$$

This system is defined by 12 coefficients.

4.3.2. *D_3/D_6-invariance*

The D_3 and the D_6-invariance lead to:

$$
\widehat{A}_{D_{(3)}} = \widehat{A}_{D_{(6)}} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 & \frac{b_{11}-b_{33}}{\sqrt{2}} - b_{35} & 0 \\ & b_{22} & 0 & b_{11} - b_{22} + b_{13} & 0 & \frac{3b_{11}-b_{33}}{\sqrt{2}} - b_{35} - \sqrt{2}b_{22} \\ & & b_{33} & 0 & b_{35} & 0 \\ & & & b_{33} + b_{22} - b_{11} & 0 & \sqrt{2}(b_{22} - b_{11}) + b_{35} \\ & & & & \frac{b_{11}+b_{33}}{2} - b_{13} & 0 \\ & & & & & \frac{-3b_{11}+b_{33}}{2} - b_{13} + 2b_{22} \end{pmatrix} \tag{49}
$$

This class is defined by 5 coefficients.

4.3.3. *D_4-invariance*

For D_4, we got:

$$
\widehat{A}_{D_{(4)}} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 & b_{15} & 0 \\ & b_{11} & 0 & b_{13} & 0 & b_{15} \\ & & b_{33} & 0 & b_{35} & 0 \\ & & & b_{33} & 0 & b_{35} \\ & & & & b_{55} & 0 \\ & & & & & b_{55} \end{pmatrix} \tag{50}
$$

This class is defined by 6 coefficients.

4.3.4. *$D_5/D_n, n \geqslant 7$-invariance*

And finally for D_5 and $D_n, n > 7$ we got the following operator:

$$\widehat{\mathrm{A}}_{\mathrm{O}(2)} = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 & \frac{b_{11}-b_{33}}{\sqrt{2}} - b_{35} & 0 \\ & b_{11} & 0 & b_{13} & 0 & \frac{b_{11}-b_{33}}{\sqrt{2}} - b_{35} \\ & & b_{33} & 0 & b_{35} & 0 \\ & & & b_{33} & 0 & b_{35} \\ & & & & \frac{b_{11}+b_{33}}{2} - b_{13} & 0 \\ & & & & & \frac{b_{11}+b_{33}}{2} - b_{13} \end{pmatrix} \tag{51}$$

which is defined by only 4 coefficients. If we substitute those coefficients with the c_i one of Mindlin second order elasticity we obtain:

$$\widehat{A}_{\mathrm{O}(2)} = \begin{pmatrix} c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_2 & 0 & c_4 & 0 & c_5 & 0 \\ 0 & c_2 & 0 & c_4 & 0 & c_5 \\ c_3 & 0 & c_5 & 0 & c_6 & 0 \\ 0 & c_3 & 0 & c_5 & 0 & c_6 \end{pmatrix} \tag{52}$$

The mirror-invariance make the hemitropic coefficient c_7 to vanish and we obtain the isotropic operator. So we have now obtained for the second order elasticity in a bidimensional space 8 different expressions for the operator $A_{\underline{(ij)k}\ \underline{(lm)n}}$ according to its different classes of symmetry. These results could be sum-up in the following table:

$\mathrm{G}_{\mathscr{M}}$	G_{A}	dim
I, Z_2	I	21
D_2	D_2	12
Z_4	Z_4	8
D_4	D_4	6
Z_3, Z_6	Z_6	6
D_3, D_6	D_6	5
$Z_5, Z_n, n \geqslant 7$	$SO(2)$	5
$D_5, D_n, n \geqslant 7$	$O(2)$	4

In a two dimensional physical space, the group of symmetry an operator $A_{\underline{(ij)k}\ \underline{(lm)n}}$ belongs to must be conjugate to an element of the following set:

$$\Sigma_{\mathrm{A}} : \{I, D_2, Z_4, D_4, Z_6, D_6, SO(2), O(2)\} \tag{53}$$

5. Discussion

Besides the fact we obtain, in a 2-D space, the explicit expression of the anisotropic second order elastic tensor in strain-gradient elasticity theory, some points concerning symmetry of the operators are worth emphasizing.

5.1. Class jump phenomenon

The results presented here are obtained considering a 2-D physical space, whereas the real physical space is 3-D. It is therefore useful to analyse the consequence of this hypothesis.

