Lattice QCD study of the heavy quark potential with the multilevel algorithm

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— 滞在型研究会「チャームバリオンの構造と生成」—

Abstract

直ちにチャームバリオンに応用できる話題ではないが... ,

- クォークの閉じ込め機構の説明や，クォーク模型や pNRQCD で使われる重いクォーク間ポテンシャルについて，マルチレベルアルゴリズムを用いた格子 QCD シミュレーションによって新たに明らかにされた性質について示す。
- マルチレベルアルゴリズムを用いると，Polyakov ループの相関関数から，クォークソースの形やゲージ固定に依らずに，精密かつ容易に基底状態のポテンシャルを計算できることを示す。
- マルチレベルアルゴリズムを用いた，３クォーク系やクォーク-ダイクォーク系などのバリオン系ポテンシャルの計算結果を示す。

Heavy quark potential in QCD

- **Hierarchy of energy scales, $m_q \gg m_q v \gg m_q v^2$ vs. $\Lambda_{\rm QCD}$**
- **Integration of the higher energy scales leads to effective field theories, HQET, NRQCD, pNRQCD, ... (Application of Wilson's renormalization group method to QCD)**
- **The matching coefficients in effective field theories are to be determined by perturbative or nonperturbative QCD depending on the matching scale**
- **pNRQCD contains matching coefficients corresponding to the heavy quark potential with relativistic corrections, which must be determined nonpertrubatively**

  [pNRQCD: Brambilla etal., Rev.Mod.Phys77(2005)1423]

Lattice QCD

- **Define QCD in four dimensional Euclidean space-time on a hyper-cubic lattice (lattice spacing a, lattice volume $L^3 \times T$)**
- **Periodic boundary conditions are imposed in all space-time directions**
- **The Wilson gauge action**

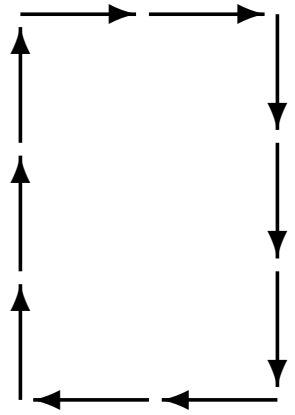
$$S = -\frac{\beta}{N_c} \mathrm{ReTr} \sum_{x,\mu\nu} [U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)]$$

- **Lattice QCD partition function and the expectation value of physical operators are computed numerically by the Monte-Carlo method**

Wilson loop

- **The heavy quark potential can be extracted from the expectation value of the Wilson loop $\langle W(r,t)\rangle$, the trace of the path-ordered product of link variables $U_\mu(x)$**

$$W(2,3) = \mathrm{Tr}$$



Wilson loop

- $W(r,t) = \mathrm{Tr}\,\mathbb{L}(0,\vec{x}_{1,2})^*_{\alpha\gamma}\{\mathbb{T}(0)\mathbb{T}(a)\cdots\mathbb{T}(t-a)\}_{\alpha\beta\gamma\delta}\mathbb{L}(t,\vec{x}_{1,2})_{\beta\delta}$
- $\mathbb{L}(0,\vec{x}_{1,2})_{\alpha\gamma}$ **and** $\mathbb{L}(t,\vec{x}_{1,2})_{\beta\delta}$

  products of spatial link variables from $\vec{x}_1$ **to** $\vec{x}_2$ **at** $x_0 = 0$ **and** t **(quark-antiquark source/sink)**
- $\mathbb{T}(x_0)_{\alpha\beta\gamma\delta} \equiv U_4(x_0,\vec{x}_1)_{\alpha\beta}U_4(x_0,\vec{x}_2)^*_{\gamma\delta}$ **(**9×9 **complex matrices)**

  the two-link correlators (direct product of two link variables separated by a distance $r = |\vec{x}_1 - \vec{x}_2|$**), which act on the color states in the** $3\otimes\bar{3}$ **rep. of SU(3) group** $|n;\vec{x}_{1,2}\rangle_{\alpha\beta}$ **:**

  $\mathbb{T}(x_0)_{\alpha\lambda\gamma\epsilon}|n;\vec{x}_{1,2}\rangle_{\alpha\gamma} = e^{-E_n(r)a}|n;\vec{x}_{1,2}\rangle_{\lambda\epsilon}$

Wilson loop

- **Transfer matrix theory:**

  inserting the complete set of eigenstates $1 = \sum_{m=0}^{\infty} |m; \vec{x}_{1,2}\rangle\langle m; \vec{x}_{1,2}|$

  at each time slices

  $$\Rightarrow \langle W(r,t)\rangle = \sum_{n=0}^{\infty} w_n(r,t) e^{-E_n(r)t}$$

  where $w_n(r,t) = \langle 0|\mathbb{L}(0,\vec{x}_{1,2})^*|n;\vec{x}_{1,2}\rangle\langle n;\vec{x}_{1,2}|\mathbb{L}(t,\vec{x}_{1,2})|0\rangle$

- $$-\frac{1}{t}\ln\langle W(r,t)\rangle = E_0(r) \underbrace{-\frac{1}{t}\ln w_0(r,t) + O(\frac{1}{t}e^{-(E_1-E_0)t})}_{\text{unwanted terms}}$$

  $$\Rightarrow V(r) \equiv E_0(r) = -\lim_{t\to\infty}\frac{1}{t}\ln\langle W(r,t)\rangle$$

