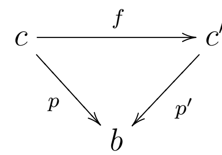
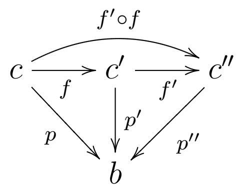
17. Slice categories and comma categories

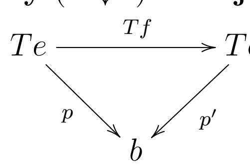
For $b \in C_0$, **slice category** $(C \downarrow b)$ or $(1_C \downarrow b)$ of C-**objects over** b has object class $\partial_1^{-1}(b)$, morphisms  (commuting),

composition , terminal object $1_b \colon b \to b$.

Dually, **slice category** $(b \downarrow C)$ or $(b \downarrow 1_C)$ of C-**objects under** b.

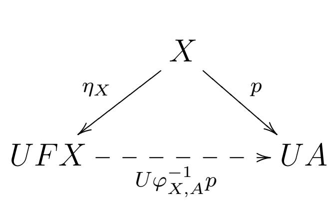
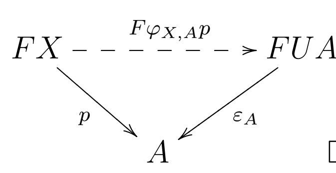
Examples: Down-sets, and up-sets (or principal filters), in posets.

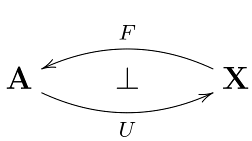
Example: For a group G and G-module A in $\mathbf{Ab}^G$,
the split extension $p \colon A \times G \to G; (a, g) \mapsto g$ in $(\mathbf{Grp} \downarrow G)$.
Here $(a, g)(a', g') = (a + ga', gg')$.

For $b \in C_0$ and $T \colon E \to C$,
comma category $(T \downarrow b)$ of **objects** T**-over** b
has morphisms .

Dually, for $b \in C_0$ and $S \colon D \to C$,
comma category $(b \downarrow S)$ of **objects** S**-under** b.

Proposition: For adjunction $(F \colon \mathbf{X} \to \mathbf{A}, U \colon \mathbf{A} \to \mathbf{X}, \eta, \varepsilon)$,
unit $\eta_X \colon X \to UFX$ is an initial object of $(X \downarrow U)$ and
counit $\varepsilon_A \colon FUA \to A$ is a terminal object of $(F \downarrow A)$.

Proof.  and  □

Cor: Given , unit and counit uniquely determined.