CS 170 Tutorial #1

Invariants and Proofs of Correctness

Why do we use induction?

Property P(n): sum of first n natural numbers is 1/2*n*(n+1)
We want to prove P(n) for all natural numbers n.

Strategy:
Prove P(0), P(1), P(2), …

Better Strategy:
Use induction!
Choose Induction Hypothesis to be P

1. Base case: Prove P(0)
2. Induction case: P(k) => P(k+1)

Reasoning about algorithms with loops

```
x = c; y = 0;
while (x > 0) {
  x--;
  y++;
}
```

Property: y equals c after the loop terminates

Strategy:
Compute state after iteration #1, iteration #2, …
Prove that state after last iteration has y = c

Better Strategy:
Use induction (over number of iterations)

- <u>Base case</u>: Prove induction hypothesis holds on loop entry
- <u>Induction case</u>: Assuming induction hypothesis holds after k iterations, prove it holds after k+1 iterations

Step 1: Construct an Inductive Hypothesis

```
x = c; y = 0;
while (x > 0) {
  x--;
  y++;
}
```

We can generalize from examples…

- On loop entry: $x = c$, $y = 0$
- After iteration 1: $x = c - 1$, $y = 1$
- After iteration 2: $x = c - 2$, $y = 2$
- …

inductive hypothesis
$x + y = c$

Inductive Hypothesis
is the loop invariant!!!

Step 2: Prove that Loop Invariant is Inductive

```
x = c; y = 0;
while (x > 0) {
  x--;
  y++;
}
```

1. Base case: loop invariant $x + y = c$ holds on loop entry
   True

2. Inductive case:
   Assume loop invariant holds after k iterations:
   $y = k, x = c - y = c - k$
   After the (k+1)st iteration, $y = k + 1, x = c - k - 1$
   Therefore, $x + y = k + 1 + c - k - 1 = c$
   True

Step 3: Proving correctness property using loop invariant

```
x = c; y = 0;
while (x > 0) {
  x--;
  y++;
}
```

- Use loop invariant to prove correctness property that $y = c$ after loop terminates

After final iteration: $x = 0$

We also know our loop invariant holds: $x + y = c$

Therefore, $y = c$.

Practice Problems

- Divide into groups of 2-3

Problem 1

Consider the following piece of code:

```
y = 0;
for (i = 0; i <= n; i++) {
  y += 2^i;
}
return y;
```

What is the value of y after the loop termination?

(Hint : Try to find a loop invariant that holds at the start of each loop iteration)

Aside: For loops

```
for (i = 0; i <= n; i++) {
    // invariant: I(i) is true
    ... loop body ...
}
```

is equivalent to:

```
i := 0
loopstart:
  // invariant: I(i) is true
  if i > n: goto end
  ... loop body ...
  i := i+1
  goto loopstart
end
```

Step 1 : Run a few iterations

```
y = 0;
for (i = 0; i <= n; i++) {
  y += 2^i;
}
```

At the start of each iteration:

- $i = 0 : y_0 = 0$
- $i = 1 : y_1 = 1$
- $i = 2 : y_2 = 1 + 2 = 3$
- $i = 3 : y_3 = 1 + 2 + 4 = 7$
- $i = 4 : y_4 = 1 + 2 + 4 + 8 = 15$
- …

Any pattern?

Step 1 : Run a few iterations

```
y = 0;
for (i = 0; i <= n; i++) {
  y += 2^i;
}
```

At the start of each iteration:

- i = 0 : y_0 = 0 = $\mathbf{2^0 - 1}$
- i = 1 : y_1 = 1 = $\mathbf{2^1 - 1}$
- i = 2 : y_2 = 1 + 2 = 3 = $\mathbf{2^2 - 1}$
- i = 3 : y_3 = 1 + 2 + 4 = 7 = $\mathbf{2^3 - 1}$
- i = 4 : y_4 = 1 +2 + 4 + 8 = 15 = $\mathbf{2^4 – 1}$

Step 1 : Run a few iterations

```
y = 0;
for (i = 0; i <= n; i++) {
  y += 2^i;
}
```

At the start of each iteration:

- $i = 0$: $y_0 = 0 = \mathbf{2^{0} - 1}$
- $i = 1$: $y_1 = 1 = \mathbf{2^{1} - 1}$
- $i = 2$: $y_2 = 1 + 2 = 3 = \mathbf{2^{2} - 1}$
- $i = 3$: $y_3 = 1 + 2 + 4 = 7 = \mathbf{2^{3} - 1}$
- $i = 4$: $y_4 = 1 + 2 + 4 + 8 = 15 = \mathbf{2^{4} - 1}$

