Infinitely many precomplete relative to parametric expressibility classes of formulas in a provability logic

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Abstract

Artificial Intelligence (AI) systems simulating human behavior are often called intelligent agents. By definition, these intelligent agents exhibit some form of human-like intelligence. Intelligent agents typically represent human cognitive states using underlying beliefs and knowledge modeled in a knowledge representation language, specifically in the context of decision making. In the present paper we investigate some functional properties of the underlying knowledge representation language based on the provability logic.

Keywords: intelligent agents, modal logic, provability logic, parametric expressibility of formulas, precomplete classes of formulas).

1 Introduction

Artificial Intelligence (AI) systems simulating human behavior are often called intelligent agents. These intelligent agents exhibit somehow human-like intelligence. Intelligent agents typically represent human cognitive states using underlying beliefs and knowledge modeled in a knowledge representation language, specifically in the context of decision making [1]. In the present paper we investigate some functional properties of the underlying knowledge representation language of intelligent agents which are based on the provability logic G [2].

The notion of parametric expressibility of formulas via a system of formulas of a given logical system, which is strongly connected with the notion of precomplete class of formulas relative to parametric expressibility, was proposed in [3]. In the present paper we state that there are infinitely many precomplete with respect to parametric expressibility classes of formulas in the propositional provability logic of Gödel-Löb.

2 Definitions and notations

Provability logic. We consider the propositional provability logic G, which formulas are based on propositinal variables $p, q, r, \ldots$ and logical connectives $\&, \vee, \supset, \neg, \Delta$, its axiomes are the classical ones together with the following Δ-formulas:

$$\Delta(p \supset q) \supset (\Delta p \supset \Delta q), \quad \Delta(\Delta p \supset p) \supset \Delta p, \quad \Delta p \supset \Delta\Delta p,$$

and the rules of inference are the rules of: 1) substitution; 2) the modus ponens, and 3) the necessitation, which allows to get formula ΔA if we already have got formula A. The normal extentions of the propositional provability logic are defined as usual [2].

Diagonalizable algebras. A diagonalizable algebra [4] is a universal algebra of the form $\mathfrak{A} = <M; \&, \vee, \supset, \neg, \Delta>$, where $<M; \&, \vee, \supset, \neg>$ is a boolean algebra, and the unary operation Δ satisfies the relations

$$\Delta(x \supset x) = (x \supset x), \qquad \Delta(\Delta x \supset x) = \Delta x,$$
$$\Delta(x \& y) = (\Delta x \& \Delta y), \qquad \Delta 1_{\mathfrak{A}} = 1_{\mathfrak{A}},$$

where $1_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, which is denoted also by 1 in case the confusion is avoided.

Diagonalizable algebras are known to be algebraic models for provability logic and its extensions [5]. Obviously we can interpret any formula of the calculus of G on any diagonalizable algebra $\mathfrak{A}$. As usual a formula F is said to be valid on $\mathfrak{A}$ if for any evaluation of variables of F with elements of $\mathfrak{A}$ the value of the formula on $\mathfrak{A}$ is $1_{\mathfrak{A}}$. The set of all valid formulas on $\mathfrak{A}$, denoted by $L\mathfrak{A}$ and referred to as the logic of the algebra $\mathfrak{A}$, forms an extension $\mathfrak{A}$ of the provability logic G [5].

We consider the diagonalizable algebra $\mathfrak{M} = (M; \&, \vee, \supset, \neg, \Delta)$ of all infinite binary sequences of the type $\alpha = (\mu_1, \mu_2, \dots), \mu_i \in \{0, 1\}, i = 1, 2, \dots$. The boolean operations $\&, \vee, \supset, \neg$ over elements of M are defined component-wise, and the operation Δ over element α we define by the equality $\Delta\alpha = (1, \nu_1, \nu_2, \dots)$, where $\nu_i = \mu_1 \& \dots \& \mu_i$. Let $\mathfrak{M}^*$ the subalgebra of $\mathfrak{M}$ generated by its zero $0_{\mathfrak{M}^*}$ element $(0, 0, \dots)$. Remark, the unit $1_{\mathfrak{M}^*}$ of the algebra $\mathfrak{M}^*$ is the element $(1, 1, \dots)$.

Parametical expressibillity [3]. They say the formula F is expressible in the logic L via a system of formulas Σ if F can be obtained from variables and Σ applying finitely many times 2 kinds of rules: a) the rule of weak substitution, b) the rule of passing to equivalent formula in L. Formula F is said to be parametrically expressible via Σ if there exist formulas $B_1, \dots, B_k, C_1, \dots, C_k, D_1, \dots, D_n$ not containing variables $\pi, \pi_1, \dots, \pi_n$ such that $B_1, \dots, B_k, C_1, \dots, C_k$ are expressible via Σ and the following first-order formulas with equalities are valid ($\wedge, \rightarrow, \sim$ are first-order connectives):

$$\begin{aligned} (F \sim \pi) \quad &\rightarrow \quad \wedge_{i=1}^{k}(B_i \sim C_i)[\pi_1/D_1, \dots, \pi_n/D_n] \\ \wedge_{i=1}^{k}(B_i \sim C_i) \quad &\rightarrow \quad (F \sim \pi) \end{aligned}$$

A system of formulas Σ is said to be complete with respect to parametric expressibility in the logic L if any formula of the calculus of L is parametrically expressible via Σ. The system Σ is precomplete with respect to parametric expressibility in L if it is not complete, but for any F, which is not parametrically expressible via Σ, the system $\Sigma \cup \{F\}$ is already parametrically complete.

3 Main result

Now we are able to formulate the main result of the present work.

Theorem 1. *The are infinitely many classes of formulas in the propositional provability logic $L\mathfrak{M}^*$ which are precompete relative to parametric expressibility in $L\mathfrak{M}^*$.*

4 Conclusion

In view of the Theorem 1 it is clear that a traditional algorithm for detecting the functional completeness with respect to parametrical expressibility in the logic $L\mathfrak{M}^*$ formulated in terms of a finite collection of precomplete classes of formulas is impossible to achieve.

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