# RING OF SUBQUOTIENTS OF A FINITE GROUP II: PURE BISETS

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Abstract. This sequel to [7] focuses on the structure of the ring $\Lambda(G)$ of subquotients of the finite group $G$. We show that this ring is isomorphic with the Grothendieck ring of the category of pure $(G, G)$-bisets, which are bisets containing no isogations. We also determine, over a field of characteristic zero, the Mackey functor structure and the primitive idempotents of $\Lambda(G)$. Main tool of this determination is the marks of subquotients on each other.
**Keywords:** Ring of subquotients, mark morphism, pure biset, primitive idempotents, orbit counting.

## 1. Introduction

In [7], we introduced the ring $\Lambda(G)$ of subquotients of a finite group $G$. As an abelian group, it is free on the set of conjugacy classes of subquotients of $G$, where by a subquotient we mean a quotient of a subgroup of $G$. More precisely, we define

$$\Lambda(G) = \sum_{H/N \preceq_G G} \mathbb{Z}[H/N]_G$$

where the sum is over a set of representatives of the conjugacy classes of subquotients of $G$. We turn this group to a ring by extending linearly the following multiplication:

$$[H/N]_G \cdot [K/M]_G = \sum_{\substack{x \in H\backslash G/K \\ {}^xMN \le H \cap {}^xK}} [H \cap {}^xK/{}^xMN]_G.$$

The aim of the first part of this paper is to show that the ring $\Lambda(G)$ has a natural structure of a $\circ$-biset functor and there is a natural morphism

$$\mathrm{lin}_{k,G} : \Lambda(G) \to \mathcal{R}_{\mu_k}(G)$$

from the functor $\Lambda$ of subquotients to the functor $\mathcal{R}_{\mu_k}$ of the representation ring of Mackey functors, which is also a $\circ$-biset functor. Here a $\circ$-biset functor is defined like a biset functor except that the composition of bisets is given by another amalgamation product dual to the one introduced by Bouc [2]. See [7] for details. Now the above morphism is defined as the linear extension of the following correspondence: Let $H/N$ be a subquotient of the group $G$. Then

$$\mathrm{lin}_{k,G}([H/N]_G) = \mathrm{Ind}_H^G \mathrm{Inf}_{H/N}^H \mathrm{Def}_{H/N}^H \mathrm{Res}_H^G B^G$$

where $B^G$ denotes the Burnside Mackey functor for $G$.

This morphism, sharing certain properties with the linearization map from the Burnside ring to the representation ring, is called the linearization map. See [7] for details.

Aim of the present paper is to have a closer look at the ring structure of the ring $\Lambda(G)$ of subquotients of $G$. Our first result is to realize the ring $\Lambda(G)$ as a Grothendieck ring of the category of *pure* $(G,G)$*-bisets*. We define a *transitive pure* $(G,G)$*-biset* as a transitive $(G,G)$-biset $(\frac{G\times G}{U})$ where $U = \{(g,h) \in H\times H \,|\, gN = hN\}$ for some subgroups $N \trianglelefteq H \le G$. Then a *pure* $(G,G)$*-biset* is a biset each of whose $(G,G)$-orbits are transitive pure $(G,G)$-bisets. With this definition $\Lambda(G)$ becomes the Grothendieck group of the category of pure $(G,G)$-bisets. To obtain the ring structure, we show that there is a categorical product of pure bisets, denoted by $\star_G$, given by

$$\Big(\frac{G\times G}{U}\Big) \star_G \Big(\frac{G\times G}{V}\Big) = \sum_{\substack{x\in H\backslash G/K\\ {}^xMN\le H\cap {}^xK}} \Big(\frac{G\times G}{U * {}^xV}\Big).$$

where both $(\frac{G\times G}{U})$ and $(\frac{G\times G}{V})$ are pure bisets and the subgroups $N \trianglelefteq H$ and $M \trianglelefteq K$ are the corresponding pairs of subgroups of $G$ and

$$U * V = \{(g,h) \in G\times G \,|\, \exists k \in G : (g,k) \in U, (k,h) \in V\}.$$

This product gives the Grothendieck group a ring structure which coincides with the ring structure of $\Lambda(G)$. See Section 2 for details.

There is a related construction by Bouc in [2]. Bouc considered the category of $(G,G)$-bisets with the categorical product given by his $\circ$-product. Then he considered the subring of the Grothendieck ring of this category generated by transitive $(G,G)$-bisets of the form $(\frac{G\times G}{U})$ where $U = \{(g,h) \in G\times G : g \in X, gh^{-1} \in N\}$. Here $N$ and $X$ are subgroups of $G$ such that $X$ normalizes $N$, not necessarily containing $N$. It is easy to show that as an abelian group, the ring of subquotients is the subgroup generated by those bisets with the condition that $X$ contains $N$. From this it is clear that the containment is strict.

In Section 3, we look at the Mackey functor structure of the ring $\Lambda(G)$ where we show that $\Lambda$ is induced from the restriction functor whose evaluation at a subgroup $H$ is free on the set of normal subgroups of $H$. We also determine the simple summands of the semisimple Mackey functor $\mathbb{Q}\Lambda$ and give a decomposition over an arbitrary field.

Next section, Section 4, deals with the ghost ring of the ring $\Lambda(G)$ and the mark homomorphism between them. The ghost ring of the Burnside ring is defined as the ring of class functions, that is, functions constant on the conjugacy classes of subgroups. Then the mark homomorphism is defined as the map from the Burnside ring to the ghost ring which associates each $G$-set the function which takes the number of fixed points of the action of the corresponding subgroup.

Similarly, we define the *ghost ring* $\Lambda^*(G)$ of the ring of subquotients as the ring of functions constant on the conjugacy classes of subquotients. Then the *(generalized) mark* of a subquotient $K/L$ on the subquotient $H/N$ is the number

$$m_{K/M,H/N} := |\{x \in G/H : K/L \leq {}^{x}(H/N)\}|.$$

This extends the definition of the mark of a subgroup on another one. We show that the table of generalized marks is lower triangular with non-zero determinant. Moreover the associated mark morphism for the ring $\Lambda(G)$ is a ring homomorphism and in particular, it has a finite cokernel which is described by the Fundamental Theorem, see Theorem 5.2.

Given a finite $G$-set $X$, the well known orbit counting lemma counts the number of orbits of $G$ on $X$ using the (ordinary) marks. In Section 4, we prove a generalization of this result, see Theorem 4.7. More precisely, we prove that given a subgroup $H$ and a subquotient $K/M$ of $G$ such that $M \leq H \leq K$, then the number of conjugates of $H$ that is contained in $K$ and contains $M$ is given by

$$\frac{|H|}{|N_G(H)|} \sum_{g \in N_G(H)} m_{<g>H/H,K/M}.$$

In the final two sections, we determine a formula for the primitive idempotents of the algebra $\mathbb{Q}\Lambda(G)$ and also determine the prime ideals of the ring $\Lambda(G)$. Our strategy in this section is to use the mark homomorphism (which is an isomorphism once we extend the coefficients to $\mathbb{Q}$) to find inverse images of primitive idempotents of the ghost ring.

Finally, at the end of the paper, we include two examples of table of generalized marks (for the groups $S_3$ and $A_4$).

## 2. The category of Pure Bisets

**2.1.** In this section we introduce the category of pure bisets. For a review of the terminology of bisets and for our notation, we refer to [7, Section 3.1]. We only recall that given finite groups $G$ and $H$ and a subgroup $U$ of the direct product $G \times H$, we denote the transitive

$(G,H)$-biset with the point stabilizer $U$ by $(\frac{G\times H}{U})$ and by Bouc's decomposition theorem, any such biset is equal to the product of five basic bisets called induction, inflation, isogation, deflation and restriction. In the notation of [7], we have

$$\Big(\frac{G\times H}{U}\Big) = \mathrm{Tin}^G_{p_1/k_1}\, \mathrm{c}^{\phi}_{p_1/k_1, p_2/k_2}\, \mathrm{Des}^H_{p_2/k_2}$$

where $p_1$ and $p_2$ are the respective projections of $U$ to $G$ and $H$ and the subgroups $k_1$ and $k_2$ are given by

$$k_1 = \{g\in G : (g,1)\in U\} \quad \text{and} \quad k_2 = \{h\in H : (1,h)\in U\}.$$

The isomorphism

$$\phi : p_2/k_2 \to p_1/k_1$$

is the one given by associating $lk_2$ to $mk_1$ where for a given element $l\in p_2$ we let $mk_1$ be the unique element in $p_1/k_1$ be such that $(m,l)\in U$. Finally for groups $H/N \preceq G$ and $\phi : G\cong K$, the bisets in the right hand side of the above decomposition are the following compositions

$$\mathrm{Tin}^G_{H/N} = \mathrm{Ind}^G_H \mathrm{Inf}^H_{H/N},\ \ \mathrm{Des}^G_{H/N} = \mathrm{Def}^H_{H/N}\mathrm{Res}^G_H,\ \ c^{\phi}_{G,K} = \mathrm{iso}^{\phi}_{G,K}$$

with the usual definitions of induction, inflation, deflation, restriction and isomorphism bisets.

**2.2. $\star$-Product of Bisets.** There are two known way of defining amalgamated products of bisets. The first one is the well-known amalgamation, denoted $\times_G$, which is defined as the quotient of the Cartesian product by the $G$-action. The second one is the $\circ$-product introduced by Bouc in [2], which we use in the first part of the paper in which we also consider the dual of this product, written $\circ^*$. Nevertheless none of these amalgamations is appropriate to amalgamate pure bisets, introduced below. As a preparation to amalgamation of pure bisets, we introduce, yet, another product of bisets as follows.

