The African School of Physics

Lecture : Particle Interactions with Matter

Version 2012

Learning Goals, Material

1. Understand the fundamental interactions of high energy particles with matter.
   1. **High Energy Physics** :
      1. Understand the HEP detector design and operation.
      2. Research in HEP
   2. Nuclear Physics
      1. Understand detector / shielding design and operation.
   3. Medical Physics
      1. Understand biological implications
      2. Understand radiation therapy
   4. Other
      1. Environmental radiation
      2. Radiation damage for Space applications
      3. Semiconductor processing
      4. Radiation Damage in Materials

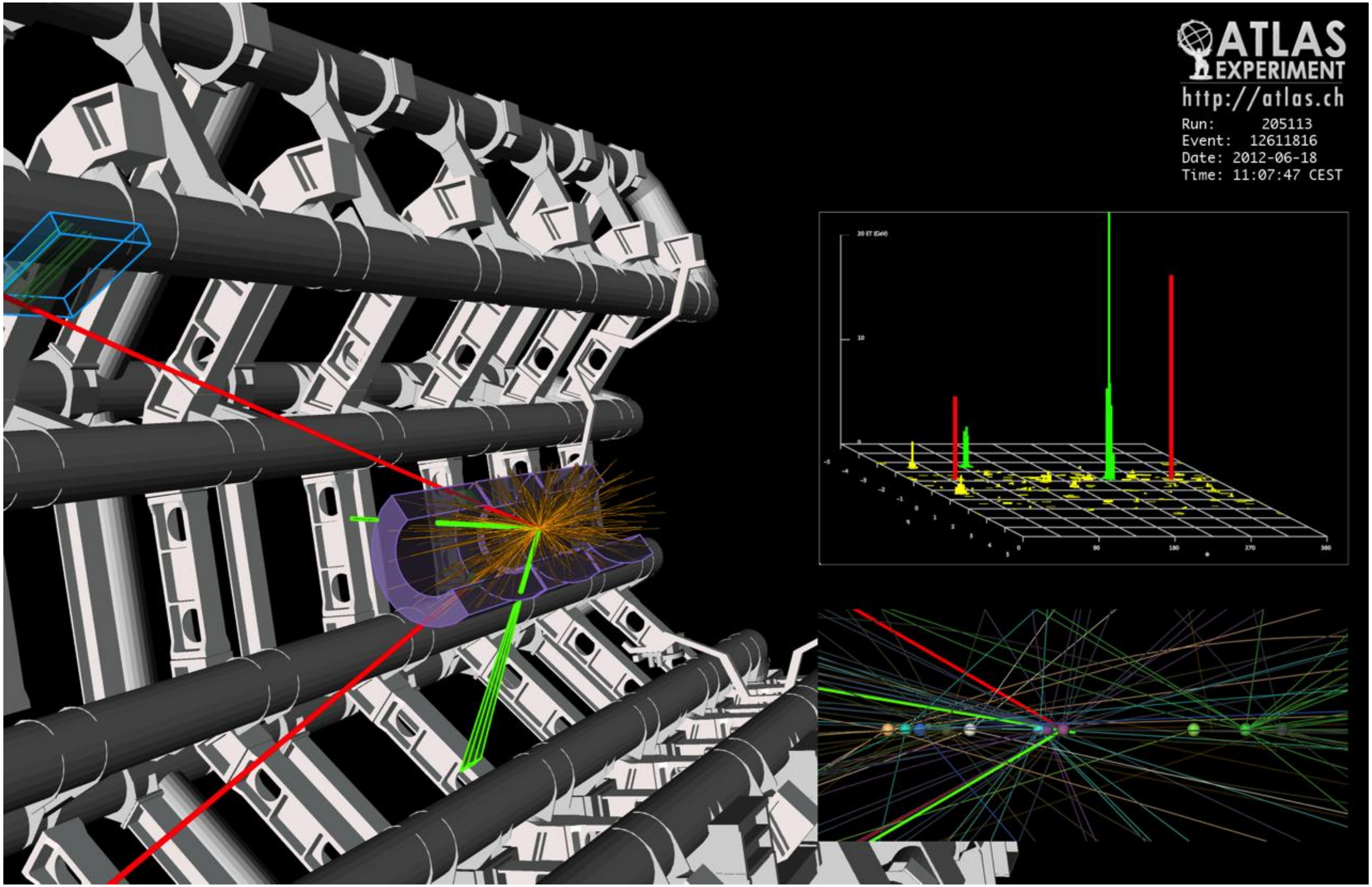
2. The core material is from “Techniques for Nuclear and Particle Physics Experiments” by WR Leo. Supplementary material from ASP2010 and ASP2012 lecture notes.

Contents

An example from the ATLAS detector

Reconstruction of a 2e2μ candidate for the Higgs boson - with $m_{2e2\mu}$= 123.9 GeV

We need to understand the interaction of particles with matter in order to understand the design and operation of this detector, and the analysis of the data.



Energy Loss Mechanisms

Heavy Charged Particles

- Inelastic collisions with atomic electrons
- Elastic scattering from nuclei
- Cherenkov radiation
- Bremsstrahlung
- Nuclear reactions
- Hadronic reactions

Light Charged Particles

- Inelastic collisions with atomic electrons
- Elastic scattering from nuclei
- Cherenkov radiation
- Bremsstrahlung
- Nuclear reactions
- Transition radiation

Photons

- Photo-electric effect
- Compton scattering
- Pair production
- Rayleigh scattering
- Photo-nuclear reactions

Neutral Particles

- Elastic nuclear scattering A(n.n)A
- Inelastic nuclear scattering A(n.n')A*
- Radiative Capture (n,γ)
- Fission (n,f)
- Other nuclear reactions
- Hadronic Showers

Introductory Comments : Interaction of Radiation with Matter

Different categories of particles have different **Energy Loss** mechanisms

Energy Loss = $\frac{dE}{dx}$ = "stopping power"

The **Energy Loss** by the particle in the detector material is what is ultimately converted into the electronic signal pulse.

Heavy Charged Particles (μ,π,p,d,α,…. (m > e)**)**

1. Coulomb Scattering by nuclei of detector material
   a) Not a significant Energy Loss Mechanism
   b) Mainly cause slight trajectory deflection (Multiple Scattering)
   c) Leads to radiation damage by creation of vacancies and interstitials.
2. Coulomb Scattering by electrons of detector material
   a) Dominant contribution to Energy Loss
   b) Expressed by Bethe-Bloch (Stopping Power) formula (derived below)
3. These particles have a well defined **range** in matter, depending on the projectile type and energy, and the material characteristics.

Light Charged Particles (e-, e^+)

1. Usually relativistic (v~c).
2. Multiple scattering angles are significant.
3. Quantum corrections to Bethe-Bloch because of exchange correlation.
4. Accompanied by *bremsstrahlung* radiation.
5. These particles also have a well defined range in matter, depending on the particle type and energy, and the material characteristics.
6. Transition radiation (when a boundary between two mediums is crossed).

Gamma Radiation

1. Primarily interacts with material via effects which transfer all or part of the (neutral) photon's energy to charged particles
    a) **Photo-electric effect** (absorbs full energy of the photon, leads to a "photo-peak")
    b) **Compton Scattering** (if the Compton scattered photon escapes, detector only records partial energy)
    c) **Pair Production** (the pair then makes an energy loss as per light charged particles). If the annihilation radiation of the positron escapes, it can lead to single or double escape peaks.
2. One does not have a concept of the **range** of photons in matter, rather, there is an exponentially decreasing **transmission probability** for the passage of photons through material.

Neutron Radiation

1. Moderation processes
    a) Elastic collisions A(n,n)A with nuclei in the material lead to fractional energy loss by a kinematic factor.
    b) The energy loss is more efficient when the stuck nucleus is light.
    c) Successive interactions lead to successively lower neutron engines until the neutron population is thermalised.
2. Absorption processes.
    1. Fast neutrons : (n,p), (n,α), (n,2n) reactions are possible
    2. Slow neutrons : (n,γ) reactions, capture leading to excitation of the capture nucleus.
    3. Absorption leads to an exponentially decreasing neutron population with material thickness traversed.
3. Detection mechanisms – neutrons produce no direct ionisation
    1. Detect secondary reaction products from the reactions (n,p), (n.α), (n,γ) or (n,fission) or (n,A_{light}).

More Introductory Comments : Reaction Cross section

In the quest to understand nature, we seek both to ***measure something*** and to ***calculate something***, (preferably the same thing !), so as to gain insight into nature, via a model.

What should this ``***something***” be ?

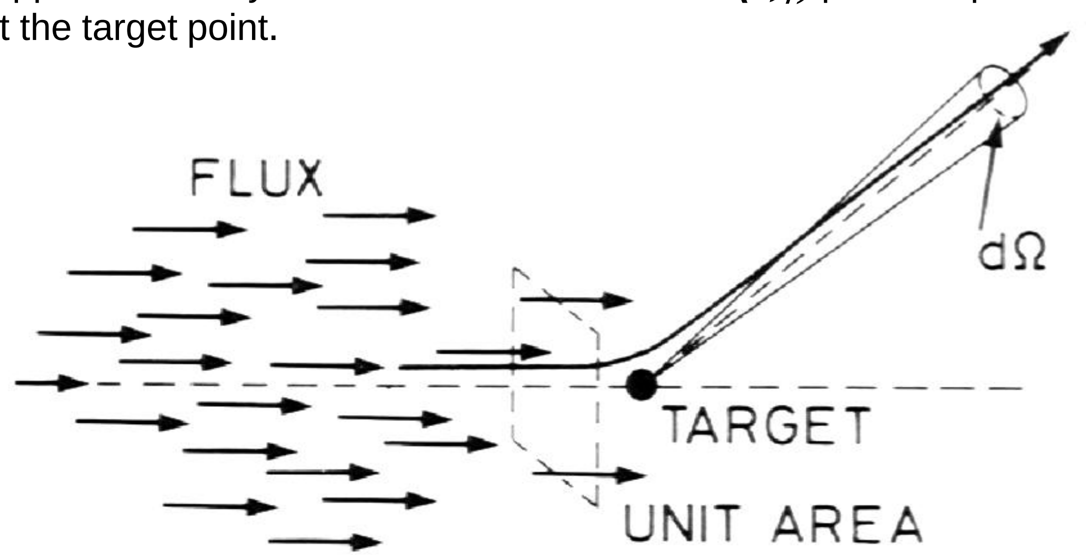
Well .... it should characterise in some clear way the probability for a given interaction to occur, and be accessible both experimentally and theoretically in a well defined way.

A long surviving concept in this regard has been the ***cross section***, which first gained widespread in the analysis of Rutherford's experiment leading to the discovery of the nucleus.

In a typical interaction between particles and matter, we can idealise the matter as a points in space, illuminated by a uniform beam flux of I_a particles (Intensity or number per unit area per unit time). The beam will see N_t scattering centres per unit area. A is either the area of the beam (if smaller than the target) or the area of the target (if smaller than the beam).
As a result of the interaction, some particles appear as if they were emitted with a rate of $r(\theta,\phi)$ particles per second into a solid angle **d**Ω from a source at the target point.
The differential cross section is ……

$$ds = \frac{r(q,j)}{I_a A} \cdot \frac{1}{N_t} \cdot \frac{dW}{4p}$$

$$\frac{ds}{dW} = \frac{r(q,j)}{4pIA_a N_t}$$



The total ***reaction cross section***, is.

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{\pi} \int_0^{2\pi} \frac{d\sigma}{d\Omega} \sin\theta \, d\theta d\phi$$

One can also define the doubly differential ***reaction cross section***
Which shows the energy dependence of the differential cross section.

$$\frac{d^2\sigma}{dE\, d\Omega}$$

The defining equation can now be turned around to give the **reaction rate** (if the cross-section) is known.

For the scattering rate into a small solid angle in the direction **(θ,ϕ)**

$$r(q,j\ ) = 4pI_a A N_t \frac{ds}{dW}$$

If the detector subtends a finite solid angle

$$r(q,j\ ) = 4pI_a A N_t \int \frac{ds}{dW} dW = 4pI_a A N_t \frac{ds}{dW} DW$$

For the total scattering rate

$$r = I_a A N_t s$$

One calculates the number of scattering centres per unit area (***N*** = surface density of nuclei).

ρ is the density of the material, $\boldsymbol{N_A}$ is Avogadro's number, $\mathcal{M}$ is the Molar mass and ***t*** is the thickness.

$$N_t = r \frac{N_A}{\mathcal{M}} t$$

The units of cross section are typically the barn. About the cross-sectional area of a nucleus with A=100

$$a_{\phi} = \pi (R_0 A^{\frac{1}{3}})^2 \approx 100\,\mathrm{fm}^2 = 10^{-24}\mathrm{cm}^2 = 1\,\mathrm{barn}$$

Suppose that we have for the number density, *N*, with *t* as the target thickness

$$Nt = N_t$$

Then, the reaction rate is

$$r = I_a A N s t$$

Considering an infinitesimal slice of the target, normalising the rate of the reaction to the incident beam rate, we get the probability for a single interaction …

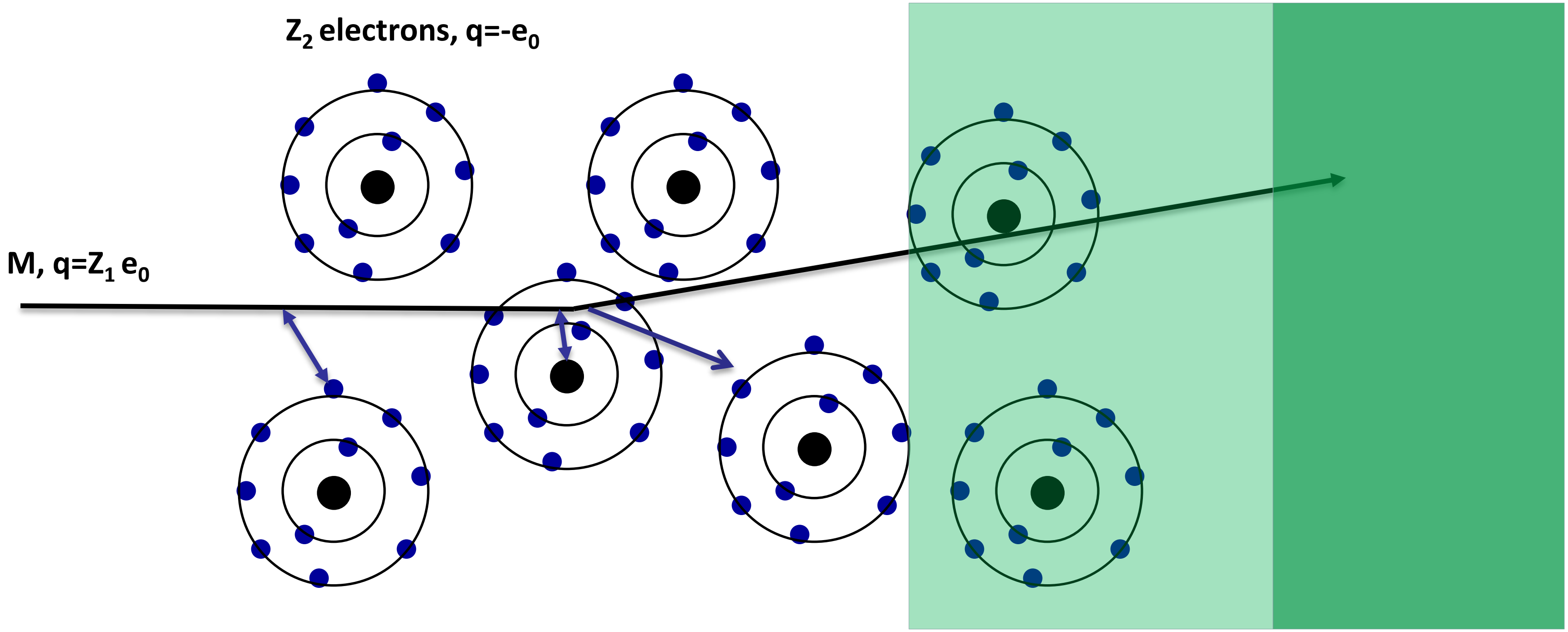
$$\mathrm{P(single\ interaction\ in}\ dx) = Ns \times dx$$

The probability of interaction per path-length is

$$m = Ns$$

We will use this last result later

Electromagnetic Interaction of Particles with Matter



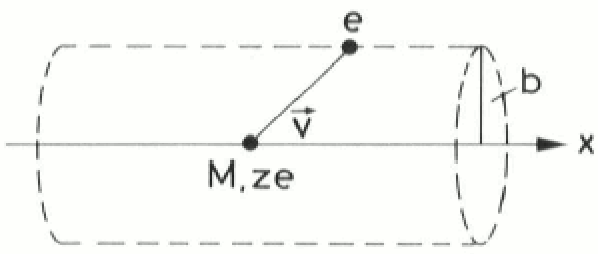
Interaction with the atomic electrons. The incoming particle loses energy and the atoms are <u>excited</u> or <u>ionised.</u>

Interaction with the atomic nucleus. The particle is deflected (scattered) causing <u>Multiple Scattering</u> of the particle in the material. During this scattering, <u>Bremsstrahlung photons</u> can be emitted.

In case the particle's velocity is larger than the velocity of light in the medium, the resulting EM shock-wave manifests itself as <u>Cherenkov Radiation</u>. When the particle crosses the boundary between two media, there is a probability of the order of 1% to produce X-ray photons, a phenomenon called <u>Transition radiation</u>.

