CERTAIN CRITERIA ON THE EXISTENCE OF A TRANSCENDENTAL ENTIRE COMMON RIGHT FACTOR

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Abstract. In this paper, we shall first prove certain criteria on the existence of a transcendental entire common right factor of two entire functions. Applying these results, we can then prove that if f is an entire function which is pseudo-prime and not of the form $H(Q(z))$, where H is a periodic entire function and Q is a polynomial, then $R(f(z))$ is also pseudo-prime for any non-constant rational function R.

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1 INTRODUCTION AND MAIN RESULTS.

Let z denote the complex variable. A meromorphic function $F(z)$ is said to have a factorization with a left factor f and right factor g provided

$$F(z) = f(g(z)) \quad ,$$

where f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be prime (pseudo-prime) if every factorization of the above form implies that either f is bilinear or g is linear (either f is rational or g is a polynomial).

Over the past thirty years or so, several interesting and general criteria for the primeness or pseudo-primeness of a meromorphic function have been established (see [2]). It seems very difficult to derive a necessary condition for an entire function to be prime or pseudo-prime. However, in a sense, prime or pseudo-prime functions constitute quite a large class of functions in the entire function space. This fact is reflected by the following result proved by Y.Noda in [6].

THEOREM A. Let f be a transcendental entire function. Then

$$\{a \in \mathbf{C} : f(z) + az \quad \text{is not prime}\}$$

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is at most a countable set.

In [9], by using Nevanlinna's value distribution theory, G.D.Song and J.Huang proved the following result :

THEOREM B. Let $f(z)$ be a pseudo-prime entire function, and $n(\geq 3)$ be an odd positive integer. Then $F(z) = f(z)^n$ is also pseudo-prime.

In the same paper, G.D.Song and J.Huang showed that for the prime function $f(z) = \sin z e^{\cos z}$, $f(z)^2 = \sin^2 z e^{2\cos z} = ((1-w^2)e^{2w}) \circ \cos z$. Hence $f(z)^2$ is not pseudo-prime. Thus, they raised the following question :

PROBLEM A : Let $f(z)$ be pseudo-prime, and $P(z)$ be a polynomial of degree ≥ 3 which has no quadratic right factor. Must $P(f(z))$ be pseudo-prime ?

The purpose of this note is to deal with the above problem by proving the following related result.

THEOREM 1. If f is an entire function which is pseudo-prime and not of the form $H(Q(z))$, where H is a periodic entire function and Q is a polynomial, then $R(f(z))$ is also pseudo-prime for any non-constant rational function R.

REMARK. The above example given by G.D.Song and J.Huang shows that the condition that f is not of the form $H(Q(z))$ in the theorem is needed.

2 LEMMA AND PRELIMINARIES.

In order to prove Theorem 1, we first derive certain criteria on the existence of a transcendental entire common right factor for two entire functions. The proof of these criteria are based on the following theorem of Grauert [4] on complex analytic equivalence relations.

THEOREM C. Let R be any equivalence relation on $\mathbf{C}$ whose graph G is an analytic subset of $\mathbf{C}^2$ containing no vertical or horizontal lines. Suppose that G is of pure dimension one (i.e. G is everywhere of the same dimension one). Then, there exists a holomorphic map h from $\mathbf{C}$ to one of the four Riemann surfaces: the whole plane, punctured plane, sphere and torus, such that xRy if and only if $h(x) = h(y)$.

In the Appendix A of [3], A.Eremenko and L.Rubel gave a more elementary and direct proof of Theorem C.

The basic terminologies and properties of complex analytic set can be found in [1]. It is surprising to note that Nevanlinna's value distribution theory, in contrast to the usual applications, does not play a role in the proof of Theorem 1.

DEFINITION : Let $F(z)$ be an entire function. We say that $g(z)$ is a generalized right factor (denoted by $g \leq F$) of F if g is a holomorphic map from $\mathbf{C}$ to a Riemann surface S and there exists a holomorphic map f from S to $\mathbf{C}$ such that $F = f \circ g$.