Wilson loop

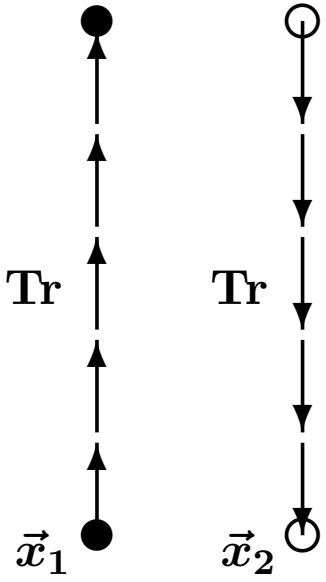
- **high simulation cost for large t, where $t < T/2$**
- **It is not straightforward to extract the ground state potential (at long distances) due to contamination from excited states as t cannot be large practically**

  $\Rightarrow$ good choice of the spatial quark source, smearing technique for better overlap with the ground state $w_0 \longrightarrow 1$ needed

PLCF

- **The heavy quark potential can also be extracted from the expectation value of the Polyakov loop correlation function (PLCF) $\langle \mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* \rangle$, a pair of Polyakov loops separated by a distance $r \equiv |\vec{x}_1 - \vec{x}_2|$**

$\mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* =$



- $\mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* = \{\mathbb{T}(0)\mathbb{T}(a)\cdots\mathbb{T}(T-a)\}_{\alpha\alpha\gamma\gamma}$

PLCF

- **Transfer matrix theory:**

  inserting the complete set of eigenstates $1 = \sum_{m=0}^{\infty} |m; \vec{x}_{1,2}\rangle\langle m; \vec{x}_{1,2}|$

  at each time slices

  $$\Rightarrow \langle \mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* \rangle = \sum_{n=0}^{\infty} w_n e^{-E_n(r)T} \qquad (w_0 = 1)$$

- $$-\frac{1}{T} \ln \langle \mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* \rangle = E_0(r) + \underbrace{O(\frac{1}{T} e^{-(E_1 - E_0)T})}_{\text{negligible}}$$

  $$\Rightarrow V(r) \equiv E_0(r) = -\frac{1}{T} \ln \langle \mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^* \rangle$$

- **Theoretically clean, but impossible to compute the PLCF accurately with ordinary simulations**

The multilevel algorithm

- **The multilevel algorithm is one of the most powerful noize reduction techniques in lattice QCD, which is applicable when the lattice QCD action consists of the sum of local variables**

  (ex.) The Wilson gauge action (sum of plaquette variables)

- **The multilevel algorithm computes the correlation functions of extremely small expectation values from the product of relatively large "sublattice average" of its components**

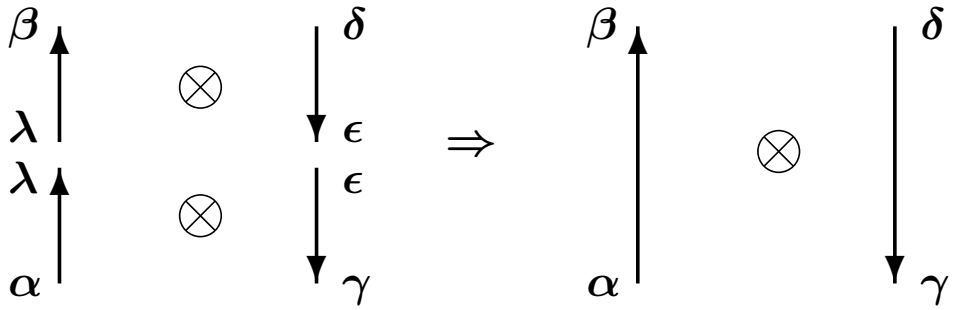
  ⇒ " hierarchical integration scheme of the partition function "

  [Lüscher&Weisz, JHEP0109(2001)010, JHEP0207(2002)049]

PLCF with the multilevel algorithm

- **(ex.) case** $T/a = 4$, $N_{\rm tsl} = 2$, $N_{\rm sub} = \dfrac{T/a}{N_{\rm tsl}} = 2$

1. **construct the Polyakov line correlators from the two-link correlators:**
   $\mathbb{T}(0)_{\alpha\lambda\gamma\epsilon}\mathbb{T}(a)_{\lambda\beta\epsilon\delta} = \{\mathbb{T}(0)\mathbb{T}(a)\}_{\alpha\beta\gamma\delta}$

