Critical Speeds for Simple Rotors

Tabular Summary

In rotordynamics, a critical speed can be most simply defined as rotational speed that coincides with one or more natural frequencies of the rotor itself. In practice, this will occur in the presence of dynamic forces induced by the operation of the rotor (ex. unbalance). Every rotor will pass through its lowest critical speed during or upon start-up and shutdown. This an especially dangerous condition for the rotor due the resonance response induced, which, if left unmitigated by damping effects or otherwise is prolonged, can result in the catastrophic failure of the rotor. It is with this mind that the below relations and tables have been compiled, so as to provide an easy reference for lateral and torsional critical speeds for simple rotor (i.e., shaft-disc) systems. It is from these simplified systems that more exact calculations for critical speeds of complicated rotors and machine trains can be "built up" and ultimately determined.

Critical speeds of rotors are often expressed in revolutions per minute (RPM). In the most general sense, this takes the following form:

$$N_c = 60 f_n = \frac{60}{2\pi}\omega_n = \frac{60}{2\pi}\sqrt{\frac{k}{m}} \qquad (1)$$

The parameters in (1) are summarized in the below table:

Table 1: Critical speed parameters

Rotor Speed (RPM)	Frequency (Hz)	Frequency (rad/s)	Stiffness (N/m)	Mass (kg)
N	f	ω	k	m

It is typically assumed in preliminary critical speed calculations that all of the rotor mass is consolidated within the disk, and that all of the rotor stiffness is consolidated within the shaft and the bearings. This assumption yields accurate predictions so long as the mass of the disc (the disc could be an impeller, for example), is at least ten times greater than the mass of the shaft. This is very often the case, so this is very often a reasonable assumption.

An even more general form of the critical speed, which is applicable to all cases is:

$$N_c = \frac{60}{2\pi}\sqrt{\frac{g}{x_{st}}} \qquad (2)$$

Where $g = 9.81\ m/s^2$ is the local acceleration due to gravity, and x_{st} is the maximum static deflection, which can almost always be looked up in standard beam deflection tables or measured directly in the field or shop.

Lateral Critical Speeds

The term "lateral vibration" in rotordynamics is taken to mean radial plane orbital motion of the rotor spin axis. A more straightforward way to think about this is to imagine yourself looking at a disc on a shaft from the side and observing it bouncing up and down as it rotates. This repeated up and down motion during rotation is the lateral vibration of the rotor. A good way of visualizing this is the mode shape:

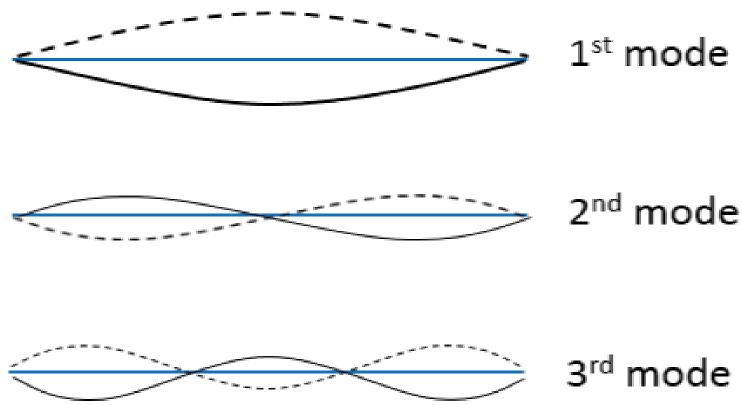


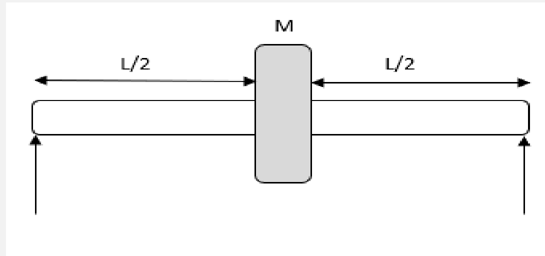
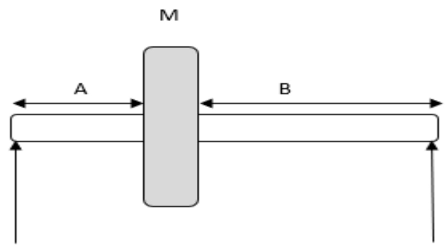
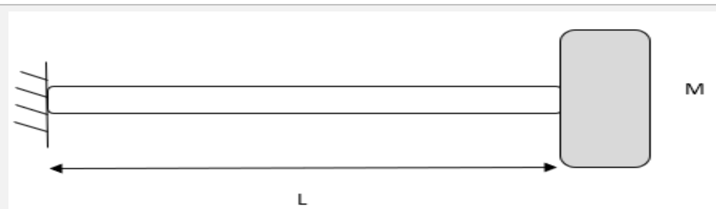
Figure 1: Representative mode shapes for single-mass rotor

The mode shape is simply the mathematical representation of the physical pattern of vibration. The two-dimensional mode shapes in Figure 1 are representative of those for a single-mass, uniformly flexible rotor mounted on rigid bearings vibrating laterally during operation.

This is a good approximation in many cases, one example being a single-stage between bearing centrifugal pump mounted on rolling-element bearings. The major factor that determines how "rigid" the bearings are is their stiffness relative to that of the shaft. Notice that a pattern is emerging here: in the case of whether to consider the mass of the shaft in calculating the rotor critical speed, we first had to stipulate that the disc was at least ten times as heavy as the shaft. Likewise, we also stipulate that in order for the bearing to be considered rigid, they must be significantly stiffer than the shaft.

It is also possible to have the situation inverted; a "rigid" rotor can operate at very high speeds in flexible bearings (i.e., fluid-film/tilting-pad journal) without ever reaching the first critical speed. Such is the case with many cryogenic turboexpanders. Experience is best teacher when it comes to deciding which conditions to apply to which type of system. Tabulated below are the formulas for the lowest (first) lateral critical speeds for three cases of single-shaft, single-disc rotors: simply supported/central mass, simply supported non-central mass, and cantilever/end-mass. These three cases, aside from having considerable utility in their own right, also form the building blocks for more complicated cases E and I are taken to be the shaft elastic modulus and the second moment of area respectively.

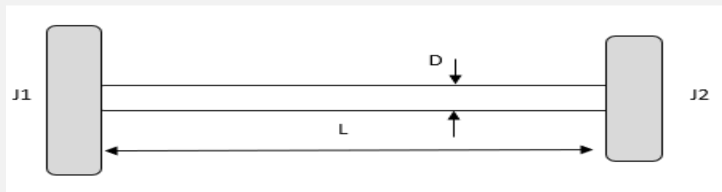
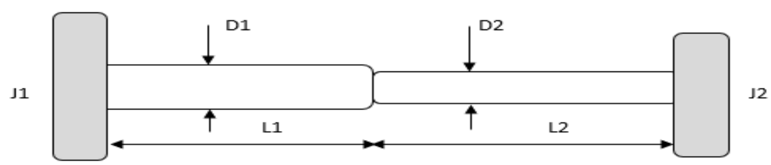
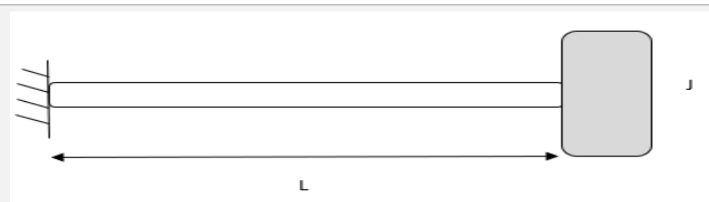
Table 2: Lowest lateral critical speeds for (top to bottom) central mass, non-central mass, & overhung rotors

Critical Speed (RPM)	Rotor Type/Case
$N_c = \frac{60}{2\pi}\sqrt{\frac{48EI}{ML^3}}$	
$N_c = \frac{60}{2\pi}\sqrt{\frac{3EI}{MA^2B^2}}$	
$N_c = \frac{60}{2\pi}\sqrt{\frac{3EI}{ML^3}}$	

