Plane Projective transformation

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1)$,$h(x_2)$,$h(x_3)$ do. (i.e. maps lines to lines in P^2)

Theorem:

A mapping h:$P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 reprented by a vector x it is true that $h(x)=\mathbf{H}x$

Definition: Planar Projective transformation: linear transformation on homogeneous 3 vectors represented by a non singular matrix H

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $\mathrm{x}' = \mathbf{H}\,\mathrm{x}$

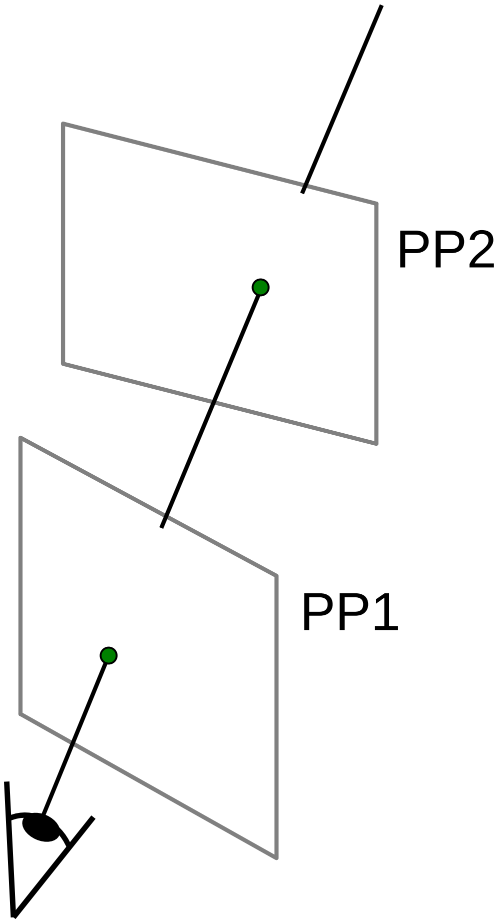
8DOF

- projectivity=collineation=projective transformation=homography
- Projectivity form a group: inverse of projectivity is also a projectivity; so is a composition of two projectivities.

Homography

- A: Projective – mapping between any two PPs with the same center of projection
  - rectangle should map to arbitrary quadrilateral
  - parallel lines aren't
  - but must preserve straight lines
  - same as: unproject, rotate, reproject
- called Homography

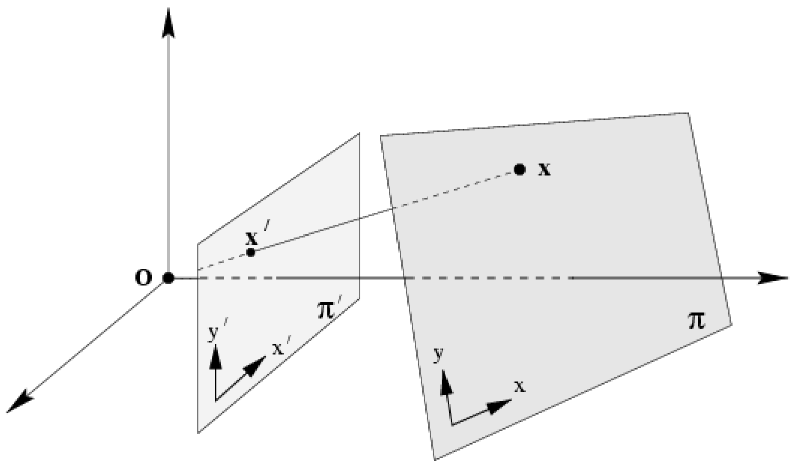
$$\underbrace{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}}_{\mathbf{p'}} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{p}}$$



To apply a homography **H**

- Compute **p'** = **Hp** (regular matrix multiply)
- Convert **p'** from homogeneous to image coordinates

Projection along rays through a common point, (center of projection) defines a mapping from one plane to another



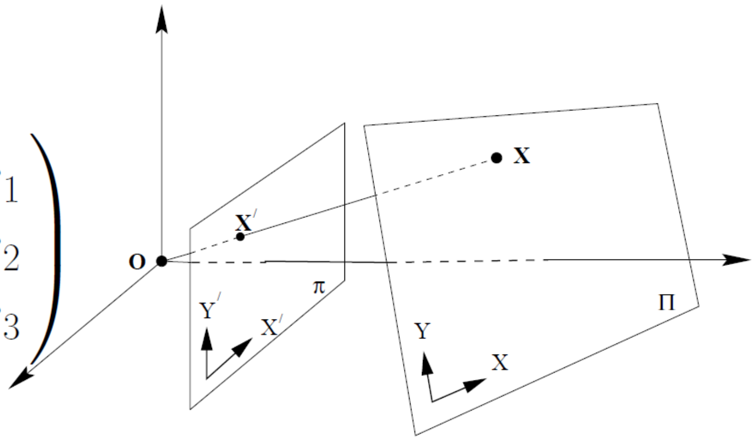
•Central projection maps points on one plane to points on another plane
•Projection also maps lines to lines : consider a plane through projection center that intersects the two planes ➔ lines mapped onto lines ➔ Central projection is a projectivity ➔

central projection may be expressed by
x’=Hx

(application of theorem)

Projective transformations continued

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



or $\mathbf{x}' = \mathtt{H}\mathbf{x}$, where $\mathtt{H}$ is a 3×3 non-singular homogeneous matrix.

- This is the most general transformation between the world and image plane under imaging by a perspective camera.
- It is often only the 3×3 form of the matrix that is important in establishing properties of this transformation.
- A projective transformation is also called a "homography" and a "collineation".
- $\mathtt{H}$ has 8 degrees of freedom.

Four points define a projective transformation

Given n point correspondences $(x, y) \leftrightarrow (x', y')$

Compute $\mathtt{H}$ such that $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$

- Each point correspondence gives two constraints

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}, \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

and multiplying out generates two linear equations for the elements of $\mathtt{H}$

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

- If $n \geq 4$ (no three points collinear), then $\mathtt{H}$ is determined uniquely.
- The converse of this is that it is possible to transform any four points in general position to any other four points in general position by a projectivity.

Example 1: Removing Perspective Distortion

Given: the coordinates of four points on the scene plane

Find: a projective rectification of the plane

- This rectification does not require knowledge of any of the camera's parameters or the pose of the plane.
- It is not always necessary to know coordinates for four points.

Removing projective distortion

Central projection image of a plane is related to the originial plane via a projective transformation ➔ undo it by applying the inverse transformation

Let (x,y) and (x’,y’) be inhomogeneous Coordinates of a pair of matching points x and x’ in world and image plane

select four points in a plane with known coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

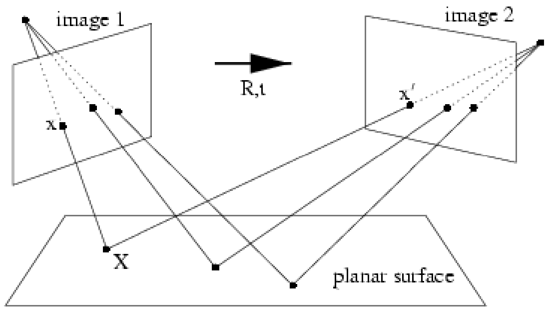
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary, better ways to compute (see later)

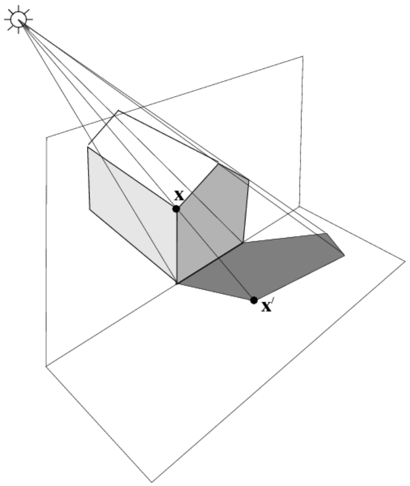
Sections of the image of the ground are subject to another projective distortion ➔

need another projective transformation to correct that.

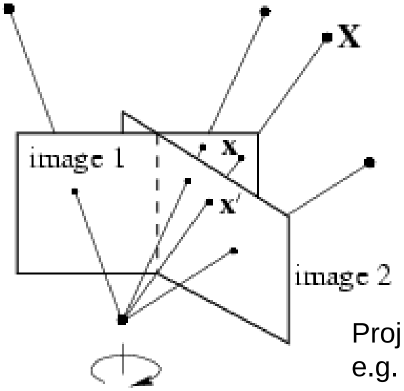
More examples



Projective transformation between two images
Induced by a world plane ➔ concatenation of two
Pojective transformations is also a proj. trans.