Derivation of the Bethe-Bloch Formula

Heavy charged particles (Bohr's calculation – classical case)



$$F = \frac{dp}{dt}$$

$$\Delta p = \int dp = \int F dt$$

Consider the momentum transfer during inelastic collisions with atomic electrons

$$\Delta p = \int F\,dt = \int eE_\perp\,dt = e\int E_\perp \frac{dx}{v}$$

and take the integral using Gauss' Law ($\int E.dA = Q/\epsilon_0$)

$$\int E_\perp 2\pi b\,dx = ze/\epsilon_0, \qquad \text{therefore} \qquad \int eE_\perp \frac{dx}{v} = \frac{2ze}{4\pi\epsilon_0 bv}.$$

and

$$\Delta p = \frac{2ze}{4\pi\epsilon_0 bv}$$

So energy transfer at impact parameter b ....

$$\Delta E(b) = \frac{(\Delta p)^2}{2m_e} = \frac{1}{(4\pi\epsilon_0)^2}\frac{2z^2e^4}{m_e v^2 b^2}.$$

Considering a volume element $dV = 2\pi b\, db\, dx$ and an electron number density n_e

$$-dE(b) = \Delta E(b) n_e\, dV = \frac{1}{(4\pi\epsilon_0)^2}\frac{4\pi z^2 e^4}{m_e v^2} n_e \frac{db}{b}\, dx.$$

Therefore the energy transfer per distance at impact parameter b ....

$$-\frac{dE(b)}{dx} = \frac{1}{(4\pi\epsilon_0)^2}\frac{4\pi z^2 e^4}{m_e v^2} n_e \frac{db}{b}.$$

Now integrate between the limits b_{min} and b_{max} to consider all impact parameters b .

$$Mv = Mv' + mv_e$$

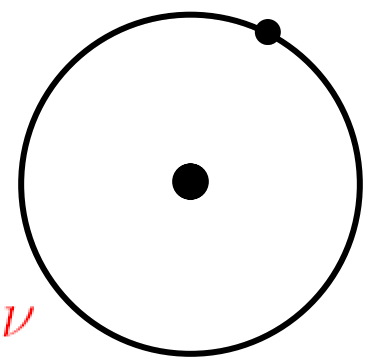
$$v_e = 2v$$

$2\gamma^2 m_e u^2$

b_{min} : A head on collision would transfer $\frac{1}{2} m_e (2u)^2$ with $(\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c)$. Substituting

Top formula, prev page

$$b_{min} = \frac{1}{4\pi\epsilon_0} \frac{ze^2}{\gamma m_e v^2}.$$



b_{max} : the electrons are not free

$$\tau_{collision} \approx b/v, \qquad \tau_{boundstate} \approx 1/\nu$$

$b/\gamma u$

Relativistic

$$b_{max} = \frac{\gamma v}{\nu}.$$

Finally, integrating over all impact parameters b ,

$$-\frac{dE}{dx} = \frac{1}{(4\pi\epsilon_0)^2}\frac{4\pi z^2 e^4}{m_e v^2} n_e \int_{b_{min}}^{b_{max}} \frac{db}{b} = \frac{1}{(4\pi\epsilon_0)^2}\frac{4\pi z^2 e^4}{m_e v^2} n_e \ln\frac{\gamma^2 m_e v^3}{z e^2 \overline{\nu}}.$$

using $n_e = \frac{\rho N_A Z}{A}$ A ~ molar mass

$$-\frac{dE}{dx} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi z^2 N_A Z \rho}{m_e v^2 A} \ln\frac{\gamma^2 m_e v^3}{z e^2 \overline{\nu}}$$

..... we get Bohr's Classical Formula

Note : .... the proportionalities .....

$$-\frac{dE}{dx} \propto \frac{z^2_{projectile}}{v^2}$$

$$\propto \frac{z^2_{projectile} m_{projectile}}{E}$$

We can use the following identities……

The classical radius of the electron is

$$r_e = \alpha^2 a_0$$

Where the fine structure constant is

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

and the Bohr radius of the atom is

$$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}$$

Then Bohr's classical formula for energy loss is

$$-\frac{de}{dx} = 4\pi N_A r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{z^2}{\beta^2} \cdot \ln \frac{\gamma^2 m_e \upsilon^3}{z e^2 \bar{\nu}}$$

The Bethe – Bloch Formula ..... (the correct quantum mechanical calculation)

$$-\frac{dE}{dx} = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{z^2}{\beta^2} \left[\ln\left(\frac{2 m_e \gamma^2 v^2 W_{max}}{I^2} \right) - 2\beta^2 - \delta - 2\frac{C}{Z} \right],$$

with

$$2\pi N_a r_e^2 m_e c^2 = 0.1535 \text{ MeVcm}^2/\text{g}$$

r_e: classical electron radius $= 2.817 \times 10^{-13}$ cm
m_e: electron mass
N_a: Avogadro's number $= 6.022 \times 10^{23}$ mol^{-1}
I: mean excitation potential
Z: atomic number of absorbing material
A: atomic weight of absorbing material

ρ: density of absorbing material
z: charge of incident particle in units of e
$\beta =$ v/c of the incident particle
$\gamma =$ $1/\sqrt{1-\beta^2}$
δ: density correction
C: shell correction
W_{max}: maximum energy transfer in a single collision.

The maximum energy transfer is that produced by a head-on or *knock-on* collision. For an incident particle of mass M, kinematics gives

$$W_{max} = \frac{2 m_e c^2 \eta^2}{1 + 2s\sqrt{1+\eta^2} + s^2},$$

$$\frac{I}{Z} = 12 + \frac{7}{Z} \text{ eV} \qquad Z < 13$$

$$\frac{I}{Z} = 9.76 + 58.8 Z^{-1.19} \text{ eV} \qquad Z \geq 13$$

where $s = m_e/M$ and $\eta = \beta\gamma$. Moreover, if $M \gg m_e$, then $W_{max} \simeq 2 m_e c^2 \eta^2$

Bethe-Bloch Formula

Bethe-Bloch formula gives the mean rate of energy loss (stopping power) of a heavy charged particle.

$$-\frac{dE}{dx} = Kz^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2m_e c^2 \beta^2 \gamma^2 T_{max}}{I^2} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

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with

A : atomic mass of absorber

$\frac{K}{A} = 4\pi N_A r_e^2 m_e c^2 / A = 0.307075 \text{ MeV g}^{-1}\text{cm}^2, \text{ for A} = 1\text{g mol}^{-1}$

z: atomic number of incident particle

Z: atomic number of absorber

T_{max} : Maximum energy transfer in a single collision

$$T_{max} = \frac{2m_e c^2 \beta^2 \gamma^2}{1 + 2\gamma m_e/M + (m_e/M)^2}$$

δ(βγ) : density effect correction to ionisation loss.

x = ρ s , surface density or mass thickness, with unit g/cm^2, where s is the length.

dE/dx has the units MeV cm^2/g

History of Energy Loss Calculations: dE/dx

1915: Niels Bohr, classical formula, Nobel prize 1922.
1930: Non-relativistic formula found by Hans Bethe
1932: Relativistic formula by Hans Bethe

Bethe's calculation is leading order in pertubation theory, thus only z^2 terms are included.

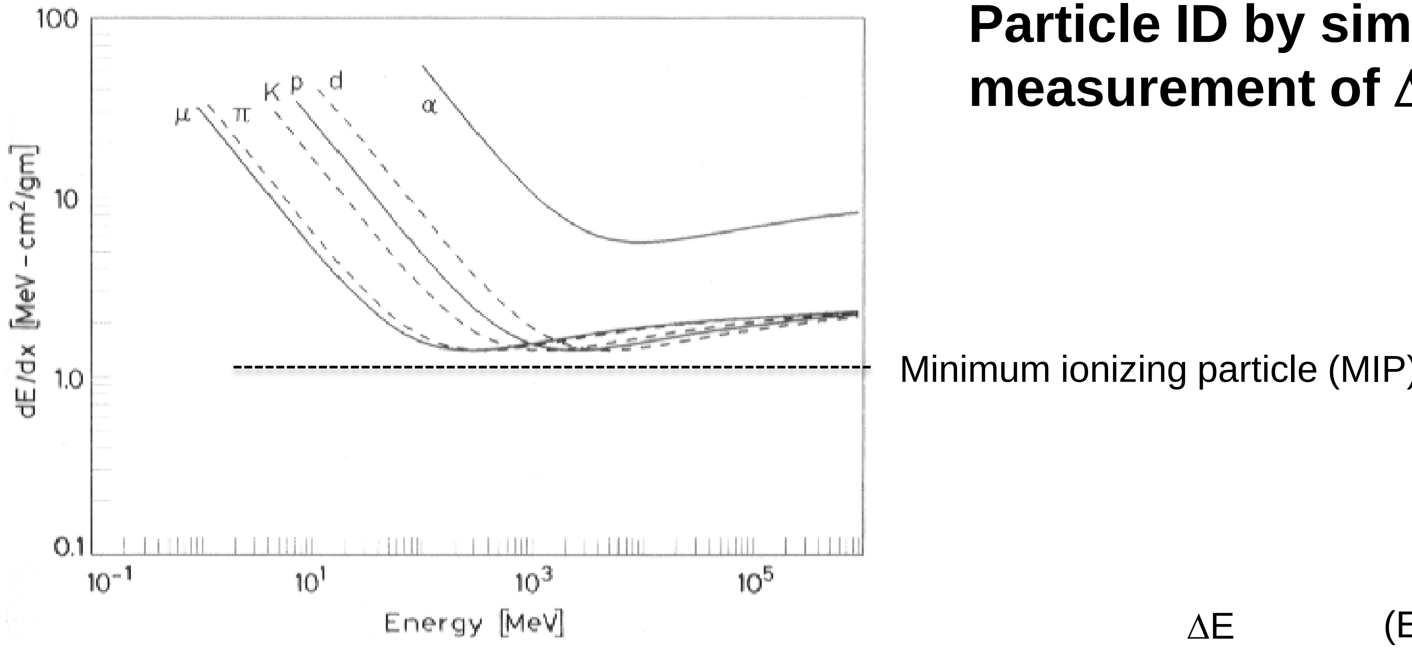
Additional corrections:

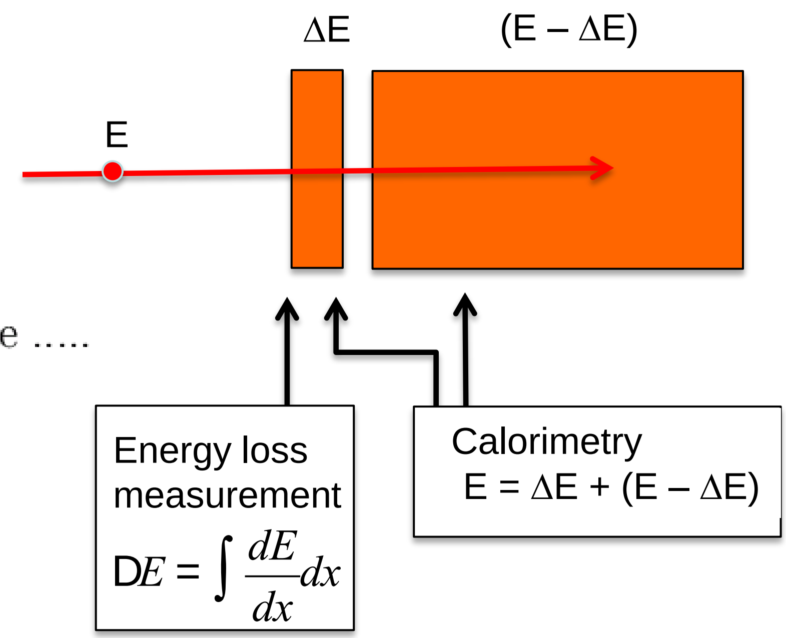
- z^3 corrections calculated by Barkas-Andersen
- z^4 correction calculated by Felix Bloch (Nobel prize 1952, for nuclear magnetic resonance). Although the formula is called Bethe-Bloch formula the z^4 term is usually not included.
- Shell corrections: atomic electrons are not stationary
- Density corrections: by Enrico Fermi (Nobel prize 1938, for discovery of nuclear reaction induced by slow neutrons).

Hans Bethe
1906-2005

Born in Strasbourg, emigrated to US in 1933.
Professor at Cornell U.
Nobel prize 1967
for theory of nuclear processes in stars.

Particle ID by simultaneous measurement of ΔE and E





Scaling Law

$$-\frac{dE}{dx} = \rho z^2 \frac{Z}{A} f(\beta, I).$$

The proportionalities for the projectile are the same .....

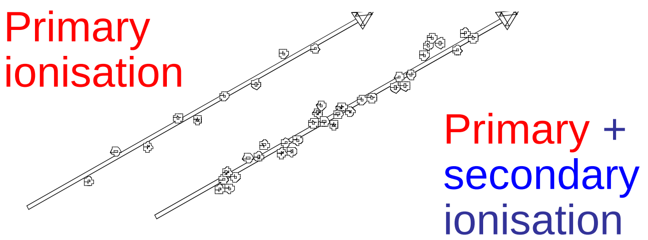
$$-\frac{dE}{dx} \propto \frac{z^2}{v^2}$$
$$\propto \frac{z^2 m}{E}$$

Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials

  - **Energy loss by ionisation**

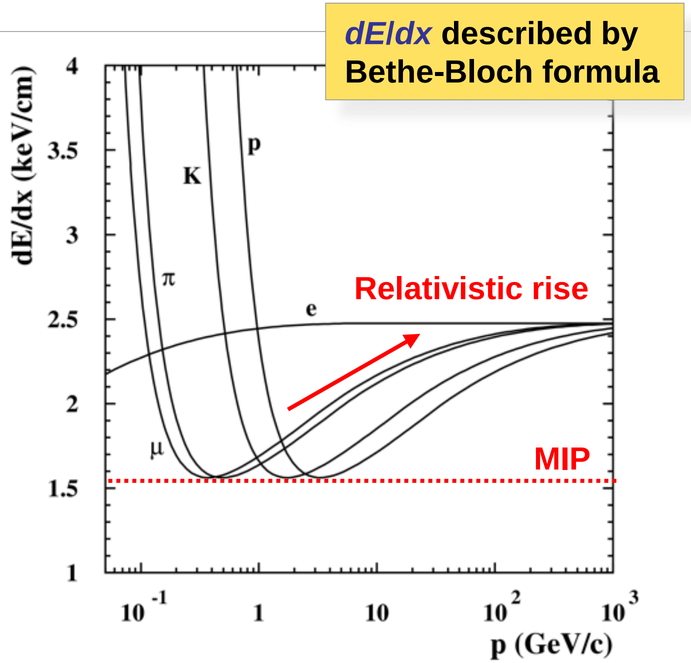
Primary ionisation can generate secondary ionisation



Typically:

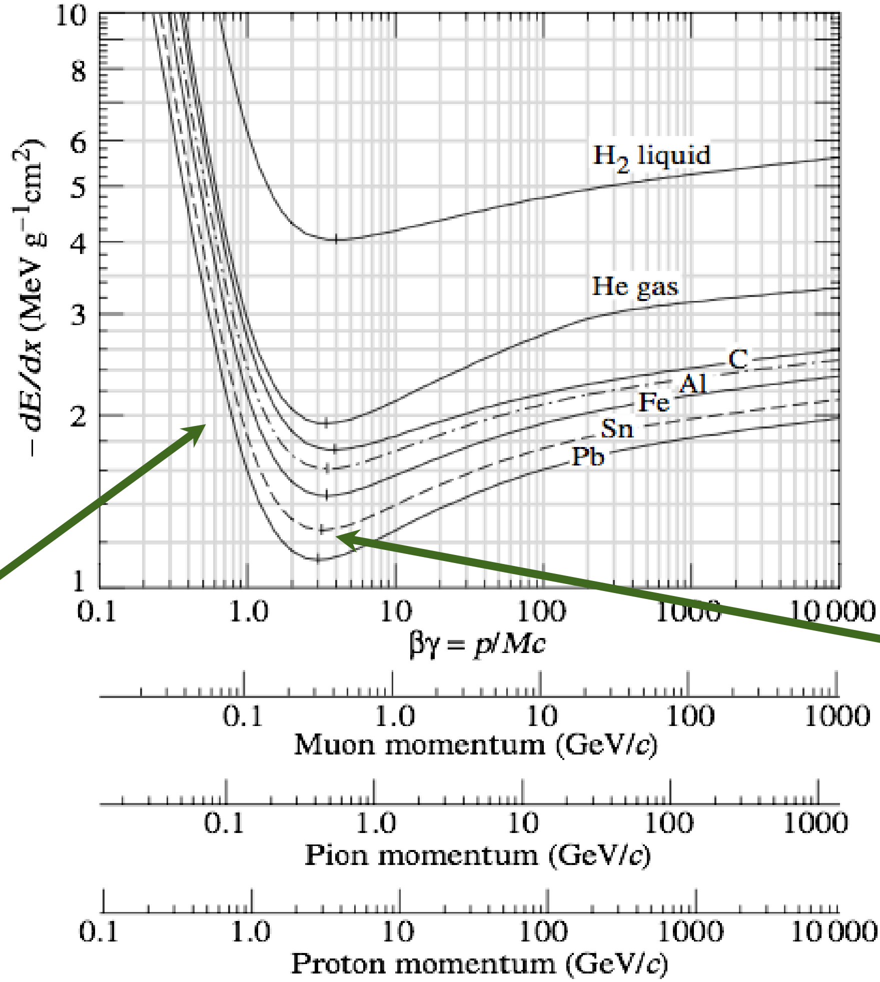
Total ionisation = 3 x primary ionisation

→ ~ 90 electrons/cm in gas at 1 bar



➡ Not directly used for PID by ATLAS/CMS

Examples of Mean Energy Loss



1/β²

Bethe-Bloch formula:

$$-\frac{dE}{dx} = Kz^2 \frac{Z}{A}\frac{1}{\beta^2}[\frac{1}{2}\ln f(\beta) - \beta^2 - \frac{\delta(\beta\gamma)}{2}]$$

Except in hydrogen, particles of the same velocity have similar energy loss in different materials.

The minimum in ionisation

occurs at βγ = 3.5 to 3.0,

as Z goes from 7 to 100

Figure 27.3: Mean energy loss rate in liquid (bubble chamber) hydrogen, gaseous helium, carbon, aluminum, iron, tin, and lead. Radiative effects, relevant for muons and pions, are not included. These become significant for muons in iron for $\beta\gamma \gtrsim 1000$, and at lower momenta for muons in higher-Z absorbers. See Fig. 27.21.