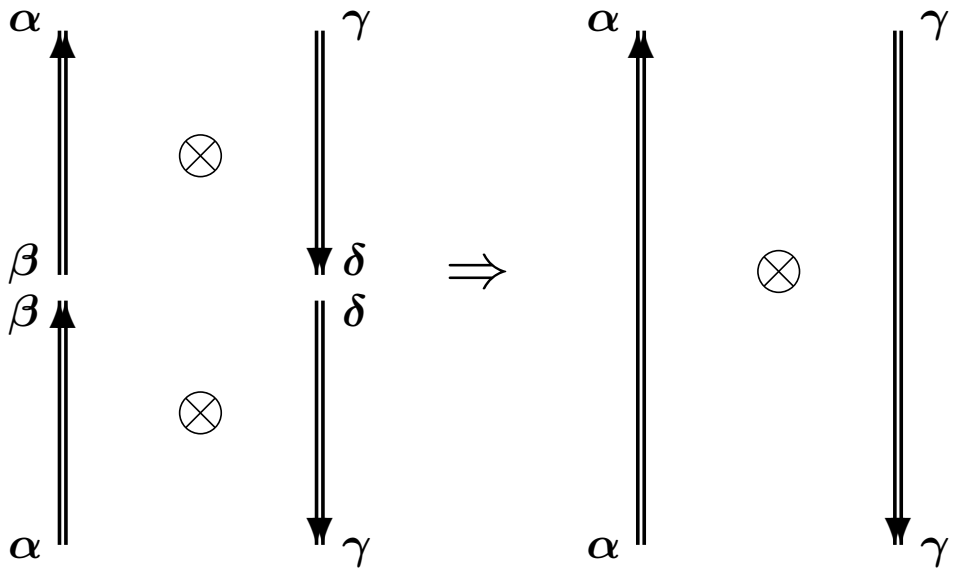


   ⇒ **compute the sublattice averages of the Polyakov line correlators,** $[\mathbb{T}(0)\mathbb{T}(a)]_{\alpha\beta\gamma\delta}$ **and** $[\mathbb{T}(2a)\mathbb{T}(3a)]_{\alpha\beta\gamma\delta}$**, by internal update of gauge configurations, where the spatial links at the sublattice boundaries remain intact during the internal update (iupd)**

PLCF with the multilevel algorithm

2. **construct the PLCF from the sublattice averages of the Polyakov line correlators:**

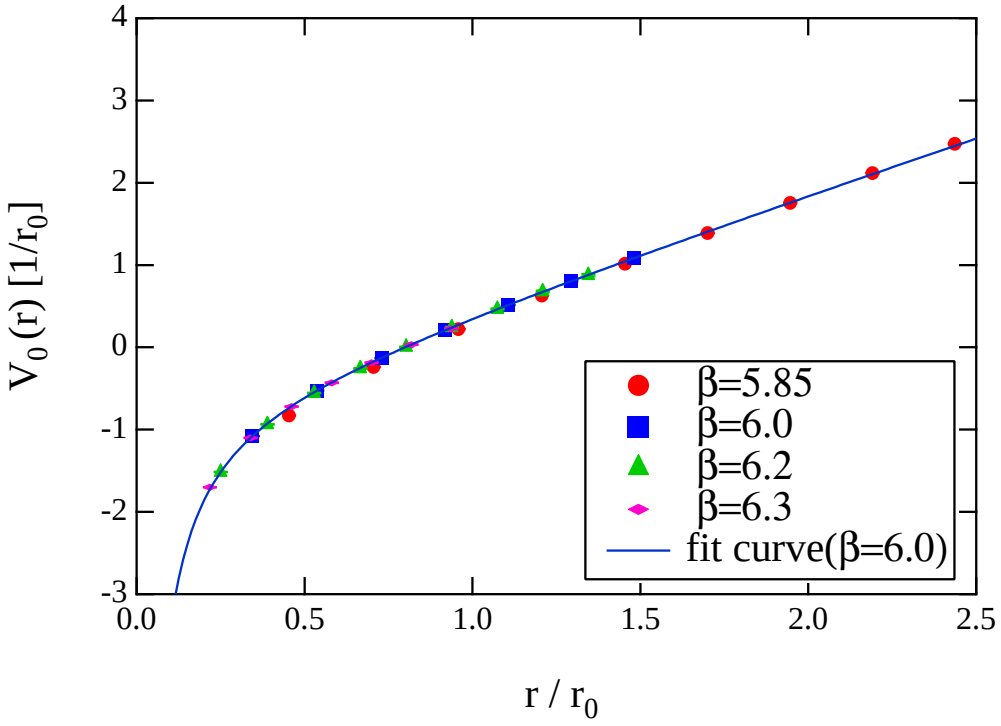
$$[\mathbb{T}(0)\mathbb{T}(a)]_{\alpha\beta\gamma\delta}[\mathbb{T}(2a)\mathbb{T}(3a)]_{\beta\alpha\delta\gamma} = [\mathbb{T}(0)\mathbb{T}(a)\mathbb{T}(2a)\mathbb{T}(3a)]_{\alpha\alpha\gamma\gamma}$$

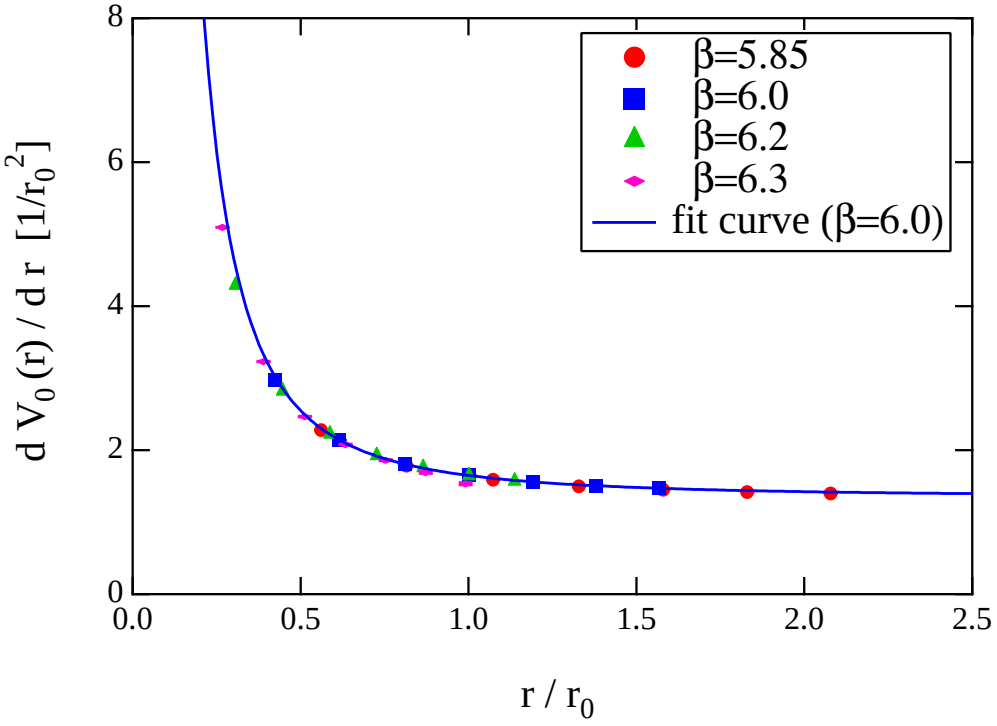


- **The number of internal updates N_{iupd} and the number of time slices in a sublattice N_{tsl} must be tuned appropriately depending on β**

