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ACTIONS OF COMPACT, CONNECTED LIE GROUPS ON SPIN MANIFOLDS

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ABSTRACT. We study actions of compact connected Lie groups on Spin manifolds with only two fixed points obtaining results about the equivalence of the representations of the group on the tangent spaces at the fixed points.

For semifree actions we get: Let X be a connected 2t-dimensional Spin manifold. Let G be a compact connected Lie group acting differentiably on X, with only two fixed points and semifreely. Then the two representations of G at the fixed points are equivalent.

1. INTRODUCTION.

As the title indicates our objective is to study actions of compact, connected Lie groups on Spin Manifolds. We consider the specific case of an action with only two fixed points attempting to obtain information on the relationship between the representations of the group on the tangent spaces at the fixed points.

The basic tools for this work are the results of Atiyah and Hirzebruch [3] (see (2.1) and (2.2)).

In the case of a semi-free actions our results generalize a well known theorem of Milnor ([1], p.478) for the particular case of connected groups.

Our notation will be essentially from [1], [2] and [3].
We also indicate by F(H,X) the fixed point set of H in X and by F(H,x,X) the connected component of F(H,X) containing the point $x \in X$.

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2. SPIN MANIFOLDS.

Let X be a compact, oriented n-manifold with a Riemannian metric. Let Q be the oriented orthonormal frame bundle of the cotangent bundle to X.

A Spin-structure of X is a principal $\mathrm{Spin}(n)$ bundle P over X with a covering map $\pi: P \longrightarrow Q$ of degree 2 such that the diagram

$$\begin{array}{ccccc} P \times \mathrm{Spin}(n) & \longrightarrow & P & \searrow & \\ \downarrow \pi \times \lambda & & \pi \downarrow & & X \\ Q \times SO(n) & \longrightarrow & Q & \nearrow & \end{array}$$

commutes, where $\lambda: \mathrm{Spin}(n) \longrightarrow SO(n)$ is the standard double covering and the horizontal arrows are the actions on the bundles. A compact, orientable manifold with a Spin-structure is called a Spin Manifold.

If we have an action of a compact Lie group G on X $(n=2t)$ by orientation preserving isometries and an action of G on P commuting with the action of $\mathrm{Spin}(2t)$ from the right and compatible with π we can consider its Spinor-index $\mathrm{Spin}(G,X)$ which is by definition an element in $R(G)$ ([1], p.479, [3], p.18).

The evaluation of $\mathrm{Spin}(G,X)$ in an element g of G is denoted by $\mathrm{Spin}(g,X)$.

We shall indicate now two results from [3] and one from [1] which are necessary for our work.

(2.1) THEOREM. (Atiyah-Hirzebruch) *Let X be a connected Spin Manifold of dimension $2t$ and G a compact connected Lie group acting non trivially on X and acting on its Spin-structure P. Then* $\mathrm{Spin}(G,X) = 0$.

(2.2) THEOREM. (Atiyah-Hirzebruch) *Let X be a compact, connected, orientable differentiable manifold with Riemannian metric on which a compact connected Lie group G operates effectively by isometries. Suppose P is a Spin-structure for X. Then there exist canonically a Lie group G_1 and a homomorphism $h: G_1 \longrightarrow G$, which is either identity or a double covering, such that G_1 acts on P inducing via h the given action of G on X.*

(2.3) THEOREM. (Atiyah-Bott) *Suppose that $f: X \longrightarrow X$ is an isome-*

try of the 2t-*dimensional compact oriented manifold* X *with only isolated fixed points* x. *Suppose further that* X *admits a Spin-structure* P *and that* f *has a lifting* $\hat{f}$ *to this Spin-structure.*

The Spin-number $Spin(\hat{f},X)$ *is then given by the expression*

$$Spin(\hat{f},X) = \sum \nu(x)$$

where x *ranges over the fixed points of* f *and*

$$\nu(x) = \pm (i/2)^t \pi_{j=1}^{j=t} \operatorname{cosec}(\theta_j/2)$$

where $\theta_1 \ldots \theta_t$ *is a coherent system of angles for* df_x ([1],p.473).

We can prove now

(2.4) THEOREM. *Let* X *be a connected* 2t-*dimensional Spin manifold and* G *be a compact, connected Lie group acting differentiably on* X *with only two fixed points* x *and* y. *Assume that each subgroup of prime order* H *of* G *satisfies the following condition:*

i) *Either* $F(H,X) = \{x,y\}$ *or* $F(H,x,X) = F(H,y,X)$

Then the representations of the group on the tangent spaces at the fixed points are equivalent.