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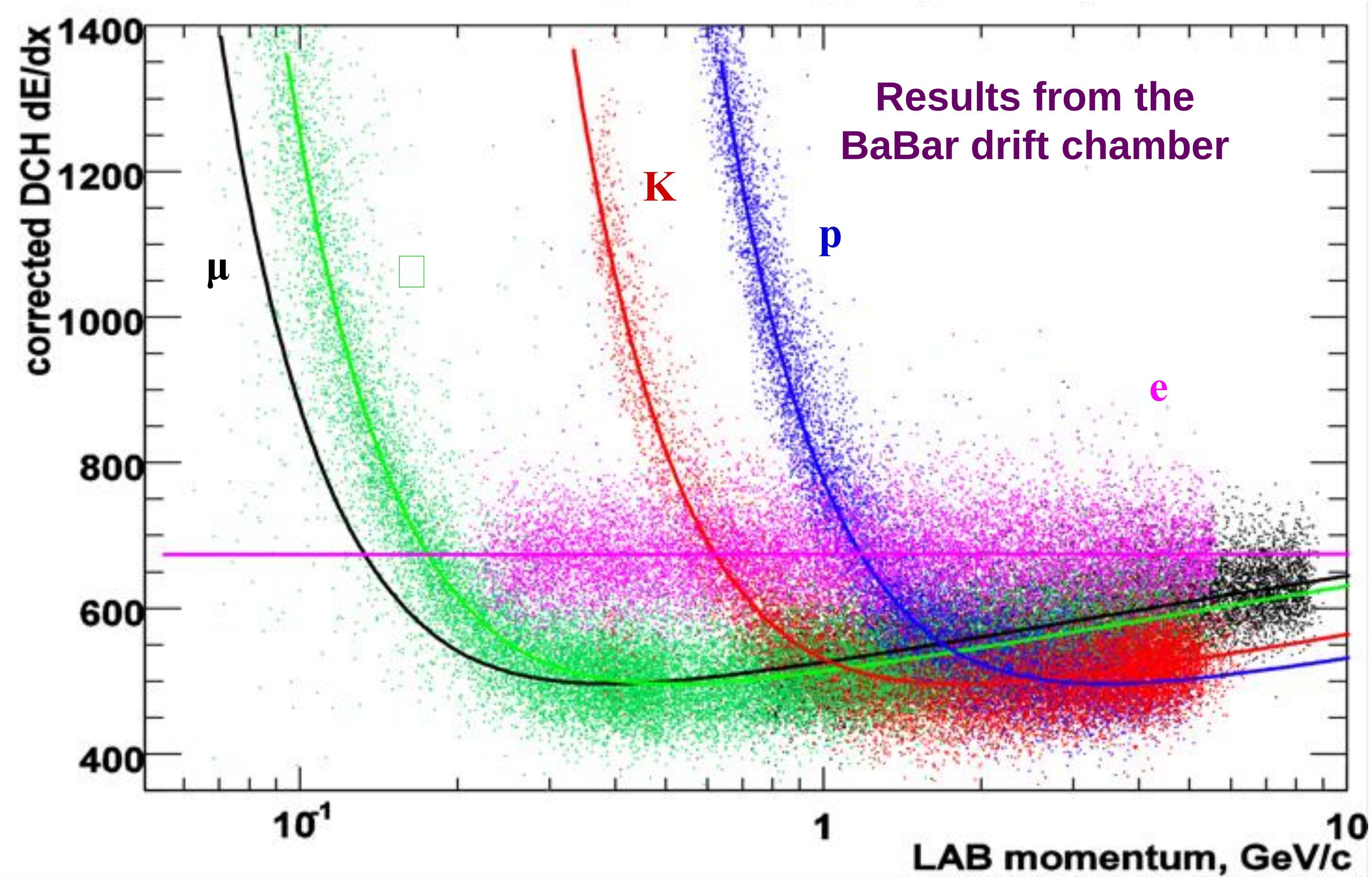
Particle identification from dE/dx and p measurements



A simultaneous measurement of dE/dx and momentum can provide particle identification.

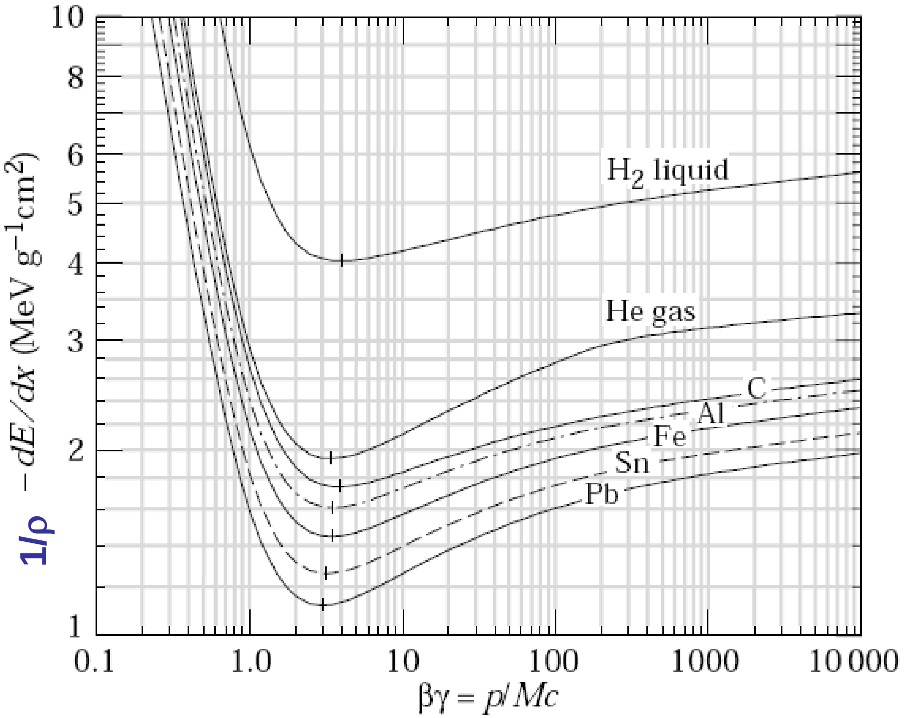
Bethe-Bloch Formula

Bethe Bloch Formula, a few numbers:

For Z ≈ 0.5 A
1/ρ dE/dx ≈ 1.4 MeV cm ^{2}/g for ßγ ≈ 3

Example :
Iron: Thickness = 100 cm; ρ = 7.87 g/cm^3
dE ≈ 1.4 * 100* 7.87 = 1102 MeV

→ A 1 GeV Muon can traverse 1m of Iron



This number must be multiplied with ρ [g/cm^3] of the material → dE/dx [MeV/cm]

Bethe-Bloch Formula

Light charged particles

The multiple scattering is severe.
QM implies indistinguishability of particles.

The modified Bethe-Bloch Formula becomes :

$$-\left(\frac{dE}{dx}\right)_c = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2\pi z^2 N_A Z\rho}{m_e c^2 \beta^2 A}\left[\ln\frac{T(T+2m_e c^2)^2\beta^2}{2I^2 mc^2} + (1-\beta^2) - (2\sqrt{1-\beta^2} - 1 + \beta^2)\ln 2 + \frac{1}{8}(1-\sqrt{1-\beta^2})^2\right]$$

where T is the kinetic energy of the particle and the subscript c stands for the Coulomb energy loss.

… however … for light charged particles …. there is something else too …

Energy loss by Bremsstrahlung

…. for light charged particles

$$\frac{dE}{dx} = \left(\frac{dE}{dx}\right)_c + \left(\frac{dE}{dx}\right)_r .$$

$$-\left(\frac{dE}{dx}\right)_r = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{Z^2 N_A (T + m_e c^2)\rho}{137\, m_e^2 c^4 A} \left[4 \ln \frac{2(T + m_e c^2)}{mc^2} - \frac{4}{3}\right]$$

where the subscript r stands for the bremsstrahlung energy loss.

Further notes on Bremsstrahlung

Bremsstrahlung arises when a charge particle decelerates in an electric field (such as the field of a nucleus, or even another similar charged particle).

$$d\sigma = 4Z^2 r_e^2 \alpha \frac{d\nu}{\nu} \left\{ (1+\varepsilon^2) \left[\frac{\phi_1(\xi)}{4} - \frac{1}{3}\ln Z - f(Z) \right] - \frac{2}{3}\varepsilon \left[\frac{\phi_2(\xi)}{4} - \frac{1}{3}\ln Z - f(Z) \right] \right\}$$

where the photon energy is $h\nu = E_0 - E$ with $\varepsilon = E/E_0$. $f(Z)$ is a Coulomb correction and $\phi_1(\xi)$ and $\phi_2(\xi)$ are screening functions depending on ξ, the Thomas-Fermi screening parameter

$$\xi = \frac{100 m_e c^2 h\nu}{E_0 E Z^{1/2}}.$$

We see that

$$\sigma \propto r_e^2 = \left(\frac{e}{mc^2}\right)^2$$

so that a muon bremsstrahls 40 000 times less than an electron.

The energy loss due to bremsstrahlung radiation ....

$$-\frac{dE}{dx} = N \int_0^{\nu_0} h\nu \frac{d\sigma}{d\nu}\, d\nu \qquad \nu_0 = E_0/h$$

which must be done numerically. The expression

It is worth noting that $h\nu \frac{d\sigma}{d\nu}$ is approximately constant, so that bremsstrahlung radiation has a flat power spectrum and

$$-\left(\frac{dE}{dx}\right)_{\mathrm{rad}} = N E_0 \Phi_{\mathrm{rad}}\,, \qquad \text{where} \qquad \Phi_{\mathrm{rad}} = \frac{1}{E_0} \int h\nu \frac{d\sigma}{d\nu}(E_0, \nu)\, d\nu$$

At higher energies, bremsstrahlung dominates the radiative energy loss for electrons

Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials
  - Energy loss by ionisation
  - **Bremsstrahlung**

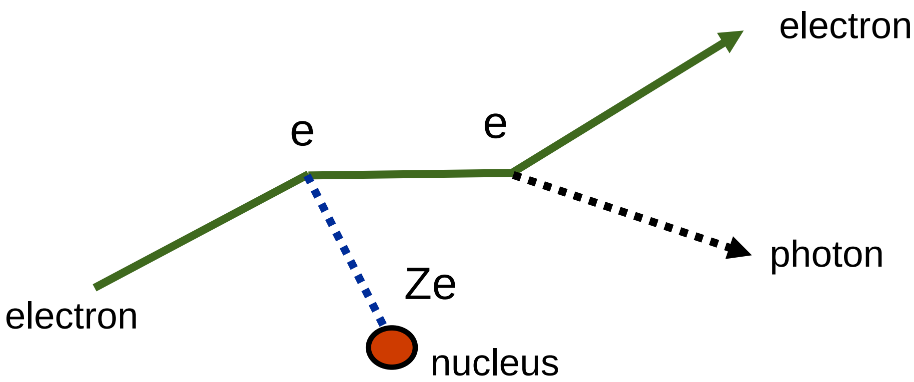
Due to interaction with Coulomb field of nucleus

Dominant energy loss mechanism for electrons down to low momenta (~10 MeV)

Initiates EM cascades (showers)

Bremsstrahlung

High energy electrons lose their energy predominantly through radiation (bremsstrahlung).



Cross section:

$$\sigma \sim (Z\, e^3)^2 \sim Z^2\ \alpha^3$$

The electron is decelerated (accelerated) in the field of the nucleus. Accelerated charges radiate photons. Thus the bremsstrahlung is strong for light charged particles (electrons), because its acceleration is large for a given force. For heavier particles like muons, bremsstrahlung effects are only important at energies of a few hundred GeV (important for ATLAS/CMS at the LHC!).

The presence of a nucleus is required to restore energy-momentum conservation. Thus the cross-section is proportional to Z^2 and α^3 (α = fine structure constant).

The characteristic length which an electron travels in material until a bremsstrahlung happens is the radiation length X_0.

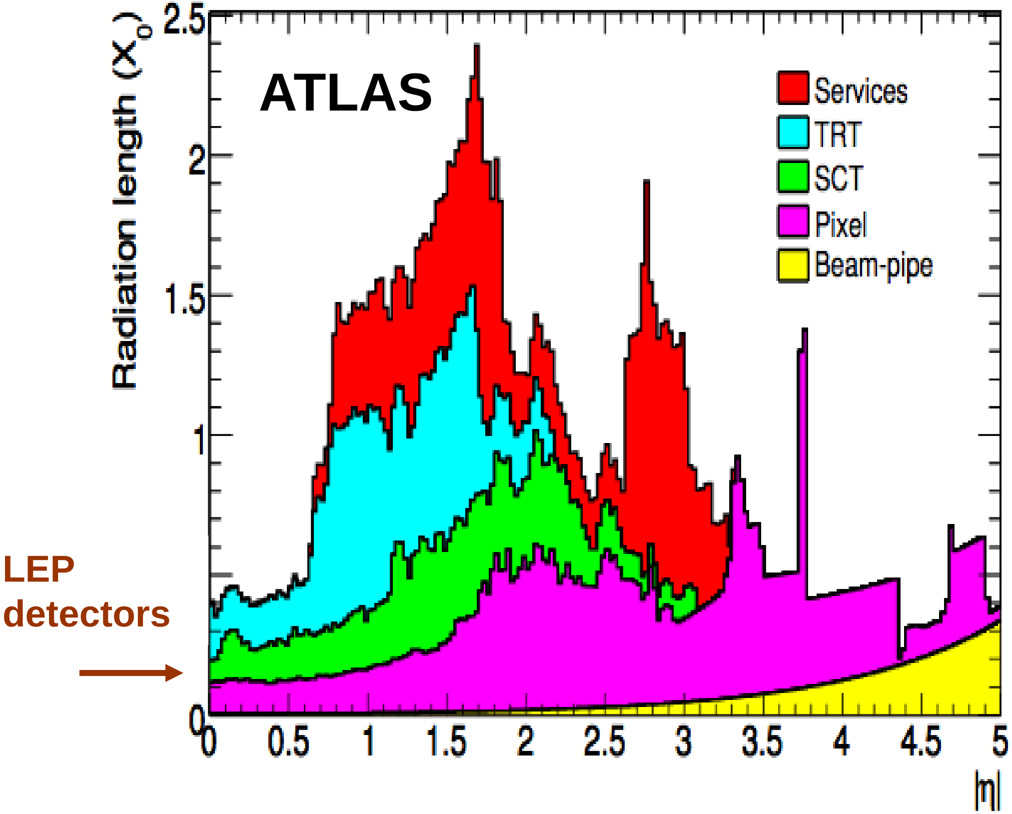
Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials

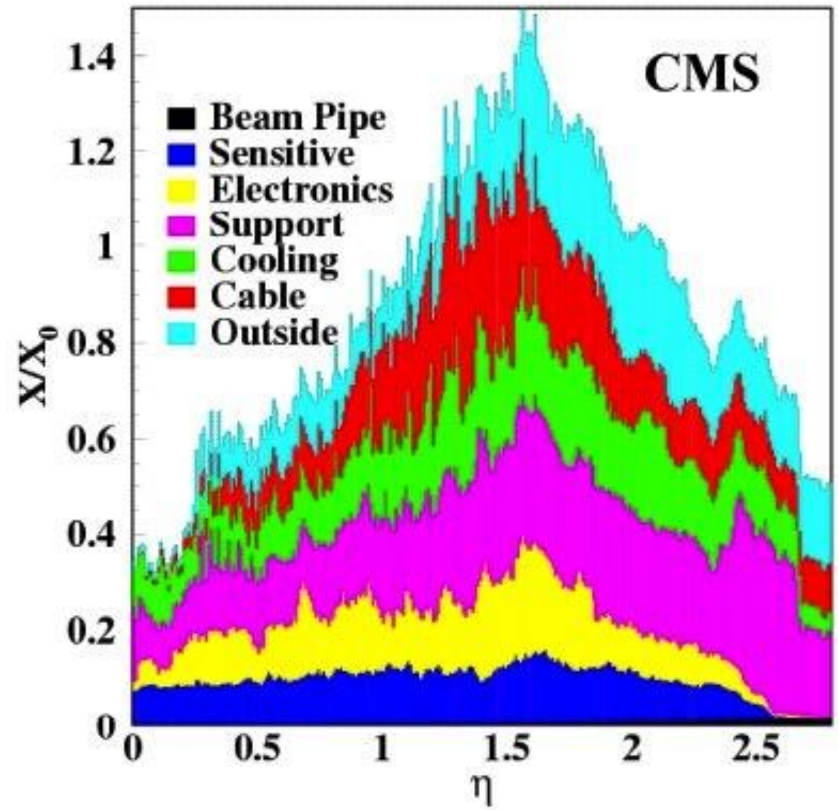
Inn… ough planning an…

- Energy lo… …strahlung
- Mult…

Weight: 4.5 tons



Weight: 3.7 tons



For ATLAS, need to add ~2 X_0 (η = 0) from solenoid + cryostat in front of EM calorimeter

Energy Loss of Charged Particles by Atomic Collisions

A charged particle passing through matter suffers
1. energy loss
2. deflection from incident direction

Main type of reactions:
1. Inelastic collisions with atomic electrons of the material.
2. Elastic scattering from nuclei.

Less important reactions are:
3. Emission of Cherenkov radiation
4. Nuclear reactions
5. Bremsstrahlung (except for electrons!)

Classification of charged particles with respect to interactions with matter:
1. Low mass: electrons and positrons
2. High mass: muons, pions, protons, light nuclei.

Energy loss:
- mainly due to inelastic collisions with atomic electrons.
- cross section $\sigma \cong 10^{-17}$ - 10^{-16} cm^2 !
- small energy loss in each collision, but many collisions in dense material. Thus one can work with average energy loss.
- Example: a proton with E_{kin}=10 MeV loses all its energy after 0.25 mm of copper.

Two groups of inelastic atomic collisions:
- soft collisions: only excitation of atom.
- hard collisions: ionisation of atom. In some of the hard collisions the atomic electron get such a large energy that it causes secondary ionisation (δ-electrons).

Elastic collisions from nuclei cause very small energy loss. They are the main cause for deflection.

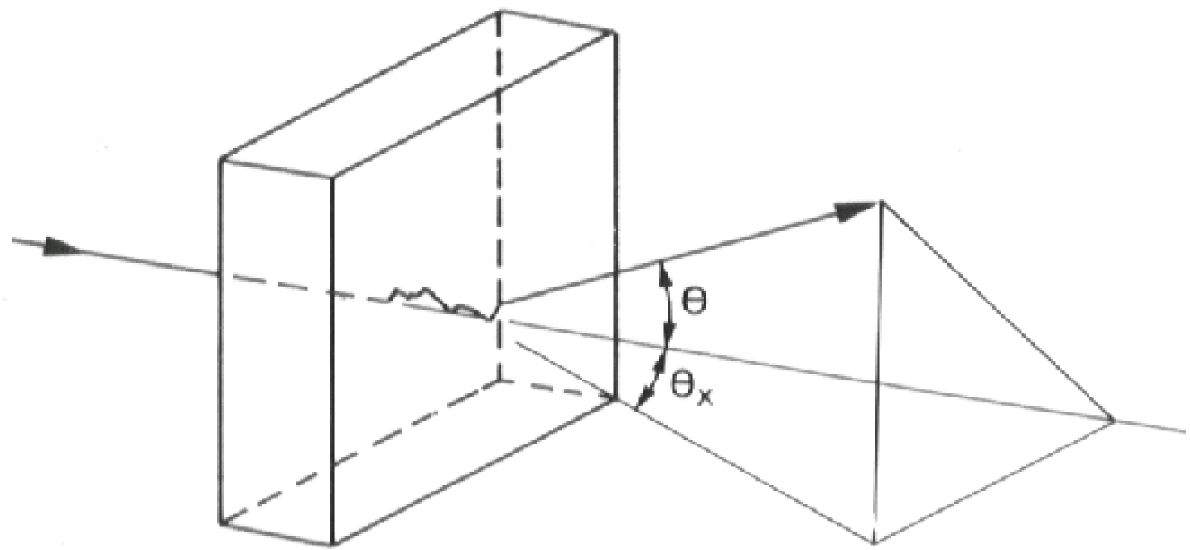
Elastic scattering from nuclei

Described by the Rutherford formula (ignoring spin and screening)

$$\frac{d\sigma}{d\Omega} = z^2 Z^2 r_e^2 \frac{(m_e c/\beta p)^2}{4\sin^4(\theta/2)}.$$

Note the very high probability for collisions resulting in a very small angle of deflection.