Efficiency of the multilevel algorithm

- **(ex.) q-$\bar{q}$ potential and force from the PLCF**





[Koma&Koma, Nucl.Phys.B769(2007)79 etc.]

Other applications

- relativistic corrections to the q-$\bar{q}$ potential (input in pNRQCD) (field strength correlators)
- qqq potential (multi-quark system) ?
- q-qq potential ?
- flux-tube profile
- glueball spectrum
- energy momentum tensor
- ...

Why the multilevel algorithm works ?

- **(ex.) The PLCF for the q-$\bar{q}$ system with 2 sublattices:**

  $\Rightarrow$ fixed spatial links at $x_0 = 0,\ T/2$ correspond to inserting the two fixed intermediate states,

  $$|\phi_1\rangle_{\alpha\beta} = \sum_{n=0}^{\infty} a_n |n; \vec{x}_{1,2}\rangle_{\alpha\beta}, \qquad |\phi_2\rangle_{\alpha\beta} = \sum_{m=0}^{\infty} b_m |m; \vec{x}_{1,2}\rangle_{\alpha\beta},$$

  where a_n and b_n are unknown

- $\mathrm{Tr} P(\vec{x}_1) \mathrm{Tr} P(\vec{x}_2)^*$

$$= \mathrm{Tr}\left[\langle\phi_1|\mathbb{T}(0)\cdots\mathbb{T}(\frac{T}{2}-a)|\phi_2\rangle\right]\left[\langle\phi_2|\mathbb{T}(\frac{T}{2})\cdots\mathbb{T}(T-a)|\phi_1\rangle\right]$$

$$= \sum_{\alpha\gamma\lambda\epsilon}\left(\sum_{n=0}^{\infty} a_n^* b_n e^{-E_n(r)(T/2)} \cdot \sum_{m=0}^{\infty} b_m^* a_m e^{-E_m(r)(T/2)}\right)$$

Why the multilevel algorithm works ?

- **If $O(e^{-(E_1-E_0)(T/2)})$ is already negligible:**

$$\mathrm{Tr}P(\vec{x}_1)\mathrm{Tr}P\,(\vec{x}_2)^* = \sum_{\alpha\gamma\lambda\epsilon} \underbrace{|a_0|^2|b_0|^2 e^{-E_0(r)T}}_{\text{independent of } \alpha\gamma\lambda\epsilon}$$

$\Rightarrow$ $E_0(r)$ can be extracted by choosing the number of time slices N_{tsl} in a sublattice appropriately

$\Rightarrow$ Each color component of the intermediate states equally contributes to the potential (practically, $|a_0|^2 = |b_0|^2 = 1$)

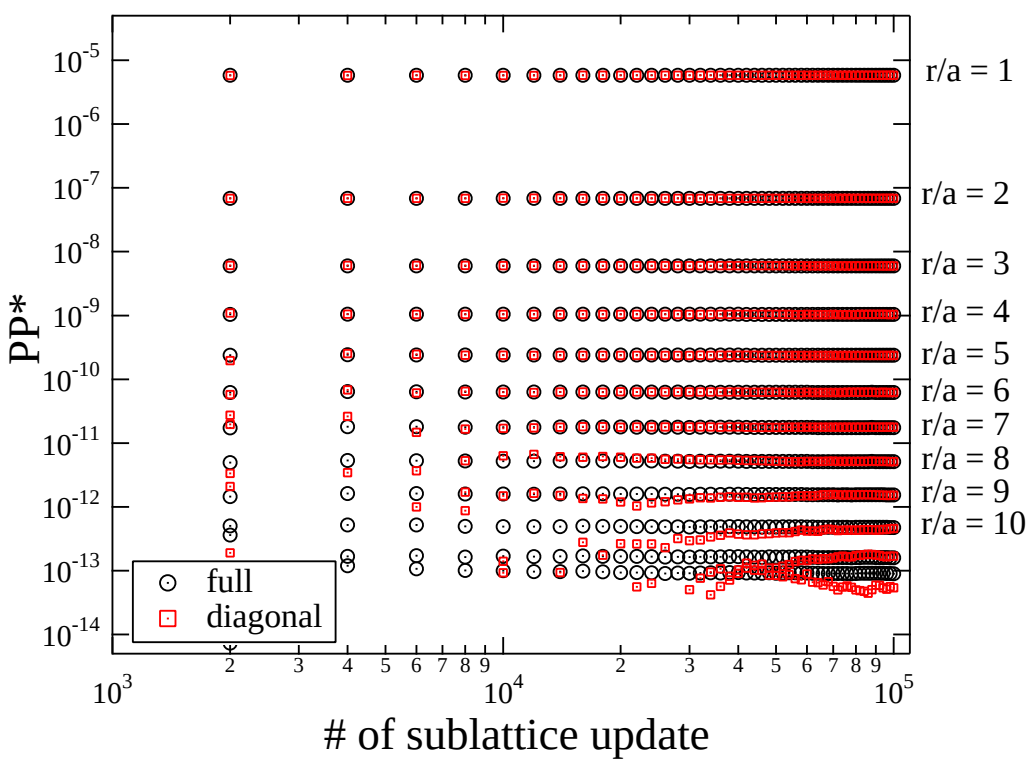
$\Rightarrow$ Possible to obtain a gauge-invariant potential from the gauge-variant PLCF with partial intermediate states ?

