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ACTIONS, PROCESSES, OBJECTS, SCHEMAS (APOS) IN MATHEMATICS EDUCATION: A CASE STUDY FOR MATRIX OPERATIONS

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ABSTRACT

Matrix algebra forms a core part of the first year mathematics curriculum at the Vietnam universities and colleges and is applicable to many other areas besides pure mathematics. Besides, the transfer of knowledge from a primarily procedural or algorithmic school approach to formal presentation of concepts is a priority for conceptualization of matrix algebra concepts. The mastery of matrix operations was a necessary step for graduate students in higher education. On the other hand, they lack adequate knowledge of advanced linear algebra, such as matrix operations, which are fundamentals in quantitative research method learning and students often find the course difficult. However, the difficulty may not be solely because of the content but also because of the transition from elementary to advanced mathematics itself.

This paper presents an application of APOS (Actions, Process, Object and Schema) theory to teach matrix operations at universities. APOS theory focuses on models of what might be going on in the mind of an individual when he or she is trying to learn a mathematical concept and uses these models to design instructional materials and/or to evaluate student successes and failures in dealing with mathematical problem situations.

Keywords: Genetic decomposition, APOS theory, Linear algebra, Matrix operations, Mathematics Education.

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1. INTRODUCTION

Linear algebra is one of the first mathematics courses that many students encounter at university because of the few prerequisites that it needs, and students often find the course difficult. However, the difficulty may not be solely because of the content but also because of the transition from elementary to advanced mathematics itself. This transition marks a move from describing to defining, from convincing to proving in a logical manner, based on those definitions, and from the coherence of elementary mathematics to the consequence of advanced mathematics [1]. Globally, many mathematics education researchers have been concerned with students' difficulties related to the undergraduate linear algebra course. There is agreement that teaching this course is a frustrating experience for both teachers and students, and despite all the efforts to improve the curriculum the learning of linear algebra remains challenging for many students. Students may cope with the procedural aspects of the course, solving linear systems and manipulating matrices but struggle to understand the crucial conceptual ideas underpinning them. The concepts are usually presented through a definition in natural language, which may be linked to a symbolic representation. These definitions are considered to be fundamental as a starting point for concept formation and deductive reasoning in advanced mathematics. Sometimes at the end of the linear algebra course many students do reasonably well in their final examinations, since most questions require knowledge of certain procedures, rather than understanding the concept [2]. Linear algebra is a required course for many science, education, technology, and mathematics (STEM) students and its abstract nature can create significant difficulties for students who struggle to grasp the more theoretical aspects of the course. Research on the pedagogy of linear algebra started in the late 1980s and early 1990s, when a number of mathematics education researchers expressed their concerns with students' difficulties in understanding linear algebra concepts. Since that time, there has been an impressive amount of research published that pushes forward the field, by documenting the cognitive resources that students do have, which are helpful for learning particular linear algebra topics, and the ways in which students use those resources to reason and solve problems. In concurrence with this trend, many empirical studies have been published that focus on instructional approaches and tools that leverage the results of this research on student reasoning. With advances in access to information through computer and social networks, the ability to develop and share resources for visualization and computational modeling, and research tools that afford more complex qualitative and quantitative methods, the time is ripe for an updated and rigorously peer-reviewed volume documenting the current state of the field of linear algebra education research [3].

In terms of APOS theory students responses revealed that many were mainly operating at an action and process stages, with few pre-service teachers operating at an object stage. Since difficulties with the learning of linear algebra by average students are universally acknowledged, this study provided a modified itemized genetic decomposition which is anticipated to help in the teaching and learning of matrix algebra concepts. The aim of providing the modified genetic decomposition is to contribute in the teaching and learning of advanced mathematics as lectures could use the modified genetic decomposition to analyze the mental constructions of their students when learning matrix algebra concepts. Besides making a contribution to the teaching and learning of some mathematical concepts, the modified genetic decomposition is a contribution to APOS theory as it is shown it can be used in other mathematical concepts in different context [4].