Let $G, H$ and $K$ be finite groups and $U$ be a subgroup of $G\times H$ and $V$ be a subgroup of $H\times K$. We define the $\star$-product of the $(G,H)$-biset $(\frac{G\times H}{U})$ and the $(H,K)$-biset $(\frac{H\times K}{V})$ by

$$\Big(\frac{G\times H}{U}\Big) \star_H \Big(\frac{H\times K}{V}\Big) := \bigsqcup_{\substack{x\in p_2(U)\backslash H/p_1(V)\\ {}^xk_1(V)k_2(U)\le {}^xp_1(V)\cap p_2(U)}} \Big(\frac{G\times K}{U * {}^xV}\Big)$$

which is clearly a $(G,K)$-biset. It is easy to check that the $\star$-product is associative. We define the $\star$-product of two arbitrary bisets by extending linearly the $\star$-product of transitive bisets.

**2.3.** It is evident from the definition that the above decomposition formula still holds with the $\star$-product. Therefore we will continue to decompose transitive bisets into products of basic ones as above. When there is no ambiguity, we shall use expressions like $\mathrm{Tin}^G_{H/N}\mathrm{Des}^G_{H/N}$ instead of $\mathrm{Tin}^G_{H/N} \star_{H/N} \mathrm{Des}^G_{H/N}$.

**2.4. Pure Bisets.** Assume the above notation. Suppose $G = H$. Then we call the biset $(\frac{G\times G}{U})$ a (transitive) *pure* $(G,G)$*-biset*[1] if we have

$$(p_1(U), k_1(U)) = (p_2(U), k_2(U)).$$

Then a $(G,G)$-biset is called *pure* if and only if any $(G,G)$-orbit of this biset is a transitive pure $(G,G)$-biset.

In terms of basic bisets, $(\frac{G\times G}{U})$ is a pure $(G,G)$-biset if and only if the equality

$$\left(\frac{G\times G}{U}\right) = \mathrm{Tin}^G_{H/N}\,\mathrm{Des}^G_{H/N}$$

holds, where we put $(H,N) := (p_1(U), k_1(U)) = (p_2(U), k_2(U))$. Also if $(\frac{G\times G}{U})$ is a pure biset, we call the subgroup $U$ of $G\times G$ a *pure subgroup*. It is clear from this identification that there is a bijective correspondence between

(1) the pure subgroups of $G\times G$ and
(2) the subquotients of $G$

given by associating a pure subgroup $U$ to the subquotient $p_1(U)/k_1(U)$. For the inverse correspondence, given a subquotient $H/N$ of $G$, we define the corresponding pure subgroup $U$ by

$$U = \{(g,h) \in H\times H \mid gN = hN\}$$

**2.5. Notation.** Given a pure subgroup $U$ of $G\times G$, we denote the transitive pure $(G,G)$-biset with point stabilizer equal to $U$ by

$$(p_1(U)/k_1(U))_G.$$

and its isomorphism class by

$$[p_1(U)/k_1(U)]_G.$$

The following proposition is self-evident.

**2.6. Proposition.** *There is a bijective correspondence between*

(1) *The isomorphism classes* $[\frac{G\times G}{U}]$ *of transitive pure* $(G,G)$*-bisets,*
(2) *the conjugacy classes* $[U]$ *of pure subgroups of* $G\times G$*,*
(3) *the conjugacy classes* $[H/N]$ *of subquotients of* $G$*.*

[1]The terminology is introduced by Ergün Yalçın in an algebra seminar.

**2.7. A categorical product of pure bisets.** Let $U$ and $V$ be two pure subgroups of $G\times G$ with the corresponding subquotients $H/N$ and $K/M$ in $G$. Then we define the *categorical product* of pure $(G,G)$-bisets by the $\star$-product of these biset, which amounts to

$$\Big(\frac{G\times G}{U}\Big)\star_G\Big(\frac{G\times G}{V}\Big)=\bigsqcup_{\substack{x\in H\backslash G/K\\ {}^xMN\le {}^xK\cap H}}\Big(\frac{G\times G}{U*{}^xV}\Big).$$

Now this definition claims that for any index $x$ of the above sum, the subgroup $U*{}^xV$ is also a pure subgroup, which shows that the $\star$-product of bisets restricts to a product of pure $(G,G)$-bisets. To justify the claim, recall that by definition,

$$U*{}^xV=\{(g,h)\in G\times G\,|\,\exists k\in G:(g,k)\in U,(k,h)\in {}^xV\}.$$

Straightforward calculations show that the equalities

$$p_1(U*{}^xV)/k_1(U*{}^xV)=H/N\sqcap {}^x(K/M)$$

and

$$p_2(U*{}^xV)/k_2(U*{}^xV)={}^x(K/M)\sqcap H/N$$

hold. Here given subquotients $H/N$ and $K/M$ of $G$, we define

$$H/N\sqcap K/M:=\frac{(H\cap K)N}{(H\cap M)N}$$

which we call the *intersection* of the subquotients $H/N$ and $K/M$. Note that, in general, this intersection is not commutative but by Zassenhaus' Butterfly Lemma, we have an isomorphism $H/N\sqcap K/M\cong K/M\sqcap H/N$. Now we need to show that under the condition that ${}^xMN\le H\cap{}^xK$, the intersection commutes which follows from the following calculations.

$$\begin{aligned}H/N\sqcap{}^x(K/M)&=\frac{(H\cap{}^xK)N}{(H\cap{}^xM)N}\\&=\frac{H\cap{}^xK}{{}^xMN}\\&=\frac{(H\cap{}^xK)^xM}{(K\cap N)^xM}\\&={}^x(K/M)\sqcap H/N\end{aligned}$$

Therefore we have proved the following theorem.

**2.8. Theorem.** (Mackey Formula) *The $\star$-product of $(G,G)$-bisets restricts to a well-defined product on the full subcategory of pure $(G,G)$-bisets. Moreover given pure subgroups $U$ and $V$ of $G \times G$ with corresponding subquotients $H/N$ and $K/M$ we have*

$$(H/N)_G \star_G (K/M)_G = \bigsqcup_{\substack{x \in H\backslash G/K \\ {}^xMN \leq {}^xK \cap H}} (H \cap {}^xK/{}^xMN)_G.$$

**2.9. Remark.** Decomposing the above pure $(G,G)$-bisets as $(H/N)_G = \mathrm{Tin}^G_{H/N}\mathrm{Des}^G_{H/N}$ and $(K/M)_G = \mathrm{Tin}^G_{K/M}\mathrm{Des}^G_{K/M}$, one can rewrite the Mackey formula in the following form

$$\mathrm{Tin}^G_{H/N}\mathrm{Des}^G_{H/N}\star_G\mathrm{Tin}^G_{K/M}\mathrm{Des}^G_{K/M} = \bigsqcup_{\substack{x \in H\backslash G/K \\ {}^xMN \leq {}^xK \cap H}} \mathrm{Tin}^G_{H\cap^x K/^x MN}\mathrm{Des}^G_{H\cap^x K/^x MN}.$$

**2.10.** Now we define the category $\mathcal{P}_G$ of pure $(G,G)$-bisets as the category where

- the objects of $\mathcal{P}_G$ are the pure $(G,G)$-bisets.
- The set of morphisms between two pure $(G,G)$-bisets is the set of $(G,G)$-biset morphisms and
- the composition is given by composition of maps.

From the above discussion, we obtain a bifunctor

$$? \star_G ? : \mathcal{P}_G \times \mathcal{P}_G \to \mathcal{P}_G$$

which is associative, up to a natural isomorphism, with a left and right identity being the object $[G/1]_G$. Therefore the category $\mathcal{P}_G$ of pure $(G,G)$-bisets is a symmetric monoidal category with the unit object the pure $(G,G)$-biset $[G/1]_G$.

**2.11. The Grothendieck ring of $\mathcal{P}_G$.** We denote by $B_{\mathcal{P}}(G)$ the Grothendieck ring of the category $\mathcal{P}_G$ of pure $(G,G)$-bisets with respect to disjoint unions and the categorical product of pure bisets.

More precisely we define $B_{\mathcal{P}}(G)$ as the quotient of the free abelian group on the set of isomorphism classes of pure $(G,G)$-bisets, by the subgroup generated by all the elements of the form $[X \sqcup Y] - [X] - [Y]$, where $X$ and $Y$ are (finite) pure $(G,G)$-bisets, and $[X]$ denotes the isomorphism class of $X$. The product on $B_{\mathcal{P}}(G)$ is defined by $[X][Y] = [X \star_G Y]$.

Clearly the ring $B_{\mathcal{P}}(G)$ is an associative commutative ring with unity $[\frac{G\times G}{\Delta(G)}]$ where $\Delta(G)$ is the diagonal inclusion of $G$ in $G \times G$, that is, $\Delta(G) = \{(g,g) \in G \times G \mid g \in G\}$. Note that the subquotient corresponding to this pure subgroup is $G/1$. Now the following proposition, although self-evident, is stated for convenience.

**2.12. Proposition.** *The Grothendieck ring $B_{\mathcal{P}}(G)$ of pure $(G,G)$-bisets is isomorphic with the ring $\Lambda(G)$ of subquotients of $G$ via the ring isomorphism*

$$\iota : B_{\mathcal{P}}(G) \to \Lambda(G), \left[\frac{G\times G}{U}\right] \mapsto [p_1(U)/k_1(U)]_G.$$

**2.13. Remark.** From the above proposition, it is clear that the Burnside ring $B(G)$ of $G$ can be identified with the subring of $\Lambda(G)$ generated as an abelian group by the left and right free pure $(G,G)$-bisets.

