Primitivity for random quantum channels

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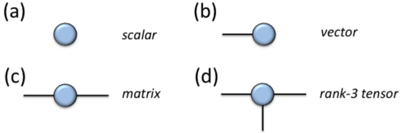
Contents

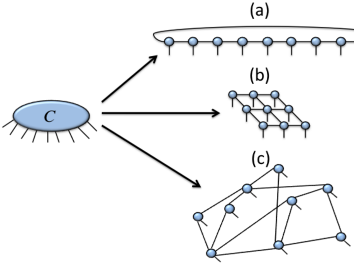
Background

- Quantum many-body system is an important part of condensed matter physics.
- Tensor networks can be used to describe quantum many-body systems and solve some problems.
- Tensor networks have many types, such as Matrix Product States(MPS), Projected Entangled Pair States (PEPS) and so on.
- MPS can be associated with quantum channels .

Tensor network



The higher rank tensor can be obtained by the contraction of the lower rank tensors.



MPS representation

Set a map

$$\mathcal{A}^{[k]} = \sum_{i=1}^{d} \sum_{\alpha,\beta=1}^{D} A_{i,\alpha,\beta}^{[k]} \mid i \rangle \langle \alpha\beta|$$

to each of the N sites, where $A_i^{[k]} \in \mathcal{M}_{D_k \times D_{k+1}}$. Then the state $|\phi\rangle$ relative to the map is given by

$$|\phi\rangle = \sum_{i_1,i_2,\cdots,i_N=1}^{d} \mathrm{Tr}\left[A_{i_1}^{[k_1]} A_{i_2}^{[k_2]} \cdots A_{i_N}^{[k_N]}\right] |i_1 i_2 \cdots i_N\rangle$$

and $|\phi\rangle$ is called a MPS with periodic boundary conditions (PBC). If $A_i^{[k]} = A_i$ is indenpendent of k, then $|\phi\rangle$ is said to be translational invariant (TI).

Proposition (Pérez-García 2007')

(Site-independent matrices) Every TI pure state with PBC on a finite chain has a MPS representation with site-independent matrices $A_i^{[m]} = A_i$, i.e.,

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} \mathrm{tr}\left(A_{i_1} \cdots A_{i_N}\right) |i_1 \cdots i_N\rangle \, .$$

Proposition (Pérez-García 2007')

(TI canonical form). Given a TI state on a finite ring, we can always decompose the matrices A_i of any of its TI MPS representations as

$$A_i = \begin{pmatrix} \lambda_1 A_i^1 & 0 & 0 \\ 0 & \lambda_2 A_i^2 & 0 \\ 0 & 0 & \cdots \end{pmatrix},$$

where $1 \geq \lambda_j > 0$ for every j and the matrices A_i^j in each block verify the conditions:

1. $\sum_i A_i^j A_i^{j*} = \mathbf{1}$.

2. $\sum_i A_i^{j*} \Lambda^j A_i^j = \Lambda^j$, for some diagonal positive and full-rank matrices Λ^j.

3. $\mathbf{1}$ is the only fixed point of the operator $\mathcal{E}_j(X) = \sum_i A_i^j X A_i^{j*}$.

It is known that the MPS relative to the Kraus operators is injective iff the map $\Gamma_N : \mathcal{M}_{D\times D} \to \left(\mathbb{C}^d\right)^{\otimes N}$ defined as

$$\Gamma_N(X) = \sum_{i_1,\cdots,i_N=1}^{d} \mathrm{Tr}[XA_{i_1}\cdots A_{i_N}]\,|i_1\cdots i_N\rangle$$

is injective. And Γ_N is injective iff

$$\mathrm{span}\{A_{i_1}\cdots A_{i_N} : 1 \leq i_1,\cdots,i_N \leq d\} = \mathcal{M}_{D\times D}.$$

Moreover, if $\sum A_i^* A_i = \mathbf{1}$, then evidently injectivity of Γ_{N_0} implies injectivity of Γ_N for all $N \geq N_0$.
The question is what is **the upper bound** of N_0 s.t. $\Gamma_{N_0}(X)$ is injective.
Conjecture: (D. Prez-Garca 2007') $N_0 \sim O(D^2)$.

Quantum Channel (CPTP Map)

$$\Phi : B(H_A) \to B(H_B)$$

Example: Let $X \in B(H_A)$ and $\sigma_B \in D(H_B)$,

$$\Phi(X) = \mathrm{Tr}(X)\sigma_B.$$

Kraus Representation:

Let $\{A_i\}_{i=1}^d$ and $\{B_i\}_{i=1}^d$ be two subsets of $B(H_A, H_B)$, define $\Phi : B(H_A) \to B(H_B)$ as

$$\Phi(X) = \sum_{i=1}^{d} A_i X B_i^*$$

It is known as the **Kraus repesentation** of Φ.