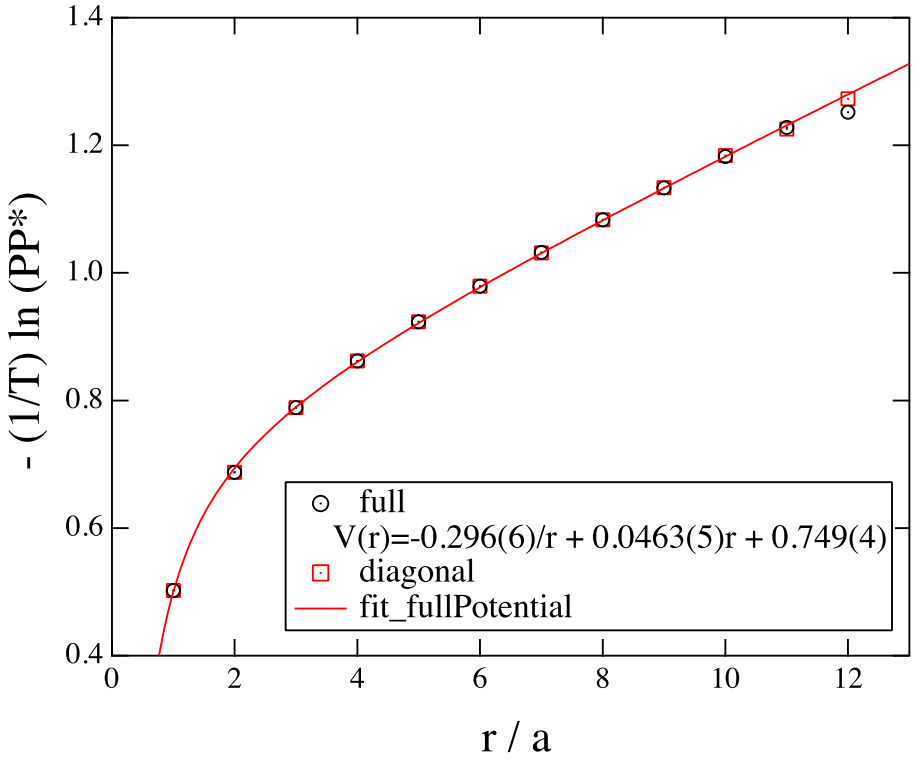
Demonstration 1

- **q-$\bar{q}$ potential ($\beta = 6.0$, 24^4 lattice, $N_{\rm sub} = 6$)**

  $[\mathbb{T}]_{\alpha\beta\gamma\delta}$(full) vs. $[\mathbb{T}]_{\alpha\alpha\gamma\gamma}$(diagonal)