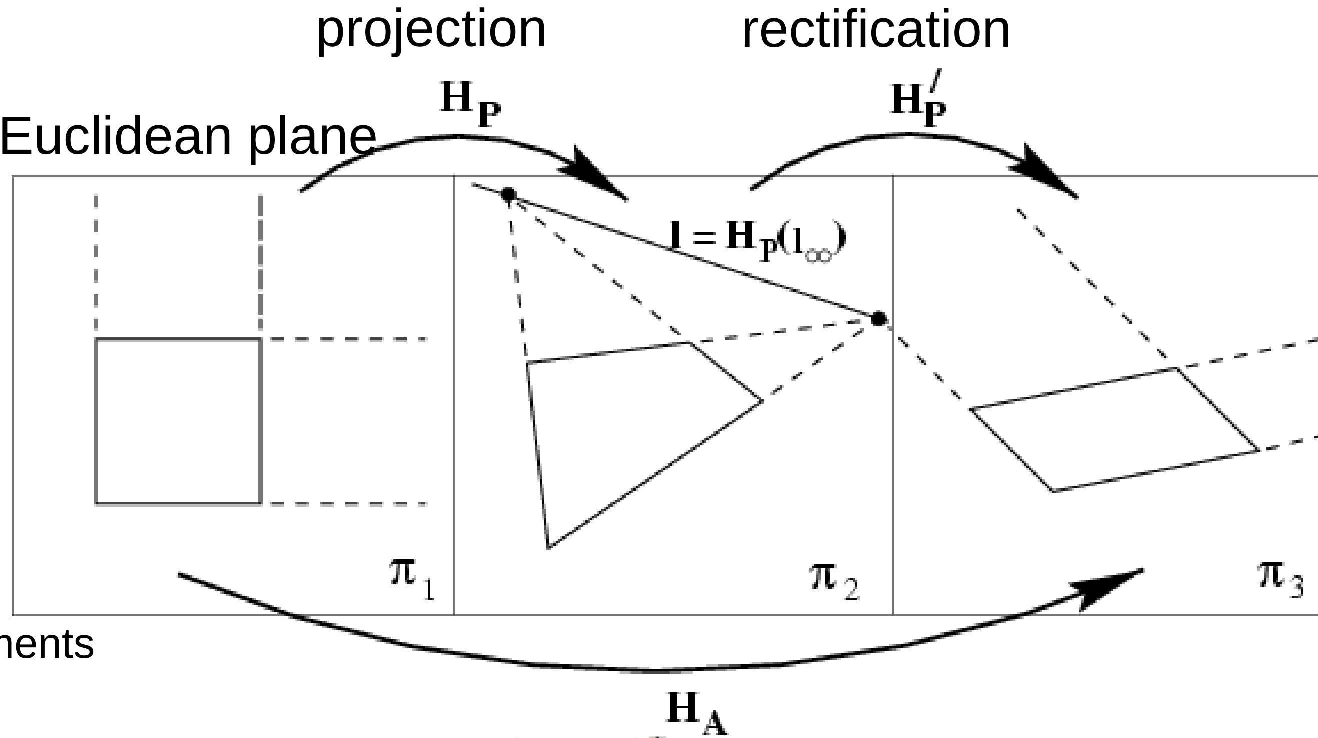


Proj. trans. Between the image of a plane
(end of the building) and the image of its
Shadow onto another plane (ground plane)



Proj. trans. Between two images with the same camera center
e.g. a camera rotating about its center

Affine properties from images



Two step process:

1. Find l the image of line at infinity in plane 2
2. Transform l to its canonical position $(0,0,1)^T$ by plugging into H'p and applying it to the entire image to get a "rectified" image
3. Make affine measurements on the rectified image

If the imaged line at infinity is the line $\mathbf{l} = (l_1, l_2, l_3)^\top$, then provided $l_3 \neq 0$ a suitable projective point transformation which will map $\mathbf{l}$ back to $\mathbf{l}_\infty = (0, 0, 1)^\top$ is

$$\text{H'p} = \mathtt{H} = \mathtt{H}_\mathtt{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \tag{2.19}$$

Fig. 2.12. **Affine rectification.** *A projective transformation maps* $\mathbf{l}_\infty$ *from* $(0,0,1)^\top$ *on a Euclidean plane* π_1 *to a finite line* $\mathbf{l}$ *on the plane* π_2. *If a projective transformation is constructed such that* $\mathbf{l}$ *is mapped back to* $(0,0,1)^\top$ *then from result 2.17 the transformation between the first and third planes* must be an affine transformation *since the canonical position of* $\mathbf{l}_\infty$ *is preserved. This means that affine properties of the first plane can be measured from the third, i.e. the third plane is within an affinity of the first.*

Affine rectification

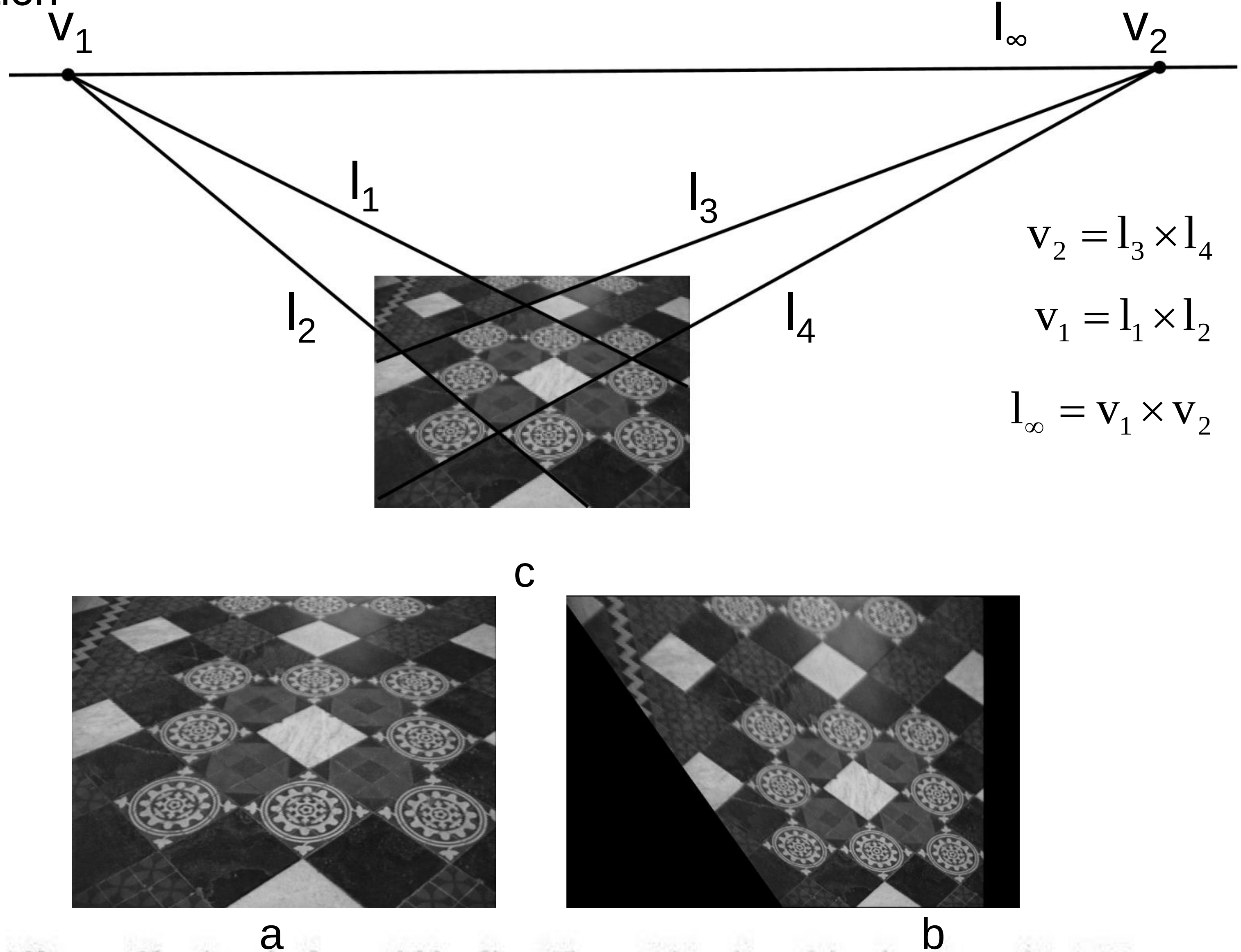
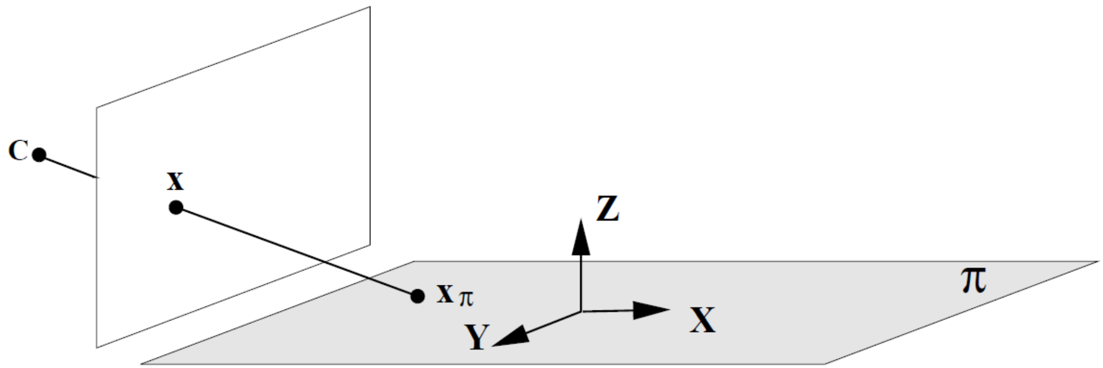