The cumulative effect of very many small angle deflections with random magnitudes and directions



Multiple Coulomb Scattering

Moliere expresses the polar angle distribution as a series

$$P(\theta)d\Omega = \eta\, d\eta \left(\underbrace{2\exp(-\eta^2)}_{\text{gaussian}} + \underbrace{\frac{F_1(\eta)}{B} + \frac{F_2(\eta)}{B^2} + \dots}_{\text{tails}} \right),$$

$$Q = \begin{cases} \sqrt{Z(Z+1)} & \text{for } e^+, e^- \\ Z & \text{for others} \end{cases}$$

$$q = \begin{cases} (Z+1)Z^{1/3} & \text{for } e^+, e^- \\ Z^{4/3} & \text{for others} \end{cases}$$

where $\eta = \theta/(\theta_1 \sqrt{B})$ and $\theta_1 = 0.3965\,(zQ/p\beta)\sqrt{(\rho\,\delta x/A)}$.

The parameter B is defined by the equation:

$$g(B) = \ln B - B + \ln \gamma - 0.154 = 0, \qquad \text{where}$$

$$\gamma = 8.831 \times 10^3 \frac{q z^2 \rho\, \delta x}{\beta^2 A\, \Delta} \qquad \text{and} \qquad \Delta = 1.13 + 3.76 \cdot \left(\frac{Zz}{137\,\beta} \right)^2 .$$

For a given γ, B may be found numerically by using, for example, Newton's Method for finding the zeros of $g(B)$. The functions $F_k(\eta)$ are defined by the integral

$$F_k(\eta) = \frac{1}{k!} \int J_0(\eta y) \exp\left(\frac{-y^2}{4} \right) \left[\frac{y^2}{4} \ln\left(\frac{y^2}{4} \right) \right]^k y\, dy\,,$$

Multiple Coulomb Scattering

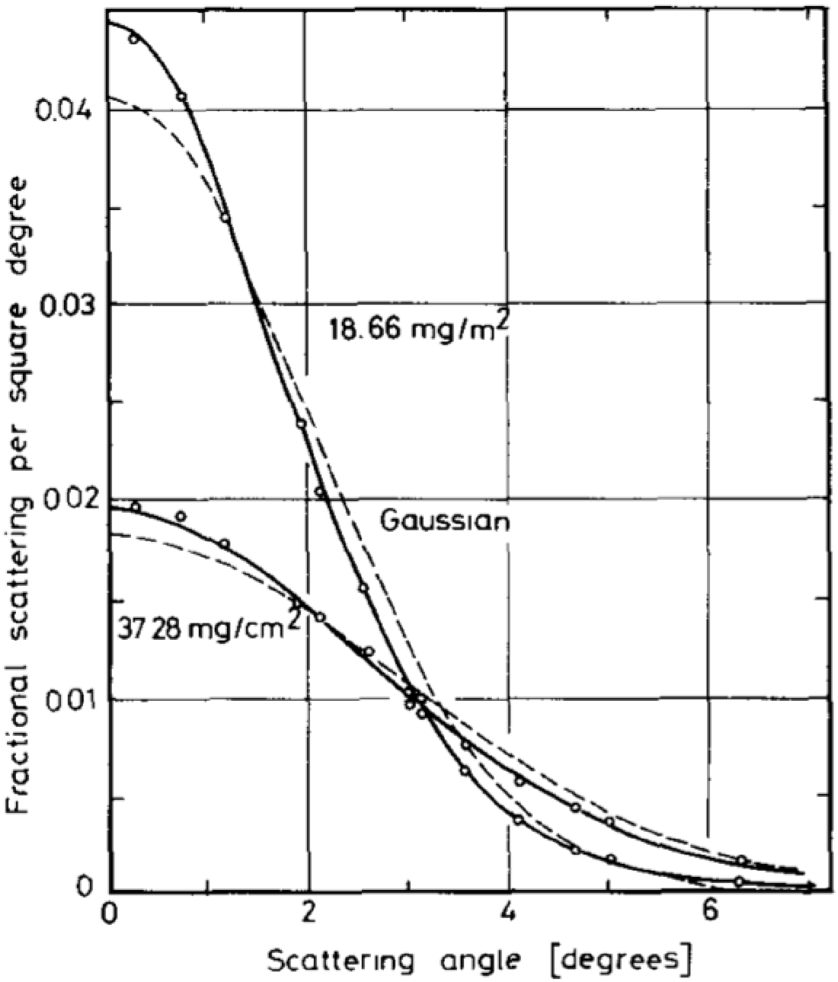


Fig. 2.15. Angular distribution of 15.7 MeV electrons scattered from a thin Au foil (from *Hanson* et al. [2.22]). The experimental values are compared with the Gaussian approximation to multiple scattering

Gaussian approximation

Relate to Moliere

$$\sqrt{\langle\theta^2\rangle} \simeq \theta_1 \sqrt{B}$$

$$P(\theta)\, d\Omega \simeq \frac{2\,\theta}{\langle\theta^2\rangle} \exp\left(\frac{-\,\theta^2}{\langle\theta^2\rangle}\right) d\theta$$

Multiple Coulomb Scattering

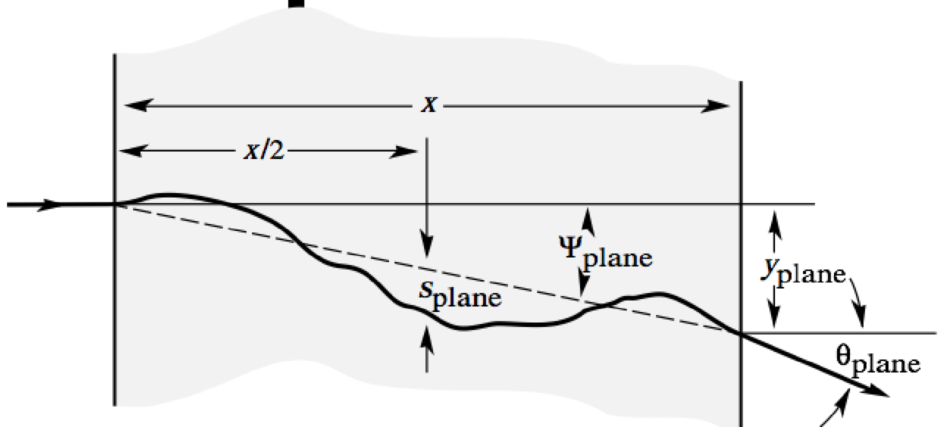


Figure 27.9: Quantities used to describe multiple Coulomb scattering. The particle is incident in the plane of the figure.

$$\psi_{\text{plane}}^{\text{rms}} = \frac{1}{\sqrt{3}}\,\theta_{\text{plane}}^{\text{rms}} = \frac{1}{\sqrt{3}}\,\theta_0 \,,$$

$$y_{\text{plane}}^{\text{rms}} = \frac{1}{\sqrt{3}}\,x\,\theta_{\text{plane}}^{\text{rms}} = \frac{1}{\sqrt{3}}\,x\,\theta_0 \,,$$

$$s_{\text{plane}}^{\text{rms}} = \frac{1}{4\sqrt{3}}\,x\,\theta_{\text{plane}}^{\text{rms}} = \frac{1}{4\sqrt{3}}\,x\,\theta_0 \,.$$

A particle which traverses a medium is deflected by small angle Coulomb scattering from nuclei. For hadronic particles also the strong interaction contributes.

The angular deflection after traversing a distance x is described by the Molière theory.
The angle has roughly a Gauss distribution, but with larger tails due to Coulomb scattering.

Defining: $\theta_0 = \theta_{\text{plane}}^{\text{rms}} = \frac{1}{\sqrt{2}}\theta_{\text{space}}^{\text{rms}}$

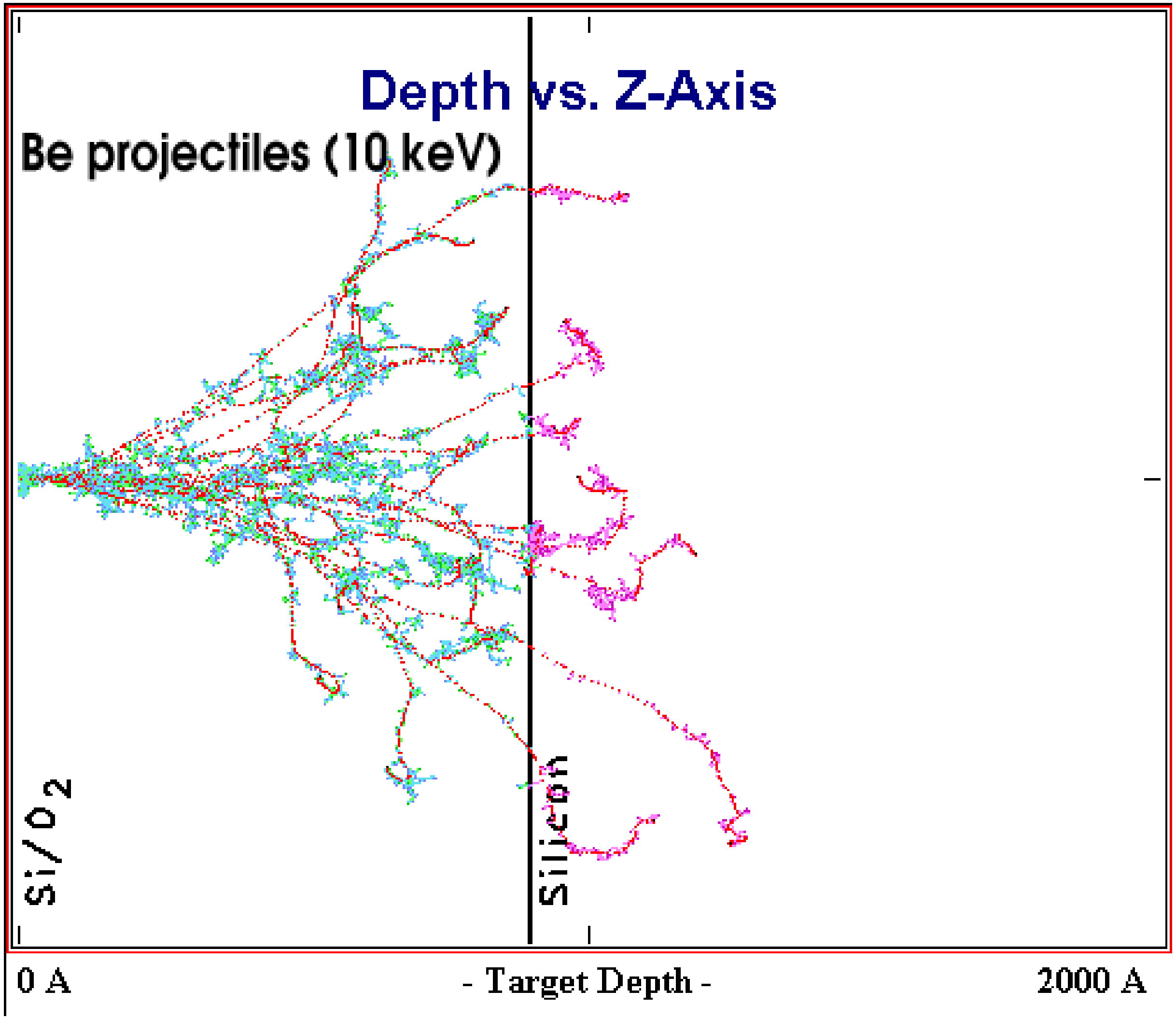
Gaussian approximation:

$$\theta_0 = \frac{13.6\ \text{MeV}}{\beta c p}\, z\, \sqrt{x/X_0}\Big[1 + 0.038\ln(x/X_0)\Big]$$

x/X_0 is the thickness of the material in radiation lengths.

Monte Carlo calculation example of

- Multiple scattering
- Range and range straggling



Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials

  - Energy loss by ionisation
  - Bremsstrahlung
  - **Multiple scattering**

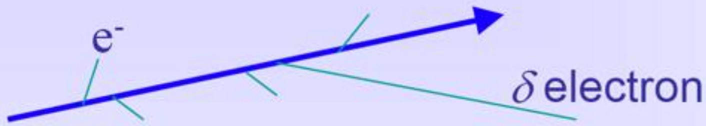
Fluctuations in Energy Loss

Real detector (limited granularity) can not measure <*dE/dx*> !

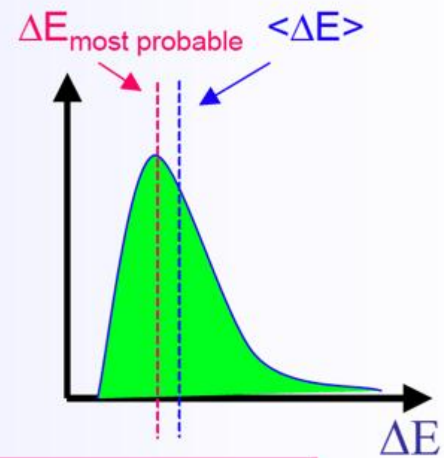
It measures the energy ΔE deposited in a layer of finite thickness δx.

For thin layers or low density materials:

→ Few collisions, some with high energy transfer.



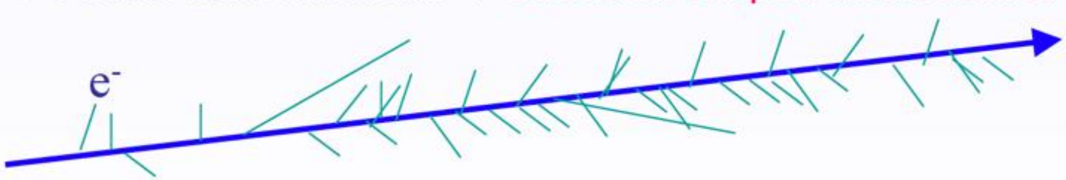
→ Energy loss distributions show large fluctuations towards high losses: "Landau tails"

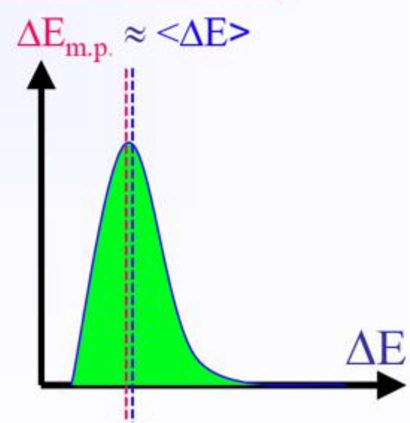


Example: Si sensor: 300 μm thick. $\Delta E_{m.p} \sim 82$ keV $\langle \Delta E \rangle \sim 115$ keV

For thick layers and high density materials:

→ Many collisions.
→ Central Limit Theorem → Gaussian shaped distributions.





