NDMI012: Combinatorics and Graph Theory 2

Lecture #12
Pólya enumeration theorem. Exponential generating functions

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1 Pólya enumeration theorem

Our goal in this section is to compute the number of different colorings of certain objects, up to symmetry. The symmetry will be determined by an appropriate group action.

A *subgroup* of a group G is a subset of G that is a group under the operation inherited from G. Note that every group is a subgroup of itself, as is the one-element group consisting only of the identity element.

Let X be a set of size n, and let G be a subgroup of $\text{Sym}(X)$. Each element of G can be represented as a composition of disjoint cycles, the sum of whose lengths is n. Now, for $g \in G$ and $k \in \{1, \dots, n\}$, we denote by $j_k(g)$ the number of cycles of length k, when g is written as a composition of disjoint cycles.[1] For $g \in G$, we set $x^{\text{cs}(g)} := x_1^{j_1(g)} x_2^{j_2(g)} \dots x_n^{j_n(g)}$. Finally, the *cycle index* of the group G is

$$\mathcal{Z}_G(x_1, \dots, x_n) \quad = \quad \tfrac{1}{|G|} \sum_{g \in G} x^{\text{cs}(g)}.$$

Example 1.1. *Compute cycle index of the group Sym*(2).

Solution. Here, using the notation from the definition of a cycle index, we have that $X = \{1, 2\}$ and $n = 2$. We have that $\text{Sym}(2) = \{(1)(2), (12)\}$, and clearly,

- $x^{\text{cs}\big((1)(2)\big)} = x_1^2 x_2^0 = x_1^2$;
- $x^{\text{cs}\big((12)\big)} = x_1^0 x_2^1 = x_2$.

[1] For example, if $n = 7$ and $g = (124)(35)(6)(7)$, then $j_1(g) = 2$, $j_2(g) = 1$, $j_3(g) = 1$, and $j_4(g) = j_5(g) = j_6(g) = j_7(g) = 0$. Do not forget to count cycles of length one!

So,

$$\mathcal{Z}_{\mathrm{Sym}(2)}(x_1, x_2) = \frac{x_1^2 + x_2}{2}.$$

□

Example 1.2. *Compute cycle index of the group Sym*(3)*.*

Solution. Here, using the notation from the definition of a cycle index, we have that $X = \{1, 2, 3\}$ and $n = 3$. Sym(3) has one element that is a composition of three 1-cycles; it has three elements that are a composition of one 2-cycle and one 1-cycle; and it has two elements that consist of one 3-cycle. So,

$$\mathcal{Z}_{\mathrm{Sym}(3)}(x_1, x_2, x_3) = \frac{x_1^3 + 3x_1x_2 + 2x_3}{6}.$$

□

Recall that for a set X, $\binom{X}{2}$ is the set of all 2-element subsets of X. For each positive integer n and permutation $\pi \in \mathrm{Sym}(n)$, we define a permutation π' on the set $\binom{\{1,\dots,n\}}{2}$ by setting $\pi'(\{i, j\}) = \{\pi(i), \pi(j)\}$, and we set $\mathrm{Sym}'(n) = \{\pi' \mid \pi \in \mathrm{Sym}(n)\}$. It is easy to check that $\mathrm{Sym}'(n)$ is a subgroup of $\mathrm{Sym}\Big(\binom{\{1,\dots,n\}}{2}\Big)$. In particular, every permutation in $\mathrm{Sym}'(n)$ can be represented as a composition of disjoint cycles, the sum of whose lengths is $\binom{n}{2}$.

Example 1.3. *Compute the cycle index of the group Sym*$'$(5)*.*

Solution. We remark that $\binom{5}{2} = 10$, and so each permutation in $\mathrm{Sym}'(5)$ can be represented as a composition of disjoint cycles, the sum of whose lengths is 10.

We analyze the cycle structure of permutations in Sym(5): given the cycle structure of a permutation $\pi \in \mathrm{Sym}(5)$, we describe the cycle structure of π'. If we, in addition, keep track of the number of permutations of each type in Sym(5), we can easily find the cycle index of $\mathrm{Sym}'(5)$.