## 3. Mackey functor structure

**3.1. Poset of subquotients.** We recall our notation from [7]. Let $G$ be a finite group. A *subquotient* of $G$ is a pair $(H^*, H_*)$ where $H_* \trianglelefteq H^* \le G$. We write the pair $(H^*, H_*)$ as $H$ and denote the subquotient relation by $H \preceq G$. Here, and afterwards, we regard the group $G$ as the subquotient $(G,1)$. When it is more convenient, we write $(H,N)$ and $H/N$ instead of $(H^*, H_*)$ and $H$.

The group $G$ acts on the set of its subquotients by conjugation. We write $H \preceq_G G$ to mean that $H$ is taken up to $G$-conjugacy. Note that we always consider $H$ as the quotient group $H^*/H_*$. Therefore, for example, what we mean by up to $G$-conjugation is that the subgroup $H^*$ is taken up to $G$-conjugacy and the normal subgroup $H_*$ of $H^*$ is taken up to $N_G(H)$-conjugacy.

The relation $\preceq$ extends to a partial order on the set of all subquotients of $G$ in the following way. Let $J$ and $H$ be two subquotients of $G$. Then we write $J \preceq H$ if and only if $H_* \le J_*$ and $H^* \ge J^*$. In this case the pair $(J^*/H_*, J_*/H_*)$ is a subquotient of $H$. The poset structure is compatible with the $G$-action, that is, the set of subquotients of $G$ is a $G$-poset.

Also we say that two subquotients $H$ and $K$ of $G$ are isomorphic if and only if they are isomorphic as groups, that is, if $H^*/H_* \cong K^*/K_*$. In this case, write $H \preceq_* G$ to mean that $H$ runs over a set of representatives of isomorphism classes of subquotients of $G$, or we simply say that $H$ is taken up to isomorphism.

**3.2. Relations with subgroups.** In this subsection, we will consider the maps between the rings of subquotients for the subgroups of $G$. There are three classical maps to consider, namely induction map, restriction map and conjugation map.

**3.2.1.** *Induction map.* Let $X \leq Y$ be subgroups of $G$. Let $H/N$ be a subquotient of $X$. Then clearly the subquotient $H/N$ embeds in to the over group $Y$ in a natural way. We still denote the embedding of $H/N$ to $Y$ by $H/N$. This induces a map on the set of conjugacy classes of subquotients and by linear extension, we obtain a map

$$\mathrm{Ind}_X^Y : \Lambda(X) \to \Lambda(Y)$$

which we call *induction*. More precisely we have

$$\mathrm{Ind}_X^Y([H/N]_X) = [H/N]_Y.$$

Note that, clearly, induction is a map of abelian groups and not a map of rings. It is also clear that the induction map is transitive.

**3.2.2.** *Restriction map.* Let $X \leq Y$ still be subgroups of $G$. Let $K/M$ be a subquotient of $Y$. As in the case of the product, there are several options for the map going from the ring $\Lambda(Y)$ to $\Lambda(X)$ and we shall chose the one that is appropriate for the rest of the paper. We define the *restriction* map

$$\mathrm{Res}_X^Y : \Lambda(Y) \to \Lambda(X)$$

by the formula

$$\mathrm{Res}_X^Y([K/M]_Y) = \sum_{\substack{x \in X\backslash Y/K \\ {}^xM \leq X}} [X \cap {}^xK/{}^xM]_X.$$

We shall prove in Section 3.4 that the restriction map, as defined above, is a map of rings and is transitive.

**3.2.3.** *Conjugation map.* Let $X$ still be a subgroup of $G$ and $g \in G$. The conjugation map

$$\mathrm{c}_H^g : \Lambda(X) \to \Lambda({}^gX)$$

is defined as the map induced by the conjugation action of $G$ on the set of its subquotients. Precisely, we define the conjugation map as

$$\mathrm{c}_X^g([K/M]_X) := [{}^gK/{}^gM]_{{}^gX}.$$

This map is also transitive and is a map of rings.

**3.3. Mackey Structure.** The groups $\Lambda(H)$ as $H$ runs over the set of subgroups of $G$ together with the above three maps is a Mackey functor for $G$. Moreover the ring structure is compatible with the Mackey structure, that is, the functor $\Lambda$ is actually a Green functor. This follows either by direct calculations or by the identification of the next section. In this part, we review the theory of Mackey functors. Details can be found in [12], [1] and [5].

A *Mackey functor* for the finite group $G$ over a commutative ring $k$ is a quadruple $(M, t, c, r)$ where $M$ is a family of $k$-modules consisting of a $k$-module $M(H)$ for each subgroup $H$ of $G$. The triple $(t, c, r)$ is a triple of families of maps between these $k$-modules and are subject to a number of relations, including the Mackey relation, see [12]. More precisely, for each pair $K \leq H$ of subgroups of $G$, we have a *transfer map* $t_K^H : M(K) \to M(H)$ and a *restriction map* $r_K^H : M(H) \to M(K)$ and for an element $g$ and a subgroup $H$ of $G$, a *conjugation map* $c_H^g$ : $M(H) \to M({}^gH)$. It is well-known that Mackey functors are modules over the *Mackey algebra* $\mu_k(G)$, see [12]. It is the quotient of the free algebra on the set of above morphisms by the ideal generated by the relations between these maps.

Simple modules of the Mackey algebra are determined by Thévenaz and Webb in [11]. The simples are parameterized by pairs $(H, V)$ where $H \leq G$ taken up to conjugation and $V$ is a simple $kN_G(H)/H$-module taken up to isomorphism. Here $N_G(H)$ denoted the normalizer of $H$ in $G$. In this case, the corresponding simple Mackey functor is denoted by $S_{H,V}$.

One way of studying the structure of Mackey functors is to consider functors with less structure, that is, to consider modules over certain subalgebras of the Mackey algebra. Classical way of obtaining subalgebras is to consider Mackey algebras for subgroups. This is done by Thévenaz and Webb in [11] and [12]. Another way of obtaining functors with less structure is to forget some of the maps, see [1],[3], [5]. Here we consider three other subalgebras, namely the conjugation algebra $\gamma_k(G)$ generated by conjugation maps, the restriction algebra $\rho_k(G)$ generated by restriction and conjugation maps and finally its opposite, the transfer algebra $\tau_k(G)$ which is generated by transfer and conjugation maps. The modules of these algebras are called conjugation functors, restriction functors and transfer functors, respectively, see [1] and [5].

In [1], Boltje introduced two plus constructions relating these functors to Mackey functors. In [5], it is shown that one of them which associates a Mackey functor to an arbitrary restriction functor and which is denoted by $-_+$ is naturally equivalent to the induction functor ${}_\rho\mathrm{mod} \to {}_\mu\mathrm{mod}$. On the other hand, the other one which associates a Mackey functor to an arbitrary conjugation functor and which is denoted by $-^+$ is naturally equivalent to the composition of the inflation functor ${}_\gamma\mathrm{mod} \to {}_\tau\mathrm{mod}$ with the coinduction functor ${}_\tau\mathrm{mod} \to {}_\mu\mathrm{mod}$. Moreover for any restriction functor $D$, the associated mark homomorphism between the plus constructions can be obtained naturally in this context.

The explicit descriptions of these functors and the mark morphism can be found in [1].

In particular, when $D$ is the constant restriction functor, the induced Mackey functor $\mathrm{ind}_\rho^\mu D$ is isomorphic to the Burnside ring functor and the above mark morphism coincides with the usual mark homomorphism for the Burnside ring. Next we show that it is possible to realize the functor of the ring of subquotients in a similar way. This provides another nice way of identifying the ring $\Lambda(G)$ and leads to the description of the primitive idempotents.

**3.4. An identification.** Our next aim is to describe a restriction functor $T$ for $G$ such that the induced Mackey functor $\mathrm{Ind}_\rho^\mu T$ is isomorphic with the functor $k\Lambda$ of subquotients for $G$ over $k$. Using this identification, it is easier to prove that the associated mark morphism is a ring homomorphism, which we discuss in the next section. We also use this to determine the inverse of this map over a field of characteristic zero.

We define, for each subgroup $H$ of $G$, the abelian group $T(H)$ by

$$T(H) = \bigoplus_{N \trianglelefteq H} \mathbb{Z}[N]_H.$$

In other words, $T(H)$ is the free abelian group on the set of normal subgroups of $H$. The conjugation maps are given by conjugating the subgroups, that is, if $g \in G$ then the conjugation map is given by $c_H^g([N]_H) = [{}^gN]_{{}^gH}$. Also for a pair $K \leq H$ of subgroups of $G$, the restriction map is given by inclusion, that is,

$$r_K^H : T(H) \to T(K), \quad [N]_H \mapsto \lfloor N \leq K \rfloor [N]_K.$$

Here the notation $\lfloor N \leq K \rfloor$ is the boolean operator which is equal to 1 when the inside proposition holds and zero otherwise. Clearly with these definitions, $T$ is a restriction functor. Moreover for any subgroup $H$ of $G$, the abelian group $T(H)$ is a commutative, associative ring with the multiplication given by

$$[N]_H \cdot [M]_H = [NM]_H.$$

Note also that the basis element $[1]_H$ corresponding to the trivial subgroup is the unit of the ring $T(H)$. Moreover the restriction maps and the conjugation maps are ring homomorphisms. A restriction functor with this property is called an *algebra restriction functor*, see [1]. Hence we have proved the following.

**3.5. Lemma.** *The functor associating a subgroup $H$ of $G$ to the abelian group $T(H)$ together with the above conjugation and restriction maps is a restriction functor. Moreover the above product turns this functor into an algebra restriction functor.*

Now we are ready to prove the following identification.