Fluctuations in Energy Loss

$$\kappa = \bar{\Delta} / W_{\max}$$

Mean energy loss

$$\bar{\Delta} \simeq \xi = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A} \left(\frac{z}{\beta}\right)^2 x$$

For Landau ….
W_{max} = ∞, electrons free, v = constant

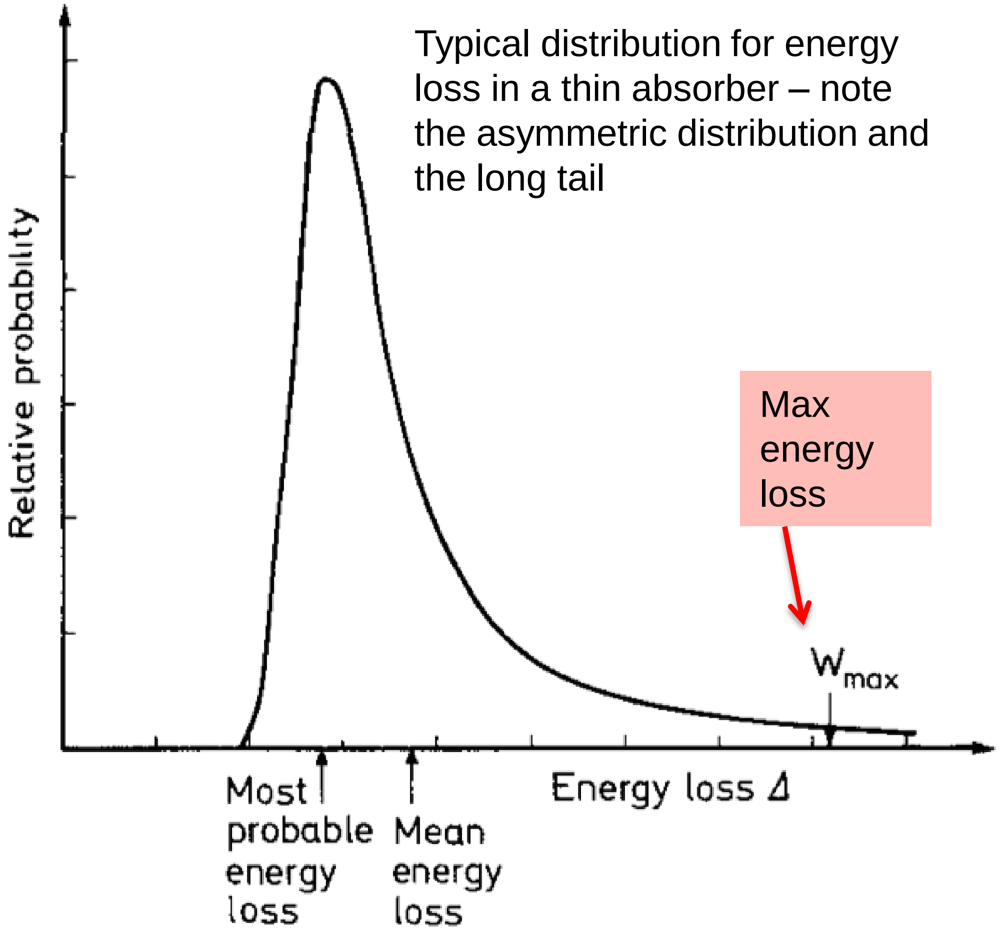
$$f(x, \Delta) = \phi(\lambda)/\xi, \quad \text{where}$$

$$\phi(\lambda) = \frac{1}{\pi} \int_0^\infty \exp(-u \ln u - u\lambda) \sin \pi u \, du$$

$$\lambda = \frac{1}{\xi}[\Delta - \xi(\ln \xi - \ln \varepsilon + 1 - C)]$$

C = Euler's Const = 0.577 … and

$$\ln \varepsilon = \ln \frac{(1-\beta^2) I^2}{2mc^2\beta^2} + \beta^2.$$



Range and Range straggle

Total Range in a thick target $$R(E) = R_0(E_{min}) + \int_{E}^{E_{min}} \left(\frac{dE}{dx}\right)^{-1} dE$$

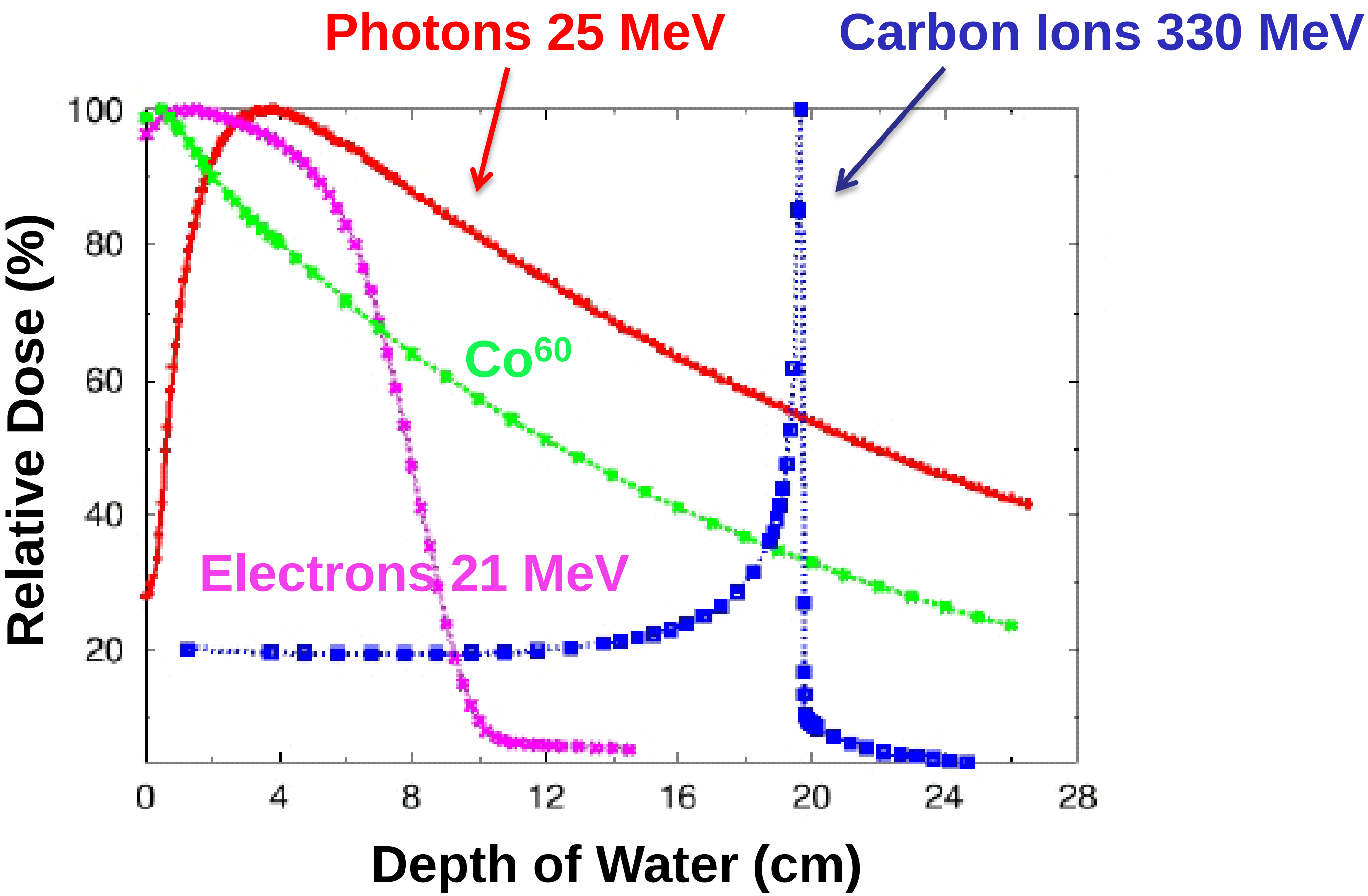
or

Energy loss in a slab $$\Delta x = \int_{E-\Delta E}^{E} \left(\frac{dE}{dx}\right)^{-1} dE$$

Range of Particles in Matter

Average Range:

Towards the end of the track the energy loss is largest → Bragg Peak → Cancer Therapy … or Archaeology!



Range of Particles in Matter

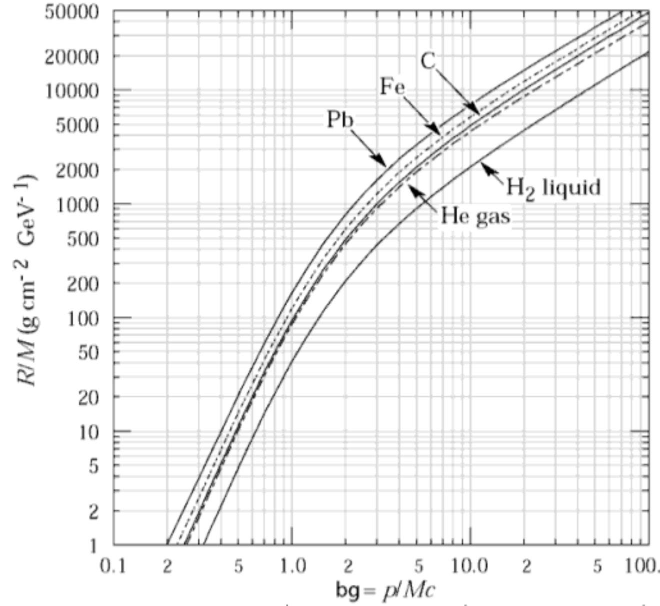
Particle of mass M and kinetic Energy E_0 enters matter and loses energy until it comes to rest at distance R.

$$R(E_0) = \int_{E_0}^{0} \frac{-1}{dE/dr}\, dE$$

$$R(\beta_0\gamma_0) = \frac{Mc^2}{\rho}\frac{1}{Z_1^2}\frac{A}{Z} f(\beta_0\gamma_0)$$

$$\frac{\rho}{Mc^2} R(\beta_0\gamma_0) = \frac{1}{Z_1^2}\frac{A}{Z} f(\beta_0\gamma_0)$$

≈Independent of the material

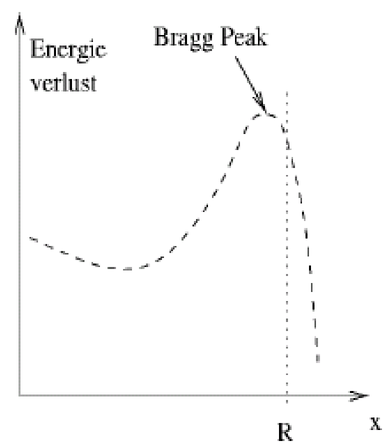


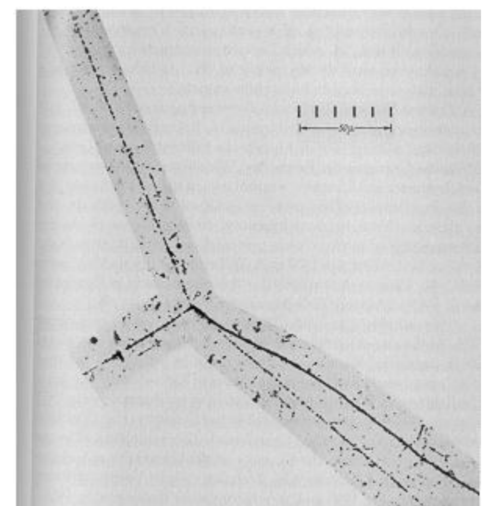
Bragg Peak:

For βγ>3 the energy loss is ≈ constant (Fermi Plateau)

If the energy of the particle falls below βγ=3 the energy loss rises as $1/\beta^2$

Towards the end of the track the energy loss is largest → Cancer Therapy.

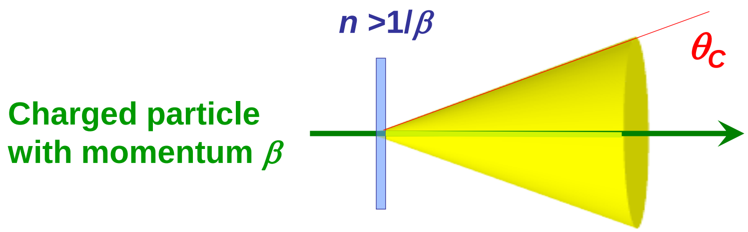




Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials
  - Energy loss by ionisation
  - Bremsstrahlung
  - Multiple scattering
  - Radiation length
  - **Cherenkov radiation**

A relativistic charge particle traversing a dielectric medium with refraction index $n > 1/\beta$, emits Cherenkov radiation in cone with angle θ_C around track: $\cos\theta_C = (n\beta)^{-1}$

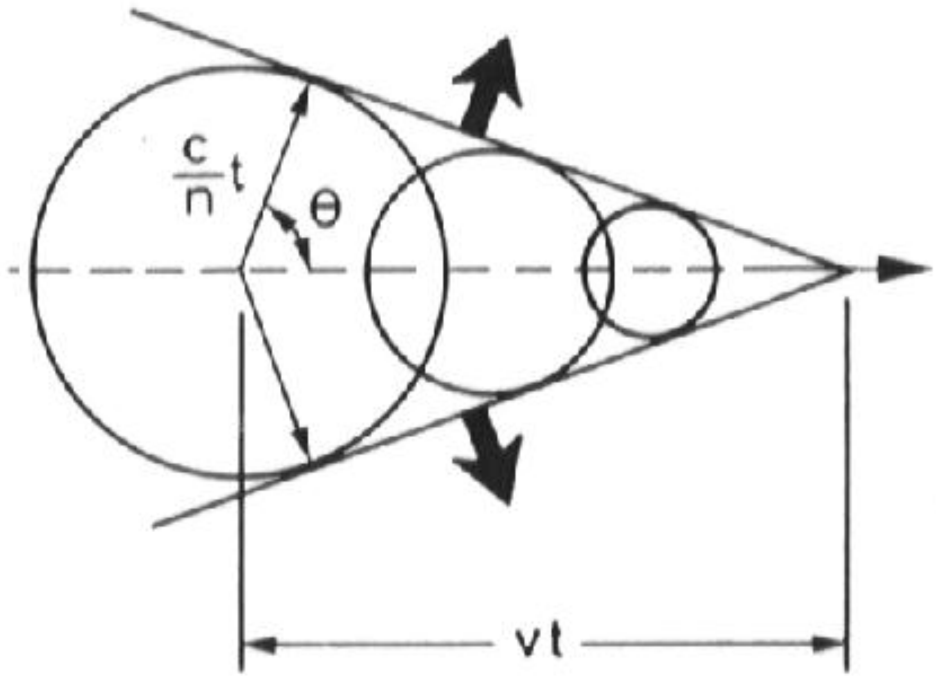


Light cone emission when passing thin medium

Detector types RICH (LHCb), DIRC, Aerogel counters (not employed by ATLAS/CMS))

Cherenkov Radiation

$$\beta c = v_{\text{particle}} > c/n$$



The well defined cone angle is (by construction)

$$\cos\theta_C = \frac{1}{\beta n(w)}$$

A more complete treatment from classical electrodynamics yields

$$\frac{d^2E}{d\omega d\Omega} = z^2\frac{\alpha\hbar}{c}n\beta^2\sin\theta\left|\frac{\omega L}{2\pi\beta c}\frac{\sin\xi(\theta)}{\xi(\theta)}\right|^2$$

and

$$\xi(\theta) = \frac{\omega L}{2\beta c}(1-\beta\cos\theta).$$

The Cherenkov light of a single particle then appears as a ring of light, projected onto a plane transverse to the particle trajectory.

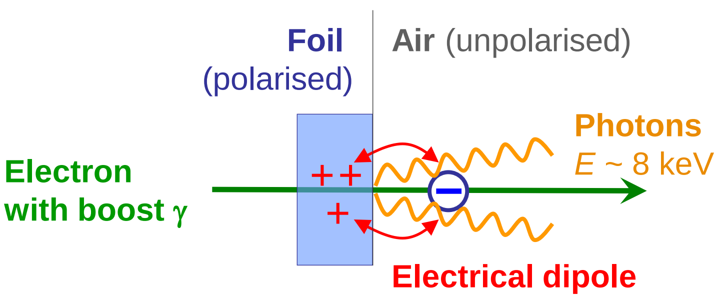
Integrating over frequencies that satisfy the Cherenkov condition, $\beta > 1/n(w)$, and using $L = dx$, we find

$$-\frac{dE}{dx} = z^2 \frac{\alpha\hbar}{c} \int \omega \, d\omega \left(1 - \frac{1}{\beta^2 n^2(\omega)} \right) .$$

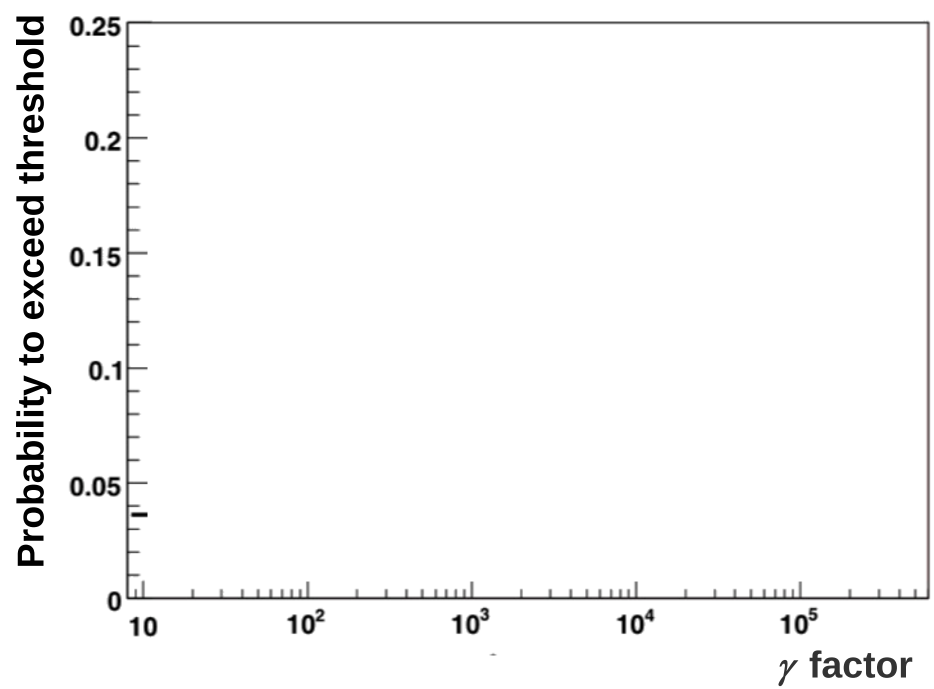
Charged Particle Interactions with Matter

- Particles are detected through their interaction with the active detector materials
  - Energy loss by ionisation
  - Bremsstrahlung
  - Multiple scattering
  - Radiation length
  - Cherenkov radiation
  - **Transition radiation**

Photon radiation when charged ultra-relativistic particles traverse the boundary of two different dielectric media (foil & air)



➡ Significant radiation for $\gamma > 1000$ and > 100 boundaries



Photon Interactions

Photons interact with a material primarily via

1. **Photo-electric effect** (absorbs full energy of the photon, leads to a photo-peak)
2. **Compton Scattering** (if the Compton scattered photon escapes, detector only records partial energy)
3. **Pair Production** (the pair then makes an energy loss as per light charged particles). If the annihilation radiation of the positron escapes, it can lead to single or double escape peaks.

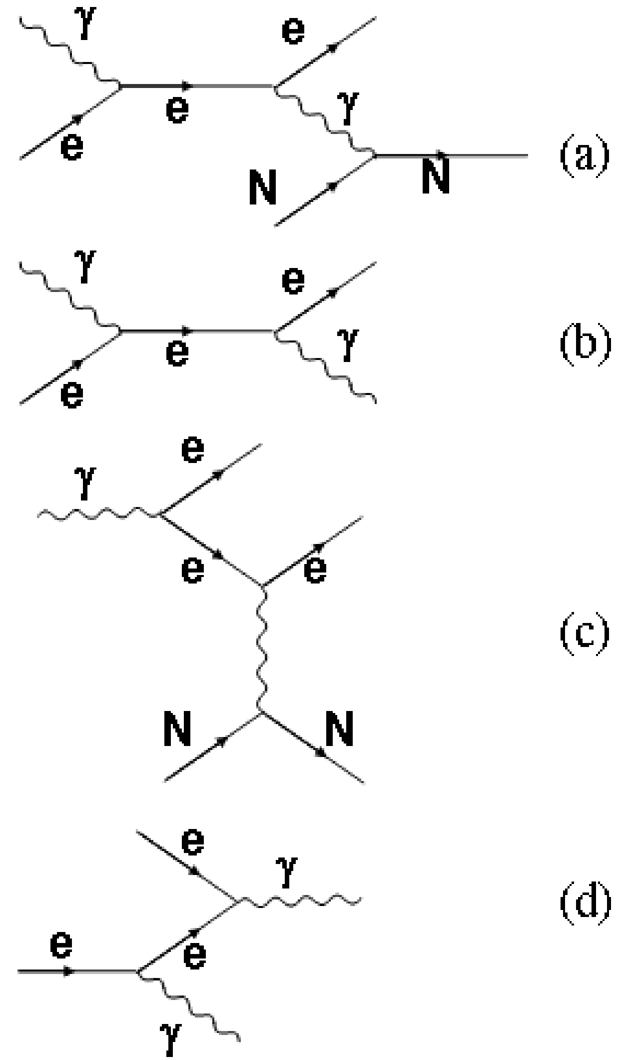


Figure 1. Feynman diagrams for (a) the photoelectric effect, (b) the Compton effect, (c) pair production, and (d) pair annihilation.