Proof. We can think that G acts effectively and by isometries. From (2.2) we obtain a Lie group G_1 (which is connected by construction [3], p.22) and a homomorphism $h:G_1 \longrightarrow G$, which is the identity or a double covering, such that G_1 acts on P and induces via h the action of G on X.

The action of G_1 on X is by orientation preserving isometries and its action on P commutes with the action of Spin(2t) from the right and is compatible with π. We can consider then the Spinor-index of the action of G_1 on X and as a consequence of (2.1) we have $Spin(G_1,X) = 0$.

Since we want to prove the equivalence of the representations and the elements of prime order form a dense set in G, it is enough to prove the equivalence of the restrictions of the representations to each cyclic subgroup of prime order p. Let H be a cyclic subgroup of prime order $p \neq 2$ of G with generator f and assume that $F(H,X) = \{x,y\}$.

Put $H_1 = h^{-1}(H)$, then $h^{-1}(f) = \{\hat{f}_1,\hat{f}_2\}$ (we could have one or two elements in this set). Take $\hat{f}_1$, then $Sign(\hat{f}_1,X) = 0$ and we can apply (2.3) to obtain a condition on the coherent systems of angles of df_x and df_y. This condition will suffice to prove the equivalence of the representations.

Let us identify H with Z_p and f with the preferred generator. The expression of $\nu(x)$ in (2.3) can be written as

$$(2.5) \qquad \nu(x) = \pm \pi_{j=1}^{j=t} \left[\frac{\exp(i\theta_j/2) - \exp(-i\theta_j/2)}{(1 - \exp i\theta_j)(1 - \exp(-i\theta_j))} \right]$$

Note that $\exp(i\theta_j) \neq 1$ since x is isolated fixed point of f.
Let us put now $h_k = \exp(-i\theta_k/2)$. We have

$$(2.6) \qquad \nu(x) = \pm \pi_{k=1}^{k=t} \left[\frac{h_k}{h_k^2 - 1} \right]$$

Now the condition $\mathrm{Sign}(\hat{f}_1, X) = 0$ gives $\nu(x) + \nu(y) = 0$ which in turn implies $|\nu(y)|^2 . |\nu(x)|^{-2} = 1$.
In $|\nu(x)|^2$ we have all the eigenvalues of df_x, therefore

$$(2.7) \qquad |\nu(x)|^2 = \pi_{s \varepsilon Z_p^*} \left[\frac{h^s}{h^{2s} - 1} \right]^{a_s^x}$$

where a_s^x is the number of eigenvalues of df_x which are equal to h^{2s} ($h = \exp(i\pi/p)$) and Z_p^* is the set of congruence classes modulo p which are prime to p.
We have then

$$(2.8) \qquad \pi_{s \varepsilon Z_p^*} \left[\frac{h^s}{h^{2s} - 1} \right]^{(a_s^y - a_s^x)} = 1$$

Put now $a_s = a_s^y - a_s^x$ and write

$$(2.9) \qquad 1 = \pi_{s \varepsilon Z_p^*} h^{sa_s} (h^{2s} - 1)^{-a_s} = h^q \pi_{s \varepsilon Z_p^*} (h^{2s} - 1)$$

where $q = \sum_{s \varepsilon Z_p^*} sa_s$.

Clearly $q \equiv 0 \pmod p$, then (2.9) can be written as

$$(2.10) \qquad \pi_{s \varepsilon Z_p^*} (h^{2s} - 1)^{-a_s} = 1$$

Put now $h^2 = t$, $b_s = -a_s$ and write

$$(2.11) \qquad (-1)^{\sum b_s} \pi_{s \varepsilon Z_p^*} (1 - t^s)^{b_s} = 1$$

Since $\sum b_s = 0$ we have

$$(2.12) \qquad \prod_{s \in Z_p^*} (1 - t^s)^{b_s} = 1$$

Now by a theorem of Kummer ([1], p.477) we have $b_s = 0$, $s \in Z_p^*$ and then $a_s = 0$.

We have proved then the equivalence of the restrictions of the representations to H in the case F(H,X) = {x,y}. If this is not the case then F(H,x,X) = F(H,y,X) and in this situation the equivalence is obvious. The theorem is then proved.

(2.13) REMARK. If the group G is a torus condition (i) of (2.4) can be replaced by

i) Either dim F(H,X) = 0 or some of the subspaces F(H,x,X), F(H,y,X) has positive dimension. (dim F(H,X) means the maximum of the dimensions of the components).

If the action of G is semifree, that is free outside the fixed point set and F(G,X) = {x,y}, the theorem can be written as

(2.14) THEOREM. *Let* X *be a connected* 2t-*dimensional Spin manifold. Let* G *be a compact connected Lie group acting differentiably on* X, *with only two fixed points and semifreely. Then the two representations of* G *at the fixed points are equivalent.*

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