Note that we use the word "map" to denote a mapping between two Rieman surfaces.

A non-constant holomorphic map k of $\mathbf{C}$ to a Rieman surface can induce an equivalence relation R in $\mathbf{C}$ defined by xRy if and only if $k(x) = k(y)$.

Let $K = \{(x, y) \in \mathbf{C}^2 | \quad k(x) = k(y)\}$, then K is a complex analytic set of pure dimension one which does not contain any vertical or horizontal line (see [1]). Such K is called the graph of equivalence relation induced by k.

Let H and K be the graphs of the equivalence relation induced by holomorphic maps h and k respectively. Then, it is not difficult to show that $h \leq k$ if and only if H is a subset of K.

LEMMA. Let f, g be two entire functions. For $i = 1, ...k, k \geq 2$, let $S_i = \{z_{in}\}_{n \in N}$ be a sequence of distinct complex numbers with limit point z_i. Suppose that all the limit points z_i are distinct and for all $n \in N$,

$$(*) \begin{cases} f(z_{1n}) = f(z_{2n}) = \cdots = f(z_{kn}) \\ g(z_{1n}) = g(z_{2n}) = \cdots = g(z_{kn}). \end{cases}$$

Then, there exists an entire function $h(z)$ (independent of k and $S_i's$) satisfying $h \leq f$, $h \leq g$ and $h(z_1) = h(z_i)$ for all $2 \leq i \leq k$.

The proof of the above lemma is contained implictly in A.Eremenko and L.Rubel's paper ([3],Theorem 1.1) For completeness, we sketch the proof below.

Let F and G be the graphs of the equivalence relation induced by f and g respectively. Then $F \cap G$ remains to be a complex analytic set (see [1], p.62), but may not have pure dimension one , so we consider its derived set H (i.e. the set of limit points). Then H will be a pure dimension one complex analytic set which does not contain any vertical or horizontal line. The non-trival fact that H is still a graph of some equivalence relation is proved in ([3], Theorem 1.1). By Theorem C, we conclude that H is a graph of the equivalence relation induced by some holomorphic map h from $\mathbf{C}$ to one of the four Riemann surfaces S stated in Theorem C. Clearly, h depends only on f and g.

Now H is a subset of both F and G, so we have $h \leq f$ and $h \leq g$. Hence, there exist holomorphic maps h_1 and h_2 from S to $\mathbf{C}$ such that $f = h_1 \circ h$ and $g = h_2 \circ h$. If S is a torus, then h_1 must be an elliptic function and $f = h_1 \circ h$ will not be entire. Therefore, S can't be a torus. If S is the whole plane or punctured plane, then h will be an entire function on $\mathbf{C}$.

If S is a sphere, then h will be a meromorphic function with at least one pole and h_1, h_2 will be rational functions. Since f and g are entire, both h_1 and h_2 can't be polynomial . Now, suppose h_1 has a pole a, then h must omit the value a, otherwise, $f = h_1 \circ h$ will not be entire. Hence, $h = a + \frac{1}{h_0}$ where h_0 is an entire function. Clearly, $h_0 \leq f$ and $h_0 \leq g$ and the graph induced by h_0 is the same as that of h. So, we may simply replace h by the entire function h_0 in the following considerations.

From the assumption $(*)$ of the lemma, we have $(z_{1n}, z_{jn}) \in F \cap G$ for all $2 \leq j \leq k$ and $n \in N$. Therefore, for all $2 \leq j \leq k$, $(z_1, z_j) \in H = (F \cap G)'$ and hence $h(z_1) = h(z_j)$. This also completes the proof of the lemma .

3 MORE THEOREMS AND THEIR PROOFS.

With the above preparations, we can now deduce the following useful criterion on the existence of a non-linear common right factor of two entire functions.