Photo-electric effect

The photo-e^- has the kinetic energy of the photon less the binding energy of the e^-.

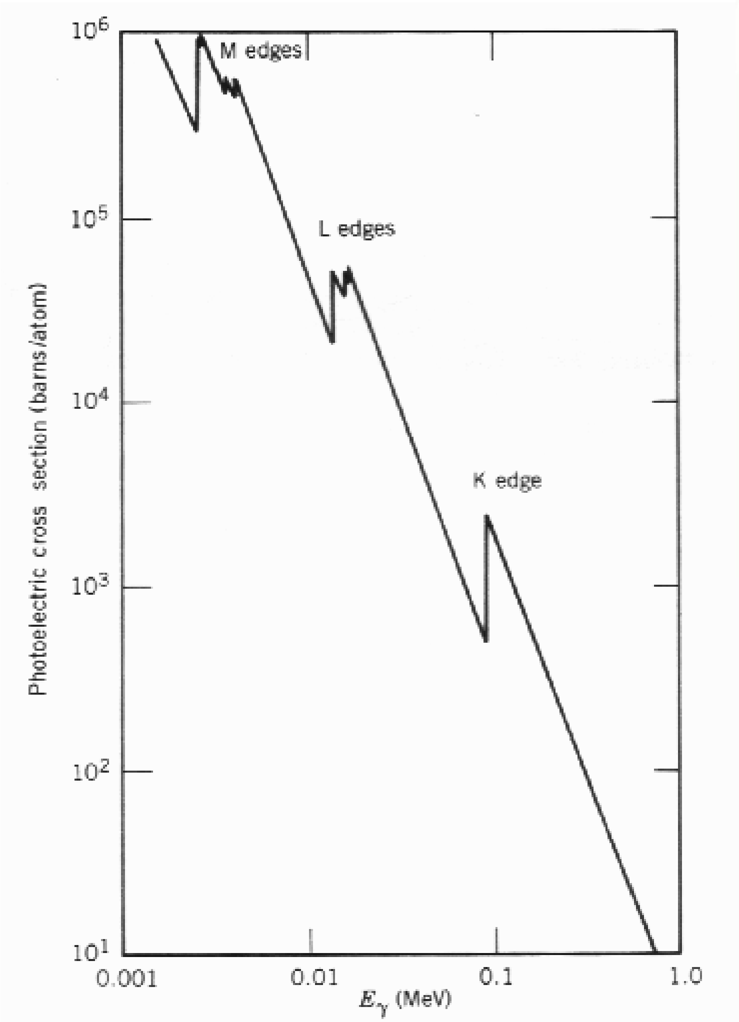
$$T_e = E_\gamma - B_e$$

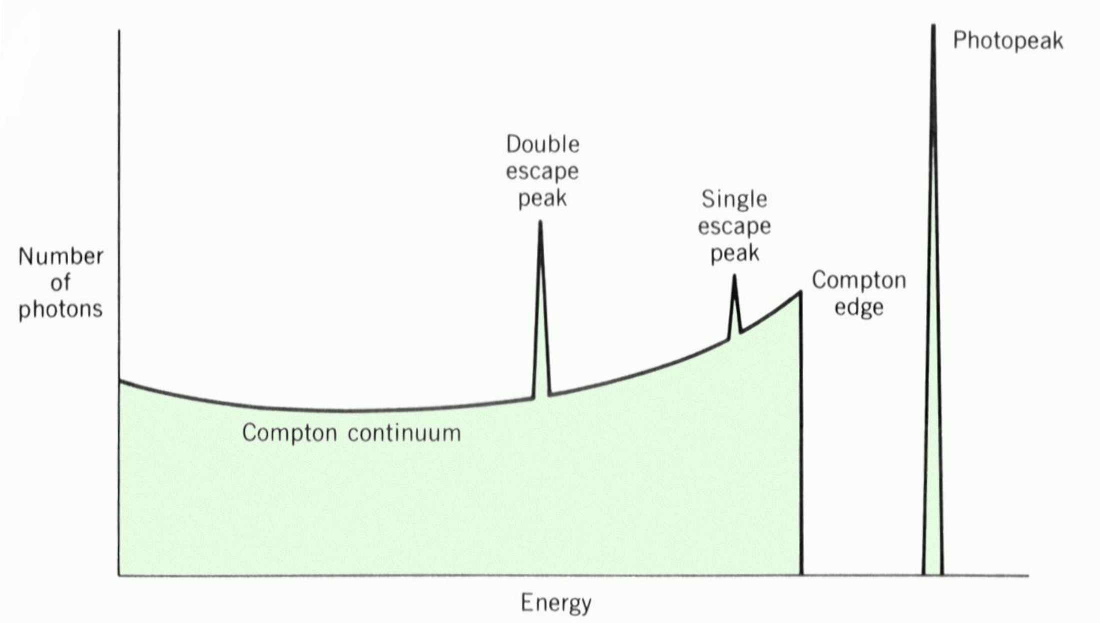
A third particle is necessary (the nucleus) which recoils away to satisfy the conservation of energy and momentum.

The probability (cross-section) for the PE is shown in the figure below. It is strongest for large $\approx Z^5$ and at low energies $\approx 1/E^{3.5}$. A non-relativistic treatment in the Born approximation gives

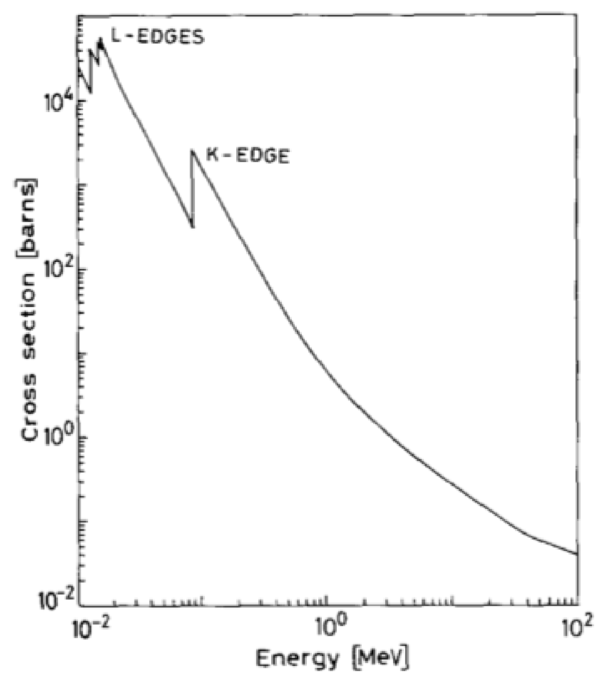
$$\sigma = 4\alpha^4\sqrt{2}Z^5 \left(\frac{8\pi r_e^2}{3}\right) \left(\frac{m_e c^2}{h\nu}\right)^{7/2} .$$

This dependence is shown in the figure below.
This process leads to the sharp "Photo Peak" in Gamma-spectroscopy





The cross section for the Photo-electric Effect for Aluminium and for Lead.



Compton Scattering

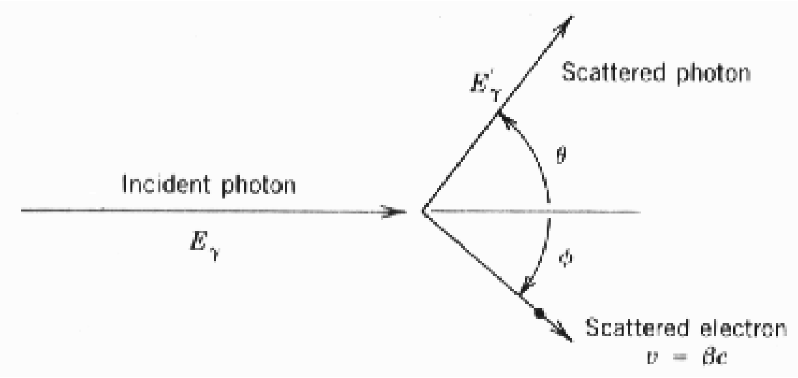
In this process the photon scatters off a free electron. It can be shown (Energy and Momentum Conservation) that the scattered photon has an energy

$$E'_\gamma = \frac{E_\gamma}{1 + (E_\gamma/mc^2)(1 - \cos\theta)}$$

The scattered photons range in energy from

$$E'_\gamma = E_\gamma \quad \text{at } \theta = 0^\circ \text{ for no scattering to}$$

$$E'_\gamma = \frac{E_\gamma}{1 + (2E_\gamma/mc^2)} \quad \text{at } \theta = 180^\circ \text{ complete backward scattering}$$



A QM calculation gives the probability for Compton Scattering at the angle θ (Klein-Nishina formula)

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} \frac{1}{[1+\gamma(1-\cos\theta)]^2} \left(1 + \cos^2\theta + \frac{\gamma^2(1-\cos\theta)^2}{1+\gamma(1-\cos\theta)} \right)$$

Integrating the angular dependence out to give the total cross section ….

$$\sigma_c = 2\pi r_e^2 \left\{ \frac{1+\gamma}{\gamma^2} \left[\frac{2(1+\gamma)}{1+2\gamma} - \frac{1}{\gamma} \ln(1+2\gamma) \right] + \frac{1}{2\gamma} \ln(1+2\gamma) - \frac{1+3\gamma}{(1+2\gamma)^2} \right\}$$

As the energy increases, the Compton Effect begins to dominate over the Photo-electric Effect

Where we have used … $\gamma = h\nu / m_e c^2$

Compton edge seen in gamma spectroscopy, as the maximum energy transferred to the electron in a single scattering process occurs when $\theta = 180°$, .

Only the electron deposits energy in the detector, the scattered photon escapes.

$$E_e = E_\gamma - E'_\gamma = E_\gamma \left(\frac{2E_\gamma / mc^2)}{1 + (2E_\gamma / mc^2)} \right)$$

The energy loss due to the Compton scattering is found by considering that the fraction of initial photon energy deposited in the material is

$$\frac{d\sigma^a}{d\Omega} = \frac{(E_\gamma - E'_\gamma)}{E_\gamma} \frac{d\sigma}{d\Omega}$$

Successive Compton events can lead to energy losses that fill in the first order Compton gap.

Pair Production

The photon transforms into a particle anti-particle pair. The most likely pair is an positron-electron pair. It is created, in the neighbourhood of a third particle (to satisfy both momentum and energy conservation)

$$\gamma \longrightarrow e^+ + e^-$$

The energy balance is given by

$$E_\gamma = T_+ + T_- + m_{e^+}c^2 + m_{e^+}c^2$$

Where the T terms are kinetic energies.
Pair production has a threshold at $E_\gamma = 2mc^2 = 1.022\,\mathrm{MeV}$.

The pair production cross section is

$$d\sigma = 4Z^2 r_e^2 \alpha \frac{dE_+}{(h\nu)^3} \left\{ (E_+^2 + E_-^2) \left[\frac{\phi_1(\xi)}{4} - \frac{1}{3}\ln Z - f(Z) \right] - \frac{2}{3}\varepsilon \left[\frac{\phi_2(\xi)}{4} - \frac{1}{3}\ln Z - f(Z) \right] \right\}.$$

Where the screening parameter is defined

$$\xi = \frac{100 m_e c^2 h\nu}{E_+ E_- Z^{1/3}}.$$

and the other functions have weak dependencies.

Note the main trends in the formula.

This process becomes dominant at energies above a few MeV.

Photon Attenuation by materials

The total probability per unit area per electron for one of the above mentioned processes to remove a photon from a photon beam as it passes through a material is

$$\sigma_\gamma = \sigma_{PE} + \sigma_E + \sigma_{PP}$$

The probability for this to happen per unit thickness of material traversed is therefore

$$\mu = \sigma_\gamma n Z$$

where n is the atomic number density of the material

$$n = \frac{\rho}{\mathcal{M}} N_A$$

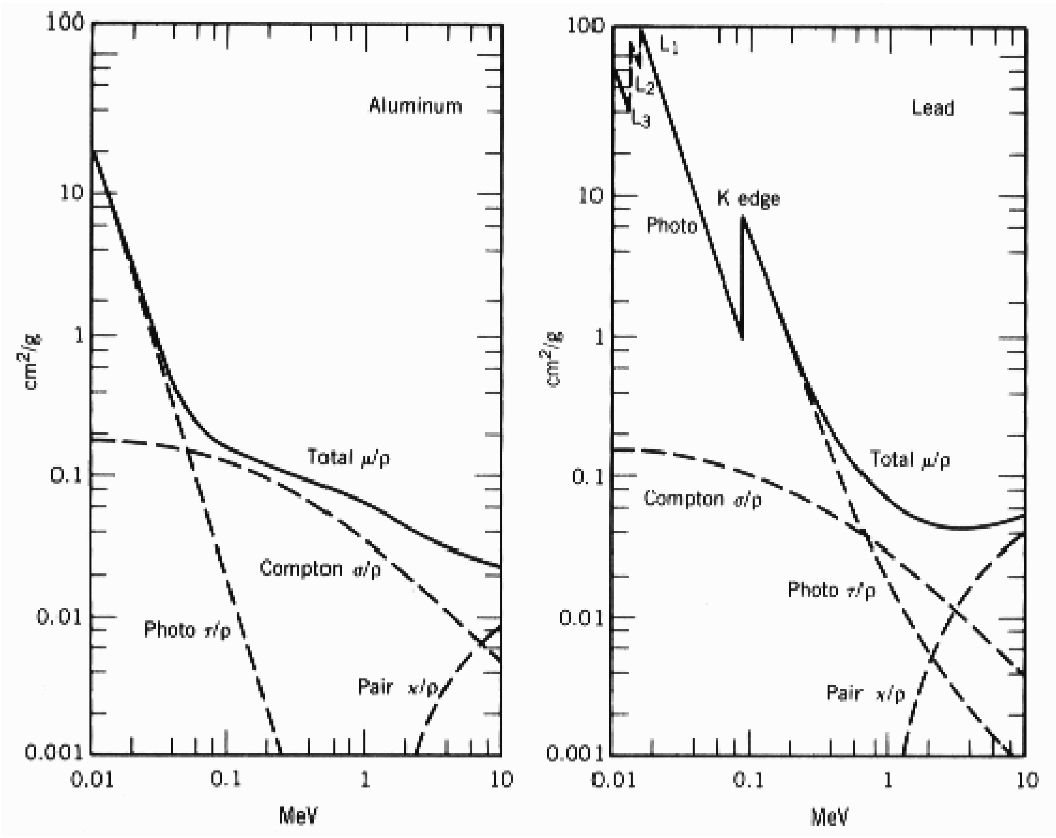
where $\mathcal{M}$ is the molar mass and N_A is Avogadro's number.

The fractional loss of intensity in crossing any thickness dx of material is

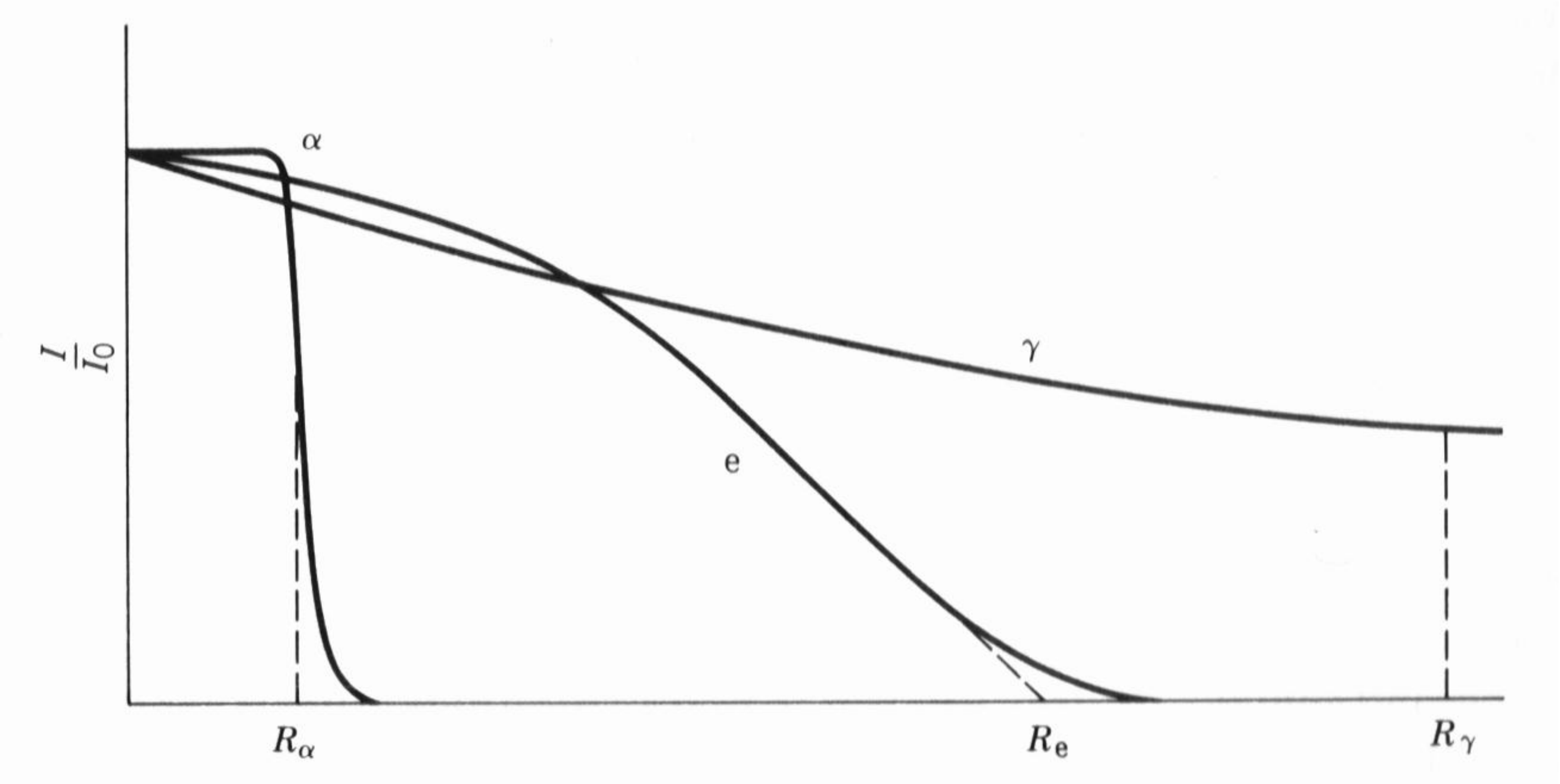
$$\frac{dI}{I} = -\mu\, dx$$

The attenuation of photons with material thickness t is

$$I = I_0 e^{-\mu t}$$



The well defined finite range of charged particles in a material and the attenuation of photons in a material



Radiation length

This parameter is defined as the distance over which the electron energy is reduced by a factor $1/e$ of its initial value, due to radiation losses only.