Fig. 2.13. **Affine rectification via the vanishing line.** *The vanishing line of the plane imaged in (a) is computed (c) from the intersection of two sets of imaged parallel lines. The image is then projectively warped to produce the affinely rectified image (b). In the affinely rectified image parallel lines are now parallel. However, angles do not have their veridical world value since they are affinely distorted. See also figure 2.17.*

Plane projective transformations



Choose the world coordinate system such that the plane of the points has zero $\mathtt{Z}$ coordinate. Then the 3×4 matrix $\mathtt{P}$ reduces to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{14} \\ p_{21} & p_{22} & p_{24} \\ p_{31} & p_{32} & p_{34} \end{bmatrix} \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ 1 \end{pmatrix}$$

which is a 3×3 matrix representing a general plane to plane projective transformation.

Action of projective camera on planes

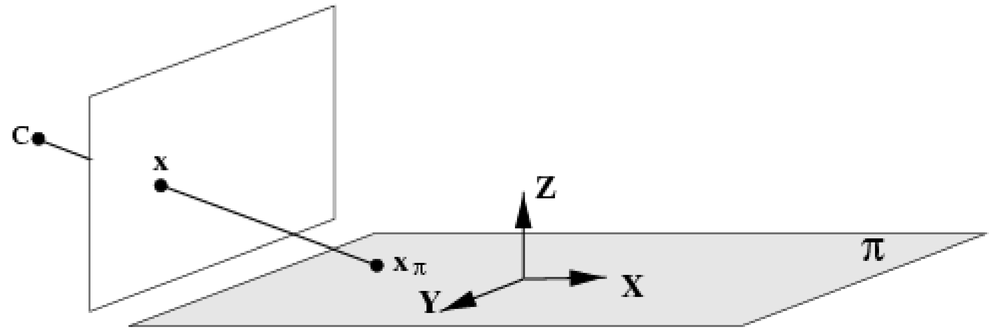


Fig. 8.1. **Perspective image of points on a plane.** *The XY-plane of the world coordinate frame is aligned with the plane π. Points on the image and scene planes are related by a plane projective transformation.*

Choose the world coordinate frome so that X-Y plane corresponds to a plane π in the scene such that points
In the scene plane have zero Z coordinate.

$$x = PX = [p_1 p_2 p_3 p_4] \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = [p_1 p_2 p_4] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

The map between the point on plane π and its image is a general planar homography: $x = H x_\pi$; H is 3x3

The most general transformation that can occur between a scene plane and an image plane under perspective imaging is a plane projective transformation

Action of projective camera on lines

forward projection

A line in 3D space projects to a line in the image.
Suppose A and B are points in space with their projections being a, and b under P; Then a point on a line that joins A and B i.e. X(μ) = A + μ B
Is the point:

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

Which is on the line joining a and b.

back-projection: Plane $\Pi = P^T l$

Set of points in space mapping to line l via the camera matrix P is plane $P^T l$

Line and camera center define a plane; image of the line is the intersection of this plane with image plane.

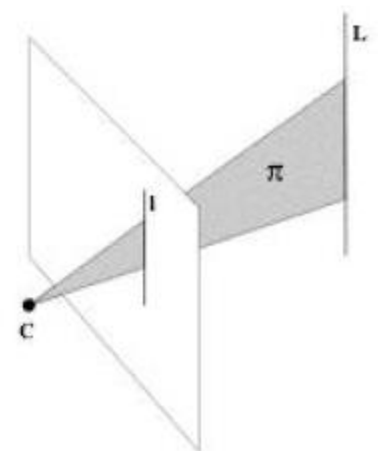


Fig. 8.2. **Line projection.** *A line* **L** *in 3-space is imaged as a line* l *by a perspective camera. The image line* l *is the intersection of the plane* π, *defined by* **L** *and the camera centre* **C**, *with the image plane. Conversely an image line* l *back-projects to a plane* π *in 3-space. The plane is the "pull-back" of the line.*

The importance of the camera center

- Object in 3 space and camera center define “set of rays”
- An image is obtained by intersecting these rays with a plane
- Images obtained with the same camera center may be mapped to one another by a plane projective transformation

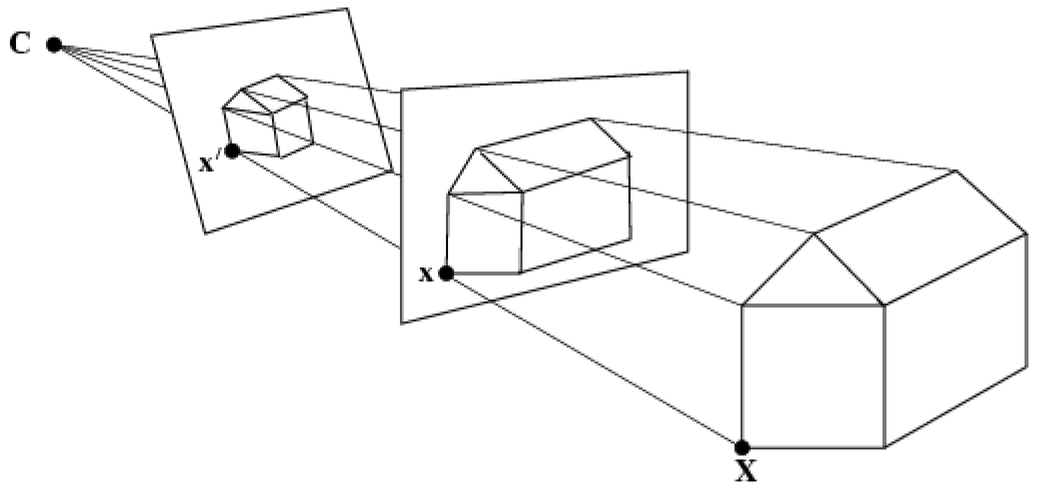


Fig. 8.5. **The cone of rays with vertex the camera centre.** *An image is obtained by intersecting this cone with a plane. A ray between a 3-space point* **X** *and the camera centre* **C** *pierces the planes in the image points* **x** *and* **x'**. *All such image points are related by a planar homography,* $\mathbf{x}' = \mathtt{H}\mathbf{x}$.

Assume two cameras
Same camera center

$$P = KR[I \mid -\tilde{C}], P' = K'R'[I \mid -\tilde{C}]$$

$$P' = K'R'(KR)^{-1}P$$

$$x' = P'X = K'R'(KR)^{-1}PX = K'R'(KR)^{-1}x$$

This is the 3x3 planar
Homography relating x and x'

$$x' = Hx \text{ with } H = K'R'(KR)^{-1}$$

Moving the image plane (zooming)

Changing the focal length, can be approximated as displacement of image plane along principal axis

x and x' are images of a point X before and after zooming respectively

Inhomogeneous principal point

$$x = K[I \mid 0]X$$
$$x' = K'[I \mid 0]X = K'\left(K\right)^{-1} x$$

x' = H x where

Assume only focal length
Is different between K and K'

$$H = K'\left(K\right)^{-1} = \begin{bmatrix} kI & (1-k)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix}$$

$$k = f / f'$$

$$K' = \begin{bmatrix} kI & (1-k)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix} K = \begin{bmatrix} kI & (1-k)\tilde{x}_0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} A & \tilde{x}_0 \\ 0^T & 1 \end{bmatrix}$$
$$= \begin{bmatrix} kA & \tilde{x}_0 \\ 0^T & 1 \end{bmatrix} = K \begin{bmatrix} kI & 0 \\ 0^T & 1 \end{bmatrix}$$

Conclusion: the effect of zooming by a factor of k is to multiply the Calibration matrix K on the right by diag(k,k,1)

Camera rotation

- Pure rotation about camera center.
- x and x' image of a point X before and after rotation.

$$x = K[I \mid 0]X$$

$$x' = K[R \mid 0]X = KRK^{-1}x$$

x'= H x where:

$$H = KRK^{-1}$$

H is called conjugate rotation: same eigen values, up to a scale factor, μ, of the rotation

Matrix: $\{\mu, \mu e^{i\theta}, \mu e^{-i\theta}\}$; So, phase of complex eigenvalues of H can be used to find rotation.