  full : diag $= 59049 : 1 = (3^{N_{\rm sub}-1})^2 : 1$



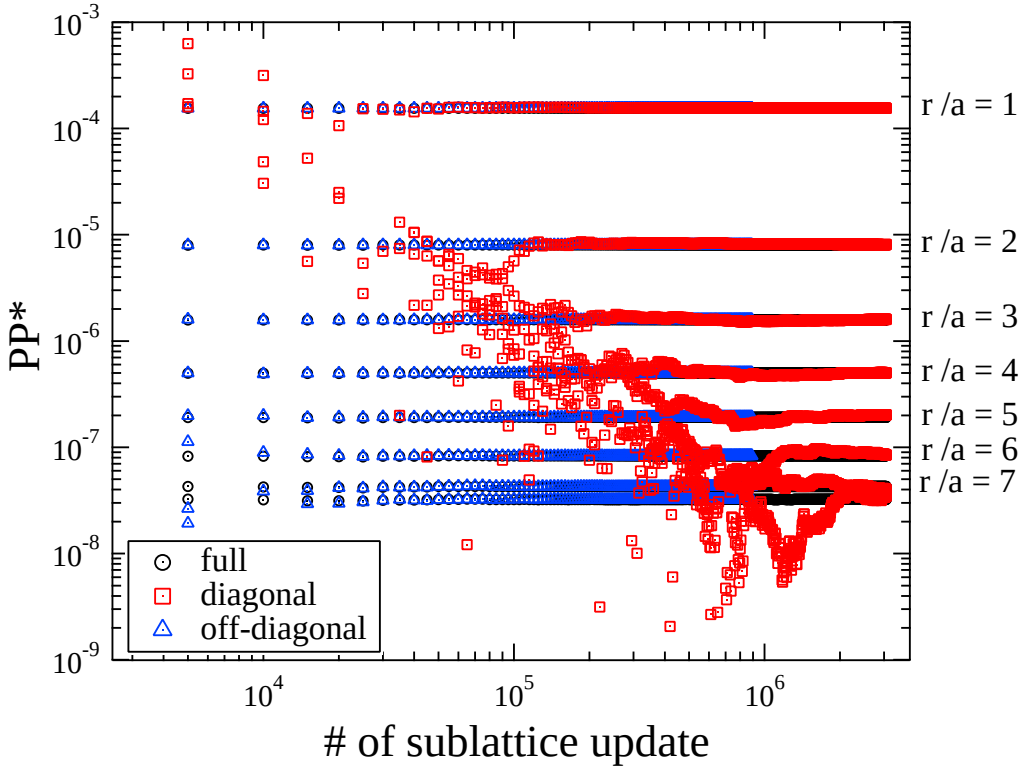


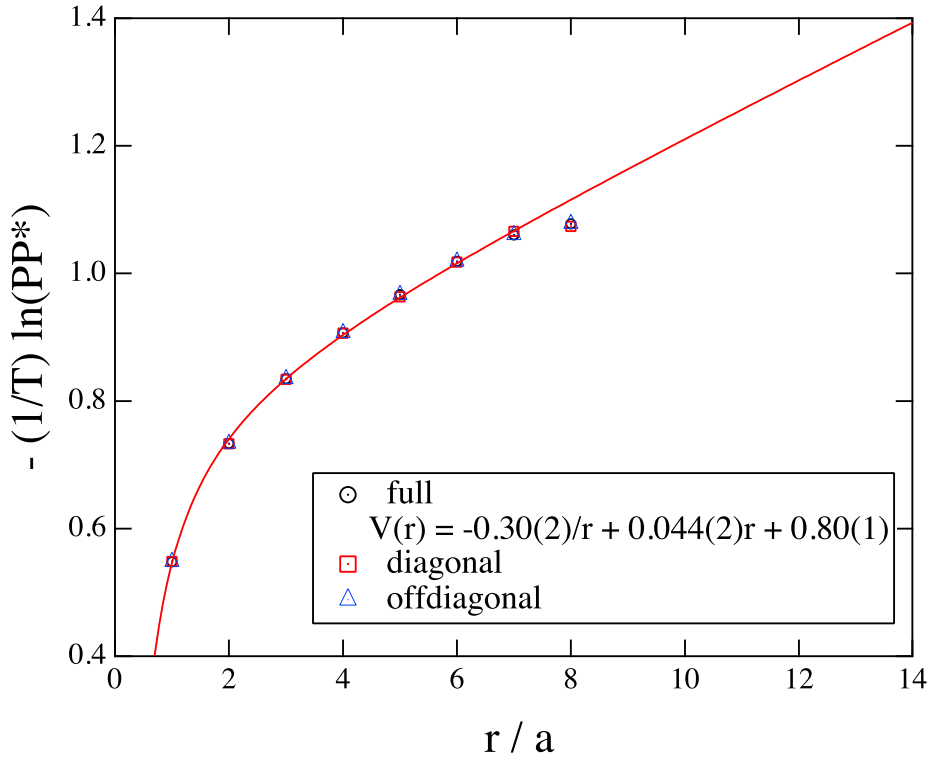
Demonstration 2

- q-$\bar{q}$ potential ($\beta = 6.0$, 16^4 lattice, $N_{\rm sub} = 4$)

$U_{4\alpha\beta}U^*_{4\gamma\delta}$(full) vs. $U_{4\alpha\alpha}U^*_{4\gamma\gamma}$(diag) vs. $U_{4\alpha\neq\beta}U^*_{4\gamma\neq\delta}$(offdiag)

$$\text{full} : \text{diag} : \text{offdiag} = 205891132094649 \ : \ 1 \ : \ 477247716$$
$$= (3^{T/a-1})^2 : 1 : \left((2^{T/a}+2(-1)^{T/a})/3\right)^2$$