From the section on Bremsstrahlung $$-dE/E = N\,\Phi_{\mathrm{rad}}\,dx$$

Solving we get the exponential dependence $$E = E_0 \exp\left(\frac{-x}{L_{\mathrm{rad}}}\right)$$

Manipulating the bremsstrahlung cross section it can be shown that the radiation length is given by

$$\frac{1}{L_{rad}} \simeq \left[4Z(Z+1)\frac{\rho N_A}{A}\right] r_e^2 \alpha [\ln(183 Z^{-1/3}) - f(Z)]$$

$L_{rad}(\mathrm{lead}) = 0.56$ cm
$L_{rad}(\mathrm{air}) = 300$ m.

Calorimetry of a shower is then very complete within 7 radiation lengths of a given material.

Radiation length

We can also calculate probability of interaction per unit path-length for Pair Production

$$\mu = N\sigma$$

Where we use the total cross section for Pair Production.

The mean free path for pair production

$$\lambda_{pair} = 1/\mu = 1/(N\sigma) \approx \tfrac{9}{7} L_{rad}$$

Leptonic showers

The cascading combined effect of pair production by high energy photons and bremsstrahlung by high energy electrons is the formation of electron-photon showers.

It is easy to see that the particle multiplicity in the shower is

$$N \simeq 2^t$$

each with an average energy

$$E(t) \simeq \frac{E_0}{2^t}$$

where t is the number of radiation lengths L_{rad} traversed.

The maximum penetration depth of the shower is reached at the critical energy E_c

$$E(t_{max}) = \frac{E_0}{2^t_{max}} = E_c$$

so that

$$t_{max} = \frac{\ln \frac{E_0}{E_c}}{\ln 2}$$

and

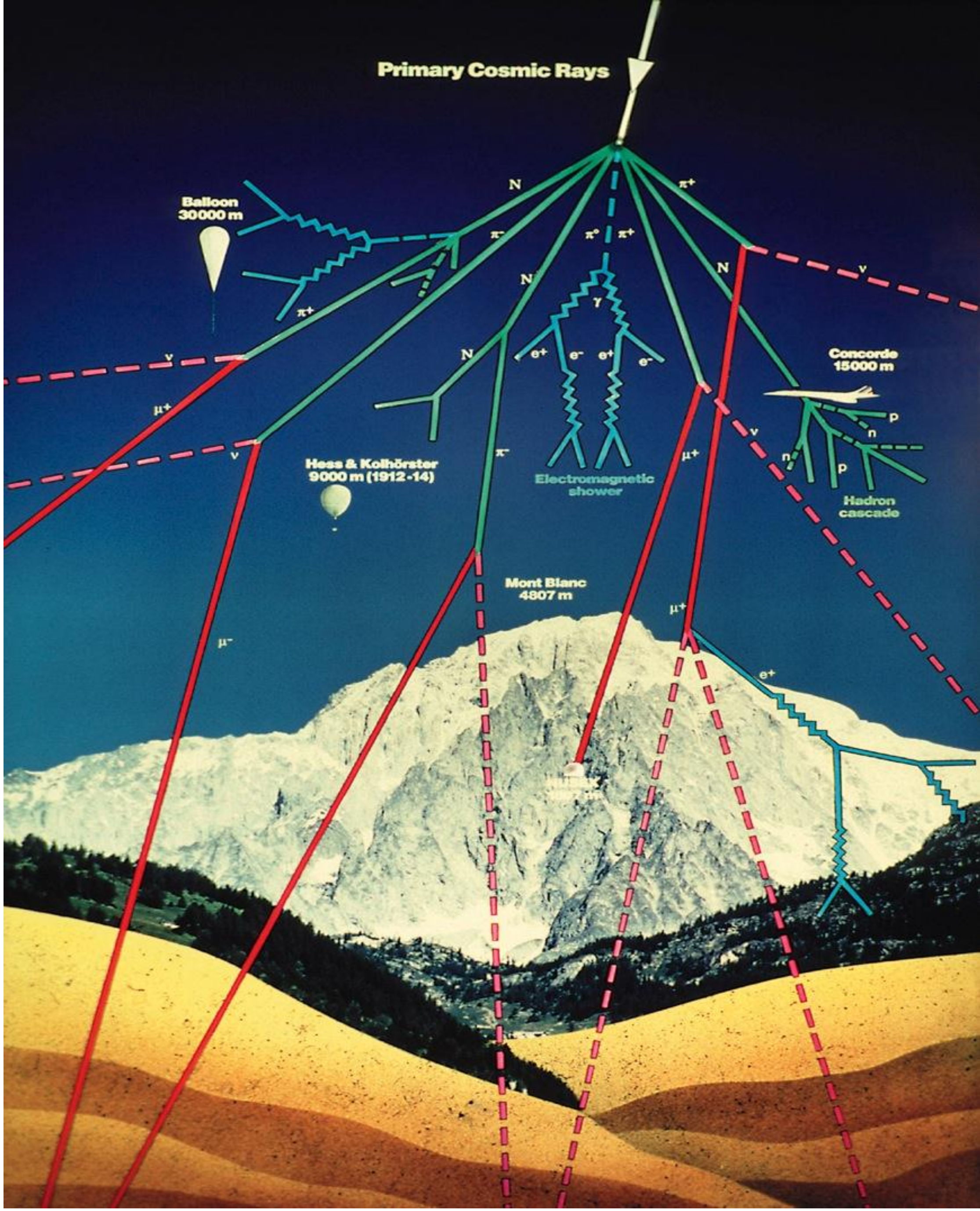
$$N_{max} = \frac{E_0}{E_c}.$$

Once again, Monte Carlo methods are required to properly understand shower development.

The earths atmosphere is a giant detector for cosmic rays.

Showers are initiated typically in the upper atmosphere (why).

Primary particles with energies of up to 10^{22} eV lead to extensive showers with a large footprint on the earth.



Electromagnetic calorimetry: radiation length

- Particles are detected through their interaction with the active detector materials
  - Energy loss by ionisation
  - Bremsstrahlung
  - Multiple scattering
  - **Radiation length**

Material thickness in detector is measured in terms of dominant energy loss reactions at high energies:

- Bremsstrahlung for electrons
- Pair production for photons

Definition:

X_0 = Length over which an electron loses all but **1/*e*** of its energy by bremsstrahlung

= 7/9 of mean free path length of photon before pair production

➡ Describe material thickness in units of X_0

Material	X_0 [cm]
Be	35.3
Carbon-fibre	~ 25
Si	9.4
Fe	1.8
$PbWO_4$	0.9
Pb	0.6

ATLAS LAr absorber (Pb)

CMS ECAL crystals ($PbWO_4$)

Electromagnetic calorimetry: radiation length

6. ATOMIC AND NUCLEAR PROPERTIES OF MATERIALS

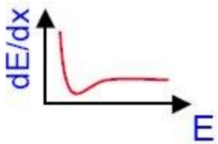
Table 6.1. Revised May 2002 by D.E. Groom (LBNL). Gases are evaluated at 20°C and 1 atm (in parentheses) or at STP [square brackets]. Densities and refractive indices without parentheses or brackets are for solids or liquids, or are for cryogenic liquids at the indicated boiling point (BP) at 1 atm. Refractive indices are evaluated at the sodium D line. Data for compounds and mixtures are from Refs. 1 and 2. Futher materials and properties are given in Ref. 3 and at `http://pdg.lbl.gov/AtomicNuclearProperties`.

Material	Z	A	$\langle Z/A \rangle$	Nuclear [a] collision length λ_T {g/cm^2}	Nuclear [a] interaction length λ_I {g/cm^2}	$dE/dx\|_{\rm min}$ [b] $\left\{\frac{\rm MeV}{\rm g/cm^2}\right\}$	Radiation length [c] X_0 {g/cm^2}	X_0 {cm}	Density {g/cm^3} ({g/ℓ} for gas)	Liquid boiling point at 1 atm(K)	Refractive index n ($(n-1)\times 10^6$ for gas)
H_2 gas	1	1.00794	0.99212	43.3	50.8	(4.103)	61.28 [d]	(731000)	(0.0838)[0.0899]		[139.2]
H_2 liquid	1	1.00794	0.99212	43.3	50.8	4.034	61.28 [d]	866	0.0708	20.39	1.112
D_2	1	2.0140	0.49652	45.7	54.7	(2.052)	122.4	724	0.169[0.179]	23.65	1.128 [138]
He	2	4.002602	0.49968	49.9	65.1	(1.937)	94.32	756	0.1249[0.1786]	4.224	1.024 [34.9]
Li	3	6.941	0.43221	54.6	73.4	1.639	82.76	155	0.534		—
Be	4	9.012182	0.44384	55.8	75.2	1.594	65.19	35.28	1.848		—
C	6	12.011	0.49954	60.2	86.3	1.745	42.70	18.8	2.265 [e]		—
N_2	7	14.00674	0.49976	61.4	87.8	(1.825)	37.99	47.1	0.8073[1.250]	77.36	1.205 [298]
O_2	8	15.9994	0.50002	63.2	91.0	(1.801)	34.24	30.0	1.141[1.428]	90.18	1.22 [296]
F_2	9	18.9984032	0.47372	65.5	95.3	(1.675)	32.93	21.85	1.507[1.696]	85.24	[195]
Ne	10	20.1797	0.49555	66.1	96.6	(1.724)	28.94	24.0	1.204[0.9005]	27.09	1.092 [67.1]
Al	13	26.981539	0.48181	70.6	106.4	1.615	24.01	8.9	2.70		—
Si	14	28.0855	0.49848	70.6	106.0	1.664	21.82	9.36	2.33		3.95
Ar	18	39.948	0.45059	76.4	117.2	(1.519)	19.55	14.0	1.396[1.782]	87.28	1.233 [283]
Ti	22	47.867	0.45948	79.9	124.9	1.476	16.17	3.56	4.54		—
Fe	26	55.845	0.46556	82.8	131.9	1.451	13.84	1.76	7.87		—
Cu	29	63.546	0.45636	85.6	134.9	1.403	12.86	1.43	8.96		—
Ge	32	72.61	0.44071	88.3	140.5	1.371	12.25	2.30	5.323		—
Sn	50	118.710	0.42120	100.2	163	1.264	8.82	1.21	7.31		—
Xe	54	131.29	0.41130	102.8	169	(1.255)	8.48	2.87	2.953[5.858]	165.1	[701]
W	74	183.84	0.40250	110.3	185	1.145	6.76	0.35	19.3		—
Pt	78	195.08	0.39984	113.3	189.7	1.129	6.54	0.305	21.45		—
Pb	82	207.2	0.39575	116.2	194	1.123	6.37	0.56	11.35		—
U	92	238.0289	0.38651	117.0	199	1.082	6.00	≈0.32	≈18.95		—

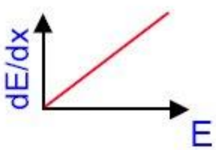
Electromagnetic showers

e⁺ / e⁻

- Ionisation

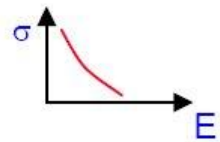


- Bremsstrahlung

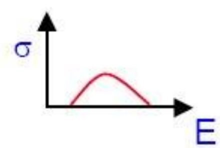


γ

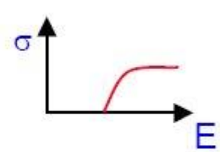
- Photoelectric effect

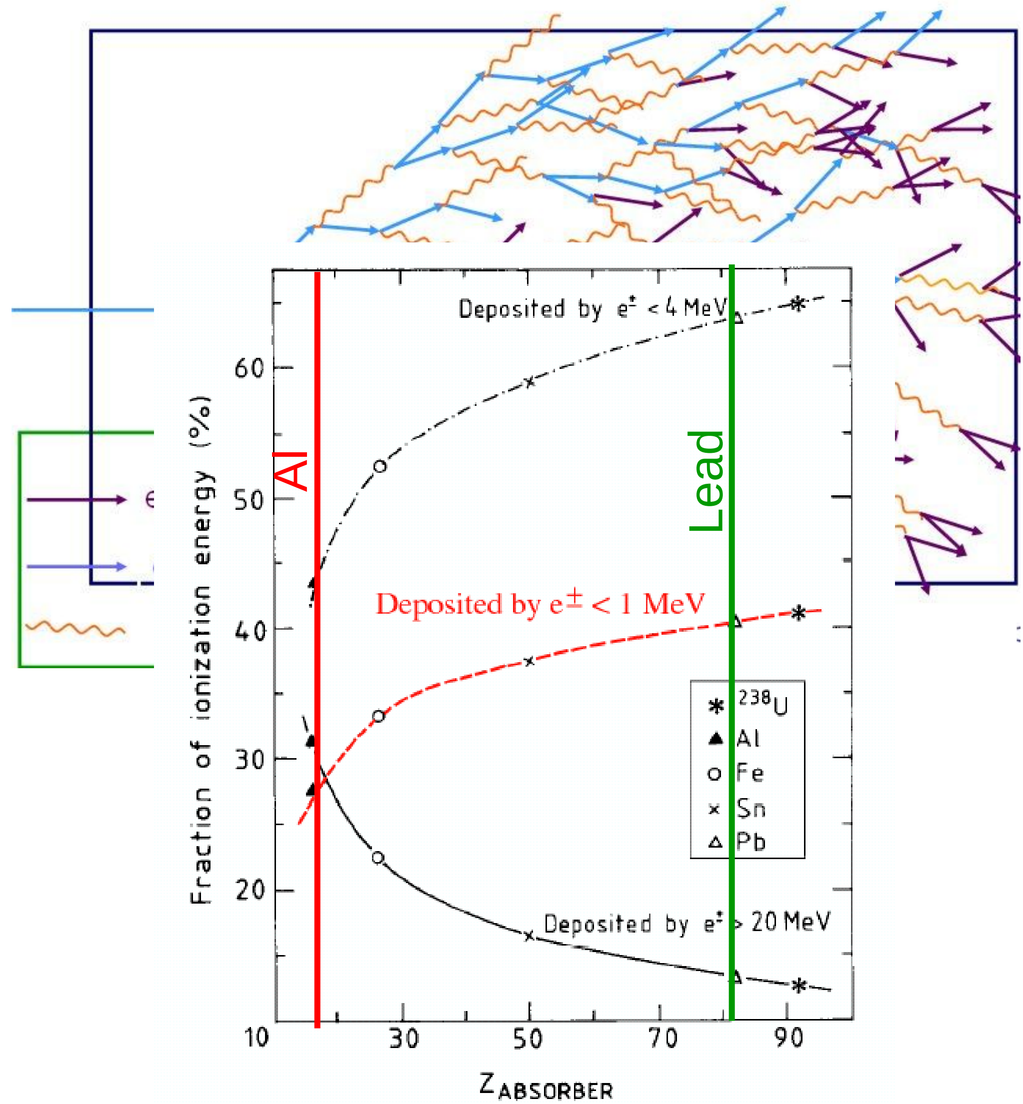


- Compton effect



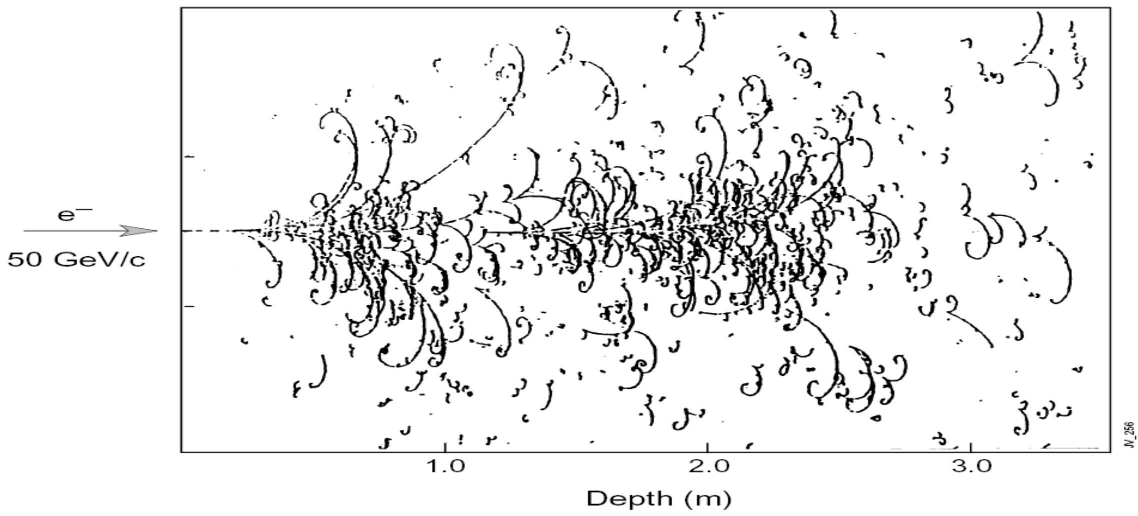
- Pair production



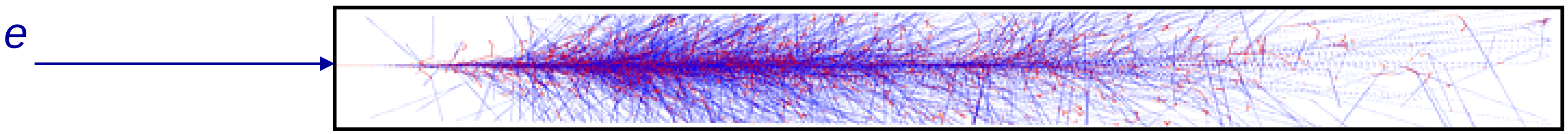


Electromagnetic showers

Big European Bubble Chamber filled with Ne:H_2 = 70%:30%,
3T Field, L=3.5 m, X_0≈34 cm, 50 GeV incident electron



PbW0$_4$ CMS, X_0=0.89 cm



Neutron Radiation
Moderation processes

Consider elastic collisions A(n,n)A with nuclei in the material.
From the Conservation of Energy and Momentum
(assuming nucleus *A* at rest)

$$\frac{E'}{E} = \frac{A^2 + 1 + 2A\cos\theta}{(A+1)^2}$$

Note : *E*' and *E* are measured in the lab frame, but θ is in the CM frame.