Can also be shown that the eigenvector of H corresponding to the real eignenvalue is the vanishing point of the rotation axis

B A C

click on 4 sets of correspondence between A and B to determine H; theta = 4.66 degrees. vanishing point of rotation axis is $(0,1,0)^T$, i.e. virtually at infinity in the y direction; so rotation axis is parallel to y direction

Fig. 8.6. *Between images (a) and (b) the camera rotates about the camera centre. Corresponding points (that is images of the same 3D point) are related by a plane projective transformation. Note that 3D points at different depths which are coincident in image (a), such as the mug lip and cat body, are also coincident in (b), so there is no motion parallax in this case. However, between images (a) and (c) the camera rotates about the camera centre* and *translates. Under this general motion coincident points of differing depth in (a) are imaged at different points in (c), so there* is *motion parallax in this case due to the camera translation.*

Image warping with homographies

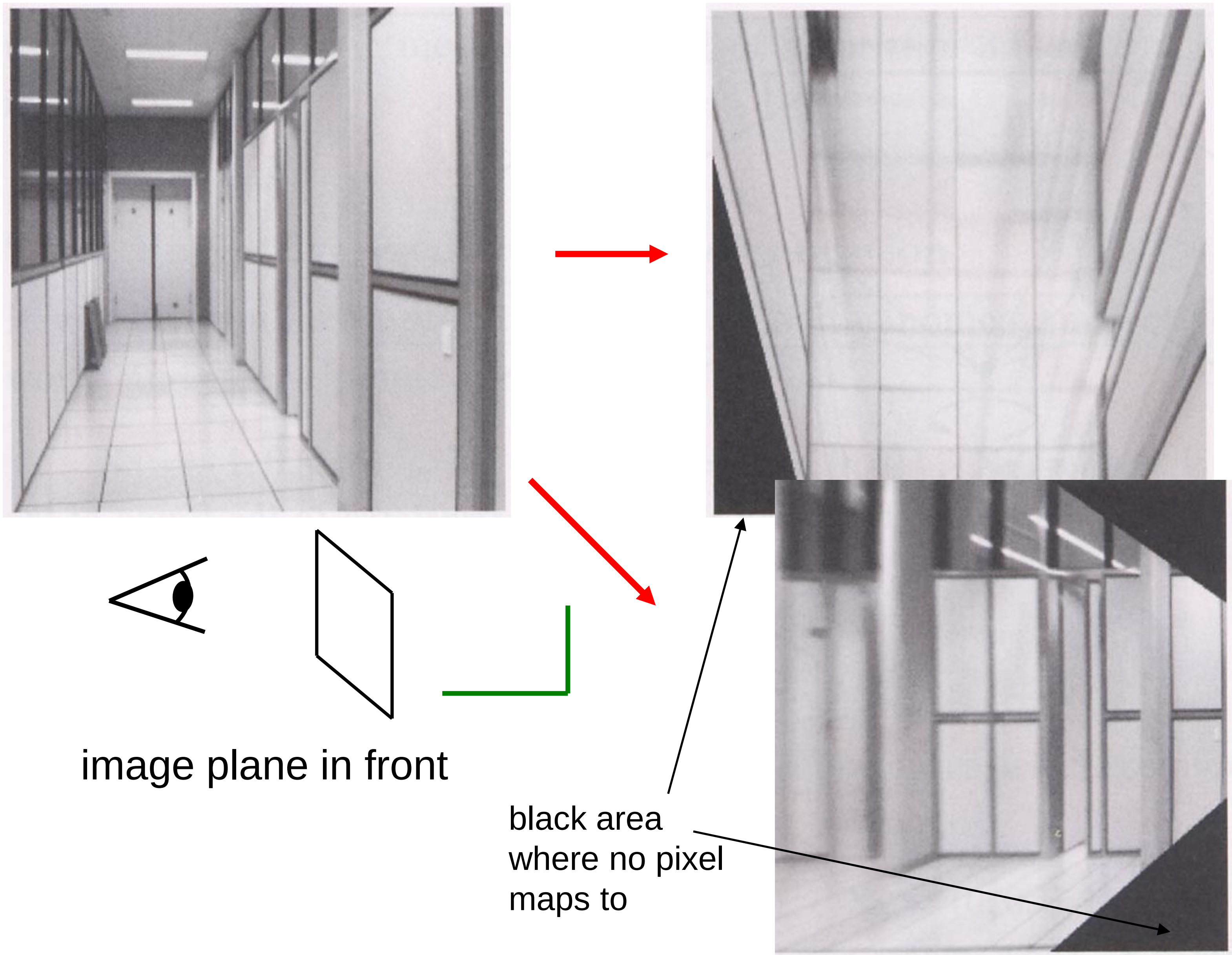
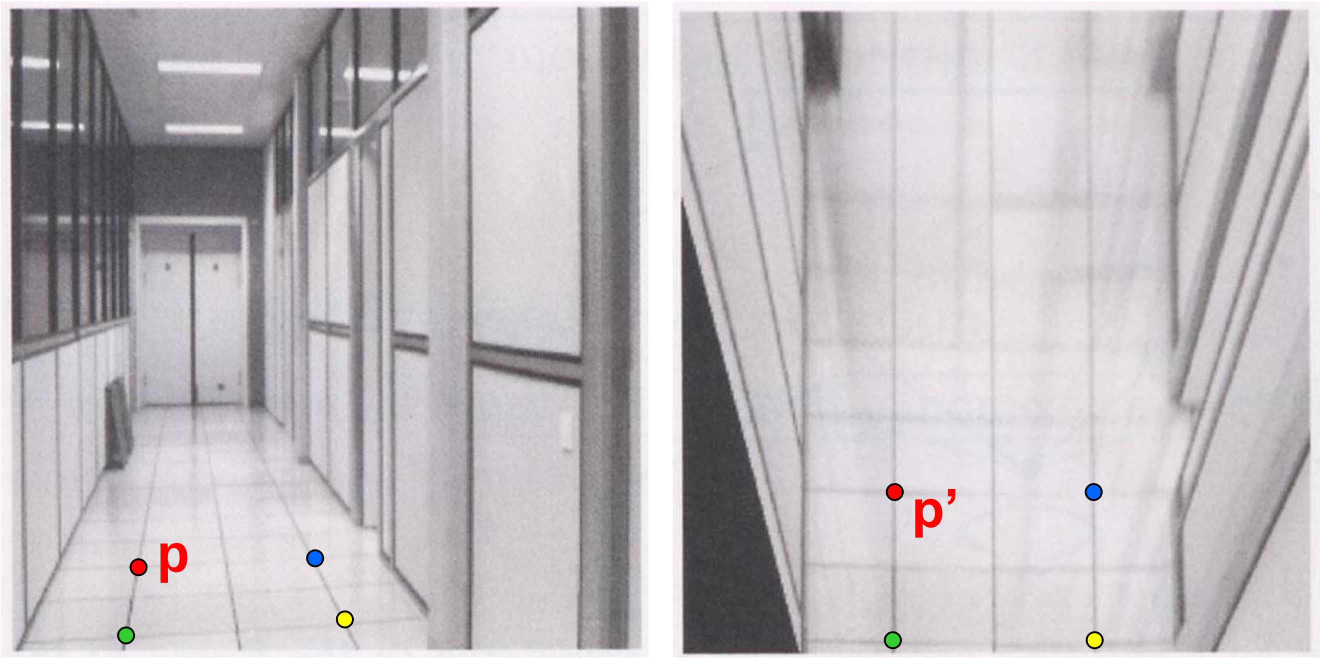


Image rectification



To unwarp (rectify) an image

- Find the homography **H** given a set of **p** and **p’** pairs
- How many correspondences are needed?
- Tricky to write H analytically, but we can <u>solve</u> for it!
  - Find such H that “best” transforms points p into p’
  - Use least-squares!