**3.6. Proposition.** *For any subgroup $H$ of $G$, there are isomorphisms of rings*

$$\mathrm{Ind}_\rho^\mu T(H) \cong \Lambda(H).$$

*Moreover this isomorphism is compatible with the induction, restriction and conjugation maps. In other words the functor $\Lambda$ becomes a Green functor via the above isomorphism.*

*Proof.* By [1, Section 2.2] and [5, Theorem 5.1], the isomorphism

$$\mathrm{Ind}_\rho^\mu T(H) \cong \Big(\bigoplus_{K\le H} T(K)\Big)_H$$

of abelian groups holds. Here for any $H$-module $M$, we denote by $M_H$ the largest quotient of $M$ on which $H$ acts trivially. Moreover it is known that the module in right-hand side of the above isomorphism is generated as an abelian group by the set $\{[K,a]_H | K \le_H H, a \in T(K)\}$. Here $[K,a]_H$ denote the image of $a \in T(K)$ in the quotient $\Big(\bigoplus_{K\le H} T(K)\Big)_H$. Now if we let $a$ run over a $N_H(K)$-basis of $T(K)$, that is over a complete set of representatives of $N_H(K)$-orbits of normal subgroups of $K$, the above set becomes a $\mathbb{Z}$-basis for $\Big(\bigoplus_{K\le H} T(K)\Big)_H$. In other words the set

$$\{[K,N]_H | K \le_H H, N \in [N_H(K)\backslash\mathfrak{N}(K)]\}$$

is a basis for $\Big(\bigoplus_{K\le H} T(K)\Big)_H$. Here we write $\mathfrak{N}(K)$ for the set of all normal subgroups of $K$ and the notation $N \in [N_H(K)\backslash\mathfrak{N}(K)]$ means that $N$ runs through a complete set of $N_H(K)$-conjugacy class representatives of normal subgroups of $K$. Note that the above isomorphism is given by associating a generator $t_K^H \otimes a$ of $\mathrm{Ind}_\rho^\mu T(H)$ to $[K,a]_H$. Therefore the induced module $\mathrm{Ind}_\rho^\mu T(H)$ has basis $\{t_K^H \otimes [N]_K | K \le_H H, N \in [N_H(K)\backslash\mathfrak{N}(K)]\}$.

Now it is clear that the correspondence

$$\mathrm{Ind}_\rho^\mu T(H) \to \Lambda(H), \quad t_K^H \otimes [N]_K \mapsto [K/N]_H$$

sets up an isomorphism of abelian groups.

Now by [1, Section 2.2], there is a ring structure on the induced functor $\mathrm{Ind}_\rho^\mu T$, given, at a subgroup $H$ of $G$, by

$$(t_K^H\otimes[N]_K)\cdot(t_L^H\otimes[M]_L) = \sum_{x\in K\backslash G/L} t_{K\cap {}^xL}^H\otimes((r_{K\cap {}^xL}^K[N]_K)\cdot(r_{K\cap {}^xL}^{{}^xL}[{}^xM]_{{}^xL}))$$

and with this ring structure, $\mathrm{Ind}_\rho^\mu T$ becomes a Green functor. Now by the definition of product in $T(H)$, we obtain the following formula.

$$(t_K^H\otimes[N]_K)\cdot(t_L^H\otimes[M]_L) = \sum_{\substack{x\in K\backslash G/L\\ {}^xMN\leq K\cap {}^xL}} t_{K\cap {}^xL}^H\otimes[{}^xMN]_{K\cap {}^xL}$$

which corresponds, under the above isomorphism, to the multiplication in the ring $\Lambda(H)$ of subquotients of $H$. In other words, the above isomorphism is an isomorphism of rings.

Now induction, restriction and conjugation maps of the Mackey functor $\mathrm{Ind}_\rho^\mu T$ is obtained by multiplication with the corresponding generator of the Mackey algebra from the left, which can easily be shown to correspond these operations defined for the ring $\Lambda(G)$. We refer to [1] or [5] for more explicit formulas of induction, restriction and conjugation maps for the functor $\mathrm{Ind}_\rho^\mu T$. □

Our next result on the structure of the ring of subquotients is its decomposition as a Mackey functor over a field $\mathbb{K}$ of characteristic zero. It is well-known that the Mackey algebra is semisimple over $\mathbb{K}$ (see [11]). In particular $\mathbb{K}\Lambda$ is semisimple as a Mackey functor, and we have the following result.

**Proposition 3.1.** *The following isomorphism of Mackey functors holds*

$$\mathbb{K}\Lambda\cong\bigoplus_{\substack{H\leq_G G\\ V\in\mathrm{Irr}_{\mathbb{K}}(N_G(H))}} m_{H,V}S_{H,V}.$$

*Here* $m_{H,V}=\sum_M \dim V^{N_G(H,N)}$ *where $M$ runs over a complete set of representatives of normal subgroups of $H$ up to $N_G(H)$-conjugacy and where* $N_G(H,M):=N_{N_G(H)}(M)$.

*Proof.* It is not difficult to prove that given a restriction functor $D$ for $G$ over $\mathbb{K}$, a simple Mackey functor $S_{H,V}$ for $G$ over $\mathbb{K}$ is a direct summand of $\mathrm{Ind}_\rho^\mu D$ with multiplicity $n$ if and only if the simple $\mathbb{K}N_G(H)/H$-module $V$ is a direct summand of $D(H)$ with multiplicity $n$. Therefore we only need to decompose $\mathbb{K}T(H)$ for each subgroup $H$ of $G$. Given a subgroup $H$ of $G$, recall that, by definition, we have

$$T(H)=\bigoplus_{N\trianglelefteq H}\mathbb{K}\,[N]_H.$$

The action of the group $N_G(H)$ on this $\mathbb{K}$-module is given by permutation of the coordinates. Therefore if we denote by $[N]_H^{N_G(H)}$ the $N_G(H)$-orbit sum of $N$, the above $\mathbb{K}\,N_G(H)$-module can be written as

$$\mathbb{K}\,T(H) = \bigoplus_{N\in[N_G(H)\backslash\mathfrak{N}(H)]} \mathbb{K}\,[N]_H^{N_G(H)}.$$

But the stabilizer of the orbit of $N$ is its normalizer in $N_G(H)$. Therefore we get

$$\mathbb{K}\,T(H) \cong \bigoplus_{N\in[N_G(H)\backslash\mathfrak{N}(H)]} \mathrm{Ind}_{N_G(H,N)}^{N_G(H)}\mathbb{K}$$

where the $N_G(H,N)$-module $\mathbb{K}$ is the trivial module. Here we write $N_G(H,N)$ for the group $N_{N_G(H)}(N)$. Clearly the action of $H$ on $T(H)$ is trivial and this decomposition is a decomposition of $\mathbb{K}\,N_G(H)/H$-modules. Now given a simple $\mathbb{K}N_G(H)/H$-module $V$, the multiplicity of $V$ in $\mathbb{K}\,T(H)$ is

$$m_{H,V} = \sum_{N\in[N_G(H)\backslash\mathfrak{N}(H)]} < \mathrm{Ind}_{N_G(H,N)}^{N_G(H)}\mathbb{K}, V >$$

which is equal to the number given in the statement of the proposition, by Frobenius reciprocity. □

**3.7. A decomposition of $\Lambda$.** The structure of the Mackey functor $\Lambda$ over an arbitrary field $k$ is more complicated. Next we obtain a decomposition of the functor of subquotients into (not necessarily indecomposable) Mackey functors, one of which is the Burnside ring functor.

Given a subgroup $H$ of $G$, define the restriction subfunctor $kT_H$ of $kT := k\otimes T$ by

$$kT_H(K) = \bigoplus_{L\trianglelefteq K, L=_G H} k[L]_K$$

with the induced restriction and conjugation maps. Then it is clear that $kT_H$ is a direct summand of $kT$. Moreover the functor $kT_H$ is indecomposable for any $H$. Indeed the functor $kT_H$ is cogenerated by its value at $H$, which is equal to $k$. Therefore the endomorphism ring of $T_H$, being equal to $k$, is local. Alternatively one can show that there is an isomorphism of restriction functors $T_H = \mathrm{Coind}_\gamma^\rho\,\mathrm{S}_{\mathrm{H},1}^\rho$, from which the indecomposability follows immediately. It is also clear that $T_H = T_L$ if, and only if, $H =_G L$. Therefore we have shown that the functor $kT$ decomposes into indecomposable summands as

$$kT = \bigoplus_{H\leq_G G} T_H$$

Now since induction is additive, this decomposition induces a decomposition of the Mackey functor $k\Lambda$ into summands $k\Lambda_H := \mathrm{Ind}_\rho^\mu T_H$ where the evaluation $k\Lambda_H(K)$ has basis consisting of subquotients of the form $[L/N]_K$ with $N =_G H$. Therefore we have proved

**3.8. Proposition.** *There is an isomorphism of Mackey functors*

$$k\Lambda \cong \bigoplus_{H \leq_G G} k\Lambda_H$$

*where $k\Lambda_H$ is the Mackey functor* $\mathrm{Ind}_\rho^\mu \mathrm{Coind}_\gamma^\rho \mathrm{S}_{\mathrm{H},1}^\gamma$.

**3.9. Mackey structure in terms of pure bisets.** It is possible to describe the Mackey structure using the identification of the group $\Lambda(G)$ as the Grothendieck group of the category of pure $(G,G)$-bisets. As far as the Mackey structure is concerned, we can slightly change the definition of the group $\Lambda(H)$ for $H \preceq G$ as follows. We define a (transitive) *pure* $(H,G)$*-biset* as a transitive $(H,G)$-biset $(\frac{H\times G}{U})$ which can be decomposed as

$$\left(\frac{H\times G}{U}\right) = \mathrm{Tin}_{K/M}^H \mathrm{Des}_{K/M}^G.$$

In other words, a pure $(H,G)$-biset is an $(H,G)$-biset which contains no isogations. By [6, Theorem 3.3], there is a bijective correspondence between the isomorphism classes of pure $(H,G)$-bisets and the conjugacy classes of subquotients of $H$.