- There is one permutation π in Sym(5) (namely, the identity permutation) of the form $(a)(b)(c)(d)(e)$. For such a π, we have that π' is the composition of ten cycles of length one. So, $x^{\mathrm{cs}(\pi')} = x_1^{10}$.

- There are 10 permutations π in Sym(5) of the form $(ab)(c)(d)(e)$. For such a π, we see that π' has three cycles of the length two (these cycles are of the form $(\{a, x\}, \{b, x\})$, with $x \notin \{a, b\}$), and it has four cycles of length one. So, $x^{\mathrm{cs}(\pi')} = x_1^4 x_2^3$.

- There are 15 permutation π in Sym(5) of the form $(ab)(cd)(e)$. For such a π, we see that π' has exactly two cycles of length one (namely, $(\{a, b\})$ and $(\{c, d\})$), and the remaining cycles of π' (four of them) are of length two. So, $x^{\mathrm{cs}(\pi')} = x_1^2 x_2^4$.

- There are 20 permutations π in Sym(5) of the form $(abc)(d)(e)$. For such a π, we see that π' has one cycle of length one (namely, $(\{d,e\})$), and the remaining cycles of π' (three of them) are of length three. So, $x^{\text{cs}(\pi')} = x_1x_3^3$.

- There are 20 permutations π in Sym(5) of the form $(abc)(de)$. For such a π, we see that π' has one cycle of length one (namely, $(\{d,e\})$), one cycle of length three (namely, $(\{a,b\},\{b,c\},\{c,a\})$), and one cycle of length six (containing all the remaining elements of $\binom{\{1,\ldots,5\}}{2}$). So, $x^{\text{cs}(\pi')} = x_1x_3x_6$.

- There are 30 permutations π in Sym(5) of the form $(abcd)(e)$. For such a π, we see that π' has two 4-cycles (namely, $(\{a,e\},\{b,e\},\{c,e\},\{d,e\})$ and $(\{a,b\},\{b,c\},\{c,d\},\{d,a\})$) and one 2-cycle (namely, $(\{a,c\},\{b,d\})$). So, $x^{\text{cs}(\pi')} = x_2x_4^2$.

- There are 24 permutations π in Sym(5) of the form $(abcde)$. For such a π, we see that π' has two 5-cycles (namely, $\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\})$ and $(\{a,c\},\{b,d\},\{c,e\},\{d,a\},\{e,b\})$). So, $x^{\text{cs}(\pi')} = x_5^2$.

Since $|\text{Sym}'(5)| = |\text{Sym}(5)| = 5! = 120$, we now see that

$$\begin{aligned} & \mathcal{Z}_{\text{Sym}'(5)}(x_1,\ldots,x_{10}) \\ = \;& \tfrac{1}{120}\Big(x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_6 + 30x_2x_4^2 + 24x_5^2\Big). \end{aligned}$$

□

We now need a couple more definitions. Suppose $C = \{c_1,\ldots,c_k\}$ is some set of colors, and that G is a subgroup of $\text{Sym}(X)$ acting on a finite set X in the natural way, i.e. for $\pi \in G$ and $x \in X$, we have $\pi \cdot x = \pi(x)$. Let $\mathcal{C}$ be the set of all colorings of X using the color set C (formally, $\mathcal{C}$ is simply the set of all functions from X to C). Then G acts on $\mathcal{C}$ in the natural way: for all $\pi \in G$, $c \in C$, and $x \in X$, we set $(\pi \cdot c)(x) = c(\pi^{-1}\cdot x)$;[2] the idea is

[2]Let us check that this is really a group action. For $c \in \mathcal{C}$ and $x \in X$, we have that $(1_G \cdot c)(x) = c(1_G^{-1}\cdot x) = c(1_G \cdot x) = c(x)$, and it follows that $1_G \cdot c = c$. Further, for $\pi_1,\pi_2 \in G$, $c \in \mathcal{C}$, and $x \in X$, we have that

$$\begin{aligned} (\pi_1\cdot(\pi_2\cdot c))(x) &= (\pi_2\cdot c)(\pi_1^{-1}\cdot x) \\ &= c(\pi_2^{-1}\cdot(\pi_1^{-1}\cdot x)) \\ &= c((\pi_2^{-1}\pi_1^{-1})\cdot x) \\ &= ((\pi_1\pi_2)^{-1}\cdot c)(x) \\ &= ((\pi_1\pi_2)\cdot c)(x); \end{aligned}$$

so, $\pi_1\cdot(\pi_2\cdot c) = (\pi_1\pi_2)\cdot c$. Thus, this is indeed a group action on $\mathcal{C}$.