The Choi Representation:

Assume $\dim(H_A) = D$ and let $(e_i)_{i=1}^{D}$ be a set of standard orthogonal bases in H_A, set

$$E_D = \sum_{i,j=1}^{D} (e_i \otimes e_i)(e_j \otimes e_j)^*,$$

then the map $\omega : B(H_A \otimes H_A) \to B(H_{A \otimes B})$ is defined as

$$\omega(\Phi) := (\mathbf{1}_A \otimes \Phi)(E_D)$$

and it is called the **Choi representation** of Φ.

Proposition (John Watrous 2018')

Let $\Phi : B(H_A) \to B(H_B)$ *be a nonzero map, TFAE:*

- Φ *is a channel.*
- $\omega(\Phi)$ *is positive and* $\mathrm{Tr}_B(\omega(\Phi)) = \mathbf{1}_A$.
- *There exists a collection* $\{A_i\}_{i=1}^d \subset B(H_A, H_B)$, *where* $d = \mathrm{rank}(\omega(\Phi))$, *satisfying*

$$\Phi(X) = \sum_{i=1}^{d} A_i X A_i^* \text{ and } \sum_{i=1}^{d} A_i^* A_i = \mathbf{1}_A$$

for all $X \in B(H_A)$.

Set Φ be a quantum channels from $B(H_A)$ to $B(H_A)$ and $\{A_k\}_{k=1}^d \subset B(H_A)$ be Kraus operators, i.e.

$$\Phi(X) = \sum_{i=1}^{d} A_i X A_i^*$$

Define $S_n(A) \subset B(H_A)$ as the linear space spanned by all possible products of exactly n Kraus operators, $A_{k_1}A_{k_2}\cdots A_{k_n}$, and by $A_k^{(n)}$ the elements of $S_n(A)$. For $\xi \in H_A$, we define $H_n(A,\xi) := S_n(A)\xi \subset H_A$.

Definition (Mikel Sanz; David Prez-Garca; Michael M. Wolf 2010')

- A quantum channel $\Phi : B(H_A) \to B(H_A)$ is called **primitive** if there exists $n \in \mathbb{N}$ such that for all $\xi \in H_A$, $H_n(A,\xi) = H_A$. The minimum n is called **the index of primitivity**, denoted by $q(\Phi)$.

Definition (Mikel Sanz; David Prez-Garca; Michael M. Wolf 2010')

- The channel Φ is said to **have eventually full Kraus rank** if there exists $n \in \mathbb{N}$ such that $\omega(\Phi^{(n)})$ is full rank, i.e.,

$$\operatorname{rank} \omega(\Phi^{(n)}) = D^2.$$

  The minimum n is denoted by $i(\Phi)$.
- We say Φ is **strongly irreducible** if the following conditions are fulfilled: (1) Φ has a unique eigenvalue λ with $|\lambda| = 1$; (2) the corresponding eigenvector, ρ, is a positive definite operator.

Proposition (Mikel Sanz; David Prez-Garca; Michael M. Wolf 2010')

Given a quantum channel $\Phi : B(H_A) \to B(H_A)$, the following statements are equivalent: (a) Φ is primitive; (b) Φ has eventually full Kraus rank; (c) Φ is strongly irreducible.

Definition (Helmut Wielandt 1950')

A Matrix $A = (a_{ij})_{i,j} \in \mathcal{M}_{D\times D}$ is called a transition probability matrix if and only if it satisfies (1) $a_{ij} \geq 0$, for all i,j; (2) $\sum_i a_{ij} = 1$.
A classical channel relative to a **transition probability matrix** A is said to be **primitive** (or transition probability matrix A is **primitive**) if there exists $n \in \mathbb{N}$, $(A^n)_{i,j} > 0$, for all i,j.
The minimum n is called **the index of primitivity** of the classical channel (transition probability matrix A), denoted by $p(A)$.

Proposition (Helmut Wielandt 1950')

(Classical Wielandt inequality) If the transition probability matrix A is primitive, then

$$p(A) \leq D^2 - 2D + 2.$$

Similarly, we call the inequality satisfied by index $q(\Phi)$ as quantum Wielandt inequality.

Theorem (Mikel Sanz; David Prez-Garca; Michael M. Wolf 2010')

Let Φ be a primitive quantum channel on $\mathcal{M}_{D\times D}$ with d Kraus operators. Then $q(\Phi) \leq i(\Phi)$ and
1) in general, $i(\Phi) \leq \left(D^2 - d + 1\right) D^2$;
2) if the span of Kraus operators $S_1(A)$ contains an invertible element, then $i(\Phi) \leq D^2 - d + 1$;
3) if the span of Kraus operators $S_1(A)$ contains a noninvertible element with at least one nonzero eigenvalue, then $i(\Phi) \leq D^2$.