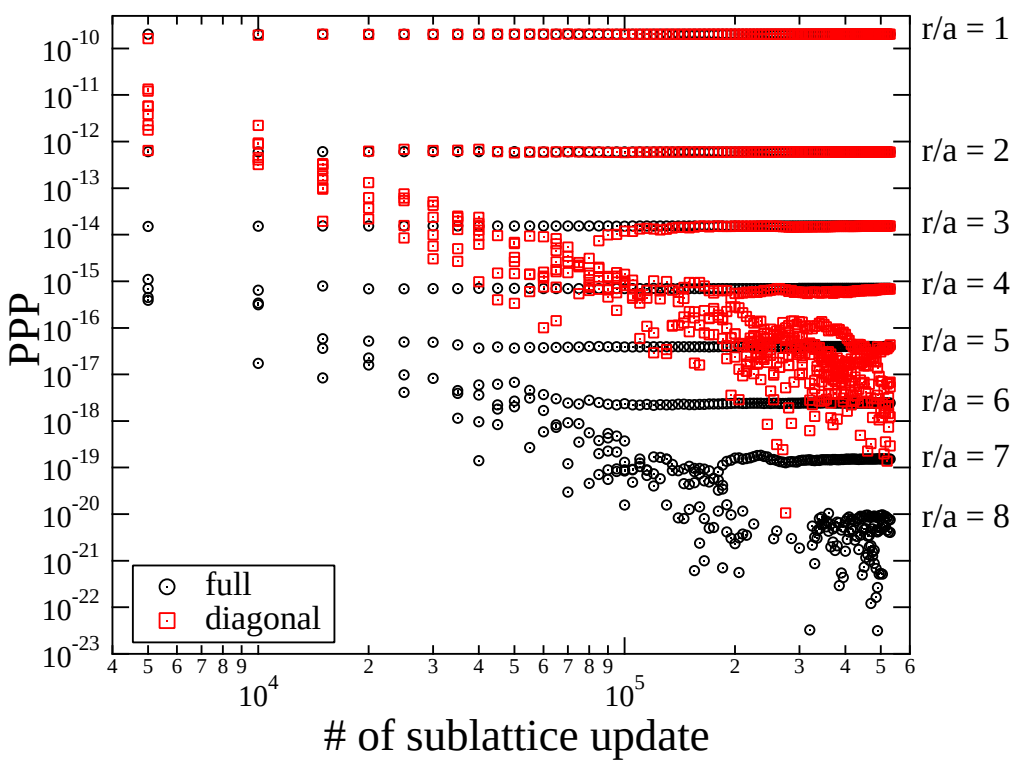
Application: qqq potential

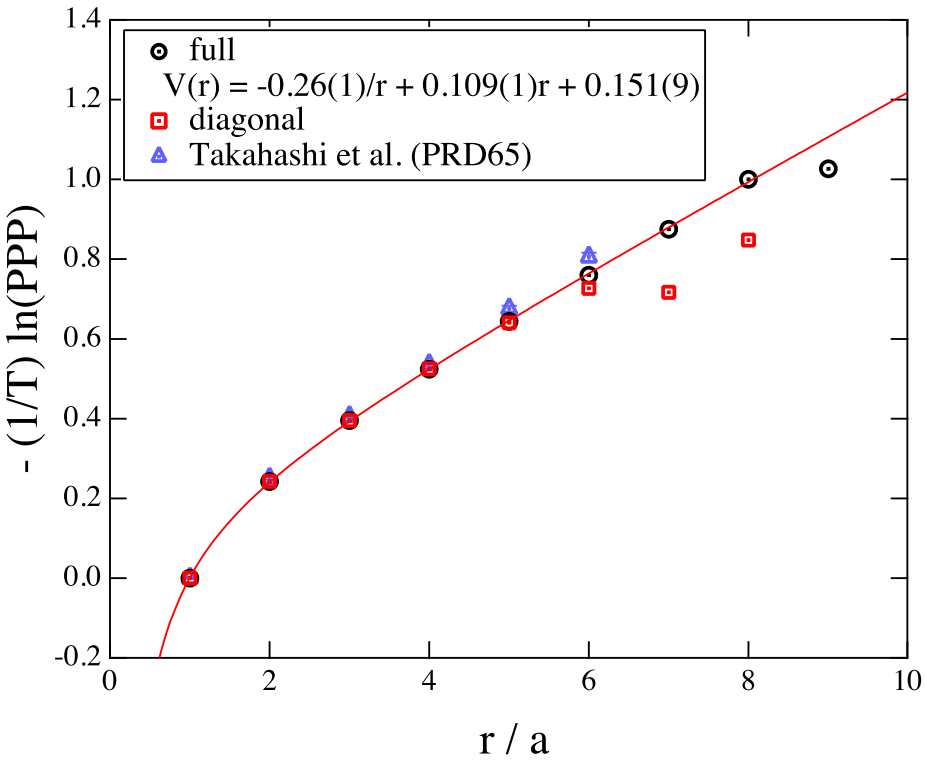
- **qqq potential ($\beta = 6.0$, 24^4 lattice, $N_{\rm sub} = 6$)**

  $\vec{x}_1 = (r, 0, 0)$, $\vec{x}_2 = (0, r, 0)$, $\vec{x}_3 = (0, 0, r)$

  $[\mathbb{T}]_{\alpha\beta\gamma\delta\epsilon\zeta}$**(full) vs.** $[\mathbb{T}]_{\alpha\alpha\gamma\gamma\epsilon\epsilon}$**(diagonal)**

  full : diag $= 14348907 : 1 = (3^{N_{\rm sub}-1})^3 : 1$





Application: qqq potential

- $R_Y = \sqrt{6}r$: (ex.) $r = 8$ corresponds to $R_Y = 1.8$ fm

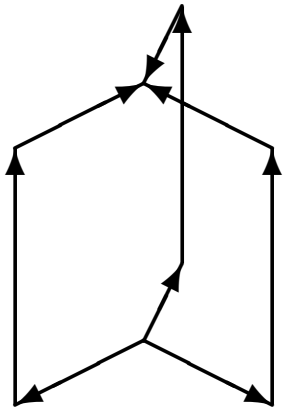
  $\Rightarrow \sigma_Y a^2 = 0.0445 \simeq \sigma_{q\bar{q}} a^2$

- $R_\Delta = 3\sqrt{2}r$: (ex.) $r = 8$ corresponds to $R_\Delta = 3.2$ fm

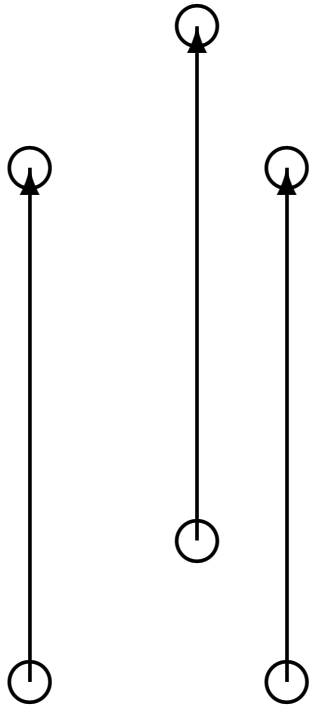
  $\Rightarrow \sigma_\Delta a^2 = 0.0257$

- 3-quark Wilson loop vs. PLCF

  [Takahashi etal., PRD65(2002)114509]