Now as above, we obtain the category of pure $(H,G)$-bisets and denoting the Grothendieck group of this category by $\Lambda(H,G)$, we obtain an isomorphism of abelian groups

$$t : \Lambda(H) \to \Lambda(H,G)$$

given by extending linearly the correspondence

$$[K/M]_H \mapsto \mathrm{Tin}_{K/M}^H \,\mathrm{Des}_{K/M}^G.$$

Now it is clear how to define induction, restriction and conjugation, (and deflation and inflation) maps. For example, if $X \leq Y$ are subgroups of $G$, then the restriction map

$$\mathrm{Res}_X^Y : \Lambda(Y) \to \Lambda(X)$$

is defined by left multiplication pre-composed by the above isomorphism and post-composed by its inverse, that is,

$$\begin{aligned}
\mathrm{Res}^Y_X([K/M]_Y) &= t^{-1}(\mathrm{Des}^Y_X \mathrm{Tin}^H_{K/M} \mathrm{Des}^G_{K/M}) \\
&= \sum_{\substack{x\in X\backslash Y/K \\ {}^xM\leq X}} t^{-1}(\mathrm{Tin}^X_{(X\cap {}^xK)/{}^xM} \mathrm{Des}^{K/M}_{(X\cap {}^xK)/{}^xM} \mathrm{c}^x_{K/M} \mathrm{Des}^G_{K/M}) \\
&= \sum_{\substack{x\in X\backslash Y/K \\ {}^xM\leq X}} t^{-1}(\mathrm{Tin}^X_{(X\cap {}^xK)/{}^xM} \mathrm{Des}^G_{(X\cap {}^xK)/{}^xM}) \\
&= \sum_{\substack{x\in X\backslash Y/K \\ {}^xM\leq X}} [(X\cap {}^xK)/{}^xM]_X
\end{aligned}$$

Here to obtain the third equality, we used compatibility of conjugation maps with destriction maps, transitivity of destriction maps and that the conjugation $\mathrm{c}^x_G$ is identity if $x\in G$. Similarly one can obtain the other maps.

## 4. Table of Generalized Marks

**4.1. Generalized marks.** In this section, we introduce marks of subquotients on each other, which we call the generalized marks. The induced mark homomorphism extends the usual one and has similar properties. In particular, it is injective and after extending the coefficients to a field of characteristic zero, it becomes an isomorphism. In the next section, we use these results to find an idempotent formula for primitive idempotents of the algebra $\mathbb{Q}\Lambda(G) = \mathbb{Q}\otimes_{\mathbb{Z}}\Lambda(G)$.

**4.2. Definition.** Given subquotients $H/N$ and $K/M$ of $G$. The *(generalized) mark* $m_{K/M,H/N}$ of $K/M$ on $H/N$ is the natural number

$$m_{K/M,H/N} := |\{x\in K\backslash G/H : {}^xN\leq M, K\leq {}^xH\}|.$$

Note that when both $N$ and $M$ are equal to the trivial subgroup of $G$, the generalized mark is equal to the ordinary one. Now we call the matrix formed by the marks of subquotients the *table of (generalized) marks*. Once we order the set of subquotients by non-decreasing order of their sizes, the table of generalized marks becomes lower triangular as the following result shows.

**4.3. Proposition.** *The table of generalized marks is lower triangular with determinant*

$$\prod_{K/M\preceq_G G} \frac{|N_G(K,M)|}{|K|}.$$

*where* $N_G(K, M) = N_G(K) \cap N_G(M)$*.*

*Proof.* Clearly an entry $m_{K/M,H/N}$ of the table of marks is non-zero if some conjugate of $K/M$ is contained in $H/N$. Therefore the matrix is upper triangular. Now a diagonal entry is given by

$$\begin{aligned} m_{K/M,K/M} &= |\{x \in K\backslash G/K : {}^xM \leq M, K \leq {}^xK\}| \\ &= |\{x \in K\backslash G/K : {}^xM = M, K = {}^xK\}| \\ &= \frac{|N_G(K, M)|}{|K|}. \end{aligned}$$

as required. □

**4.4. Remark.** It is well-known that the table of (ordinary) marks does not determine the group. Examples of non-isomorphic groups with isomorphic table of marks are given by Thévenaz [10]. Unfortunately the table of generalized marks could not identify these groups, too. However the table of marks determines the table of ordinary marks, as the following proposition shows.

**4.5. Proposition.** *The table of generalized marks determines the table of ordinary marks.*

*Proof.* Given a table of generalized marks without the labels of columns and rows, it is possible to determine the columns that corresponds to subgroups. First, the order of the group $G$ appears only once in the diagonal and this entry corresponds to the mark $m_{1/1,1/1}$. Indeed the diagonal entries are

$$m_{K/M,K/M} = \frac{|N_G(K, M)|}{|K|}$$

and $m_{K/M,K/M} = |G|$ is satisfied only if $K = 1$ is the trivial group. Now $m_{1/1,H/N}$ is non-zero if and only if $N = 1$. Therefore the columns corresponding to non-zero entries of the row corresponding to the trivial group are the one that are indexed by subgroups of $G$. Since generalized marks for subgroups coincide with the ordinary marks, the table of marks is determined. □

**4.6. The orbit counting lemma revisited.** Now we come to the main result of this section. The well known orbit counting lemma (also known as the Cauchy-Frobenius-Burnside relation or Burnside's Theorem) states that given a finite $G$-set $X$, one has the number $|X\backslash G|$ of $G$-orbits of $X$ satisfies

$$|X\backslash G| = \frac{1}{|G|} \sum_{g \in G} |X^{<g>}|$$

where $X^{<g>}$ denotes the set of $<g>$-fixed points of $X$. Recall that given a subgroup $K$ of $G$, the *mark* $m_K(X)$ of $K$ on $X$ is defined as the number $|X^K|$ of $K$ fixed points of $X$. Thus we can rewrite the above equation as

$$|X\backslash G| = \frac{1}{|G|}\sum_{g\in G} m_{<g>}(X).$$

In particular, if $X = G/H$ is transitive, then this equation becomes

$$|G| = \sum_{g\in G} m_{<g>}(G/H).$$

The generalized mark version of this equation is the following theorem. It counts the number of certain conjugacy classes of a given subgroup.

**4.7. Theorem.** *Let $H$ be a subgroup of $G$ and $K/M$ be a subquotient of $G$ such that $M \le H \le K$. Then the number of $K$-conjugacy classes of the $G$-conjugacy class of $H$ that contains $M$ is*

$$\frac{|H|}{|N_G(H)|}\sum_{g\in N_G(H)} m_{<g>H/H,K/M}.$$

*Proof.* Recall that

$$m_{<g>H/H,K/M} = |\{x \in <g> H\backslash G/K | {}^xM \le H, <g> H \le {}^xK\}|$$

which can be written as

$$m_{<g>H/H,K/M} = |\{x \in G/K | {}^xM \le H, H \le {}^xK, g \in {}^xK\}|.$$

Therefore we obtain the following equalities.

$$\begin{aligned}\sum_{g\in N_G(H)} m_{<g>H/H,K/M} &= \sum_{g\in N_G(H)} |\{x \in G/K | {}^xM \le H, H \le {}^xK, g \in {}^xK\}| \\ &= \sum_{\substack{g\in N_G(H)\\ x\in G/K}} \lfloor {}^xM \le H \le {}^xK, g \in {}^xK\rfloor\end{aligned}$$

where the notation $\lfloor p\rfloor$ is the boolean operator which is equal to 1 if $p$ is true and is equal to zero otherwise. Changing the order of the summation we get

$$\begin{aligned}\sum_{\substack{g\in N_G(H)\\ x\in G/K}} \lfloor {}^xM \le H \le {}^xK, g \in {}^xK\rfloor &= \sum_{\substack{x\in G/K\\ {}^xM\le H\le {}^xK}} |\{g \in N_G(H) | g \in {}^xK\}| \\ &= \sum_{\substack{x\in G/K\\ {}^xM\le H\le {}^xK}} |N_{{}^xK}(H)|\end{aligned}$$

Now denote by $X$ the indexing set of the last summation, that is, put

$$X := \{xK \in G/K \mid {}^xM \leq H \leq {}^xK\} = \{x \in G/K \mid M \leq H^x \leq K\}.$$

Then $X$ is an $N_G(H)$-set via left multiplication and as an $N_G(H)$-set, it decomposes as

$$X \cong \bigsqcup_{x \in [N_G(H)\backslash X]} N_G(H)/N_{{}^xK}(H).$$

Indeed $h \in N_G(H)$ is in the stabilizer in $N_G(H)$ of any left coset $xK$ of $K$ in $G$ if and only if $hxK = xK$ if and only if $h \in {}^xK \cap N_G(H) = N_{{}^xK}(H)$.