that $\pi \cdot c$ should assign to x the color that c assigned to the element of X that got "moved" to x via π, i.e. to the element $\pi^{-1} \cdot x$. Two colorings are *equivalent* if one can be transformed into the other via our group action, i.e. if they belong to the same orbit of our action. Now, let $\mathcal{D} \subseteq \mathcal{C}$.

- The *coloring inventory* of $\mathcal{D}$ is a polynomial in $c_1, \dots, c_k$, which is the sum of terms of the form $c_1^{d_1} \dots c_k^{d_k}$, and the coefficient in front of the term $c_1^{d_1} \dots c_k^{d_k}$ is the number of colorings in $\mathcal{D}$ that, for each $i \in \{1, \dots, k\}$, assign color c_i to precisely d_i elements of X.

- The *pattern inventory* of $\mathcal{D}$ is a polynomial in $c_1, \dots, c_k$, which is the sum of terms of the form $c_1^{d_1} \dots c_k^{d_k}$, and the coefficient in front of the term $c_1^{d_1} \dots c_k^{d_k}$ is the number of **non-equivalent** colorings in $\mathcal{D}$ that, for each $i \in \{1, \dots, k\}$, assign color c_i to precisely d_i elements of X.

Lemma 1.4. *Let $C = \{c_1, \dots, c_k\}$ be a set of colors, let X be a finite set of size n, and let G be a subgroup of $Sym(X)$, acting on X in the natural way.*[3] *Let $\mathcal{C}$ be the set of all colorings of X with colors from C, and let G act on $\mathcal{C}$ in the natural way.*[4] *Then for all $\pi \in G$, the coloring inventory of $\mathcal{C}^\pi$ (the set of fixed points of π in $\mathcal{C}$) is the polynomial $p_\pi(c_1, \dots, c_k)$ obtained by substituting $\sum_{i=1}^k c_i^r$ for each x_r in $x^{cs(\pi)}$.*[5]

Proof. We write π as a product of disjoint cycles, and we set up a correspondence between the cycles of π and the terms in the product $x^{\mathrm{cs}}(\pi)$,[6] in such a way that a cycle of length r corresponds to an x_r term.[7] Then a coloring $c \in \mathcal{C}$ is a fixed point of π if and only if, for each cycle of π, c assigns the same color to each element of X in the cycle. We can choose colors independently for each cycle. Now if we substitute $\sum_{i=1}^k c_i^r$ for each x_r in $x^{\mathrm{cs}(\pi)}$, then each r-cycle of π has a corresponding term of the form $\sum_{i=1}^k c_i^r$; selecting color c_i for all elements of the r-cycle is equivalent to choosing the summand c_i^r from the corresponding term $\sum_{i=1}^k c_i^r$ in the product defining the polynomial $p_\pi(c_1, \dots, c_k)$. It follows that the number of ways that we can color X in such a way that π fixes the coloring, and that there are precisely d_i elements of X colored c_i (for each $i \in \{1, \dots, k\}$) is precisely the coefficient in front of the summand $c_1^{d_1} \dots c_k^{d_k}$ in the polynomial $p_\pi(c_1, \dots, c_k)$. The result now follows. $\square$

[3]This means that for all $\pi \in \mathrm{Sym}(X)$ and $x \in X$, we have that $\pi \cdot x = \pi(x)$.

[4]That is, for all $\pi \in G$, $c \in C$, and $x \in X$, we set $(\pi \cdot c)(x) = c(\pi^{-1} \cdot x)$.

[5]For example, if $C = \{c_1, c_2\}$, $X = \{1, \dots, 7\}$, $G = \mathrm{Sym}(7)$, and $\pi = (125)(36)(47)$, then $x^{\mathrm{cs}(\pi)} = x_2^2 x_3$; if we substitute $\sum_{i=1}^k c_i^r = c_1^r + c_2^r$ for each x_r in $x^{\mathrm{cs}(\pi)}$, then we get $p_\pi(c_1, c_2) = (c_1^2 + c_2^2)^2 (c_1^3 + c_2^3) = c_1^7 + 2c_1^5 c_2^2 + c_1^4 c_2^3 + c_1^3 c_2^4 + 2c_1^2 c_2^5 + c_2^7$.