Theorem (Mateusz Michałek; Tim Seynnaeve; Frank Verstraete 2019')

Let Φ be a primitive quantum channel on $\mathcal{M}_{D\times D}$ with d Kraus operators. Then $i(\Phi) \leq 2D^2\,(6 + \log 2(D))$.

Theorem (Mizanur Rahaman 2020')

A positive linear map $\Phi : \mathcal{M}_{D\times D} \to \mathcal{M}_{D\times D}$ *is said to be a Schwarz map if it satisfies the Schwarz inequality* $\Phi(XX^*) \geq \Phi(X)\Phi(X^*)$, *for every element* $X \in \mathcal{M}_{D\times D}$.
For a trace preserving primitive Schwarz map Φ *acting on* $\mathcal{M}_{D\times D}$, *then*

$$q(\Phi) \leq 2(D-1)^2.$$

Proposition (Stinespring Dilation, John Watrous 2018')

A quantum channel $\Phi : \mathcal{M}_{D\times D} \to \mathcal{M}_{D\times D}$ *can be written as*

$$\Phi(X) = \mathrm{Tr}_{\mathcal{K}}[U(X \otimes Y)U^*], \quad \forall X \in \mathcal{M}_{D\times D}, \tag{1}$$

where $\mathcal{K} = \mathbb{C}^d$ *is a finite dimensional Hilbert space,* $Y \in \mathcal{M}_{d\times d}$ *is a density matrix and* $U \in \mathcal{U}_{dD}$ *is a unitary matrix.*

Theorem (Bai; Wang; Yin)

Let Φ *be a random quantum channel given by Equation* (1)*, then almost surely*

$$i(\Phi) \geq \frac{2\log D}{\log d} = O(\log D).$$

Research Contents:

Let $\Phi_1, \cdots, \Phi_n$ be random quantum channels given by Equation (1), where $U_i, i = 1, \ldots, n$ are Haar-distributed random unitary matrices in $\mathcal{U}_{dD}$. Let us consider the Choi matrix $Z := \omega(\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1)$ in the following two cases:

- $U_1, U_2, \ldots, U_n$ are independent;
- $U_1 = U_2 = \cdots = U_n := U$.

Note that in the second case, we have the Choi matrix of n-fold composition of Φ.

Research Method:

1: Calculate $\mathbb{E}\left[\mathrm{Tr}(Z^p)\right]$;

2: (Borel-Canteli Lemma) We need get $\mathrm{Tr}(Z^p) \to \mathbb{E}\left[\mathrm{Tr}(Z^p)\right]$;

3: Analyze the eigenvalue of Z in the sense of limit.

Definition (Don Weingarten 1978')

The (unitary) Weingarten function $\mathrm{Wg}(D, \sigma)$ is given by the inverse of the function $\sigma \mapsto D^{\#\sigma}$ with respect to the following convolution formula

$$\sum_{\tau \in \mathcal{S}_p} \mathrm{Wg}(D, \sigma\tau^{-1}) D^{\#(\tau\pi^{-1})} = \delta_{\sigma,\pi}$$

for all $\sigma, \pi \in \mathcal{S}_p$.

For the Weingarten function, it has the following estimate:

$$\mathrm{Wg}(D, \sigma) = D^{-(p+|\sigma|)}(\mathrm{Mob}(\sigma) + O(D^{-2})), \tag{2}$$

where $\mathrm{Mob}(\sigma)$ is the Möbius function on σ.

Proposition (The Weingarten formula, Don Weingarten 1978')

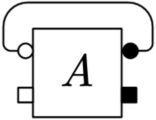
Let $i = (i_1, \cdots, i_p)$, $i' = (i'_1, \cdots, i'_p)$, $j = (j_1, \cdots, j_p)$, $j' = (j'_1, \cdots, j'_p)$ *be* p*-tuples of positive integers from* $\{1, 2, \cdots, D\}$*. Then*

$$\begin{aligned}&\int_{\mathcal{U}_D} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U}_{i'_1 j'_1} \cdots \overline{U}_{i'_p j'_p} d\mu(U) \\ &= \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \cdots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \cdots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(D, \alpha\beta^{-1}).\end{aligned} \tag{3}$$

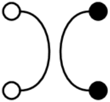
If $p \neq p'$*, then*

$$\int_{\mathcal{U}_D} U_{i_1 j_1} \cdots U_{i_p j_p} \overline{U}_{i'_1 j'_1} \cdots \overline{U}_{i'_p j'_p} d\mu(U) = 0.$$