It is clear that the function $|N_{{}^?K}(H)| : X \to \mathbb{Z}$ is constant on the $N_G(H)$-orbits of the set $X$. Hence we get

$$\begin{aligned}
\sum_{\substack{x \in G/K \\ {}^xM \leq H \leq {}^xK}} |N_{{}^xK}(H)| &= \sum_{x \in [N_G(H)\backslash X]} \frac{|N_G(H)|}{|N_{{}^xK}(H)|} |N_{{}^xK}(H)| \\
&= \sum_{x \in [N_G(H)\backslash X]} |N_G(H)| \\
&= |N_G(H)\backslash X||N_G(H)|
\end{aligned}$$

Finally, it is also clear that the number $|N_G(H)\backslash X|$ is the number of $K$-conjugacy classes that contains $M$ of the $G$-conjugacy class of $H$, as required. $\square$

**4.8. Remark.** To obtain the orbit counting lemma, one takes $H = 1$ the trivial subgroup. Indeed in this case the sum in the statement of the above theorem is equal to 1 if $M$ is the trivial subgroup and equal to zero otherwise and the sum becomes

$$\frac{1}{|G|} \sum_{g \in G} m_{<g>,K}$$

which is the same summation as in the statement of the orbit counting lemma.

## 5. Ghost Ring and the Mark Homomorphism

The generalized marks can also be interpreted as ring homomorphisms. Given a subquotient $K/M$, denote by $\beta^G_{K/M}$ the map

$$\beta^G_{K/M} : \Lambda(G) \to \mathbb{Z}$$

given by $\beta^G_{K/M}([H/N]_G) = m_{K/M,H/N}$. As in the case of the Burnside ring, we also consider the product map

$$\beta^G := \prod_{K/M \preceq_G G} \beta^G_{K/M} : \Lambda(G) \to \prod_{K/M \preceq_G G} \mathbb{Z}$$

which we call the *mark homomorphism*. It is clear from this definition that the mark homomorphism is a homomorphism of abelian groups. Next we show that it is a ring homomorphism.

First note that by the identification of Proposition 3.6 and by [1, Section 2.3], we obtain a mark homomorphism

$$\tilde{\beta}^G : \Lambda(G) \to (\prod_{K \leq_G G} T(K))^G$$

defined as follows. Given a subquotient $H/N$ of $G$, the $K$-th coordinate $\tilde{\beta}^G_K([H/N]_G)$ of $\tilde{\beta}^G([H/N]_G)$ is

$$\tilde{\beta}^G_K([H/N]_G) = \sum_{\substack{x \in G/H \\ K \leq {}^xH}} r^{{}^xH}_K([{}^xN]_{{}^xH})$$

where restriction on the right hand side is the one defined for the restriction functor $T$. Applying this definition we obtain

$$\tilde{\beta}^G_K([H/N]_G) = \sum_{\substack{x \in G/H \\ {}^xN \leq K \leq {}^xH}} [{}^xN]_K.$$

Now by its definition, $\tilde{\beta}^G_K([H/N]_G)$ lies in $T(K)^{N_G(K)}$ which has a basis consisting of the $N_G(K)$-orbits $[N]^G_K$ of normal subgroups of $K$. Moreover, for a normal subgroup $M$ of $K$, there is a map

$$\bar{\beta}^K_M : T(K) \to \mathbb{Z}$$

given by

$$\bar{\beta}^K_M([N]_K) = \begin{cases} 1, & \text{if } N \leq M; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that for any normal subgroup $M$ of $G$, the map $\bar{\beta}^G_M$ is a ring homomorphism and we have

$$\beta^G_{K/M} = \bar{\beta}^K_M \circ \tilde{\beta}^G_K.$$

Therefore the mark homomorphism, $\beta^G$, being a composition of two ring homomorphisms is a ring homomorphism. Now we define the *ghost ring* $\Lambda^*(G)$ by

$$\Lambda^*(G) := \prod_{K/M \preceq_G G} \mathbb{Z}.$$

Note that this ring can be identified by the ring of class functions constant on the conjugacy classes of subquotients of $G$. Moreover by the above discussion, there is an isomorphism

$$\Lambda^*(G) \cong (\prod_{K \leq G} T(K))^G \cong \prod_{K \leq_G G} T(K)^{N_G(K)}$$

of rings. Hence we have proved

**5.1. Proposition.** *The map*

$$\beta^G : \Lambda(G) \to \Lambda^*(G)$$

*given by associating a subquotient $H/N$ of $G$ to the class function*

$$f^G_{H/N} : K/M \mapsto |\{x \in K\backslash G/H : \ K/M \preceq {}^x(H/N)\}|$$

*is a ring homomorphism.*

**5.2. Fundamental theorem.** *Let $G$ be a finite group. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow \Lambda(G) \xrightarrow{\beta^G} \Lambda^*(G) \longrightarrow \mathrm{Obs}(G) \longrightarrow 0$$

*where $\beta^G$ is the mark homomorphism and*

$$\mathrm{Obs}(G) = \prod_{K/M \preceq_G G} \mathbb{Z}/|N_G(K,M) : K|\mathbb{Z}.$$

Here one can interpret the group $\mathrm{Obs}(G)$ as the group of the obstructions for an element of the ghost ring to be an element of the ring of subquotients.

*Proof.* By the above identification of the mark morphism and by Theorem 4.1 of [8], the cokernel of the mark homomorphism is given by

$$\mathrm{Coker}\beta^G = \bigoplus_{K \leq_G G} \widehat{H}^0(N_G(K)/K, T(K))$$

where $\widehat{H}^0(N_G(K)/K, T(K))$ is the 0-th Tate cohomology of the $N_G(K)/K$-module $T(K)$. Now we also have

$$T(K) \cong \bigoplus_{M \trianglelefteq_{N_G(K)} K} \mathrm{Ind}_{N_G(K,M)/K}^{N_G(K)/K} \mathbb{Z}$$

where $\mathbb{Z}$ is the trivial $N_G(K,M)/K$-module, since $T(K)$ is a permutation $N_G(K)/K$-module with permutation basis the set of normal subgroups of $K$. Now the result follows from Shapiro's Lemma since

$$\begin{aligned}\widehat{H}^0(N_G(K)/K, T(K)) &= \widehat{H}^0(N_G(K)/K, \bigoplus_{M \trianglelefteq_{N_G(K)} K} \mathrm{Ind}_{N_G(K,M)/M}^{N_G(K)/K}\mathbb{Z}) \\ &= \bigoplus_{M \trianglelefteq_{N_G(K)} K} \widehat{H}^0(N_G(K,M)/K, \mathbb{Z})\end{aligned}$$

Similarly, by the same theorem, Theorem 4.1 of [8], and by similar calculations, we obtain that the kernel of the mark morphism is given by

$$\mathrm{Ker}\beta^G = \bigoplus_{M \in [N_G(K)\backslash \mathfrak{N}(K)]} \widehat{H}^{-1}(N_G(K,M)/K, \mathbb{Z})$$

which is equal to zero as the Tate cohomology groups at the right hand side are well-known to be zero. □

**5.3. Remark.** Note that the mark morphism is not compatible with the $\circ$-biset functor structure of the functor $\Lambda$ described in [7]. However there is an alternative definition, described below, of a mark of subquotient on another one which is also compatible with this structure. Our reason of choosing the above one is that it is compatible with the Mackey functor structure and hence it is easier to determine the inverse of it. The alternative description goes as follows.

Given subquotients $H/N$ and $K/M$ of $G$. The *$\circ$-mark* $n_{K/M,H/N}$ of $K/M$ on $H/N$ is the natural number

$$m_{K/M,H/N} := |\{x \in K\backslash G/H : {}^xN \le M, K \le {}^xHM\}|.$$

This definition also extends the ordinary marks and also the table of $\circ$-marks is upper triangular with the following order on the set of subquotients of $G$.

We write $K/M < H/N$ if and only if either $|K/M| < |H/N|$ or $|K/M| = |H/N|$ and $|M| > |N|$. Now denote by $\mathcal{S} := \{H_0, H_1, \cdot, H_n\}$ the ordered set of conjugacy classes of subquotients of $G$ indexed in such a way that for $i < j$, we have $H_i < H_j$ or $|H_i| = |H_j|$ and $|H_{i*}| = |H_{j*}|$. Then the *table of $\circ$-marks* $M$ is the matrix

$$M := \left(m_{H_i,H_j}\right)_{H_i,H_j \in \mathcal{S}}.$$

To prove that the matrix is upper triangular, note that, given subquotients $K/M, H/N$ of $G$, the mark $m_{K/M,H/N}$ is non-zero only if ${}^xN \le M$ and $K \le {}^xHM$ for some $x \in G$. Now multiplying both sides of the second inequality by ${}^xN$, we get $K{}^xN \le {}^xHM$. In particular, $\frac{|K|}{|M|} \le \frac{|H|}{|N|}$.

If the inequality is strict we are done. Otherwise if $|K/M| = |H/N|$ then the condition ${}^xN \leq M$ should be satisfied. Again we are done if the inequality is strict. But when the equality is satisfied we have $|K| = |H|$ and $|M| = |N|$. In this case ${}^xN = M$ and $K = {}^xH$, that is, $K/M = {}^xH/N$, as required.

## 6. Primitive Idempotents over $\mathbb{K}$

In this section, we find formula for the primitive idempotents of the algebra $\mathbb{K}\Lambda(G)$ where $\mathbb{K}$ denotes a field of characteristic zero. Since $\mathbb{K}\Lambda(G)$ is commutative, there is an isomorphism of $\mathbb{K}$-vector spaces

$$\mathbb{K}\Lambda(G) \cong \prod_{H/N \preceq_G G} \mathbb{K}$$

and hence there are as many primitive idempotents as the number of conjugacy classes of subquotients of $G$. Note that by the Fundamental Theorem of the last section, over $\mathbb{K}$, the mark homomorphism

$$\beta_G : \mathbb{K}\Lambda(G) \to \mathbb{K}\Lambda^*(G)$$

is an isomorphism of rings. Therefore we can determine the primitive idempotents of the algebra $\mathbb{K}\Lambda(G)$ of subquotients of $G$ by determining the inverse images under $\beta_G$ of the primitive idempotents of the ghost $\mathbb{K}\Lambda^*(G)$.