[6]Here, $x_i^{d_i}$ is understood as a term of d_i different terms (namely, d_i copies of x_i), and not as a single term.

[7]For example, if $\pi = (125)(36)(47)$, then $x^{\mathrm{cs}(\pi)} = x_2^2 x_3$, and we can set up a correspondence $(125) \mapsto x_3$, $(36) \mapsto x_2$, and $(47) \mapsto x_2$. (So, two different cycles of length two get mapped to two "different" x_2's.)

Pólya enumeration theorem. *Let $C = \{c_1, \dots, c_k\}$ be a set of colors, let X be a finite set of size n, and let G be a subgroup of Sym(X), acting on X in the natural way.*[8] *Let $\mathcal{C}$ be the set of all colorings of X with colors from C, and let G act on $\mathcal{C}$ in the natural way.*[9] *Then the pattern inventory of $\mathcal{C}$ is $\mathcal{Z}_G(\sum_{i=1}^k c_i, \sum_{i=1}^k c_i^2, \dots, \sum_{i=1}^k c_i^n)$.*

Proof. Fix a vector $\mathbf{d} = (d_1, \dots, d_k)$ with non-negative integer entries, and let $\mathcal{C}_\mathbf{d}$ be the set of all colorings in $\mathcal{C}$ in which, for each $i \in \{1, \dots, n\}$, the number of elements of X receiving color c_i is precisely d_i.[10] Then $\mathcal{C}_\mathbf{d}$ is the union of some orbits of the action of G on $\mathcal{C}$, and so in fact, G acts on $\mathcal{C}_\mathbf{d}$ as well. By Burnside's lemma, we have that

$$|\mathcal{C}_\mathbf{d}/G| \quad = \quad \tfrac{1}{|G|} \sum_{\pi \in G} |\mathcal{C}_\mathbf{d}^\pi|.$$

and consequently,

$$|\mathcal{C}_\mathbf{d}/G| \; c_1^{d_1} \dots c_k^{d_k} \quad = \quad \tfrac{1}{|G|} \sum_{\pi \in G} |\mathcal{C}_\mathbf{d}^\pi| \; c_1^{d_1} \dots c_k^{d_k},$$

Now we sum up over all possible choices of the vector $\mathbf{d} = (d_1, \dots, d_k)$, and we get[11]

$$\sum_\mathbf{d} |\mathcal{C}_\mathbf{d}/G| \; c_1^{d_1} \dots c_k^{d_k} \quad = \quad \sum_\mathbf{d} \tfrac{1}{|G|} \sum_{\pi \in G} |\mathcal{C}_\mathbf{d}^\pi| \; c_1^{d_1} \dots c_k^{d_k}.$$

and consequently,

$$\sum_\mathbf{d} |\mathcal{C}_\mathbf{d}/G| \; c_1^{d_1} \dots c_k^{d_k} \quad = \quad \tfrac{1}{|G|} \sum_{\pi \in G} \sum_\mathbf{d} |\mathcal{C}_\mathbf{d}^\pi| \; c_1^{d_1} \dots c_k^{d_k},$$

Clearly, the left-hand-side of this last equality is precisely the pattern inventory of $\mathcal{C}$. On the other hand, for each $\pi \in G$, $\sum\limits_\mathbf{d} |\mathcal{C}_\mathbf{d}^\pi| \; c_1^{d_1} \dots c_k^{d_k}$ is precisely the coloring inventory of $\mathcal{C}^\pi$, which (by Lemma 1.4) is precisely $p_\pi(c_1, \dots, c_k)$, where $p_\pi(c_1, \dots, c_k)$ is the polynomial obtained by substituting $\sum_{i=1}^k c_i^r$ for each x_r in $x^{\mathrm{cs}(\pi)}$. So, the pattern inventory of $\mathcal{C}$ is $\frac{1}{|G|} \sum\limits_{\pi \in G} p_\pi(c_1, \dots, c_k)$, which (by the definition of cycle index) is precisely $\mathcal{Z}_G(\sum_{i=1}^k c_i, \sum_{i=1}^k c_i^2, \dots, \sum_{i=1}^k c_i^n)$. This completes the argument. $\square$