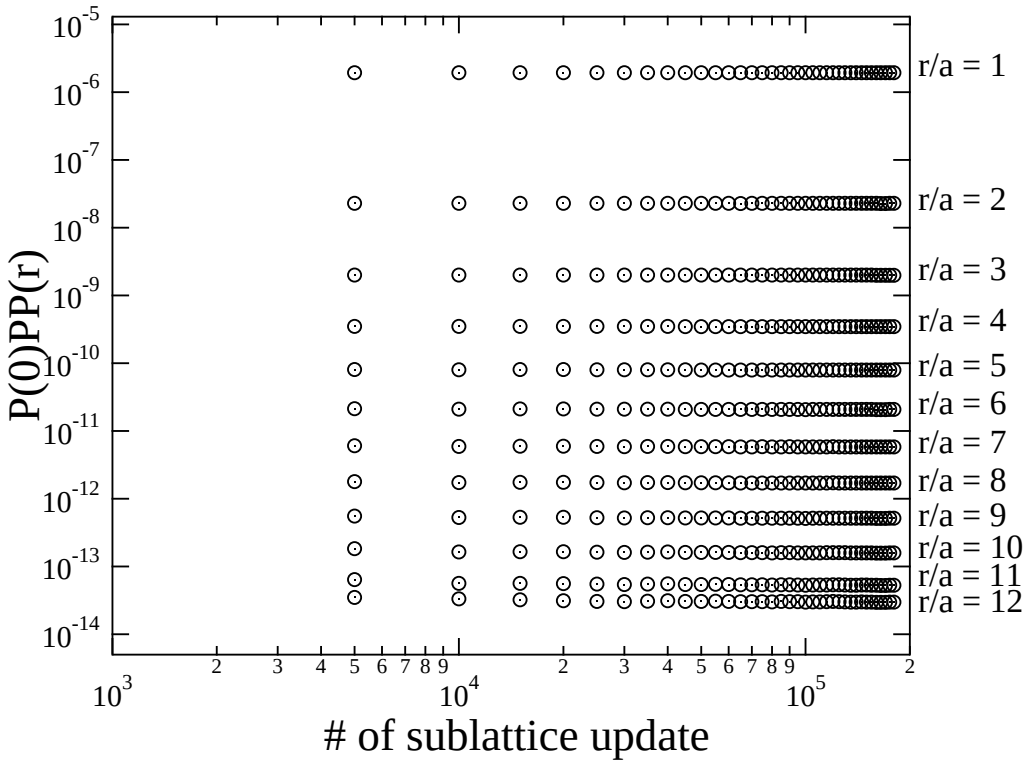
vs.

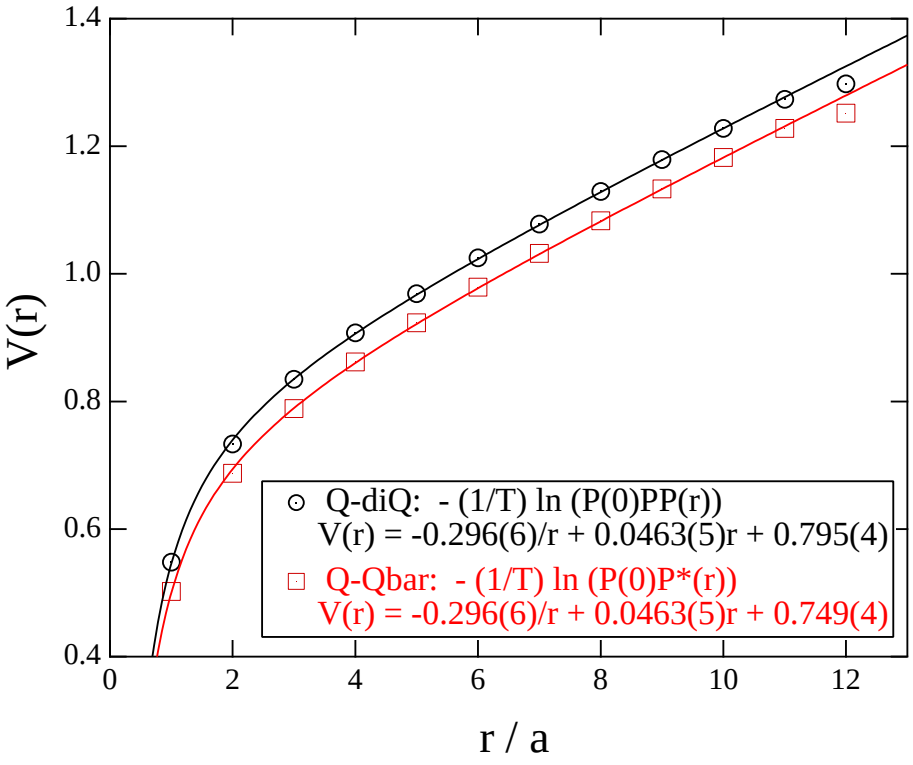


Application: q-qq potential

- **q-qq potential ($\beta = 6.0$, 24^4 lattice, $N_{\rm sub} = 6$)**

  $\mathbf{3} \otimes (\bar{\mathbf{3}} \oplus \mathbf{6}) \Rightarrow \mathbf{3} \otimes \bar{\mathbf{3}}$





- **q-$\bar{q}$ and q-qq potential are exactly the same (up to constant)**

  [cf. Bissey,Signal&Leinweber, PRD80(2009)114506]

Summary

- By employing the multilevel algorithm, we have investigated the static inter-quark potentials from the Polyakov loop correlation function (PLCF) (constructed from the selected intermediate states)
- While the use of partial intermediate states cannot guarantee gauge invariance of the PLCF, we have found that the functional forms of the PLCF and the static potential are unchanged from the gauge invariant ones, which is due to the fact that each color component of the intermediate states "equally" contributes to the PLCF
- The method is applicable not only to the q-$\bar{q}$ potential but also to other inter-quark potentials such as the qqq and q-qq systems