Solving for homographies

p' = Hp

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Can set scale factor *i*=1. So, there are 8 unkowns.
- Set up a system of linear equations:
  - **Ah = b**
- where vector of unknowns h = $[a,b,c,d,e,f,g,h]^T$
- Need at least 8 eqs, but the more the better…
- Solve for h. If overconstrained, solve using least-squares:

$$\min \|Ah - b\|^2$$

- Can be done in Matlab using "\" command
  - see "help lmdivide"

Application: generate synthetic view

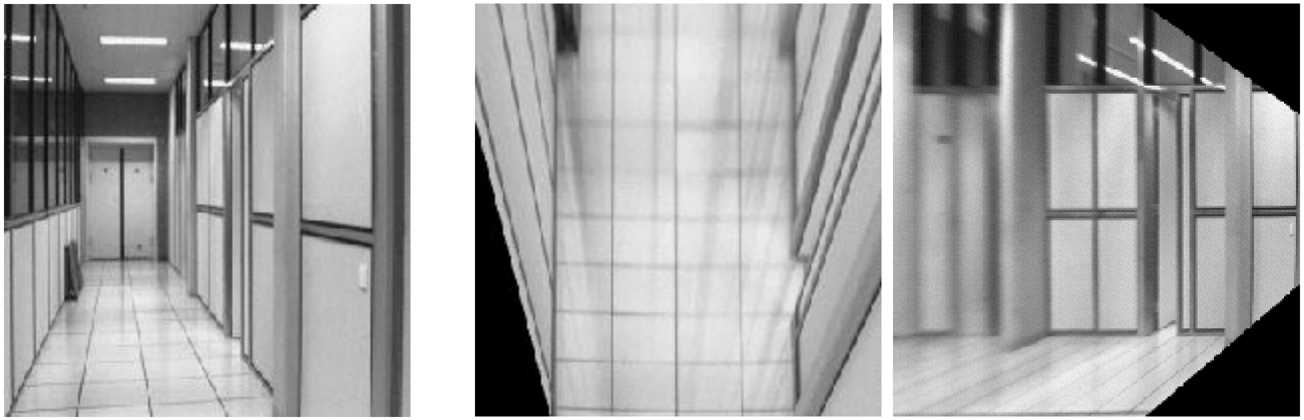


Fig. 8.7. **Synthetic views.** *(a) Source image. (b) Fronto-parallel view of the corridor floor generated from (a) using the four corners of a floor tile to compute the homography. (c) Fronto-parallel view of the corridor wall generated from (a) using the four corners of the door frame to compute the homography.*

•In fronto parallel view (1): a rectangle is imaged as a rectangle; (2) world and image rectangle have same aspect ratio.

•Approach: synthesize fronto parallel view by warping an image with the homography that maps a rectangle imaged as a quadrilateral to a rectangle with the correct aspect ratio.

(i) Compute the homography H that warps the image quadrilateral to a rectangle with the known aspect ratio

(ii) Projectively warp the source image with this homography

Fun with homographies

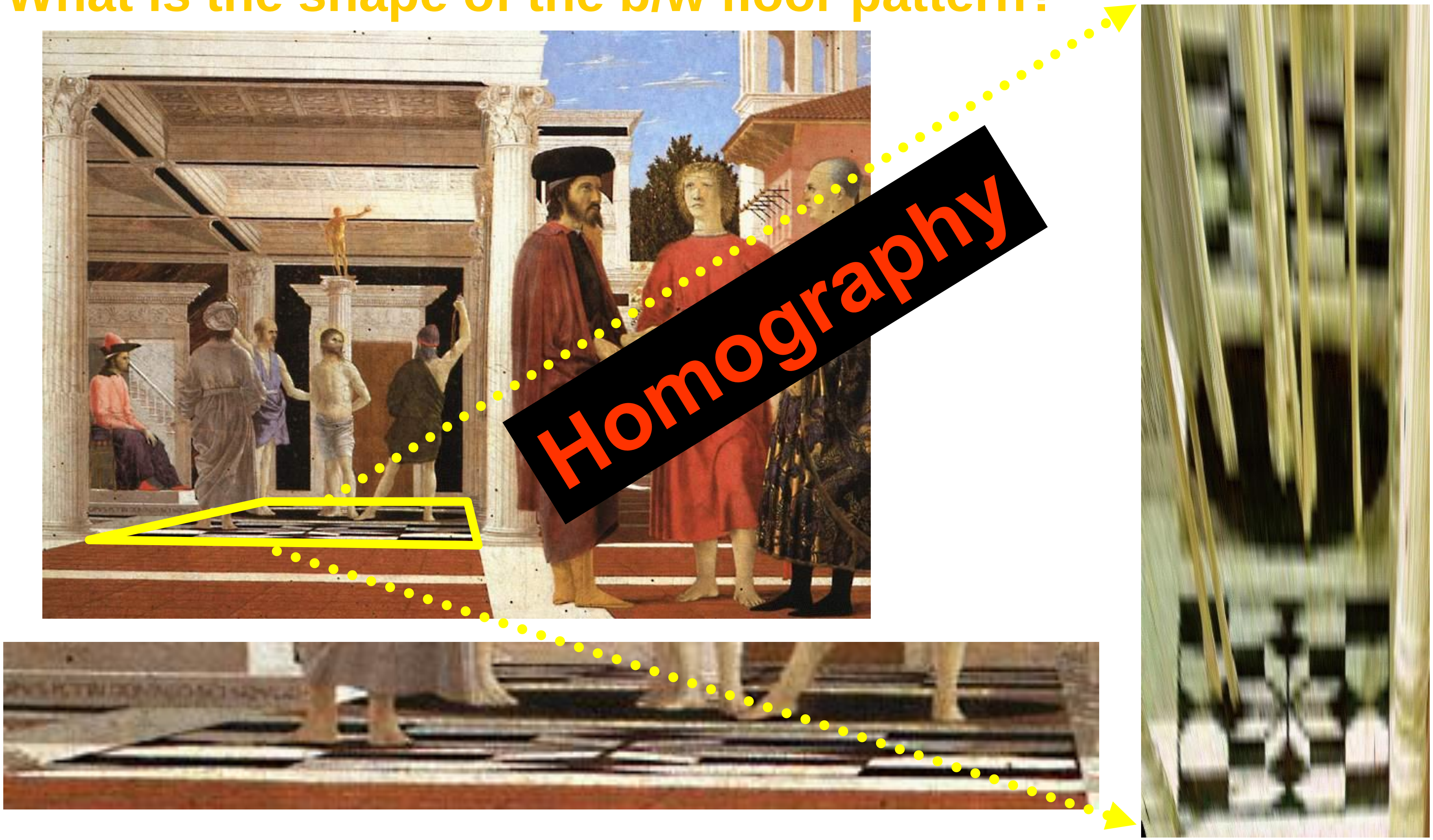
Original image

St.Petersburg
photo by A. Tikhonov

Virtual camera rotations

Analysing patterns and shapes

What is the shape of the b/w floor pattern?



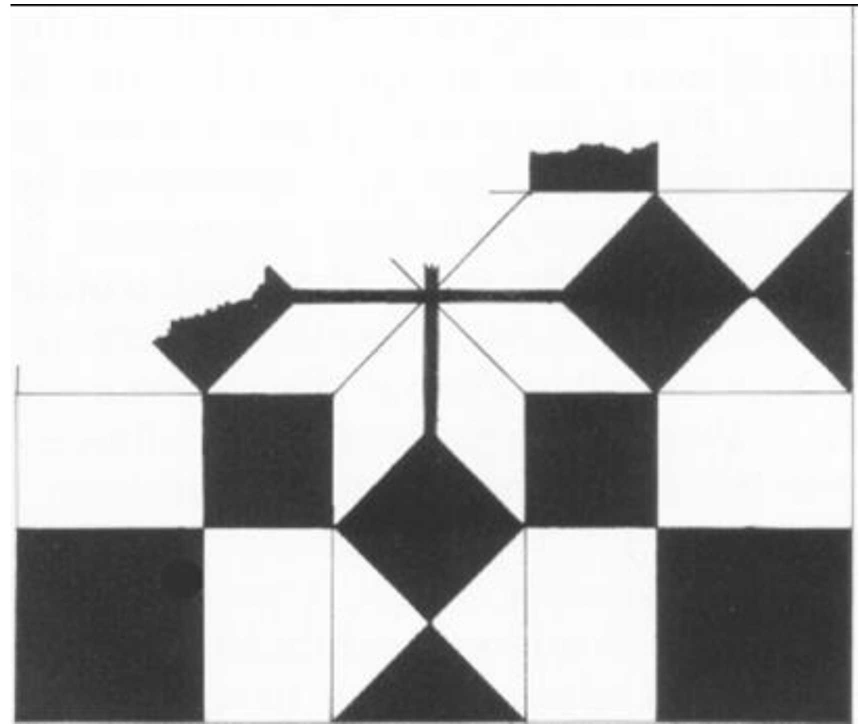
The floor (enlarged)

Automatically rectified floor

Slide from Criminisi

Analysing patterns and shapes

Automatic rectification



From Martin Kemp *The Science of Art*
(manual reconstruction)

2 patterns have been discovered !