Example 1.5. *Compute the number of non-equivalent colorings of a bracelet with four beads, using colors black and white for the beads. (Two colorings are equivalent if one can be transformed into the other via a rotation or a reflection.)*

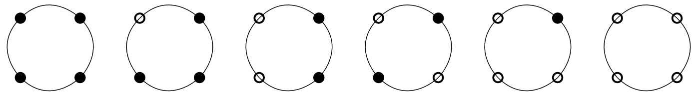
[8]This means that for all $\pi \in \mathrm{Sym}(X)$ and $x \in X$, we have that $\pi \cdot x = \pi(x)$.

[9]That is, for all $\pi \in G$, $c \in C$, and $x \in X$, we set $(\pi \cdot c)(x) = c(\pi^{-1} \cdot x)$.

[10]Note that if $d_1 + \dots + d_k \neq n$, then $\mathcal{C}_\mathbf{d} = \emptyset$.

[11]Note that our sums are in fact finite because if $d_1 + \dots + d_k \neq n$, then $\mathcal{C}_\mathbf{d} = \emptyset$.

Solution. In this particular case, it is easy to see that there are exactly six non-equivalent colorings, represented below.



However, let us apply the Pólya enumeration theorem in order to illustrate the principle. We label the beads $1, 2, 3, 4$ counterclockwise. The group acting on the beads is simply the dihedral group D_8 (symmetries of the square). The elements of the group are:

- (1)(2)(3)(4) - identity;
- (1234) - rotation by 90° ccw;[12]
- (13)(24) - rotation by 180°;
- (1432) - rotation by 270° ccw;
- (12)(34) - reflection about the axis through the centers of edges $12, 34$;
- (14)(23) - reflection about the axis through the centers of edges $14, 23$;
- (1)(24)(3) - reflection about the axis through vertices/beads $1, 3$;
- (13)(2)(4) - reflection about the axis through vertices/beads $2, 4$.

So,

$$\mathcal{Z}_{D_8}(x_1, \ldots, x_4) \quad = \quad \tfrac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4),$$

and we have

$$\begin{aligned} & \mathcal{Z}_{D_8}(b+w, b^2+w^2, b^3+w^3, b^4+w^4) \\ = \; & \tfrac{1}{8}\Big((b+w)^4 + 2(b+w)^2(b^2+w^2) + 3(b^2+w^2)^2 + 2(b^4+w^4)\Big) \\ = \; & b^4 + b^3 w + 2b^2 w^2 + b w^3 + w^4. \end{aligned}$$

The total number of colorings is equal to the sum of coefficients of the polynomial above: $1 + 1 + 2 + 1 + 1 = 6$.

We also remark that the polynomial above allows us to do more, namely, to count the number of non-equivalent colorings with a fixed number of black and white beads. So, there are two non-equivalent colorings with two beads colored black and two colored white. For any other (fixed) combination of black and white beads, where the total number of beads adds up to four, we only have one non-equivalent coloring. □

[12] ccw = counterclockwise

Proposition 1.6. *Let $n \geq 2$ and $k \geq 0$ be integers. Then the number of non-isomorphic graphs on n vertices and k edges is equal to the coefficient in front of the term x^k in the polynomial $\mathcal{Z}_{Sym'(n)}(1+x, 1+x^2, \ldots, 1+x^{\binom{n}{2}})$.*