$\widehat{\mathrm{A}}$ tensor could be represented, in 3-D space, by the following block matrix:

$$\begin{pmatrix} [x \otimes x] & [x \otimes y] & [x \otimes z] & [x \otimes c] \\ & [y \otimes y] & [y \otimes z] & [y \otimes c] \\ & & [z \otimes z] & [z \otimes c] \\ & & & [c \otimes c] \end{pmatrix} \tag{54}$$

where x, y, z stand for mechanisms along the different direction, and c stands for a coupling between all of those mechanisms. This full matrix is square and of dimension 18. A sub-operator modeling effect along the x and y direction could be extracted. This sub operator will be of the following form:

$$\begin{pmatrix} [x \otimes x] & [x \otimes y] \\ & [y \otimes y] \end{pmatrix} \tag{55}$$

This matrix is obviously square, and it could be shown that its dimension is 10. What we called x and y are kind of vectors containing indexes. We got:

$$x = \begin{bmatrix} 111 \\ 221 \\ 331 \\ 122 \\ 133 \end{bmatrix}, \quad y = \begin{bmatrix} 222 \\ 332 \\ 112 \\ 233 \\ 121 \end{bmatrix} \tag{56}$$

And so if we get rid of elements with indices equal to 3, we obtain x_{2D} and y_{2D} each of length 3. And so we could construct the following sub-operator:

$$\begin{pmatrix} [x_{2D} \otimes x_{2D}] & [x_{2D} \otimes y_{2D}] \\ & [y_{2D} \otimes y_{2D}] \end{pmatrix} \tag{57}$$

This last operator is the operator of Mindlin's elasticity in a 2D-space, the operator we have been working on since the beginning of that paper.

The operator we obtain is so the one we would obtain by suppressing rows and columns with an out-of-plane indice in the expression of the 3-D operator. This operation implies a loss of information. The most noticeable consequence is the existence for 2-D modeling of a "class-jump" phenomena. An example of such phenomenon is the following.

In Section 4.2 we notice that the Hermann's theorem implies that for an order of symmetry that exceed 6 the operator of strain-gradient elasticity must possesses a continuous group of symmetry. We also show in the same subsection that it was also the case for an order of symmetry equal to 5. The fact a 5-fold axe induce a continuous symmetry is a dimensional anomaly specific to bi-dimensional space. In Fig. 4, we plot, for the cyclic group, the material invariance group against the physical invariance group. We observe that for A, that is an even-order tensor, an odd-order material-invariance imply a physical invariance of twice order: a Z_{2p+1}-material invariance implies a $Z_{2(2p+1)}$-physical invariance. This fact which is specific to bi-dimensional space can be formally proved working on the harmonic decomposition of the operator (Auffray et al., accepted for publication). This phenomena could also be observed for classical elasticity. So the difference induced on an operator by a Z_{2p+1}-material symmetry or a $Z_{2(2p+1)}$ one just concern out-of-plane coefficients. This explain why continuous symmetry class appear for an 5-fold symmetry whereas we are not in the case of Hermann's theorem. In case of bi-dimensional space an 5-fold symmetry is seen as a 10-fold symmetry and this time we are in the case of the former theorem. Finally, it's well known since Mindlin that isotropic strain-gradient elasticity depend on five independent coefficients. But as we showed here this number of coefficient depend on the dimension of the physical space, for a two dimensional space this number decreases to 4. This fact depend on the operator, for the conventional elasticity the number of isotropic coefficients is the same in two and three dimensions (Zou et al., 2001).