Slide from Criminisi

Analysing patterns and shapes

St. Lucy Altarpiece, **D. Veneziano**

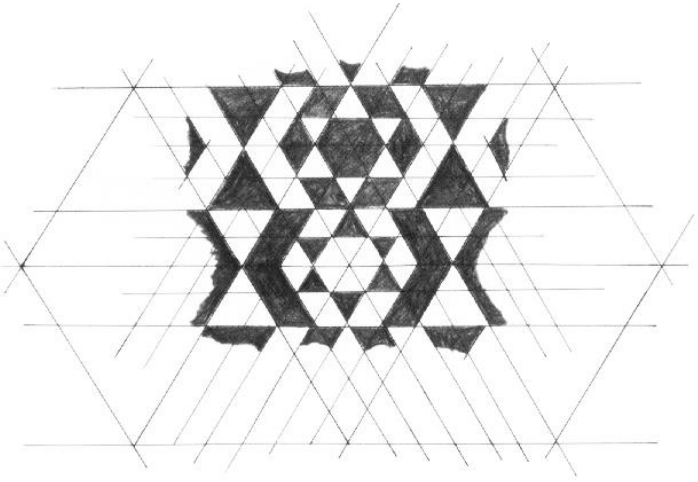
What is the (complicated) shape of the floor pattern?

Automatically rectified floor

Slide from Criminisi

Analysing patterns and shapes

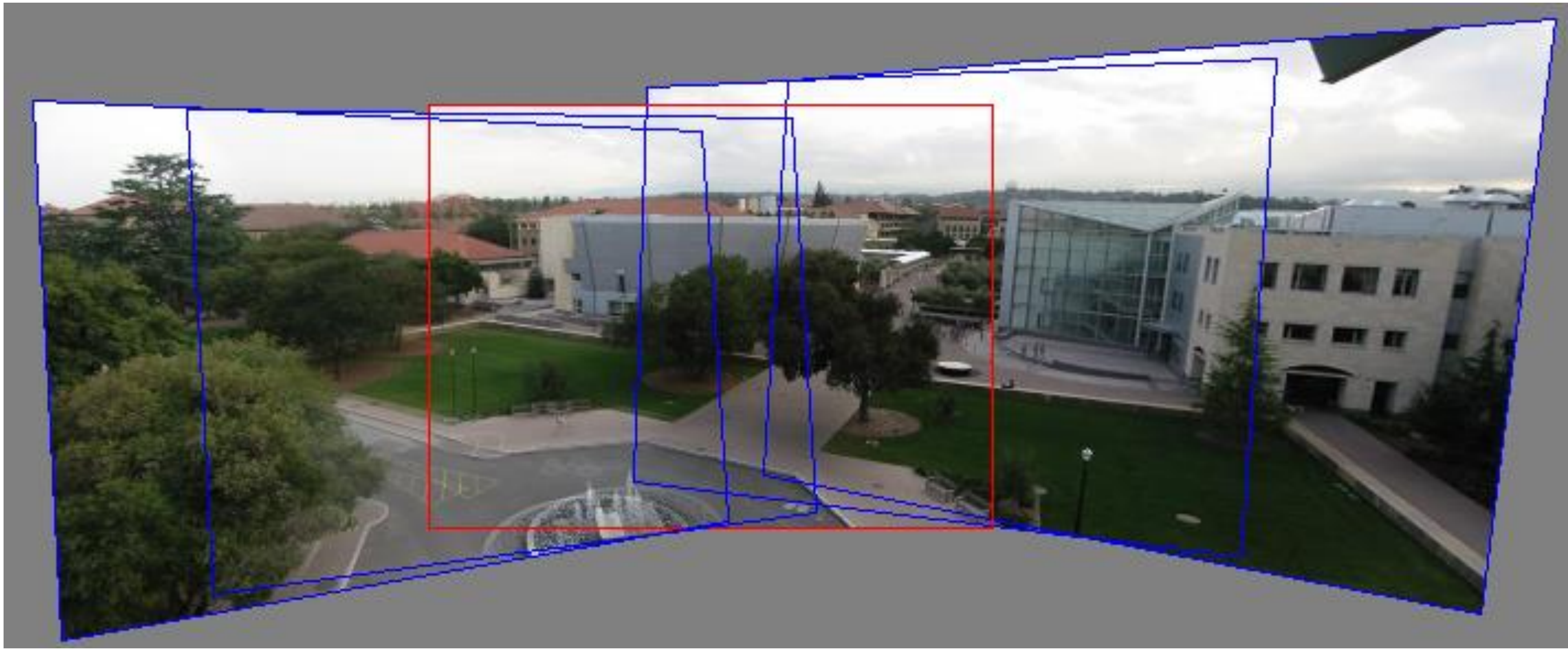
Automatic rectification



From Martin Kemp, *The Science of Art*
(manual reconstruction)

Slide from Criminisi

Panoramas



1. Pick one image (red)
2. Warp the other images towards it (usually, one by one)
3. blend

Planar homography mosaicing

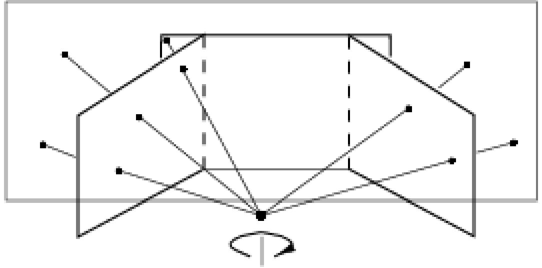


Fig. 8.8. *Three images acquired by a rotating camera may be registered to the frame of the middle one, as shown, by projectively warping the outer images to align with the middle one.*

(i) Choose one image of the set as a reference.
(ii) Compute the homography H which maps one of the other images of the set to this reference image.
(iii) Projectively warp the image with this homography, and augment the reference image with the non-overlapping part of the warped image.
(iv) Repeat the last two steps for the remaining images of the set.

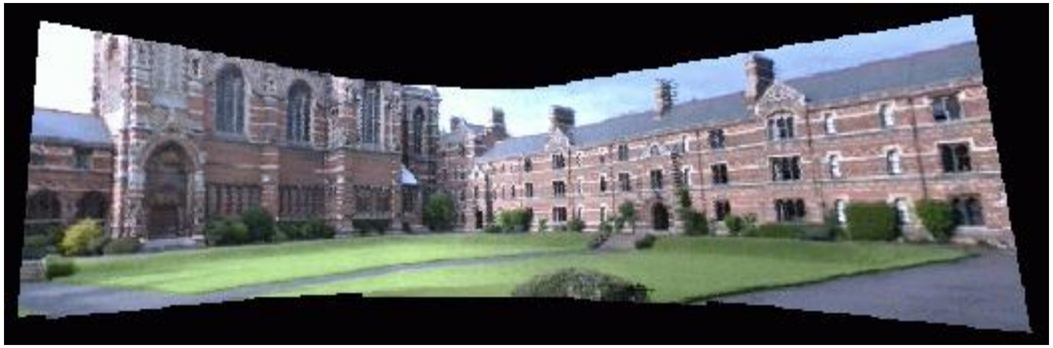
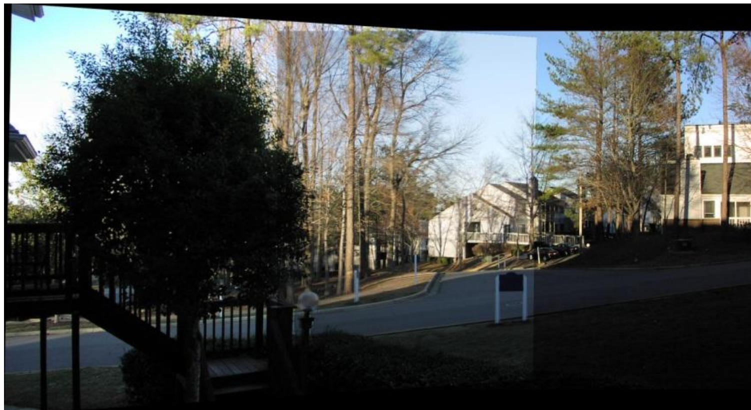


Fig. 8.9. **Planar panoramic mosaicing.** *Eight images (out of thirty) acquired by rotating a camcorder about its centre. The thirty images are registered (automatically) using planar homographies and composed into the single panoramic mosaic shown. Note the characteristic "bow tie" shape resulting from registering to an image at the middle of the sequence.*

close-up: interlacing
can be important problem!