THEOREM 2. Let f and g be two entire functions . Suppose that there exist two non-constant complex functions h_1 and h_2 such that $F(z) = h_1(f(z)) = h_2(g(z))$ is meromorphic. Suppose further that there exist $k \geq 2$ distinct points $z_1, \ldots\ldots, z_k$ such that $F'(z_i) \neq 0, \infty$ for all i and

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_k) \\ g(z_1) = g(z_2) = \cdots = g(z_k). \end{cases}$$

Then, there exists an entire function $h(z)$ (independent of k and $z_i's$) with $h \leq f$, $h \leq g$ and $h(z_1) = h(z_i)$ for all $2 \leq i \leq k$.

Proof of Theorem 2. We first note that for all $1 \leq i \leq k, f'(z_i)$ and $g'(z_i)$ are finite and non-zero as $F'(z_i) \neq 0, \infty$. For $i \neq 1$, define $v_i(s,t) = f(z_i + t) - f(z_1 + s)$. Then $v_i(0,0) = 0$ and $\frac{\partial v_i}{\partial t}(0,0) = f'(z_i) \neq 0$. According to the Implict Function Theorem, there exits a unique analytic function ϕ_i on a neighborhood A_i of $s = 0$ such that $v_i(s, \phi_i(s)) = 0$ on A_i , i.e.

$$f(z_1 + s) = f(z_i + \phi_i(s)). \tag{1}$$

Similarly, for each $2 \leq i \leq k$, there exist neighborhoods B_i, C_i of $s = 0$ and unique analytic functions φ_i (on B_i) and ψ_i (on C_i) such that

$$g(z_1 + s) = g(z_i + \varphi_i(s)), \tag{2}$$

$$F(z_1 + s) = F(z_i + \psi_i(s)). \tag{3}$$

It follows from (1),(2),(3) and $h_1 \circ f = h_2 \circ g$ that on $D_i = A_i \cap B_i \cap C_i \neq \phi$,

$$F(z_1 + s) = F(z_i + \phi_i(s)) = F(z_i + \varphi_i(s)) = F(z_i + \psi_i(s)).$$

Due to the uniqueness of ϕ_i, φ_i and ψ_i, we have on a neighborhood D_i of $s = 0$ that $\phi_i = \varphi_i = \psi_i$. Hence, we have on $E = \cap_{i=2}^{k} D_i \neq \phi$ of $s = 0$,

$$\begin{cases} f(z_1 + s) = f(z_2 + \phi_2(s)) = \cdots = f(z_k + \phi_k(s)) \\ g(z_1 + s) = g(z_2 + \phi_2(s)) = \cdots = g(z_k + \phi_k(s)). \end{cases}$$

Clearly, the sequences S_i required in the lemma exist and by that lemma , we are done.

THEOREM 3. Let f and g be two entire functions. Suppose that there exist two non-constant complex functions k and R such that $F = R \circ f = k \circ g$ is meromorphic. If g is transcendental and R is rational, then there exists a transcendental entire function h satisfying $h \leq f$ and $h \leq g$.

REMARK. Let $f(z) = ze^{z^2}$, $g(z) = z^2$, $R(z) = z^2$ and $k(z) = ze^{2z}$. Then $R \circ f = z^2 \circ (ze^{z^2}) = z^2 e^{2z^2} = ze^{2z} \circ z^2 = k \circ g$. Note that there doesn't exist any transcendental entire h with $h \leq f$ and $h \leq g$. Therefore, the condition that g is transcendental is needed.

REMARK. Let $f(z) = e^z + z, g(z) = e^z, R(z) = e^z$ and $k(z) = ze^z$. Then $R \circ f = e^z \circ (e^z + z) = e^z e^{e^z} = ze^z \circ (e^z) = k \circ g$. Note that there doesn't exist any transcendental entire h with $h \leq f$ and $h \leq g$. Therefore, the condition that R is rational is also needed.

