Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 1

Due date: October 9, 2015

Remark: Let X, Y be Banach spaces. Remember that an operator $T \in \mathcal{L}(X,Y)$ is called compact if $T(\overline{B_1(0)})$ has compact closure, where $\overline{B_1(0)} = \{x \in X \mid \|x\|_X \leq 1\}$.

Problem 1.1. Let X, Y be Banach spaces, $K \in \mathcal{L}(X,Y)$ be compact. Show:

- If K has closed range, then $X/ker(K)$ is finite dimensional.
- If K is surjective, then Y is finite dimensional.

(Hint: Quotient spaces)

Problem 1.2. Let X be a Banach space. An operator $P \in \mathcal{L}(X)$ is called (linear continuous) projection if $P^2 = P$. Let U be a closed subspace of X. Show that the following are equivalent:

i) There exists a closed subspace V of X such that $X = U \oplus V$ (i.e. $X = U + V$ and $U \cap V = \{0\}$.)

ii) There exists a linear continuous projection $P \in \mathcal{L}(X)$ with $\text{rg}(P) = U$ ($\text{rg}(P)$ denotes the range of P).

Problem 1.3. If i) of Problem 1.2 above holds true, we say that U (again being a closed subspace) admits a complement in X.

- Show that, if U is finite dimensional, it admits a complement.
- We say that U is of finite codimension, if there exists a finite dimensional subspace V of X such that $X = U + V$. Show:
  - If U is of finite codimension, it admits a (finite dimensional) complement.
  - If U admits a finite dimensional complement V, for any subspace W of X with $X = U \oplus W$ it follows that $\dim(W) = \dim(V)$.
- Show that the complement is not unique in general (e.g. by providing a counterexample).

Problem 1.4. For $f \in C^1([0,1])$ (the real valued continuously differentiable functions) define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that the (embedding) operator

$$\begin{aligned} i : C^1([0,1], \|\cdot\|) &\to C([0,1], \|\cdot\|_\infty) \\ f &\mapsto f \end{aligned}$$

is compact. (Hint: Arzela-Ascoli)

Problem 1.5. For $k \in C([0,1]^2)$ define (the Volterra integral operator) $T_k : C([0,1], \|\cdot\|_\infty) \to C([0,1], \|\cdot\|_\infty)$ as

$$T_k x(s) = \int_0^s k(s,t)x(t) \, \mathrm{d}t.$$

Show that T_k is well defined and compact.

Problem 1.6. (Bonus question): Let X be a Banach space and U a closed subspace. With $U^\perp = \{x^* \in X^* \,|\, \langle x^*, u\rangle_{X^*,X} = 0 \text{ for all } u \in U\}$, show

$$U \text{ if of finite dimension} \quad \Leftrightarrow \quad U^\perp \text{ is of finite codimension}$$

and, in the above case, $\dim(U) = \operatorname{codim}(U^\perp)$.