2. BACKGROUND

2.1. APOS Theory

In APOS theory, an Action is a transformation of a mathematical Object that is perceived as external. It may be the rigid application of an explicit algorithm or the application of a memorized fact or procedure. The Action is external in the sense that it is relatively isolated from other mathematical knowledge of the individual, so that the individual will not be able to justify the Action. When an Action is repeated and the individual reflects on it, it may be interiorized as a Process. The Process is a transformation based on an internal construction, no longer directed by external stimuli, so that the individual can imagine the steps involved in the transformation without having to explicitly perform them, skip some, and even coordinate and invert them. Being able to imagine and omit steps enables the individual to have corresponding dynamic imagery of the transformation. This is made possible by the establishment of significant connections to other mathematical knowledge so that the individual will also be able to justify the Process. When the individual becomes aware of the Process as a totality, understands that transformations can act on that totality and can do or imagine doing such transformations, then one says that the individual encapsulated the Process into a cognitive Object. The individual with an Object conception can see a dynamic structure (i.e., Process) as a static structure to which Actions can be applied. A Schema is a complex cognitive construction formed by Actions, Processes, Objects, other Schemas and their interrelations. These structures are related in the mind of the individual, consciously or unconsciously, and allow him or her to face various problem situations. The Schema must be coherent in the sense that it gives a way to determine when a given problem situation falls within the scope of the Schema [5].

In APOS theory, the main mental mechanisms for building the mental structures of action, process, object, and schema are called interiorization and encapsulation. The mental structures of action, process, object, and schema constitute the acronym APOS. APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. Figure 1 shows how student's mental structure and the mental mechanism works in APOS theory [6].

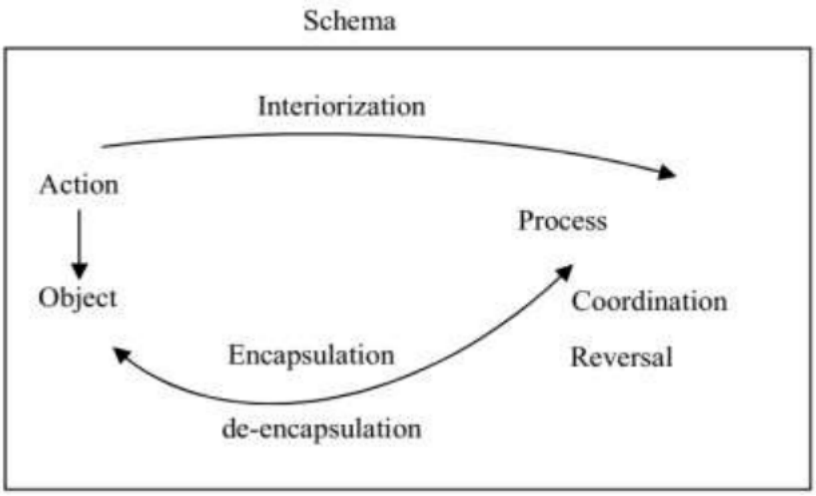

Figure 1 Mental Structure and Mental Mechanism in APOS theory.

In this paper, one of the major tools used in APOS-based research is the genetic decomposition (GD). A genetic decomposition is a hypothetical model of mental constructions that a student may need to make in order to learn a mathematical concept. Until it is tested experimentally, a genetic decomposition is a hypothesis and is referred to as preliminary genetic decomposition [7]. To teach mathematics using APOS theory, an implementation of the GD is needed. A GD is a model that intends to predict the mental constructions needed to learn the concepts of interest. It is proposed by researchers and needs to be tested experimentally. A preliminary GD is constructed as a first approximation to model the construction of a mathematical concept. It is then used to design didactical materials and research instruments to aid in the analysis of students' constructions. The results obtained are used in the refinement or validation of the GD [8]. More specifically APOS theory could lead us towards pedagogical strategies that in turn lead to marked improvement in the understanding of the matrix operations, through the GD. The aim of applying APOS theory was to reveal the nature of students' mental constructions, not to compare students' performances in matrix algebra concepts. The four stages of learning a mathematics concept used in APOS theory defined below are derived from Dubinsky and McDonald [9] for a clear understanding of the genetic decomposition of matrix algebra. Examples to illustrate each of the mental constructions arise from Figure 2.