Torsional Critical Speeds

The term “torsional vibration” in rotordynamics is taken to mean the angular oscillatory twisting of a rotor about its centerline and superimposed on its angular spin velocity. One way to visualize this is to stretch a rubber band and twist it at opposite ends while holding it taught. The resulting repeated cyclical twisting observed is the torsional vibration. Long/flexible rotors, such as ship propeller shafts, and rotors coupled to variable frequency drives (VFDs) are especially vulnerable to the destructive effects of torsional resonance during operation. Accordingly, a torsional rotordynamic analysis (RDA) should always be carried out during the design phase to identify torsional critical speeds for the machine train in question. Tabulated below are the formulas for the lowest torsional critical speeds for three cases: two discs on a uniform shaft, two discs on a stepped solid shaft, and a single disc on the end of a cantilever shaft.

Table 3: Lowest torsional critical speeds for (top to bottom) uniform shaft, stepped shaft, & overhung shaft

Critical Speed (RPM)	Rotor Type/Case
$N_c = \frac{60}{2\pi}\sqrt{\frac{GJ_p}{L}\left(\frac{J_1+J_2}{J_1J_2}\right)}$	
$N_c = \frac{60}{2\pi}\sqrt{\frac{GJ_{p1}}{L_1+L_2\left(\frac{D_1}{D_2}\right)^4}\left(\frac{J_1+J_2}{J_1J_2}\right)}$	
$N_c = \frac{60}{2\pi}\sqrt{\frac{GJ_p}{LJ}}$	

Where for Table 3:

$$J_p = \frac{\pi D_s^4}{32} \quad (Solid\ Shaft)$$

$$J_p = \frac{\pi}{32}(D_{so} - D_{si})^4 \quad (Hollow\ Shaft)$$

$$J = \frac{1}{8}MD^2 \quad (Disc\ Mass\ Moment\ of\ Inertia)$$

$$I = \frac{\pi D_s^4}{64} \quad (Second\ Moment\ of\ Area, Solid\ Shaft)$$

$$I = \frac{\pi}{64}(D_{so} - D_{si})^4 \quad (Second\ Moment\ of\ Area, Hollow\ Shaft)$$

Where the "o" and "i" subscripts correspond to the outer and inner shaft diameters, respectively. For convenience, the parameters that define lateral/torsional critical speeds in Tables (2) and (3) are collected below in Table 4:

Table 4: Summary of critical speed parameters

Elastic Modulus (N/m^2)	**Shear Modulus (N/m^2)**	**Disc Mass (kg)**	**Shaft/Section Length (m)**	**Mass Moment of Inertia ($kg*m^2$)**	**Polar Second Moment of Area (m^4)**	**Second Moment of Area for Shaft (m^4)**	**Disc Diameter (m, mm)**	**Shaft Diameter (m, mm)**
E	G	M	L	J	J_p	I	D	D_s

It is often the case for the end user and/or in the field that the values for E and I in the above table will not be easily retrievable. In this case the undamped lateral critical speeds for the three cases in Table 1 can be calculated based upon the weight of the disc and local length factors using the formulas below in Table 5 (U.S. Customary Units):

Figure 5: Lowest lateral critical speeds for Table 1 rotors in terms of weight/length factors

Critical Speed (RPM)	**Corresponding Case**
$N_c = 1{,}550{,}500 \frac{D_s^2}{L\sqrt{WL}}$	Simply Supported/Central Mass
$N_c = 387{,}000 \frac{D_s^2}{AB}\sqrt{\frac{L}{W}}$	Simply Supported/Non-Central Mass
$N_c = 387{,}000 \frac{D_s^2}{L\sqrt{WL}}$	Cantilever/End-Mass

Where the shaft diameter (D), distance between bearing centers (L), and distances from bearing to load (A,B) are in inches, and the applied load (W) is in pounds. The last case we'll take up, following from Table 5, is the lowest lateral critical speed for that of the shaft itself, assumed to be simply supported:

$$N_c = 4{,}760{,}000 \frac{D_s}{L^2} \qquad (Shaft\ Critical\ Speed, RPM)$$

Used properly, the above formulas can be valuable tools in estimating critical speeds, either as a precursor to a full-fledged rotodynamics analysis (RDA), or, as is often the case, as a "sanity check" to verify computational results and/or in the preliminary design stages.