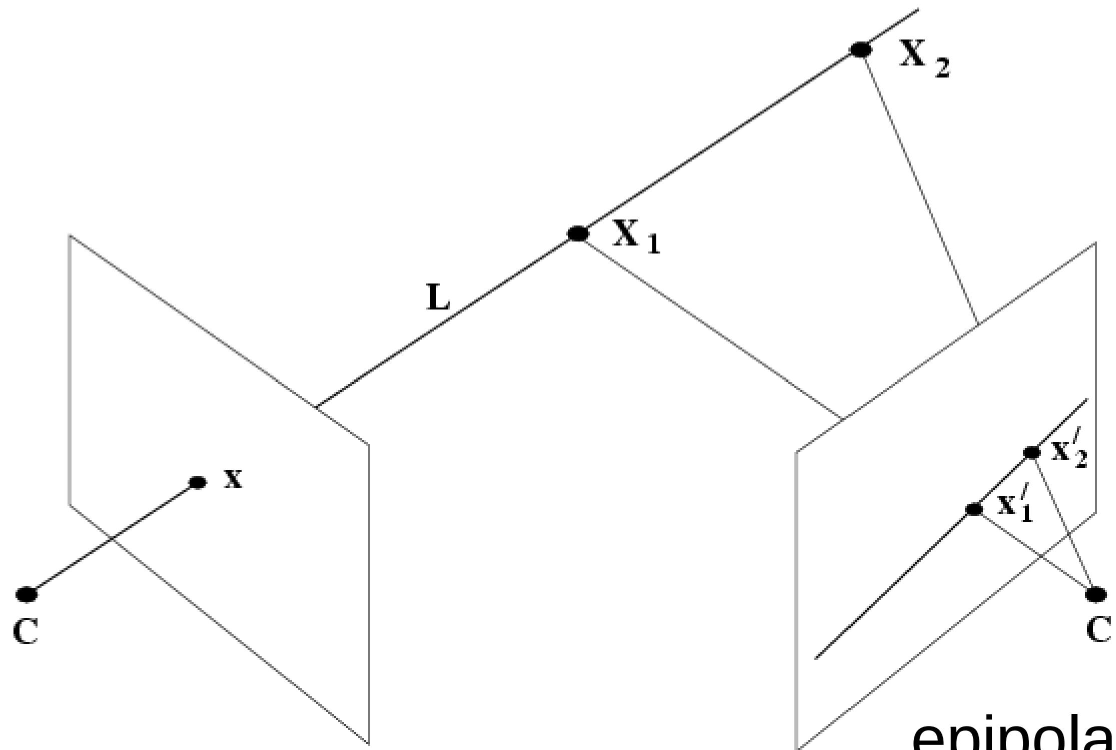
Planar homography mosaicing more examples



Moving the camera center

If C is fixed, but camera rotates or zooms, transformation of image only depends on the image plane motion NOT on 3D space structure
If C moves, map between images depends on 3 space structure it is looking at:

motion parallax



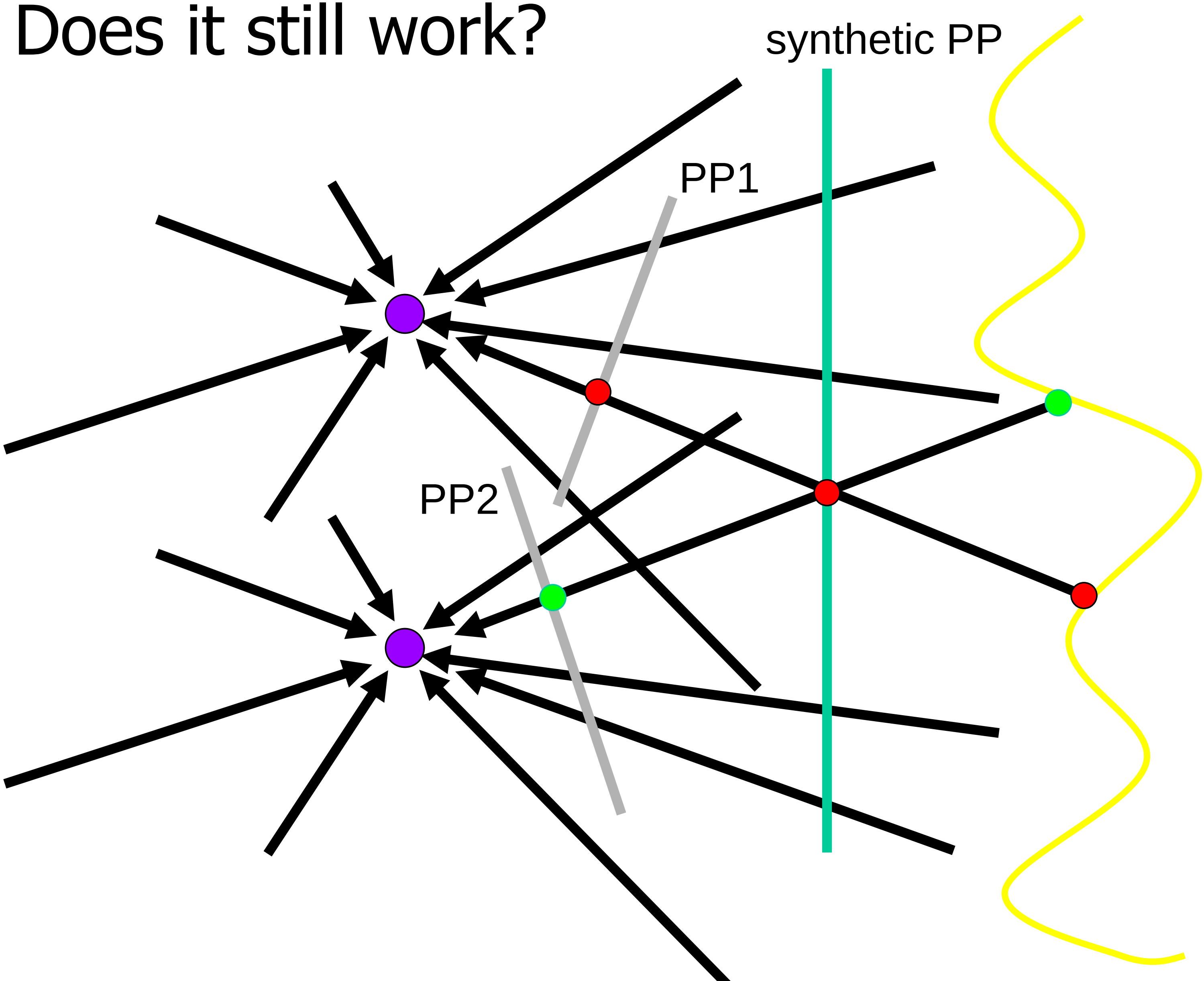
epipolar line

Fig. 8.10. **Motion parallax.** *The images of the space points* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are coincident when viewed by the camera with centre* $\mathbf{C}$. *However, when viewed by a camera with centre* $\mathbf{C}$, *which does not lie on the line* $\mathbf{L}$ *through* $\mathbf{X}_1$ *and* $\mathbf{X}_2$, *the images of the space points are not coincident. In fact the line through the image points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *is the image of the ray* $\mathbf{L}$, *and will be seen in chapter 9 to be an* **epipolar line**. *The vector between the points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *is the parallax.*