It looks like $y_i = 2^i - 1$ is a good candidate for loop invariant

Step 2 : Prove that loop invariant is inductive

- **Base case**

  $i = 0 : y_0 = 2^0 - 1 = 0$ ✔

- **Inductive step**

  Assume that at the start of the i-th iteration $y_i = 2^i - 1$

  Then, at the start of the (i+1)-th iteration we will have:

  $y_{i+1} = y_i + 2^i = 2^i - 1 + 2^i = 2 \times 2^i - 1 = 2^{i+1} - 1$ **Q.E.D.**

Step 3 : Loop invariant at the last iteration

- When the loop terminates i = n + 1. Thus after the loop execution we have:

$$\mathbf{y = 2^{n+1} - 1}$$

Problem 2: Binary Search

Binary Search

```
def binary_search(A, target):
  lo = 0
  hi = len(A) - 1
  while lo <= hi:
    mid = (lo + hi) / 2
    if A[mid] == target:
      return mid
    elif A[mid] < target:
      lo = mid + 1
    else:
      hi = mid - 1
```

You've all seen this a billion times.

But how do we prove that it's correct?

Given that A is sorted and A contains target, prove that binary_search(A, target) always returns target's index within A

Use Loop Invariants!!

Step 1: Hypothesize a Loop Invariant

```
def binary_search(A, target):
  lo = 0
  hi = len(A) - 1
  while lo <= hi:
    mid = (lo + hi) / 2
    if A[mid] == target:
      return mid
    elif A[mid] < target:
      lo = mid + 1
    else:
      hi = mid - 1
```