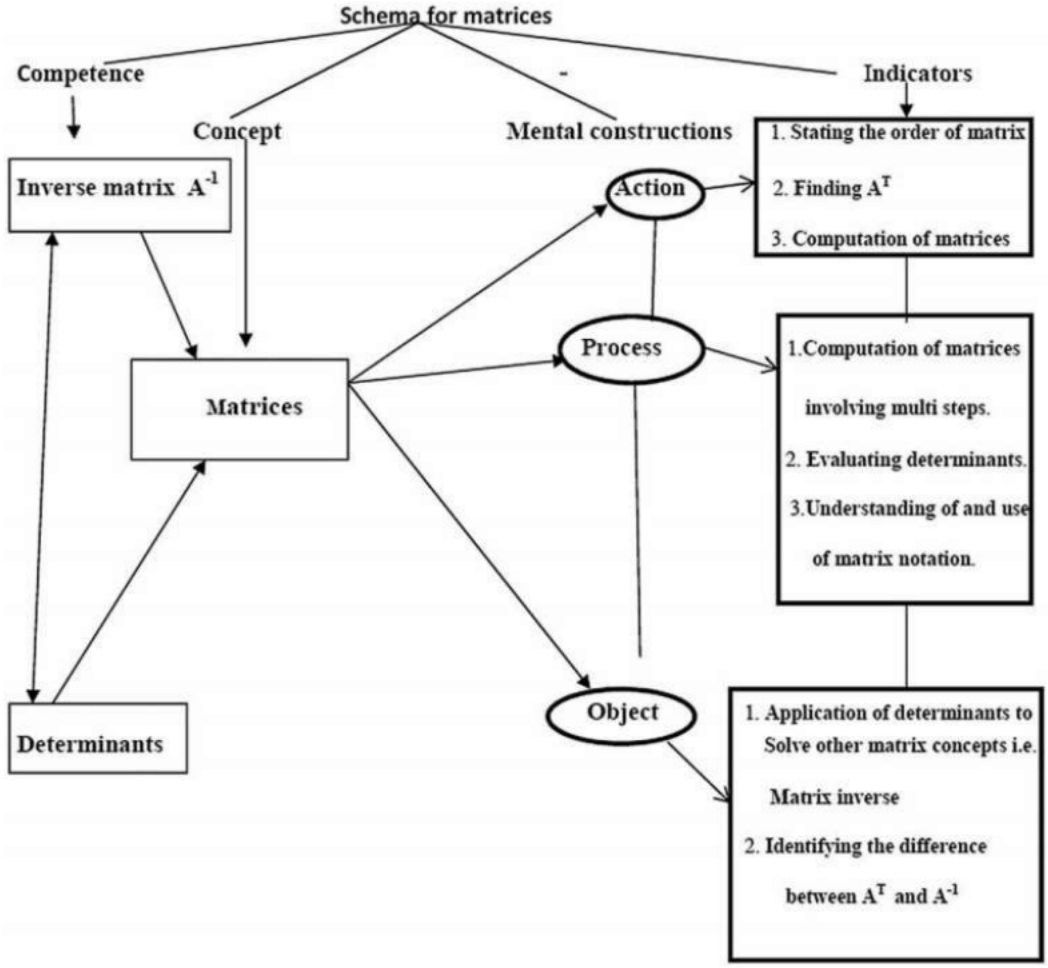

Figure 2 A preliminary genetic decomposition for the concepts of matrix algebra.