Proof of Theorem 3. Define $E = \{g(z) | F'(z) = 0 \quad \text{or} \quad \infty\}$. Then E is a countable set. Therefore, by the Little Picard Theorem, we can choose $A \in \mathbf{C} - E$ so that the equation $g(z) = A$ has infinitely many distinct roots $\{z_n\}_{n \in N}$. Since $k(A) = k(g(z_n)) = R(f(z_n))$, $g(z_n)$ are roots of the equation $R(z) = k(A)$ which has only finitely many zeros. So, there exists an infinite subsequence of $\{z_n\}_{n \in N}$ (which we denote by the same $\{z_n\}_{n \in N}$) such that $f(z_1) = f(z_n)$ for all z_n. Note that $g(z_1) = g(z_n) = A$ for all n and $F'(z_n) \neq 0 \quad \text{or} \quad \infty$. By Theorem 2, there exists an entire function h with $h \leq f$, $h \leq g$ and $h(z_1) = h(z_n)$ for all $n \in N$. As all z_n are distinct, h must be transcendental.

As an application of Theorem 3, we can obtain immediately a generalized result of Alfred Renyi and Catherine Renyi ([7],Theorem 2) as below. Note that in [8], H.S. Shapiro also obtainsd the same result by a completely different argument.

COROLLARY. If $R(z)$ is a non-constant rational function and g is an transcendental entire function which is not periodic, then $R(g(z))$ can not be periodic.

Proof of Corollary. Suppose $R(g(z))$ is periodic with period (say) $2\pi i$. Then $R(g(z)) = k(e^z)$ for some k meromorphic on $\mathbf{C}$-$\{0\}$. By Theorem 3, there exists a transcendental entire function h with $h \leq e^z$ and $h \leq g$. Hence, $e^z = h_1 \circ h$ and $g = h_2 \circ h$, where h_1, h_2 are analytic on the image of h.

If the image of h is $\mathbf{C} - \{a\}$, then $h = a + e^q$ for some entire function q. We may assume $a = 0$ so that $e^z = h_1(e^w) \circ q(z)$. The pseudo-primeness of e^z will force $q(z)$ to be a polynomial. Since the derivative of e^z doesn't take zero, $q(z)$ must be linear. Hence h is periodic and so is g. This contradicts the assumption on g. Therefore, the image of h must be the whole plane. This implies that both h_1, h_2 are entire. Again, the pseudo-primeness of e^z will force h_1 to be a polynomial. By the Little Picard Theorem, it is easy to see that $h_1(z) = (z-b)^n$ and $h(z) = b + e^q$ for some entire function q and complex number b. This reduces to the first case and we will again get a contradiction. Therefore, $R(g(z))$ is not periodic.

Proof of Theorem 1. Suppose $R \circ f = k \circ g$ for some meromorphic function k and transcendental entire function g. We need to show that k is rational. By Theorem 3, there exists a transcendental entire function h so that $h \leq f$ and $h \leq g$. Hence, $f = h_1 \circ h$ and $g = h_2 \circ h$, where h_1, h_2 are analytic on the image of h. If the image of h is $\mathbf{C} - \{a\}$, then $h = a + e^q$ for some entire function q. We may assume $a = 0$ so that $f(z) = h_1(e^w) \circ q(z)$. The psuedo-primeness of f will force q to be a polynomial. This contradicts the assumption that f is not the composition of a periodic function with a polynomial. So the image of h must be the whole plane. This implies that both h_1, h_2 are entire and $R \circ h_1 = k \circ h_2$ on $\mathbf{C}$. Since $f = h_1 \circ h$ is pseudo-prime, h_1 must be a polynomial. From $R \circ h_1 = k \circ h_2$, k must be a rational function. Hence, $R \circ f$ is pseudo-prime.

4 FINAL REMARK.

It is worth mentioning that the following question (proposed by He-Yang in [5] , p.124), which is closely related to problem A, remains unsolved for more than 2 decades.

PROBLEM B : Let f be a pseudo-prime transcendental meromorphic function, and p a polynomial of degree ≥ 2. Must $f(p(z))$ be pseudo-prime ?

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