The maximum energy loss is therefore

$$\left(\frac{E'}{E}\right)_{\min} = \left(\frac{A-1}{A+1}\right)^2$$

What would be the best materials for a neutron moderator ?

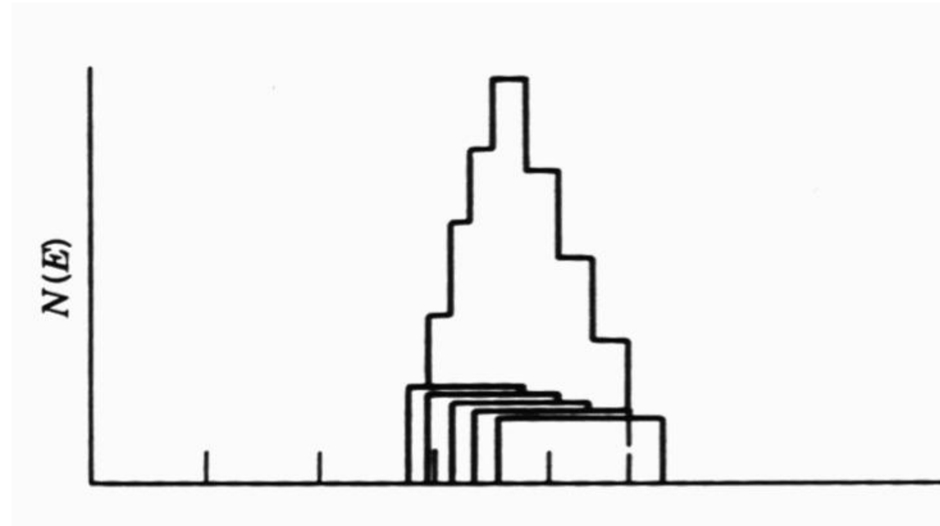
1. For energies below 10 MeV, scattering is isotropic in the CM frame.
2. One may expect a first generation scattered energy in the range E' ~ (E, E'_{min}).
3. This is represented by the rectangle in the figure below
4. A second generation scattered energy would be represented by a set of rectangles starting from the highest point of the first rectangle to the lowest, leading to a net triangular distribution.
5. Successive scattering events lead to broader and lower energy triangular distributions.
6. Eventually the neutron will have a thermal energy distribution, we say the neutrons a re thermalised.

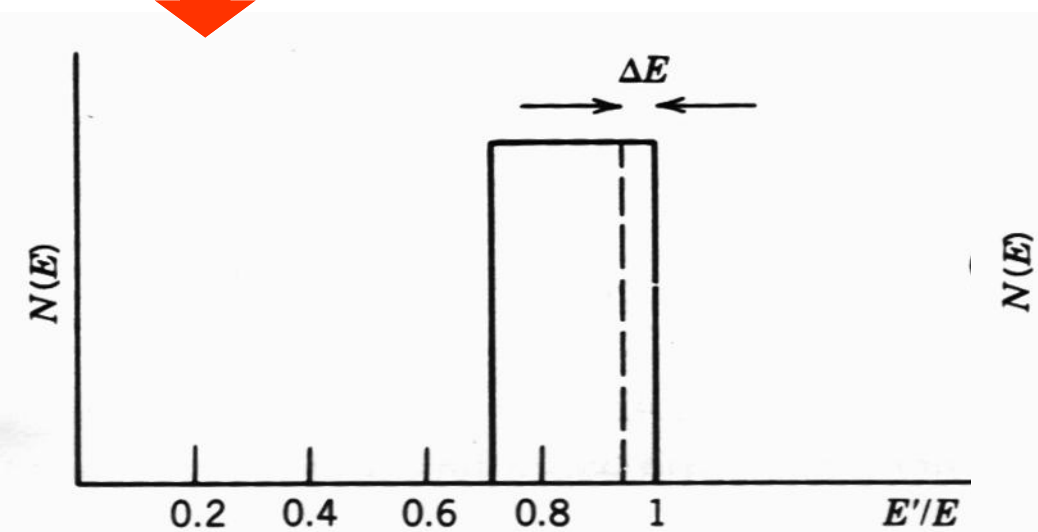
Schematic of neutron energy distributions

Consider first the distribution resulting from the first energy scattering beginning with a mono-energetic neutron

The next picture approximates the energy distribution following the second generations scattering.

Four neutron generations are depicted based on an accurate calculation in the last graph.





We define the moderating power of a particular material by the quantity ξ, defined as logarithm of the average fractional residual energy after a single collision

$$
\begin{aligned}
\xi &= \left[\log \frac{E'}{E}\right]_{av} \\
&= \frac{\int \log\left[\dfrac{(A+1)^2}{(A^2+1+2A\cos\theta}\right] d\Omega}{\int d\Omega} \\
&= 1+\frac{(A-1)^2}{2A}\log\frac{A-1}{A+1}
\end{aligned}
$$

After n collisions, the average value of E' is E'_n

$$\log E'_n = \log E - n\xi$$

A comparison of moderators, and the number of scattering to thermalisation

Nucleus	ξ	n
^{1}H	1.00	18
^{2}H	0.725	25
^{4}He	0.425	43
^{12}C	0.158	110
^{238}U	0.0084	2200

Thermal energies for room temperature

***E* = *kT* = 25 meV**

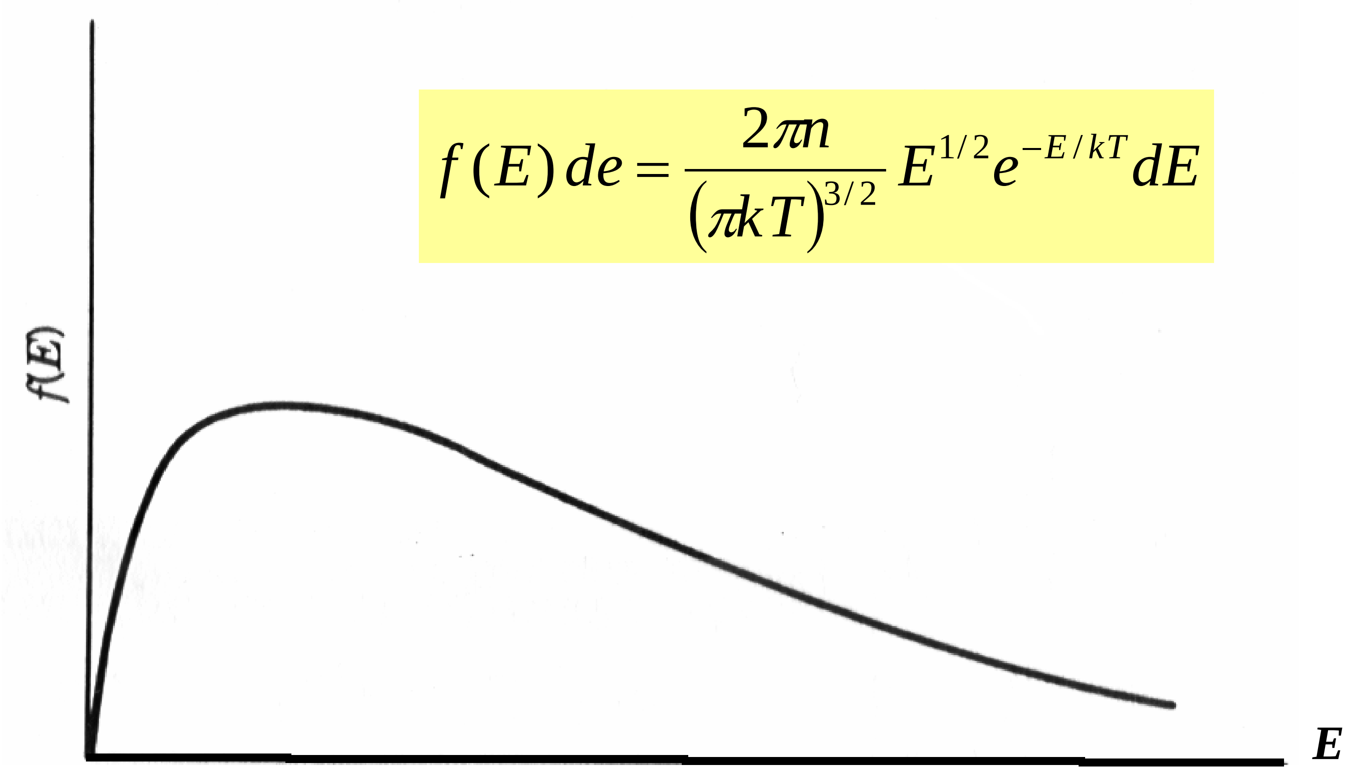
Some neutron detectors make use of the fact that the neutron absorption cross section is higher at thermal energies.

Accordingly, they contain a moderator component as well as a detector component

In fact, thermal energies actually means an energy distribution.

In the field of statistical mechanics, this distribution is derived as a speed distribution and known as the Maxwellian Speed Distribution.

We represent it here converted into an energy distribution.



$$f(E)\,de = \frac{2\pi n}{(\pi k T)^{3/2}} E^{1/2} e^{-E/kT} dE$$

Absorption processes.

Fast neutrons : (n,p), (n,α), (n,2n) reactions are possible
Slow neutrons : (n,γ) reactions, capture leading to excitation of the capture nucleus.

Absorption leads to an exponentially decreasing neutron population with material thickness traversed.
(One may think of the analogy with the attenuation of photons by a material)

$$dI = -I\sigma_t n\, dx$$

Here σ_t is the total neutron reaction cross-section, except for elastic scattering, and n is the number density of atoms in the material, calculated as before.

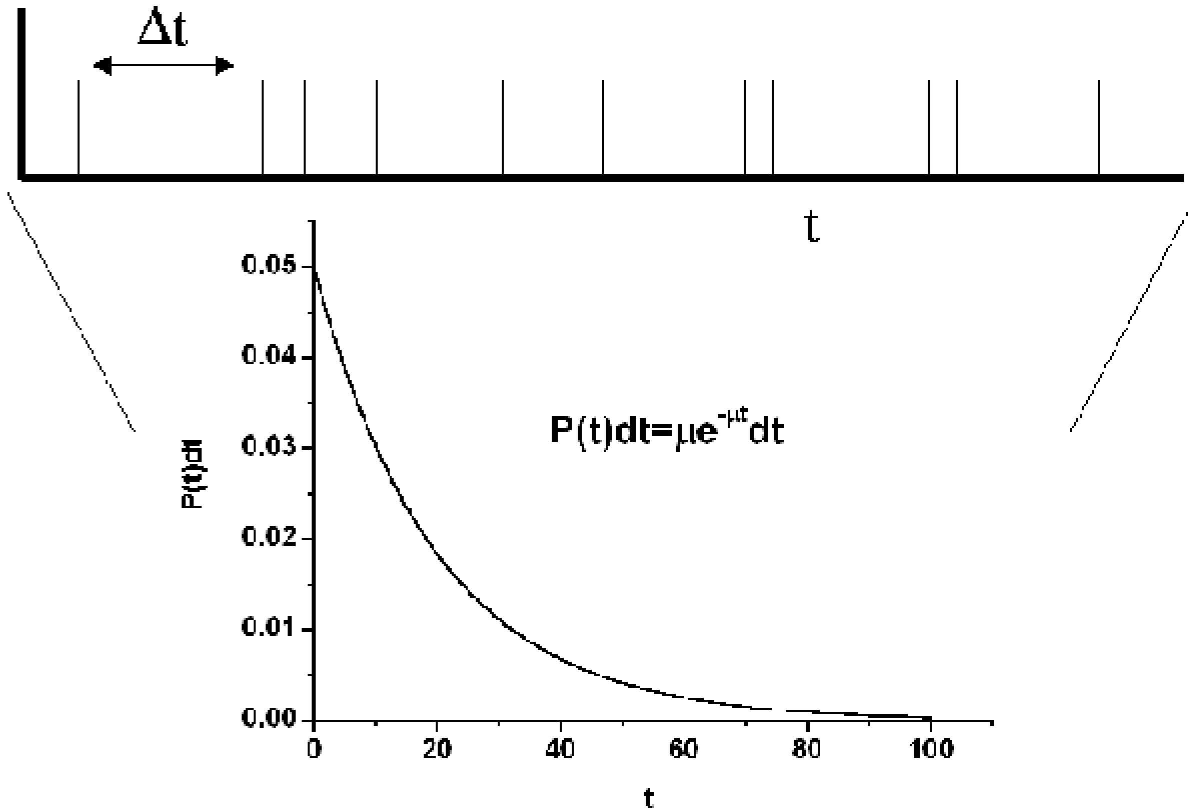
Integrating ….

$$I = I_0 e^{-\sigma_t n x}$$

This expression would be modified for the energy loss, as the cross-sections are energy dependent, and the neutron is usually being thermalised at the same time it is exposed to the possibility of inelastic reactions.

Statistics in counting experiments

The exponential time(space) interval distribution between random events



Events occur with a mean rate μ. The probability for an event in time dt is $\mu.dt$.

Consider probability function $F_{no}(t)$ for no event in time t. Then

$$F_{no}(t + dt) = F_{no}(t)(1 - \mu dt)$$

solving this equation we get

$$F_{no}(t) = Ce^{-\mu t},$$

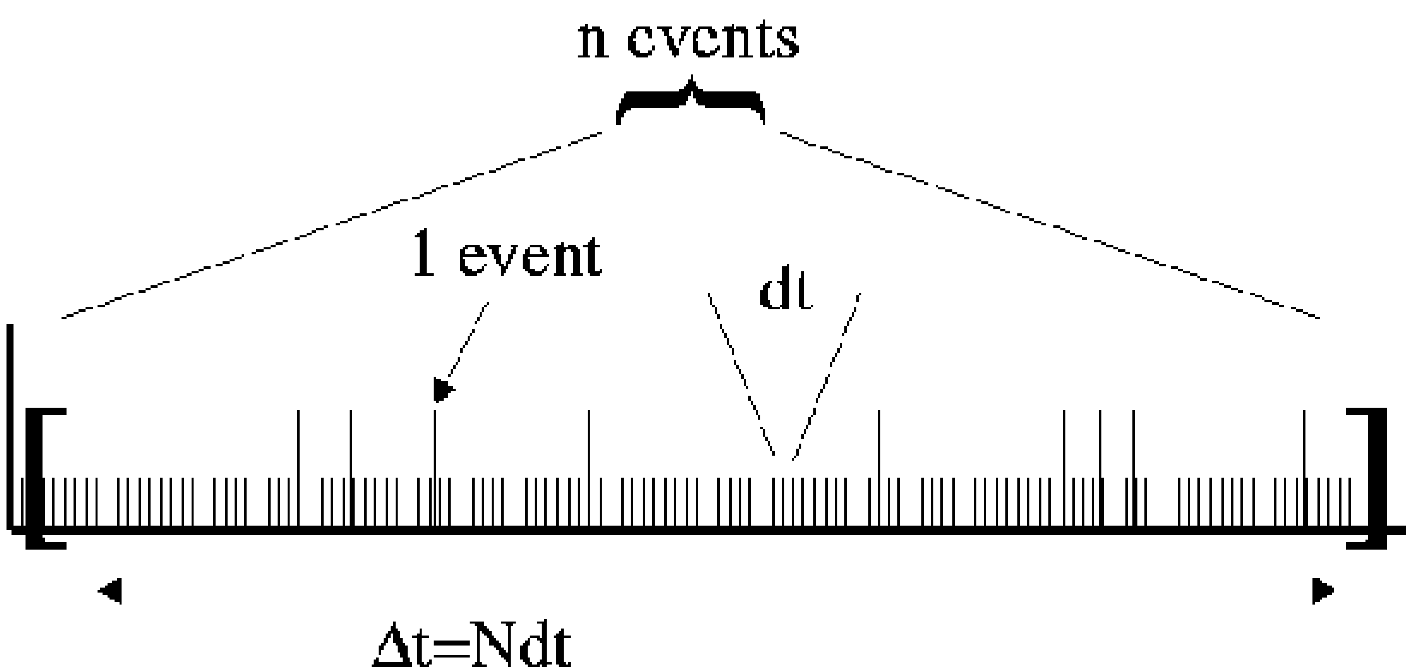
The probability density function $P(t)$ of the random event not occurring for a time t, and then occurring in the next dt.

$$P(t)dt = F_{no}(t)\,\mu dt = \mu\, e^{-\mu t}\, dt$$

C=1 by normalisation

Thus, the waiting times between random events are exponentially distributed.

Poisson counting statistics for binned random events



If there are on average $\overline{n} = \mu\Delta t$ events in the tome interval Δt, then the probability of an event in the subinterval will be

$$p = \overline{n}/N \ll 1.$$

The number of events n in the interval Δt then follows a Binomial distribution

$$P(n) = \binom{N}{n} p^n (1-p)^{(N-n)}.$$

Decreasing the size of the subintervals, ie, $N \longrightarrow \infty$ and $p \longrightarrow 0$ but $\overline{n} = Np$ remains finite, we find that the Binomial distribution may be replaced by its approximation for small p, the Poisson distribution.

$$P(n) = \frac{\overline{n}^n e^{-\overline{n}}}{n!}$$

Consequently, the standard deviation of the number of events per bin is

$$\sigma = \sqrt{\overline{n}}$$

So that the relative error in counting random events diminishes as $\sim \frac{1}{\sqrt{\overline{n}}}$.

Ionisation statistics

If the incident particle energy loss is ΔE and the average energy for ionisation is w, then one would expect on average $N = \Delta E/w$ ionisation events. However, there will be a distribution of momenta transfer in the collisions, and a fraction of the energy loss goes to other excitations. The element of randomness introduces a spread in the relationship between energy deposition and ionisation current, which is characterised by the Poisson distribution. The fraction of successful ionisations and other effects is swallowed up in the Fano factor F. We have

$$\text{Resolution} = R = 2.35\frac{\Delta N}{N} = 2.35\sqrt{\frac{Fw}{\Delta E}}.$$