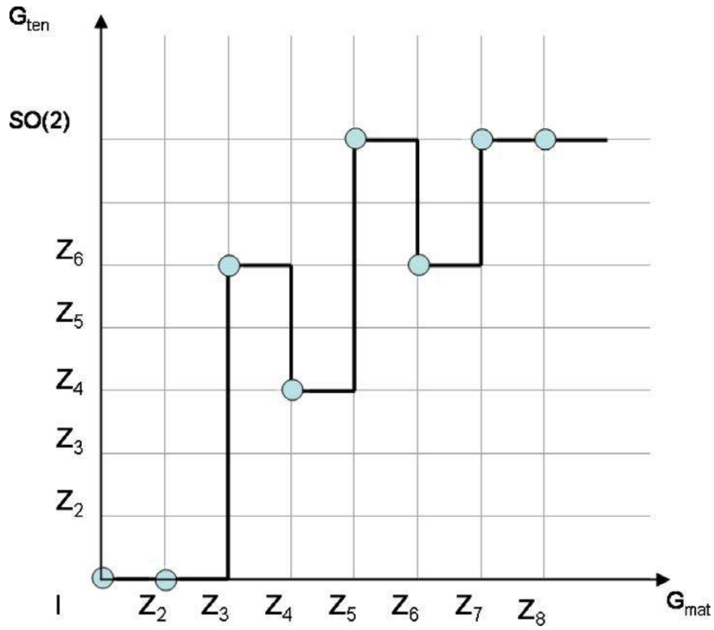


Fig. 4. Jump of symmetry classes between material and physical invariances.

If we get back to the expression of matrices (55) and (57), we understand that the last one don't take into account for out-of plane coupling. In (Auffray et al., accepted for publication) it has been proved that, in 3-D, different material invariances lead to different physical behaviors. This fact means that the matrix (55) is different for even and odd-material invariance, meanwhile its submatrix (57) remains the same. This remark, made through the study of operators in 2-D space, has a deep meaning about the physical consequence of material symmetry in 3-D. In, 3-D, the difference between a Z_{2p+1} and a $Z_{2(2p+1)}$ material invariance will just concerns out-of-plane coupling coefficients. This remark holds true for any kind of linear behavior.

5.2. Chiral-sensitivity

The second is the fact that strain-gradient elasticity is a chiral-sensitive behavior. For conventional elasticity, for example, the Z_4-invariance and the D_4 lead to the same elastic operator expressed in two different basis; as shown by Forte and Vianello (1996) you can always find a angle of rotation to turn the Z_4-invariant operator into the D_4 one. As shown in,Section 4 for the second order elasticity after reduction the operators for the two different class remain distinct. The existence of an hemitropic class of symmetry, class which does not exist for classical elasticity, shows that the sensitivity to chirality is independent of the choice of an appropriate basis. The chirality coupling can be easily illustrated in the following way. In Section 3 we introduce the following matrix representation for $\widehat{\mathrm{A}}$:

$$\begin{pmatrix} S_{111} \\ S_{222} \\ S_{221} \\ S_{112} \\ \sqrt{2}S_{122} \\ \sqrt{2}S_{121} \end{pmatrix} = \begin{pmatrix} A_{111111} & A_{111222} & A_{111221} & A_{111112} & \sqrt{2}A_{111122} & \sqrt{2}A_{111121} \\ A_{222111} & A_{222222} & A_{222221} & A_{222112} & \sqrt{2}A_{222122} & \sqrt{2}A_{222121} \\ A_{221111} & A_{221222} & A_{221221} & A_{221112} & \sqrt{2}A_{221122} & \sqrt{2}A_{221121} \\ A_{112111} & A_{112222} & A_{112221} & A_{112112} & \sqrt{2}A_{112122} & \sqrt{2}A_{112121} \\ \sqrt{2}A_{122111} & \sqrt{2}A_{122222} & \sqrt{2}A_{122221} & \sqrt{2}A_{122112} & 2A_{122122} & 2A_{122121} \\ \sqrt{2}A_{121111} & \sqrt{2}A_{121222} & \sqrt{2}A_{121221} & \sqrt{2}A_{121112} & 2A_{121122} & 2A_{121121} \end{pmatrix} \begin{pmatrix} K_{111} \\ K_{222} \\ K_{221} \\ K_{112} \\ \sqrt{2}K_{122} \\ \sqrt{2}K_{121} \end{pmatrix} \tag{58}$$

We can rewrite this operator separating the strain-gradient mechanisms along the x-direction and the y-direction, leading to:

$$\begin{pmatrix} S_{111} \\ S_{221} \\ \sqrt{2}S_{122} \\ S_{222} \\ S_{112} \\ \sqrt{2}S_{121} \end{pmatrix} = \begin{pmatrix} A_{111111} & A_{111221} & \sqrt{2}A_{111122} & A_{111222} & A_{111112} & \sqrt{2}A_{111121} \\ A_{221111} & A_{221221} & \sqrt{2}A_{221122} & A_{221222} & A_{221112} & \sqrt{2}A_{221121} \\ \sqrt{2}A_{122111} & \sqrt{2}A_{122221} & 2A_{122122} & \sqrt{2}A_{122222} & \sqrt{2}A_{122112} & 2A_{122121} \\ A_{222111} & A_{222221} & \sqrt{2}A_{222122} & A_{222222} & A_{222112} & \sqrt{2}A_{222121} \\ A_{112111} & A_{112221} & \sqrt{2}A_{112122} & A_{112222} & A_{112112} & \sqrt{2}A_{112121} \\ \sqrt{2}A_{121111} & \sqrt{2}A_{121221} & 2A_{121122} & \sqrt{2}A_{121222} & \sqrt{2}A_{121112} & 2A_{121121} \end{pmatrix} \begin{pmatrix} K_{111} \\ K_{221} \\ \sqrt{2}K_{122} \\ K_{222} \\ K_{112} \\ \sqrt{2}K_{121} \end{pmatrix} \tag{59}$$

We can rewrite now the matrix we obtain in this system. We shall do that here just for the hemitropic and the isotropic cases, but this result stands for any Z_k-invariance and D_k-invariance: Z_k-invariance behave likes the hemitropic case meanwhile D_k-invariance is stimulate to the isotropic one. For the isotropic-invariance, and for any D_k-invariance, in the former system of vectors the matrix operators are block-diagonal. We have, on one hand:

$$\widehat{\mathrm{A}}_{\mathrm{O}(2)} = \begin{pmatrix} c_1 & c_2 & c_3 & 0 & 0 & 0 \\ & c_4 & c_5 & 0 & 0 & 0 \\ & & c_6 & 0 & 0 & 0 \\ & & & c_1 & c_2 & c_3 \\ & & & & c_4 & c_5 \\ & & & & & c_6 \end{pmatrix} \tag{60}$$

for hemitropic-invariance, and for any Z_k-invariance, we have, on the other hand:

$$\widehat{\mathrm{A}}_{\mathrm{SO}(2)} = \begin{pmatrix} c_1 & c_2 & c_3 & 0 & c_7 & -\frac{c_7}{\sqrt{2}} \\ & c_4 & c_5 & -c_7 & 0 & \frac{c_7}{\sqrt{2}} \\ & & c_6 & -\frac{c_7}{\sqrt{2}} & -\frac{c_7}{\sqrt{2}} & 0 \\ & & & c_1 & c_2 & c_3 \\ & & & & c_4 & c_5 \\ & & & & & c_6 \end{pmatrix} \tag{61}$$

In this system the upper right block matrix represent the coupling effect between strain gradient in the x-direction and in the y-direction. If the material is invariant under a mirror-symmetry this coupling effect vanish. Otherwise the coupling effect appears in the form of a skew-symmetric matrix. This skew-symmetric coupling is a chiral-sensitive mechanism.

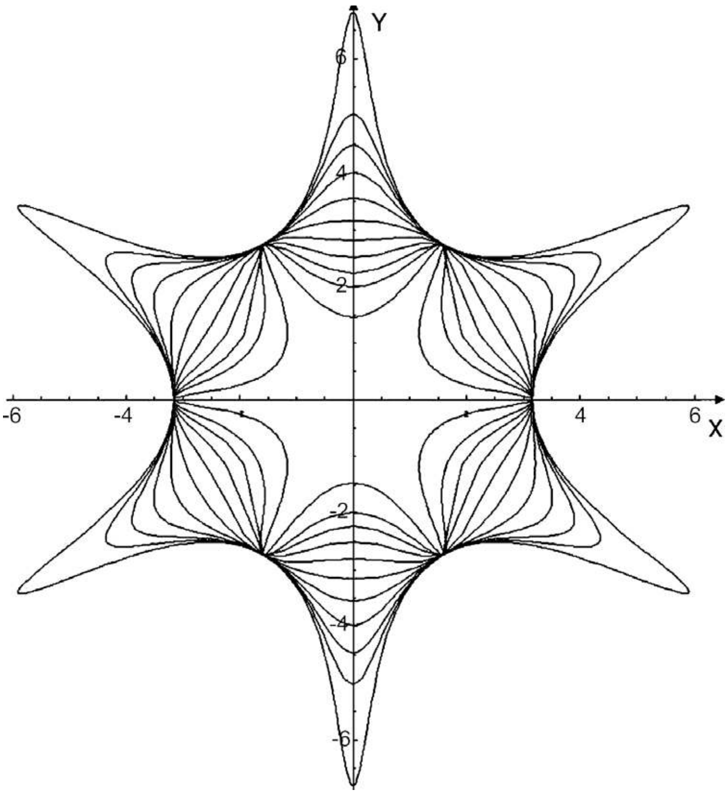


Fig. 5. Different isoenergy curves for different tensor parameters b_{11} and b_{22}.