2.2. Preliminary genetic decomposition for matrix operations

The specific constructions relating to concepts of scalar matrix multiplication, addition of matrices, matrix transpose and matrix multiplications are detailed below. This paper drew upon the discussion by Arnon et al. [7] on examples of what a genetic decomposition is not, to refine our genetic decomposition. This was done to avoid the common errors which can confound a sound description of a genetic decomposition with description of teaching sequence or mathematical description of a concept. Note that the genetic decomposition does not explicitly cover linear combinations of matrices, which are the focus of the first question in the research instrument. However, a linear combination is a schema which involves the coordination of one's conceptions of matrix addition and scalar multiplication and it is these operations that are described in the genetic decomposition.

2.2.1. Addition of matrices

2.2.1.1 Action

The individual performs single additions (resulting in a new entry of the required matrix or row or column) at a time, without thinking beyond the addition of the numbers being added.

2.2.1.2. Process

The individual can imagine what the sums of the corresponding elements will be without carrying out step-by-step procedures. Addition of multiples of matrices can be done in one step, without first having to work out the result of the scalar multiples of the matrices. At this level, the individual is able to predict whether it is possible to add given matrices.

2.2.1.2.3 Object

The individual can see the effect of the matrix addition as a totality on any given matrix n by m. He or she is able to explain why it possible or not possible for given matrices. The individual will be able to apply processes or further transformations to matrix sums.

2.2.2. Scalar matrix multiplication

2.2.2.1 Action

The individual multiplies each element at a time by k, limited to an action conception. An individual cannot think beyond the single multiplication being carried out.

2.2.2.2 Process

An individual reflects on the rule and thinks about the effect of the scalar k on all the elements of the row or column or matrix A to form kA, by imagining that each element has been multiplied by the scalar k. The individual has interiorized the scalar multiplication and can carry out operations without doing step-by-step procedures. He or she is able to express the result of the scalar multiple symbolically using algebraic notation.

2.2.2.3 Object

The individual can see the effect of the scalar multiplication as a totality. The individual will be able to apply processes or further transformations on a scalar multiple of a matrix or scalar multiple of a row or column.

2.2.3 Matrix transpose

2.2.3.1 Action

The individual performs a single transformation of a row to a column, by systematically considering each row and transforming it into a column in a step-by step manner without thinking beyond the rearrangement of each row.

2.2.3.2 Process

The individual can imagine the effect of transposing each row into a column and can also see how the reversal of the transpose operation can result in the original matrix.

2.2.3.3 Object

The individual can see the effect of the transpose as a totality on any given matrix. The individual will be able to apply processes or further transformations on matrix addition. The individual can see the A^T as an object in its own right and can carry out further actions on A^T and recognize that two consecutive transpose operations have the effect of returning to the original matrix.

2.2.4 Matrix multiplication

Although some students may be at the object conception, the analysis focuses only on action and process conceptions, which are described below.

2.2.4. 1 Action

In working out the product AB = C of two matrices, the individual is able to multiply out one row by one column at a time, by multiplying each element in a row from the first matrix by the corresponding element of a column from B and then adding them up, in the same way as a vector dot product is computed. The individual is able to identify the i^{th} row of matrix A that must be multiplied by the j^{th} column of matrix B that results in the ij^{th} element c_{ij} of the product C.

2.2.4.2 Process

The individual is able to imagine the effect of finding the dot product of the i^{th} row of the first matrix with the j^{th} column of the second matrix to generate a new specific c_{ij} element.

He or she does not necessarily have to go through the pair-wise multiplication of each element of the row with each element of the corresponding column but is able to recognize the corresponding elements of the rows and columns that are paired.