Say we're searching for 14 in the following array A

0	1	2	3	4	5	6
-5	***10***	***14***	***33***	***42***	***42***	***42***

1^{st} step: lo = 0, hi = 6, mid = 3

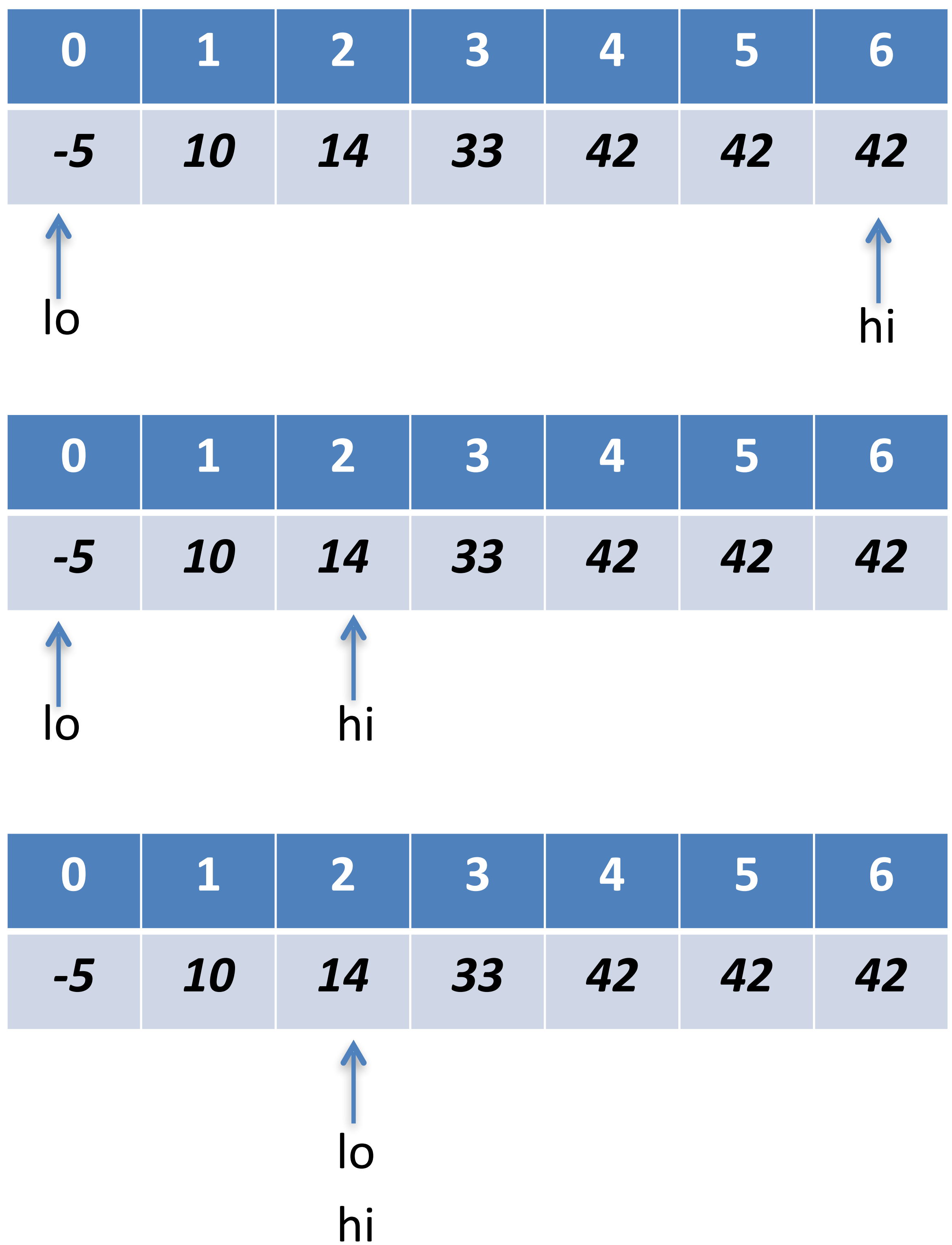
2^{nd} step: lo = 0, hi = 2, mid = 1

3^{rd} step: lo = 2, hi = 2, mid = 2

At each step of the while loop, **lo** and **hi** *surrounded* the actual location of where 14 is! This was always true!

THIS IS OUR LOOP INVARIANT.

Step 1: Construct Loop Invariant



At each iteration of the while loop,
lo and **hi** are such that:

A[lo] ≤ target ≤ A[hi]

Step 2: Prove that loop invariant is inductive

- Base Case: when the algorithm begins, lo = 0 and hi = len(A) - 1. **lo** and **hi** enclose ALL values, so **target** must be between **lo** and **hi**.
- Inductive Hypothesis: suppose at any iteration of the loop, **lo** and **hi** still enclose the **target** value.
- Inductive Step:
  – Case 1: If **A[mid] > target**, then the target must be between **lo** and **mid**
    - We update **hi = mid – 1**
  – Case 2: If **A[mid] < target**, then the target must be between **mid** and **hi**
    - we update **lo = mid + 1**
  – In either cases, we preserve the inductive hypothesis for the next loop

Step 3: Prove correctness property using loop invariant

- Notice for each iteration, **lo** always increases and **hi** always decreases. These value will converge at a single location where lo = hi.

- By the induction hypothesis, **A[lo] ≤ target ≤ A[hi].**

0	1	2	3	4	5	6
-5	***10***	***14***	***33***	***42***	***42***	***42***

↑ (index 2)

lo

hi

Food for thought: How will the proof change if **target** *isn't* in the array?

Problem 3: array reversal

In-place Array Reversal

```
//inputs: array A of size n
void reverse_array(int *A, int n):
  int i = (n - 1) / 2;
  int j = n / 2;
  int tmp;
  while (i >= 0 && j <= (n - 1))
    tmp = A[i];
    A[i] = A[j];
    A[j] = tmp;
    i--;
    j++;
```

Prove that array A of size n is reversed as a result of invoking reverse_array(A, n)

Step 1: Hypothesize a Loop Invariant

0	1	2	3	4	5	6
-5	*10*	*14*	*33*	*42*	*42*	*42*

i j

0	1	2	3	4	5	6
-5	*10*	*14*	*33*	*42*	*42*	*42*

i j

0	1	2	3	4	5	6
-5	*10*	*42*	*33*	*14*	*42*	*42*

i j

Before iteration of the while loop,
i and **j** are such that:

A[i+1 : j-1] is reversed

Or more formally,

new_A[i+1 : j-1] = reverse(old_A[i+1 : j-1])
where,
reverse([]) = []
reverse([a0]) = [a0]
reverse([a0, a1, ...]) = [reverse([a1,...]), a0]

Step 2: Prove that loop invariant is inductive

- Loop invariant: **A[i+1 : j-1] is reversed**

- Base Case: Upon loop entry, **j - 1 < i + 1**. Invariant holds trivially.

- Inductive case:
  At the start of k-th iteration, assume that A[i+1 : j-1] is reversed.
  The loop body swaps A[i] and A[j], decrements i and increments j.
  Therefore, at the start of (k+1)-th iteration, we can prove that A[i+1 : j-1] is reversed.

Step 3: Prove correctness property using loop invariant

0	1	2	3	4	5	6
-5	*10*	*14*	*33*	*42*	*42*	*42*

i j

- After the loop terminates, i = -1 and j = n.
- Loop invariant tells us that A[i+1 : j-1] is reversed.
- Therefore, A[0:n-1] is reversed.
  QED