**6.1.** First we determine the primitive idempotents of the ghost $\mathbb{K}\Lambda^*(G)$. Since

$$\mathbb{K}\Lambda^*(G) \cong \Big(\prod_{H \leq G} \mathbb{K}T(H)\Big)^G \cong \prod_{H \leq_G G} \mathbb{K}T(H)^{N_G(H)}$$

and addition is coordinate-wise, the primitive idempotents of $\mathbb{K}\Lambda^*(G)$ are all of the form $(x_H)_{H \leq G}$ where $x_H = 0$ for all subgroups $H$ of $G$ except for a unique conjugacy class, say with representative $K$. Moreover $x_K$ is a primitive idempotent of the algebra $\mathbb{K}T(K)^{N_G(K)}$. Therefore we need to determine the primitive idempotents of the algebra $\mathbb{K}T(K)^{N_G(K)}$. By definition,

$$\mathbb{K}T(K) = \bigoplus_{N \trianglelefteq K} \mathbb{K}[N]_K$$

with the ring structure given by the linear extension of the following multiplication of the basis elements

$$[M]_K[N]_K = [MN]_K$$

where $M$ and $N$ are both normal subgroups of $K$.

Clearly $\mathbb{K}T(K)^{N_G(K)}$, with the inherited ring structure, is commutative and is generated by $N_G(K)$-orbit sums of the normal subgroups of

$K$. Hence the primitive idempotents of the algebra $\mathbb{K}T(K)^{N_G(K)}$ are determined by the species of $\mathbb{K}T(K)^{N_G(K)}$, that is by the algebra maps $\mathbb{K}T(K)^{N_G(K)} \to \mathbb{K}$. Next we classify these species.

Now for any normal subgroup $N \trianglelefteq K$, the map $S_N^K$ given by

$$S_N^K([M]_K) = \begin{cases} 1, & \text{if } M \leq N \\ 0, & \text{otherwise.} \end{cases}$$

for any $M \trianglelefteq K$, is a species of $\mathbb{K}T(K)$ and any species of $\mathbb{K}T(K)$ is of this form for some normal subgroup $N$. Now the species of $\mathbb{K}T(K)^{N_G(K)}$ are the $N_G(K)$-orbit sums of these species. Explicitly, these are given as follows.

**6.2. Proposition.** *Let $N$ be a normal subgroup of $K$. Then the map*

$$s_N^K : \mathbb{K}T(K)^{N_G(K)} \to \mathbb{K}, \quad x \mapsto \sum_{\substack{M \trianglelefteq K \\ M \leq N}} \text{coefficient of } [M]_K^G \text{ in } x$$

*is a species. Here $[M]_K^G$ denotes the $N_G(K)$-orbit sum of $[M]_K$. Moreover two species $s_N^K$ and $s_M^K$ are equal if, and only if, $N =_{N_G(K)} M$ and any species of $\mathbb{K}T(K)^{N_G(K)}$ is of the form $s_N^K$ for some normal subgroup $N$ of $K$.*

*Proof.* It is clear that for a given normal subgroup $N$, the map $s_N^K$ is the $N_G(K)$-orbit sum of the species $S_N^K$ of $\mathbb{K}T(K)$ and that given two normal subgroups $N$ and $M$ of $K$, the species $s_N^K$ and $s_M^K$ are equal if and only if $N =_{N_G(K)} M$. Therefore as $N$ runs over the set of $N_G(K)$ orbits of the set of normal subgroups of $K$, the species $s_N^K$ are pairwise distinct. Moreover since $\dim \mathbb{K}T(K)^{N_G(K)}$ is equal to the number of $N_G(K)$-orbits of normal subgroups of $K$, there is no other species. $\square$

**6.3.** Now also the primitive idempotents of $\mathbb{K}T(K)^{N_G(K)}$ are the $N_G(K)$-orbit sums of the primitive idempotents of $\mathbb{K}T(K)$. We first determine the primitive idempotents of $\mathbb{K}T(K)$. For a given normal subgroup $N$ of $K$, we define the element $\epsilon_N^K$ of $\mathbb{K}T(K)$ by the property that

$$S_M^K(\epsilon_N^K) = \delta_{(M,N)},$$

where $\delta_{(M,N)}$ is the Kronecker symbol. It is clear that these are primitive mutually orthogonal idempotents of $\mathbb{K}T(K)$ which sum up to 1. Then any element $x$ in $\mathbb{K}T(K)$ can be written as

$$x = \sum_{N \trianglelefteq K} x_N \epsilon_N^K$$

for some $x_N$ in $\mathbb{K}$. By the above property, we get

$$x_N = S_N^K(x).$$

In particular, for any normal subgroup $M$ of $K$, we have

$$[M]_K = \sum_{N \trianglelefteq K} S_N^K([M]_K)\epsilon_N^K = \sum_{\substack{N \trianglelefteq K \\ M \leq N}} \epsilon_N^K$$

by the definition of the species $S_N^K$. Now we apply Möbius inversion to the poset of normal subgroups of $K$ with incidence function given by

$$\zeta(N, M) = \begin{cases} 1, & \text{if } N \leq M; \\ 0, & \text{otherwise.} \end{cases}$$

The inversion gives

$$\epsilon_M^K = \sum_{\substack{N \trianglelefteq K \\ M \leq N}} \mu_{\trianglelefteq K}(M, N)[N]_K$$

where $\mu_{\trianglelefteq K}$ is the Möbius function of the poset of normal subgroups of $K$.

Now the primitive idempotents of $\mathbb{K}T(K)^{N_G(K)}$ are given by

$$\epsilon_{[M]}^K = \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M}} \epsilon_{M'}^K$$

where $M$ runs over $N_G(K)$-orbits of normal subgroups of $K$. More explicitly, we have

$$\epsilon_{[M]}^K = \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M}} \sum_{\substack{N \trianglelefteq K \\ M' \leq N}} \mu_{\trianglelefteq K}(M', N)[N]_K$$

Hence we have proved

**6.4. Proposition.** *Let $M$ be a normal subgroup of $K$. Then the element*

$$\epsilon_{[M]}^K = \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M}} \sum_{\substack{N \trianglelefteq K \\ M' \leq N}} \mu_{\trianglelefteq K}(M', N)[N]_K$$

*is a primitive idempotent of $\mathbb{K}T(K)^{N_G(K)}$. Moreover letting $M$ run over a complete set of $N_G(K)$-orbit representatives of normal subgroups of $K$, the idempotents $\epsilon_{[M]}^K$ are pairwise orthogonal and sum up to the identity element.*

As a corollary we obtain the primitive idempotents of the ghost $\mathbb{K}\Lambda^*(G)$.

**6.5. Corollary.** *Let $K$ be a subgroup of $G$ and $N$ be a normal subgroup of $K$. Then the element $e^G_{K,N} = (x_H)_{H\leq G} \in \Lambda^*(G)$ where $x_H = 0$ unless $H =_G K$ and $x_K = \epsilon^K_{[N]}$ is a primitive idempotent of $\mathbb{K}T(K)^{N_G(K)}$. Moreover as $K$ runs over subgroups of $G$ and $N$ runs over a complete set of $N_G(K)$-orbit representatives of normal subgroups of $K$, these idempotents are pairwise orthogonal and the sum is equal to the identity in $\mathbb{K}\Lambda^*(G)$.*

Finally we obtain a formula for the primitive idempotents of the algebra $\mathbb{K}\Lambda(G)$ of subquotients of $G$.

**6.6. Theorem.** [Idempotent Formula] *Let $G$ be a finite group and $\mathbb{K}$ be a field of characteristic zero. For a given subquotient $K/M$ of $G$, define*

$$e^G_{K,M} = \sum_{\substack{M'\trianglelefteq K\\ M'=_{N_G(K)}M}} \sum_{\substack{L/N\preceq K/M'\\ N\trianglelefteq K}} \frac{|L|}{|N_G(K)|}\mu(L,K)\mu_{\trianglelefteq K}(M',N)[L/N]_G$$

*where $\mu$ denotes the Möbius function of the poset of (all) subgroups of $G$ and $\mu_{\trianglelefteq K}$ denotes the Möbius function of the poset of normal subgroups of $K$. Then the set $\{e^G_{K,M} | K/M \preceq_G G\}$ is a complete set of orthogonal primitive idempotents of $\mathbb{K}\Lambda(G)$ such that*

$$1_{\Lambda(G)} = \sum_{K/M\preceq_G G} e^G_{K,M}.$$

*Proof.* By [1, Proposition 2.4], over the field $\mathbb{K}$, the mark morphism is an isomorphism and its inverse is given by

$$\beta_G^{-1}((x_H)_{H\leq G}) = \frac{1}{|G|}\sum_{L\leq H\leq G}|L|\mu(L,H)\sum_{\substack{N\trianglelefteq H\\ N\leq L}} a_N[L/N]_G$$

where the equality

$$x_H = \sum_{N\trianglelefteq H} a_N[N]_H$$

is satisfied in $T(H)$. Now we calculate $\beta^{-1}(e^G_{K,M})$, where $e^G_{K.M}$ is a primitive idempotent of $\Lambda^*(G)$ defined above. We have

$$\beta^{-1}(e^G_{K,M}) = \sum_{\substack{L\leq H\leq G\\ H=^gK}} \frac{|L|}{|G|}\mu(L,H)\sum_{\substack{N\trianglelefteq H\\ N\leq L}} \sum_{\substack{M'\trianglelefteq H\\ M'=_{N_G(H)}{}^gM,M'\leq N}} \mu_{\trianglelefteq H}(M',N)[L/N]_G$$