Proof. Let $\mathcal{C}$ be the set of all colorings of the set $\binom{\{1,\ldots,n\}}{2}$ using the color set $\{b, w\}$. We let $\mathrm{Sym}'(n)$ act on $\mathcal{C}$ in the natural way. Now, colorings in $\mathcal{C}$ correspond to n-vertex graphs in the natural way: the vertex-set is $\{1, \ldots, 5\}$, and edges are pairs colored b ("black"), where as the non-edges are the pairs colored w ("white"). The number of non-isomorphic five-vertex graphs with k edges is precisely the number of non-equivalent colorings in $\mathcal{C}$ (with respect to our group action) in which exactly k elements of $\binom{\{1,\ldots,n\}}{2}$ are colored b (and the remaining $\binom{n}{2} - k$ elements are colored white). By the Pólya enumeration theorem, the latter is precisely the coefficient in front of $b^k w^{\binom{n}{2}-k}$ in the polynomial $\mathcal{Z}_{\mathrm{Sym}'(5)}(b+w, b^2+w^2, \ldots, b^{\binom{n}{2}} + w^{\binom{n}{2}})$. But this is exactly the coefficient in front of x^k in the polynomial $\mathcal{Z}_{\mathrm{Sym}'(n)}(1+x, 1+x^2, \ldots, 1+x^{\binom{n}{2}})$ (we replace b by x and w by 1). □

Example 1.7. *For each non-negative integer k, find the number of non-isomorphic k-edge graphs on five vertices.*

Solution. We apply Proposition 1.6. By Example 1.3, we know that

$$\begin{aligned} & \mathcal{Z}_{\mathrm{Sym}'(5)}(x_1, \ldots, x_{10}) \\ = \ & \tfrac{1}{120}\Big(x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_6 + 30x_2x_4^2 + 24x_5^2\Big), \end{aligned}$$

and so

$$\begin{aligned} & \mathcal{Z}_{\mathrm{Sym}'(5)}(1+x, \ldots, 1+x^{10}) \\ = \ & \tfrac{1}{120}\Big((1+x)^{10} + 10(1+x)^4(1+x^2)^3 + 15(1+x)^2(1+x^2)^4 + \\ & +20(1+x)(1+x^3)^3 + 20(1+x)(1+x^3)(1+x^6) + \\ & +30(1+x^2)(1+x^4)^2 + 24(1+x^5)^2\Big), \\ = \ & 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + 2x^8 + x^9 + x^{10}. \end{aligned}$$

Thus, up to isomorphism,

- there is one edgeless graph on five vertices;
- there is one graph on five vertices with one edge;
- there are two graphs on five vertices with two edges;
- there are four graphs on five vertices with three edges;

- there are six graphs on five vertices with four edges;
- there are six graphs on five vertices with five edges;
- there are six graphs on five vertices with six edges;
- there are four graphs on five vertices with seven edges;
- there are two graphs on five vertices with eight edges;
- there is one graph on five vertices with nine edges;
- there is one graph on five vertices with ten edges;
- there are no graphs on five vertices with more than ten edges.

□

2 Exponential generating functions

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real (or complex) numbers. The *ordinary generating function* (abbreviated *ogf*) of $\{a_n\}_{n=0}^{\infty}$ is the function

$$f(x) \quad = \quad \sum_{n=0}^{\infty} a_n x^n \quad = \quad a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

The *exponential generating function* (abbreviated *egf*) of $\{a_n\}_{n=0}^{\infty}$ is the function

$$g(x) \quad = \quad \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \quad = \quad \frac{a_0}{0!} + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \ldots$$

Ordinary generating functions (or simply “generating functions”) were studied in Combinatorics & Graph Theory 1. Here, we give a brief introduction to exponential generating functions. We begin with a simple example, in which we contrast the use of ogf’s and egf’s.

Example 2.1.

(a) Find the number of ways that three letters from the word SEQUENCE can be selected (order does not matter).[13]

(b) Find the number of ways that three letters from the word SEQUENCE can be arranged (order matters).

[13] Note that the letter E appears three times, and so we may select between zero and three copies of E. The three E’s are considered the same: so, if we select (say) two E’s, we do not care which particular two we selected.

Solution. (a) The number of ways we can select three letters from the word SEQUENCE is the coefficient in front of x^3 in the polynomial

$$f(x) \quad = \quad (1 + x + x^2 + x^3)(1 + x)^5,$$

which is 26. (Here, the polynomial $1 + x + x^2 + x^3$ corresponds to the letter E, and the five terms $1 + x$ correspond to the remaining five letters of the word SEQUENCE.)

More generally, the coefficient in front of x^k in $f(x)$ is the number of ways we can select k letters from the word SEQUENCE (when order does not matter). So in fact, $f(x)$ is the ogf for the sequence $\{a_k\}_{k=0}^{\infty}$, where a_k is the number of ways of selecting k letters from the word SEQUENCE (when order does not matter).