5.3. Anisotropy of plane periodic tilling

We observe that a material symmetry of order 6 will lead to an anisotropic elastic behavior. So whereas 1st-order elasticity is isotropic for a 6-fold invariant material, its 2nd-order term will become anisotropic. This fact makes sense since that the order of rotation group that allow a bi-dimensional-media to be periodic is finite and must be either 1, 2, 3, 4 or 6. This fact is known as the crystallographic restriction. So it can be concluded that for any bidimensional periodic material, the strain-gradient elasticity must be anisotropic.[2] This effect can be shown by expressing the 2nd-order elastic energy W in the plane (x, y) as a function of $K_{(ij)k}(x, y)$. This energy could be expressed through the Voigt's formalism:

$$W(x, y) = \frac{1}{2}\widehat{S}_{\alpha}\widehat{K}_{\alpha} = \frac{1}{2}\widehat{A}_{(\alpha\beta)}\widehat{K}_{\beta}\widehat{K}_{\alpha} \tag{62}$$

We consider now the following special strain-gradient field[3] $\underline{\hat{K}}$:

$$\underline{\hat{K}}(x, y) = \begin{pmatrix} x^3 \\ y^3 \\ xy^2 \\ yx^2 \\ \sqrt{2}xy^2 \\ \sqrt{2}yx^2 \end{pmatrix} \tag{63}$$

For such a special field the second order elastic energy is a symmetric homogeneous polynomial:

$$\begin{aligned} 2W(x, y) = {} & b_{11}x^6 + b_{22}y^6 + 2(b_{14} + \sqrt{2}b_{16})x^5y + 2(b_{23} + \sqrt{2}b_{25})xy^5 + (2(b_{13} + b_{66} + \sqrt{2}(b_{15} + b_{46})) + b_{44})x^4y^2 \\ & + (2(b_{24} + b_{55} + \sqrt{2}(b_{26} + b_{35})) + b_{33})x^2y^4 + 2(b_{12} + b_{34} + 2b_{56} + \sqrt{2}(b_{36} + b_{45}))x^3y^3 \end{aligned} \tag{64}$$

[2] This result is obviously also true in 3-D space. In 3-D space, Mindlin's elasticity defined over a Z_6-invariant media is not transverse isotropic.

[3] The elastic energy expressed through this field with that special spatial dependence depend only on the full symmetrical part of the tensor A. Such a choice of a strain-gradient field allows us to represent the anisotropic part of the tensor in the plane.

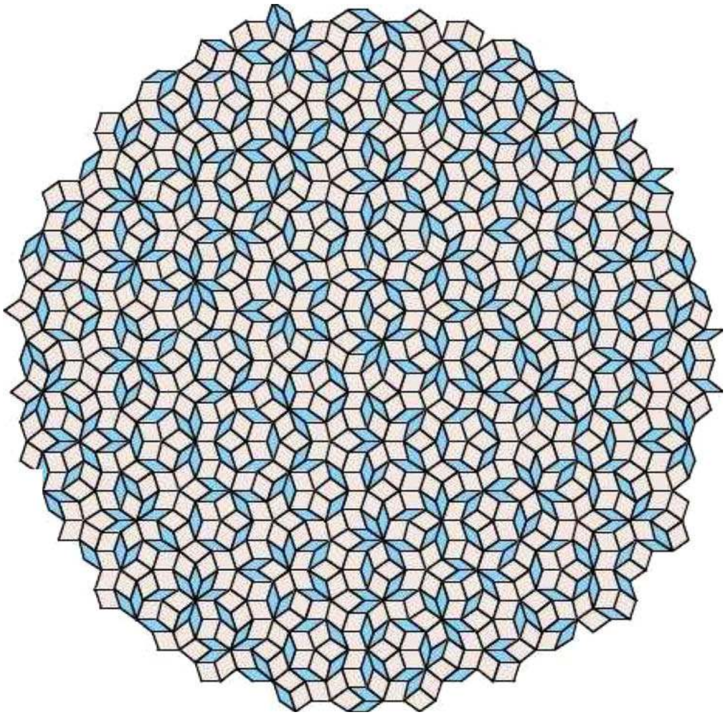


Fig. 6. D_5-invariant Penrose tilling.