3. LITERATURE REVIEW

Across the world, many researchers in mathematics education have been alarmed by the extent of students' difficulties related to the undergraduate linear algebra course [10]. These authors argue that in linear algebra there are challenges in the constructions of curriculum which make the subject cognitively and conceptually difficult. Some researchers attribute the students' perceptions of the difficulties to the different ways in which they (students) understand the concepts. Likewise, Ndlovu and Brijlall [11] note that students cope with the procedural aspects of the course, such as manipulating matrices and solving linear systems, but struggle to understand the crucial conceptual ideas underpinning them. Sometimes at the end of the linear algebra course many students do reasonably well in their final examinations, since most mathematical questions require knowledge of certain procedures, rather than conceptual understanding of the concept [12]. However, Hiebert [13] cautions that it is important to recognize that the relationships between procedural and conceptual knowledge change over time and are influenced by numerous factors which are both external and internal to the learner. Hiebert [13] elaborates that the relationships between procedural and conceptual knowledge enable the unpacking of some avenues that sometimes pose significant problems in mathematics, including linear algebra. Star [14] suggests there is a need for further research in mathematics education about deep procedural knowledge. He argues that the methods of assessing students' procedural knowledge are inadequate since they focus only on students' ability or inability to do mathematical problems. Procedural knowledge should act as the bedrock of all levels of mathematical learning especially in linear algebra concepts since many of the concepts are introduced first as procedures. As the understanding of these procedures deepens, students are able to discern properties and relationships between objects that are embedded in the procedures. It is clear that much research is needed that focuses on how students' understanding of foundational concepts such as matrix operations in linear algebra is developed. In a recent study Maharaj [15] used APOS theory in conjunction with instrumental and relational understanding to analyse undergraduate mathematics students' understanding of addition of matrices in linear algebra. He interviewed two students about their responses to two problems involving addition of matrices comprising algebraic terms. The results revealed that one of the students could not interpret the equality of two matrices indicated in the symbolic form even though the student was able to carry out the subtraction of the corresponding entries. The same student was also unable to expand matrices in algebraic form (A B +)2 and did not know the property that multiplication of matrices is not commutative. The student displayed particular difficulties with using the symbolic notation to communicate the equality relationship between two matrices. The student expressed the result of the subtraction of two equal matrices using the zero-number symbol instead of the zero matrix [15]. The author highlights the communicative function of symbolic notation in mathematics and asserts that the correct use of symbolic notation to communicate relevant mathematics concepts and relationships is an indicator of the students' understanding of the concept [15]. Findings from a study with undergraduate students on mental constructions in matrix algebra by Ndlovu and Brijlall [11] concur with those of Siyepu [12] that most participants were confident when applying algorithms but had difficulties in answering the questions requiring them to provide reasons for particular observations. Other studies have also confirmed that students carry out procedures easily but their limited previous knowledge of basic algebra have a negative impact on the construction of necessary matrix algebra mental constructions [11, 12,13]. De Lima and Tall [16] highlighted the role of knowledge of previously encountered concepts when learning new concepts, by introducing the phrase 'metbefores' to describe the previous experience. They argue that previous experiences may cause serious conflicts when learning new concepts. Tall, De Lima, and Healy [17] add that the divergence between success and failure might increase as supportive and difficult 'met-befores' affect successive learning in increasingly complicated mathematical contexts. Furthermore, Tall et al. [17] advocate that present mathematics educators should develop an approach that takes into account the concepts that each student has met before.

4. CONCLUSION

Further research is required to enable us to identify participants who have developed object level conceptions of these topics. In order to distinguish between individuals who are working on higher levels suitable tasks set at object levels will be required in order to elicit the necessary data. Interviews conducted with students while they work on the items could also have provided useful evidence in that regard. Further research will be conducted and these limitations will be addressed in the design of the follow up studies. By modeling student understanding of implicit differentiation using the triad of Schema development, this study contributes to improving the understanding of Schemas, thus filling a key literature gap as mentioned by Arnon et al. [7]. The majority of students in the study seemed not to have constructed a Process of implicit function. This suggests that many students are learning implicit differentiation procedures without understanding implicit function, the notion that supports such computations. Hence, our study underscores the importance of helping students construct a Process of implicit function. An inter-chain rule Schema and some prerequisite constructions such as elementary algebra structures, a Schema of equations, and the product rule, are also needed.

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