(b) Here, we use an egf. The number of ways we can arrange three letters from the word SEQUENCE is the coefficient in front of $\frac{x^3}{3!}$ in the polynomial

$$g(x) \quad = \quad (1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!})(1 + x)^5,$$

which is 136.

Let us explain why this is correct. For each $k \in \{0, 1, 2, 3\}$, we select k E's and $3 - k$ of the remaining five letters. The number of ways of selecting those $3-k$ other letters is precisely the coefficient in front of x^{3-k} in $(1+x)^5$, and then the number of ways of arranging our three chosen letters (k E's and $3 - k$ other letters) is $\frac{3!}{k!}$. So, the total number of ways of arranging three letters from the word SEQUENCE is precisely the coefficient in front of $\frac{x^3}{3!}$ in $g(x)$.

More generally, the coefficient in front of $\frac{x^k}{k!}$ in $g(x)$ is the number of ways we can arrange k letters from the word SEQUENCE (when order matters). So in fact, $g(x)$ is the ogf for the sequence $\{b_k\}_{k=0}^{\infty}$, where b_k is the number of ways of arranging k letters from the word SEQUENCE (when order matters). □

Example 2.2. *Find the ogf and egf of the constant sequence* $1, 1, 1, 1, \ldots$.

Solution. The ogf of the sequence is

$$f(x) \quad = \quad \sum_{n=0}^{\infty} x^n \quad = \quad \tfrac{1}{1-x},$$

whereas the egf of the sequence is

$$g(x) \quad = \quad \sum_{n=0}^{\infty} \tfrac{x^n}{n!} \quad = \quad e^x.$$

□

For some sequences, it is possible to find a closed formula for the egf, but not for the ogf. For instance, consider the sequence $\{n!\}_{n=0}^{\infty}$. The ogf of this sequence is

$$f(x) \quad = \quad \sum_{n=0}^{\infty} n!x^n,$$

which has radius of convergence 0,[14] i.e. the series only converges for $x = 0$. On the other hand, the egf of the sequence is

$$g(x) \quad = \quad \sum_{n=0}^{\infty} \tfrac{n!x^n}{n!} \quad = \quad \sum_{n=0}^{\infty} x^n \quad = \quad \tfrac{1}{1-x},$$

with the radius of convergence 1 (the series converges when $|x| < 1$).

The formulas for the basic operations with egf's are as follows. (Here, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences, and c is a constant).

- $\Big(\sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}\Big) \pm \Big(\sum\limits_{n=0}^{\infty} \frac{b_n x^n}{n!}\Big) = \sum\limits_{n=0}^{\infty} \frac{(a_n \pm b_n)x^n}{n!}$
- $c\Big(\sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}\Big) = \sum\limits_{n=0}^{\infty} \frac{c a_n x^n}{n!}$.
- $\Big(\sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}\Big)\Big(\sum\limits_{n=0}^{\infty} \frac{b_n x^n}{n!}\Big) = \sum\limits_{n=0}^{\infty} \big(\sum\limits_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\big) \frac{x^n}{n!}$
- $\frac{d}{dx}\Big(\sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}\Big) = \sum\limits_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!}$

The first two formulas above are obvious. For the third, we observe that the coefficient in front of x^n is $\sum\limits_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} = \sum\limits_{k=0}^{n} \binom{n}{k} \frac{a_k b_{n-k}}{n!}$, and the formula follows. Finally, for the fourth formula, we compute:

$$a'(x) \quad = \quad \sum_{n=1}^{\infty} \tfrac{n a_n x^{n-1}}{n!} \quad = \quad \sum_{n=1}^{\infty} \tfrac{a_n x^{n-1}}{(n-1)!} \quad = \quad \sum_{n=1}^{\infty} \tfrac{a_{n+1} x^n}{n!}.$$

Example 2.3. *A* derangement *of a set X is a permutation of X that has no fixed points.*[15] *For all integers $n \geq 0$, let d_n be the number of derangements of an n-element set. Find a recursive formula for the sequence $\{d_n\}_{n=0}^{\infty}$.*

Solution. Clearly, $d_0 = 1$ and $d_1 = 0$.[16] Now, fix an integer $n \geq 0$, and let X be a set of size $n + 2$. Fix any $a \in X$. Then a derangement of X can

[14] This can be shown using, for example, the Ratio Test.