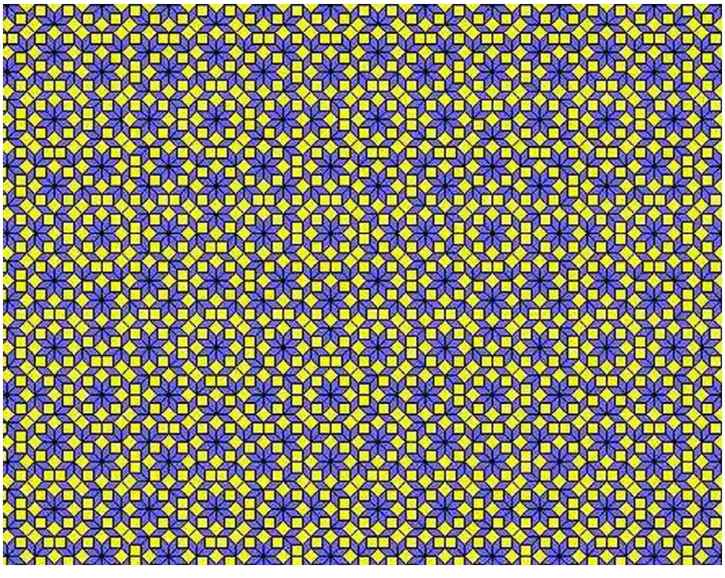


Fig. 7. D_8-invariant Ammann–Beenker tilling.

For both hemitropy or isotropy symmetry the polynomial (64) will reduce to:

$$2W_{iso}(x,y) = b_{11}(x^2 + y^2)^3 \tag{65}$$

This implies that the iso-energy lines are concentric circles. That was for at least hemitropic behavior. For the Z_6/D_6-invariance the polynomial will reduce to:

$$2W_{D_6}(x,y) = b_{11}(x^3 - 3xy^2)^2 + b_{22}(-3x^2y + y^3)^2 \tag{66}$$

As shown in Fig. 5 iso-energy lines for such a material would be represented by different "stars" according the values of the parameters b_{11} and b_{22}. This fact clearly show that the physical response of an Z_6/D_6-invariant material depends, for 2nd-order elasticity, on the direction of the space, and so that the tensor corresponding to this symmetry is anisotropic.

So a natural question one can wonder, is "what kind of geometry a media should have for Mindlin's elasticity to be isotropic on?". As it would be shown in Section 5.1 the answer depend on the dimension of the physical space, but in both case it deals with quasi-periodic tilling (Gratias et al., 2000). In 2-D space, for instance, Mindlin's elasticity define over Penrose tilled media should be isotropic (cf. Fig. 6). As shown Section 5.1 and explained by Auffray et al. (accepted for publication) in 2-D space this a Z_{2p+1}-material invariance induce a $Z_{2(2p+1)}$-physical invariance.[4] So for Mindlin's elasticity a D_5-invariant media is seen as a D_{10}-invariant one.

But, even if it is not the scope of this paper, it should be note that in 3-D space the Penrose tilling will not induce a transverse isotropic behavior. The reason is that in 3-D, we don't have jump of class phenomenon, and so the order of rotation should be strictly greater than the tensor's number of index to imply a continuous class of symmetry. So in that case we could consider, for example, the D_8 Ammann–Beenker tilling Fig. 7.

6. Conclusion

We derived all the expressions the 2nd-order tensor of Mindlin elasticity could have in a 2-D physical space depending of the material symmetry. To achieve this goal we introduced an algebraically transformation of space to change a 2-dimensional 6th-order tensor in a 6-dimensional 2nd-order tensor. This goal was reached using the formalism introduce by Mehrabadi and Cowin (1990). That allowed us to show that the tensor $A_{\underline{(ij)k}\ \underline{(lm)n}}$ could be of eight different types. The two main results concerning second order elasticity are that all the periodic media are anisotropic and that this elasticity depend on the chirality of the material. The particularity of two dimensional physical space was finally pointed out, showing that in two dimension some anomaly, like the class-jump, appears.

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[4] For behavior described by even order tensors.