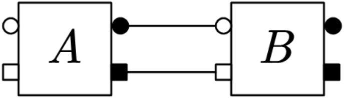
The graphical Weingarten calculus



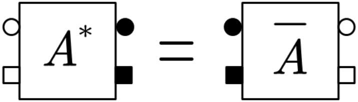
(a) Partial trace: $\mathrm{Tr}_1(A)$



(b) Maximum mixed state



(c) Matrix multiplication: AB



(d) Matrix transpose: $(A^*)^\top = \bar{A}$

Figure: some examples of the graphical formalism

Recall: The quantum chennal $\Phi : \mathcal{M}_D \to \mathcal{M}_D$ is defined as

$$X \mapsto \mathrm{Tr}_{\mathcal{K}}[U(X \otimes Y)U^*], \quad \forall X \in \mathcal{M}_D$$

and $\omega(\Phi) := (\mathbf{1}_A \otimes \Phi)(E_D)$, where

$$E_D = \sum_{i,j=1}^{D} (e_i \otimes e_i)(e_j \otimes e_j)^*$$

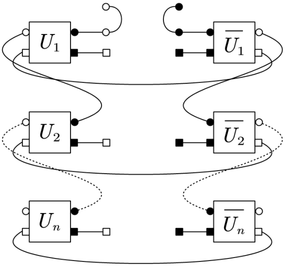


Figure: Diagram for $Z = \omega(\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1)$.

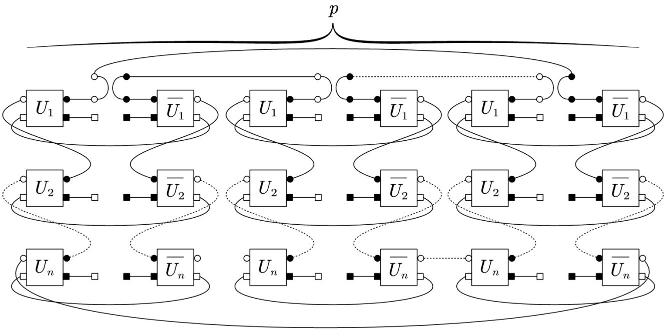


Figure: Diagram for $\mathrm{Tr}[Z^p]$

Proposition (Bai; Wang; Yin)

Let $p \geq 1$, and suppose that $U_1 = U_2 = \cdots = U_n := U$, we have

$$\mathbb{E}(\mathrm{Tr}[Z^p]) = \sum_{\alpha,\beta \in S_{pn}} d^{\#(\alpha)} D^{\#(\eta^{-1}\alpha'\delta\beta'^{-1})-2p} Wg(dD, \alpha\beta^{-1}). \tag{4}$$

We will consider the following two asymptotic regimes: (a). D is fixed and $d \to \infty$; (b). d is fixed and $D \to \infty$.

Proposition (Bai; Wang; Yin)

Let $U_1 = U_2 = \cdots = U_n := U$, then in the regime (a), the random matrix $Z = \omega(\Phi^{(n)})$ converges to the chaotic state

$$\rho_* = \frac{\mathbf{1}_{D^2}}{D^2}.$$

In regime (b), the asymptotic eigenvalues of Z are $1/d^n$ with multiplicity d^n and 0 with multiplicity $D^2 - d^n$.

As $D \to \infty$, $\mathbb{E}(\mathrm{Tr}[Z^p]) = d^{n-np} + O(D^{-2})$

Proposition (Bai; Wang; Yin)

For random quantum channel Φ, then the convergence of the eigenvalues of $Z = \omega(\Phi^{(n)})$ is almost sure in regime (b).

Let Φ be a random channel, then almost surely, we conclude that the eigenvalues of $Z = \omega(\Phi^{(n)})$ converge to $1/d^n$ with multiplicity d^n and 0 with multiplicity $D^2 - d^n$. Hence Z can not be full rank if

$$D^2 - d^n > 0 \implies n < \frac{2\log D}{\log d}.$$

Therefore, if one asks Φ to be primitive, then $i(\Phi) \geq 2\log D/\log d$. In summary,

Theorem (Bai; Wang; Yin)

Let Φ be a random quantum channel, then almost surely

$$i(\Phi) \geq \frac{2\log D}{\log d} = O(\log D).$$

Conclusion:

- In regime (a), our result can be understood as a thermalization of the maximal mixed state via the n-fold random quantum channels. Namely, the maximal entangled state thermalizes to an equilibrium state (the chaotic state) when the dimension (d) of the bath goes large.
- In regime (b), we have obtained a necessary condition for $\omega(\Phi^{(n)})$ to be full rank. As a corollary, we have derived a lower bound for the primitive index of Φ.
- D. Pérez-García claimed that it can be verified numerically that Γ is generically injective if $N \geq 2\log D/\log d$ for randomly chosen A_i's. And they also claimed a rigorous proof for the case $d = D = 2$.

Thank You