[15] In other words, a *derangement* of X is a permutation π of X such that for all $x \in X$, $\pi(x) \neq x$.

[16] Indeed, the empty function is a permutation of the empty set, and it has no fixed points (so, it is a derangement). On the other hand, any one-element set admits only one permutation (namely, the identity), and this permutation has one fixed point (and so it is not a derangement).

map a to any element of $b \in X \setminus \{a\}$ (so, there $n+1$ choices for b). Now, suppose we have chosen b. Then our derangement of X either does or does not map b to a. If it does map b to a, then our derangement swaps a and b and then deranges $X \setminus \{a, b\}$; for fixed b, there are d_n choices for this type of derangement.[17] Suppose now that our derangement π does not map b to a. The number of such derangement is equal to the number of derangements of $X \setminus \{b\}$,[18] which is d_{n+1}. So, $d_{n+2} = (n+1)(d_n + d_{n+1})$.

We have now obtained the desired recursive formula:

- $d_0 = 1$, $d_1 = 0$;
- $d_{n+2} = (n+1)(d_n + d_{n+1})$ for all integers $n \geq 0$.

□

In our next example, we use egf's to find a non-recursive formula for d_n (from Example 2.3).

Example 2.4. *Let the sequence $\{d_n\}_{n=0}^{\infty}$ be defined recursively as follows:*

- $d_0 = 1$, $d_1 = 0$*;*
- $d_{n+2} = (n+1)(d_n + d_{n+1})$ *for all integers* $n \geq 0$*.*

Find a closed formula for the egf of the sequences $\{d_n\}_{n=0}^{\infty}$, and then find a non-recursive formula for d_n.

Solution. Let $d(x) = \sum\limits_{n=0}^{\infty} \frac{d_n x^n}{n!}$ be the egf of the sequence $\{d_n\}_{n=0}^{\infty}$. We first differentiate $d(x)$, and then we apply the recursive formula, as follows.

$$
\begin{aligned}
d'(x) &= \sum_{n=0}^{\infty} \frac{d_{n+1}x^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{d_{n+1}x^n}{n!} && \text{because } d_1 = 0 \\
&= \Big(\sum_{n=1}^{\infty} \frac{nd_{n-1}x^n}{n!}\Big) + \Big(\sum_{n=1}^{\infty} \frac{nd_n x^n}{n!}\Big) && \text{by the recursive formula} \\
&= x\Big(\sum_{n=0}^{\infty} \frac{d_n x^n}{n!}\Big) + x\Big(\sum_{n=0}^{\infty} \frac{d_{n+1}x^n}{n!}\Big) \\
&= xd(x) + xd'(x).
\end{aligned}
$$

[17] We are using the fact that $|X \setminus \{a, b\}| = n$.

[18] Indeed any derangement π of X such that $\pi(a) = b$ and $\pi(b) \neq a$ corresponds to a derangement of $X \setminus \{a\}$ that maps a to $\pi(b)$.

So, we have obtained a differential equation:

$$d'(x) = xd(x) + xd'(x).$$

The differential equation above is equivalent to $\frac{d'(x)}{d(x)} = \frac{x}{1-x}$, i.e.

$$\frac{d'(x)}{d(x)} = \frac{1}{1-x} - 1.$$

By integrating both sides, we get

$$\ln(d(x)) = -\ln(1-x) - x + C,$$

and since $d(0) = d_0 = 1$, we have that $C = 0$. So, $\ln(d(x)) = -\ln(1-x) - x$. By exponentiating both sides, we get

$$d(x) = \frac{e^{-x}}{1-x}.$$

We have now obtained a closed formula for the exponential generating function $d(x)$. To obtain a formula for d_n, we note that $e^{-x} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ and $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n = \sum\limits_{n=0}^{\infty} \frac{n! x^n}{n!}$. By the formula for the product of egf's, we now have that, for all integers $n \geq 0$,

$$d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!,$$

and we are done. □