Here $g \in G$. We can write the above equality as

$$e^G_{K,M} = \sum_{\substack{g \in G/N_G(K) \\ L \leq {}^gK}} \frac{|L|}{|G|}\mu(L, {}^gK) \sum_{\substack{N \trianglelefteq K \\ N \leq L^g}} \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M, M' \leq N}} \mu_{\trianglelefteq K}(M', N)[L^g/N]_G$$

Here nothing depends on $g$, so we get

$$e^G_{K,M} = \sum_{L \leq K} \frac{|L|}{|N_G(K)|}\mu(L, K) \sum_{\substack{N \trianglelefteq K \\ N \leq L}} \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M, M' \leq N}} \mu_{\trianglelefteq K}(M', N)[L/N]_G$$

Finally rearranging the sums we obtain

$$e^G_{K,M} = \sum_{\substack{M' \trianglelefteq K \\ M' =_{N_G(K)} M}} \sum_{\substack{L/N \preceq K/M' \\ N \trianglelefteq K}} \frac{|L|}{|N_G(K)|}\mu(L, K)\mu_{\trianglelefteq K}(M', N)[L/N]_G$$

as required. The other claims follows easily. □

## 7. Prime ideals

**7.1.** In this section, we determine the prime ideals of the ring $\Lambda(G)$. Since both the ring $\Lambda(G)$ and its ghost $\Lambda^*(G)$ are finitely generated as abelian groups, the extension $\Lambda(G) \leq \Lambda^*(G)$ is integral. Therefore by [9, Chapter 28] any prime ideal of $\Lambda(G)$ is of the form $\beta^{-1}(P)$ where $P$ is a prime ideal of the ghost. Moreover since the ring extension $\Lambda^*(G) \leq \prod_{H \leq_G G} T(H)$ is integral, the prime ideals of $\Lambda^*(G)$ are all of the form $P' \cap \Lambda^*(G)$ for some prime ideal $P'$ of the product $\prod_{H \leq_G G} T(H)$. Finally, the non-zero prime ideals of the product are of the form $(P_H)_{H \leq_G G}$ where $P_H \neq 0$ only for a unique representative $K$ of conjugacy classes of subgroups of $G$ and $P_K$ is a prime ideal of $T(K)$. Therefore it suffices to determine the prime ideals of $T(K)$ for any subgroup $K$ of $G$.

Fix a subgroup $K$ of $G$. Then the map

$$\alpha_K : T(K) \to \prod_{N \trianglelefteq K} \mathbb{Z}, \quad [M]_K \mapsto (\lfloor M \leq N \rfloor)_{N \trianglelefteq K}$$

is a unital ring homomorphism where $\lfloor M \leq N \rfloor$ is equal to one if $M \leq N$ and zero otherwise. Clearly $\alpha_K$ is injective and with a suitable ordering of the normal subgroups of $K$, the matrix of $\alpha_K$ is upper-triangular with 1's in the diagonal. Therefore any prime ideal of $T(K)$ is the inverse image of a prime ideal of $\mathbb{Z}$.

**7.2.** We can describe the inverse image by tracing the maps in the above discussion. Any prime ideal of $\Lambda(G)$ is obtained by determining the preimage of a prime ideal of $\mathbb{Z}$ under the following composite map:

$$\Lambda(G) \overset{\beta}{\hookrightarrow} \Lambda^*(G) \hookrightarrow \prod_{K\leq_G G} T(H) \overset{\alpha}{\hookrightarrow} \prod_{K\leq_G G}\prod_{N\trianglelefteq K}\mathbb{Z} \overset{\pi_N^K}{\hookrightarrow} \mathbb{Z}$$

Here $\beta$ is the mark morphism, $\alpha = (\alpha_K)_{K\leq_G G}$ is the map defined above and $\pi_N^K$ is the projection to the $N$-th coordinate. Let us denote this composite by $s^G_{K,N}$. Then since all the maps are ring homomorphisms, the map $s^G_{K,N}$ is a species of $\Lambda(G)$. Moreover the equality $s^G_{K,N} = s^G_{K',N'}$ is satisfied if and only if $K =_G K'$ and assuming $K = K'$, $N =_{N_G(K)} N'$. The species $s^G_{K,N}$ is given explicitly by

$$s^G_{K,N}([H/M]_G) = |\{x \in G/H | K/N \preceq {}^x(H/M)\}|$$

where $[H/N]_G \in \Lambda(G)$. Hence we have proved the following theorem.

**7.3. Theorem.** *Given a subquotient $K/N$ of $G$ and a prime ideal $\mathfrak{p}$ of $\mathbb{Z}$. Define*

$$P(K, N, \mathfrak{p}) = \{x \in \Lambda(G) |\ s^G_{K,N}(x) \in \mathfrak{p}\}.$$

*Then $P(K, N, \mathfrak{p})$ is a prime ideal of $\Lambda(G)$. Moreover any prime ideal of $\Lambda(G)$ is of the form $P(K, N, \mathfrak{p})$ for some subquotient $K/N \preceq_G G$ and some prime ideal $\mathfrak{p}$ of $\mathbb{Z}$.*

## Appendix A. Two examples of table of generalized marks

We include the table of generalized marks for the groups $S_3$ and $A_4$. For comparison, we also include the table of (ordinary) marks.

**A.1. The group $S_3$.** Let $G = S_3$ be the symmetric group on three letters. It has a unique subgroup $H$ of order 3, and three conjugate subgroups $K_1, K_2, K_3$ of order 2. We denote by $K$ a fixed representative of this class. In this case the ordered set $\mathcal{S}$ is

$$\mathcal{S} = \{G/G, H/H, K/K, 1/1, G/H, K/1, H/1, G/1\}.$$

Now using the definition of the generalized marks, we calculate the table of marks as follows. Note that the $\cdot$'s are all zeros.

| | $G/G$ | $H/H$ | $K/K$ | $1/1$ | $G/H$ | $K/1$ | $H/1$ | $G/1$ |
|---|---|---|---|---|---|---|---|---|
| $G/G$ | 1 | · | · | · | 1 | · | · | 1 |
| $H/H$ | | 2 | · | · | 1 | · | 2 | 1 |
| $K/K$ | | | 1 | · | · | 1 | · | 1 |
| $1/1$ | | | | 6 | · | 3 | 2 | 1 |
| $G/H$ | | | | | 1 | · | · | 1 |
| $K/1$ | | | | | | 1 | · | 1 |
| $H/1$ | | | | | | | 2 | 1 |
| $G/1$ | | | | | | | | 1 |

On the other hand, the table of ordinary marks of the symmetric group $S_3$ is the following matrix.

| | $1/1$ | $K/1$ | $H/1$ | $G/1$ |
|---|---|---|---|---|
| $1/1$ | 6 | 3 | 2 | 1 |
| $K/1$ | | 1 | · | 1 |
| $H/1$ | | | 2 | 1 |
| $G/1$ | | | | 1 |

**A.2. The group $A_4$.** Now let $G = A_4$, alternating group of order 12. This group has a unique subgroup of order 4, isomorphic to $V_4$, and has two conjugacy classes of subgroups of order 3. We denote the copy of $V_4$ by $H$ and representatives of the conjugacy classes of the cyclic groups of order 3 by $K$ and $L$. The non-trivial proper subgroups of $H$ are all $G$-conjugate to each other and we denote by $T$ a representative of this conjugacy class. With this notation, $H$ is normal in $G$, $K$ and $L$ are self-normalizing and the normalizer of $T$ is $H$. Therefore the ordered set $\mathcal{S}$ is the following set

$$\{G/G, H/H, K/K, L/L, T/T, 1/1, H/T, T/1, G/H, K/1, L/1, H/1, G/1\}.$$

The table of marks of the alternating group $A_4$ is the following.

| | $1/1$ | $T/1$ | $K/1$ | $L/1$ | $H/1$ | $G/1$ |
|---|---|---|---|---|---|---|
| $1/1$ | 12 | 6 | 4 | 4 | 3 | 1 |
| $T/1$ | | 2 | · | · | 3 | 1 |
| $K/1$ | | | 1 | · | · | 1 |
| $L/1$ | | | | 1 | · | 1 |
| $H/1$ | | | | | 3 | 1 |
| $G/1$ | | | | | | 1 |

On the other hand the table of generalized marks is calculated as follows.

| | $G/G$ | $H/H$ | $K/K$ | $L/L$ | $T/T$ | $1/1$ | $H/T$ | $T/1$ | $G/H$ | $K/1$ | $L/1$ | $H/1$ | $G/1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $G/G$ | 1 | · | · | · | · | · | · | · | 1 | · | · | · | 1 |
| $H/H$ | | 3 | · | · | · | · | 3 | · | 1 | · | · | 3 | 1 |
| $K/K$ | | | 1 | · | · | · | · | · | · | 1 | · | · | 1 |
| $L/L$ | | | | 1 | · | · | · | · | · | · | 1 | · | 1 |
| $T/T$ | | | | | 2 | · | 1 | 2 | · | · | · | 3 | 1 |
| $1/1$ | | | | | | 12 | · | 6 | · | 4 | 4 | 3 | 1 |
| $H/T$ | | | | | | | 1 | · | · | · | · | 3 | 1 |
| $T/1$ | | | | | | | | 2 | · | · | · | 3 | 1 |
| $G/H$ | | | | | | | | | 1 | · | · | · | 1 |
| $K/1$ | | | | | | | | | | 1 | · | · | 1 |
| $L/1$ | | | | | | | | | | | 1 | · | 1 |
| $H/1$ | | | | | | | | | | | | 3 | 1 |
| $G/1$ | | | | | | | | | | | | | 1 |

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