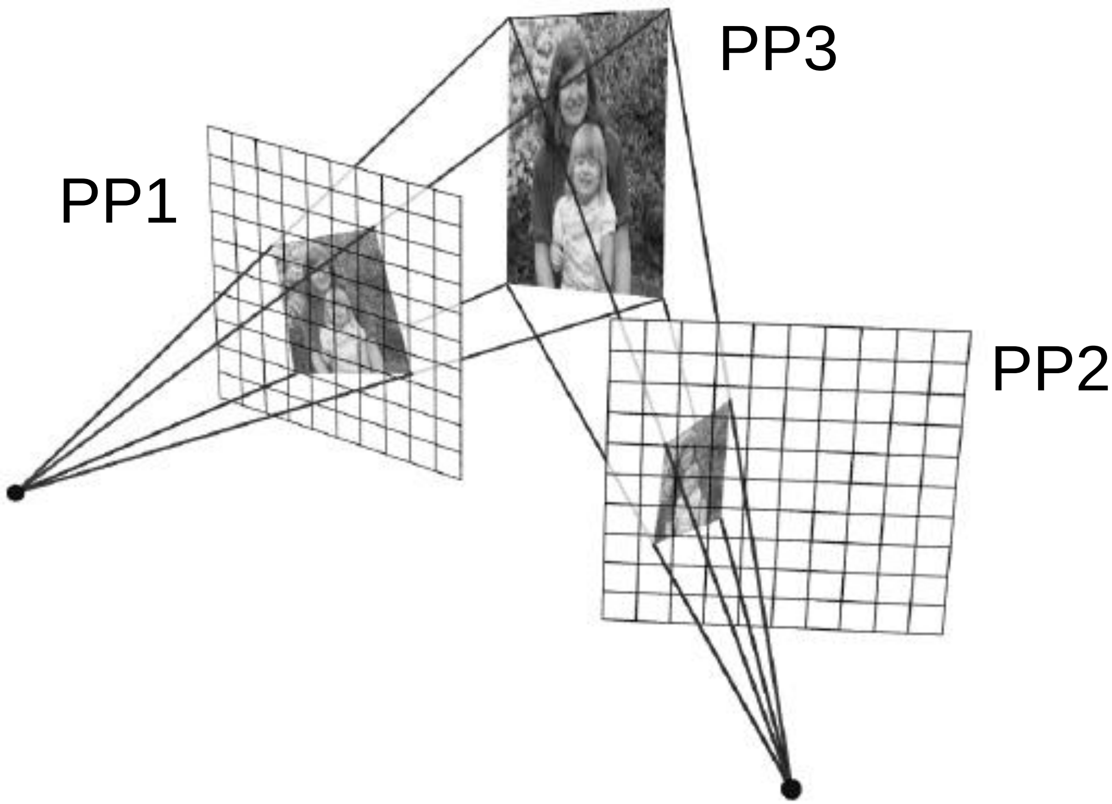
Important special case of 3-space structure: all scene points are co-planar:
2 images with motion parallax (camera center moving) are still related by planar homography

changing camera center

- Does it still work?

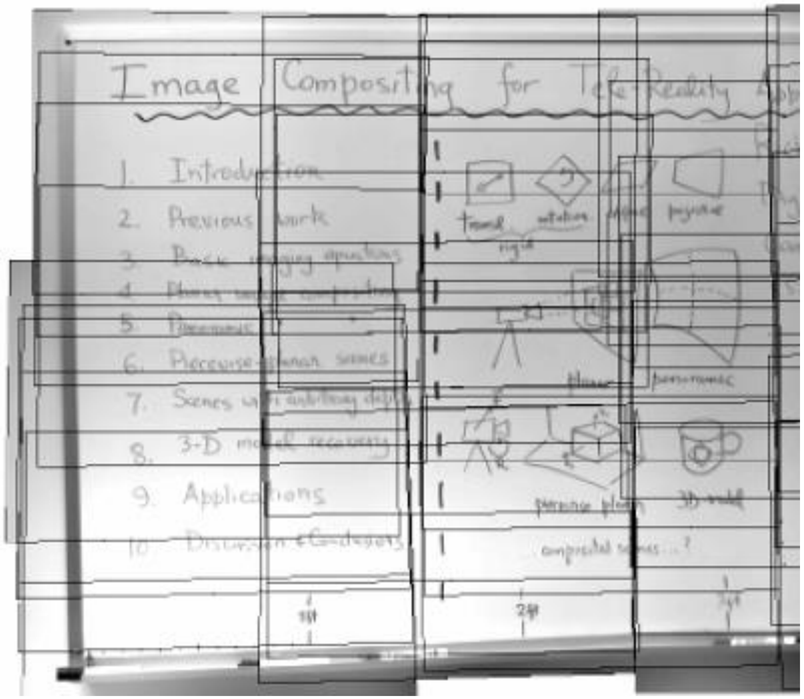


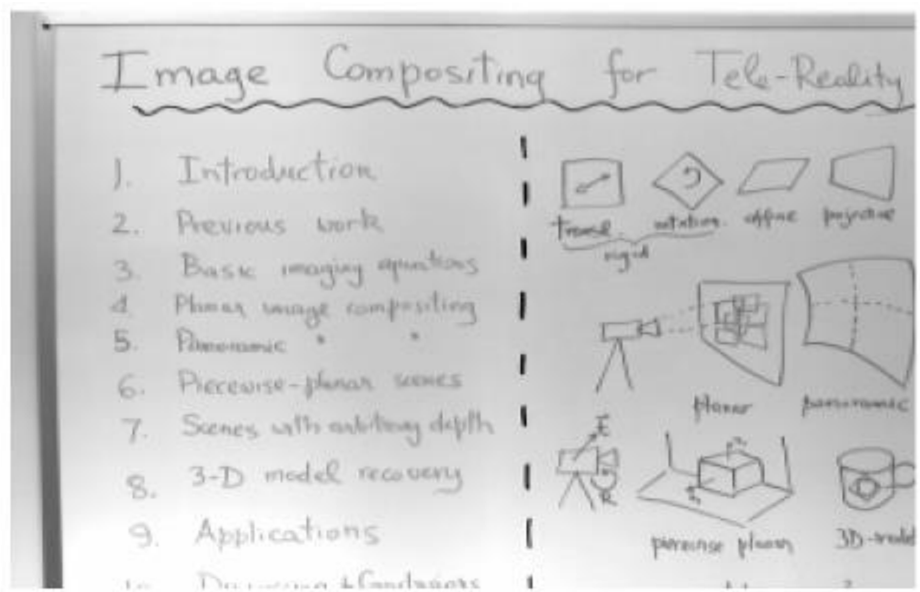
Planar scene (or far away)



- PP3 is a projection plane of both centers of projection, so we are OK!
- This is how big aerial photographs are made

Planar mosaic



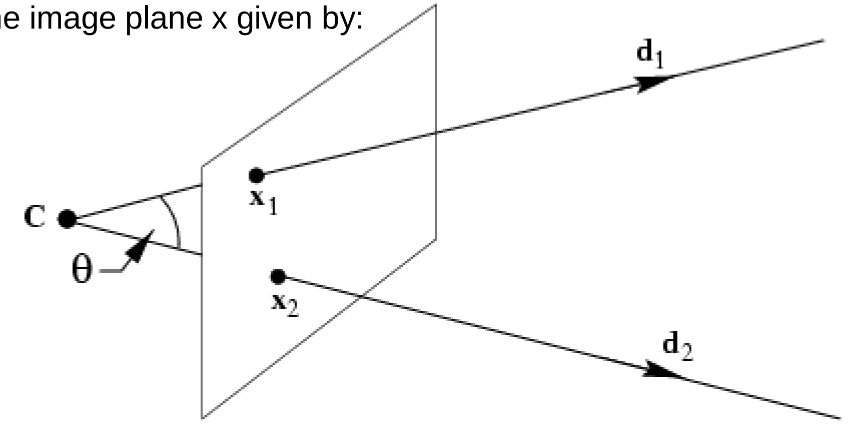


What does calibration give?

Image point x backprojects to a ray defined by x and camera center ➔ calibration relates image point to the ray's direction

Points on a ray X = λ d with d being a direction, map to a point on the image plane x given by:

$$x = K[I \mid 0]\begin{bmatrix} d \\ 0 \end{bmatrix}$$



Get d from x by :

$$d = K^{-1}x$$

Fig. 8.11. *The angle* θ *between two rays.*

Result 8.15. *The camera calibration matrix K is the (affine) transformation between x and the ray's direction $d = K^{-1}x$ measured in the camera's Euclidean coordinate frame.*

Angle between two rays with directions d_1 and d_2 corresponding to x_1 and x_2

$$\cos\theta = \frac{d_1^T d_2}{\sqrt{(d_1^T d_1)(d_2^T d_2)}} = \frac{x_1^T (K^{-T}K^{-1})x_2}{\sqrt{(x_1^T (K^{-T}K^{-1})x_1)(x_2^T (K^{-T}K^{-1})x_2)}}$$

if K is known, can measure angle between rays from their image points.

A camera for which K is known is called calibrated;

A calibrated camera is a direction sensor able to measure the direction of rays like a 2D protractor

An image line l defines a plane through the camera center with normal $n=K^T l$ measured in the camera's Euclidean frame