

17. SLICE CATEGORIES AND COMMA CATEGORIES

For $b \in C_0$, **slice category** $(C \downarrow b)$ or $(1_C \downarrow b)$ of C -**objects over** b has object class $\partial_1^{-1}(b)$, morphisms $c \xrightarrow{f} c'$ (commuting),

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ & \searrow p & \swarrow p' \\ & & b \end{array}$$

composition $c \xrightarrow{f} c' \xrightarrow{f'} c''$, terminal object $1_b: b \rightarrow b$.

$$\begin{array}{ccccc} & & f' \circ f & & \\ & \curvearrowright & & \curvearrowleft & \\ c & \xrightarrow{f} & c' & \xrightarrow{f'} & c'' \\ & \searrow p & \downarrow p' & \swarrow p'' & \\ & & b & & \end{array}$$

Dually, **slice category** $(b \downarrow C)$ or $(b \downarrow 1_C)$ of C -**objects under** b .

Examples: Down-sets, and up-sets (or principal filters), in posets.

Example: For a group G and G -module A in \mathbf{Ab}^G , the split extension $p: A \times G \rightarrow G; (a, g) \mapsto g$ in $(\mathbf{Grp} \downarrow G)$. Here $(a, g)(a', g') = (a + ga', gg')$.

For $b \in C_0$ and $T: E \rightarrow C$, **comma category** $(T \downarrow b)$ of **objects** T -**over** b

has morphisms $Te \xrightarrow{Tf} Te'$.

$$\begin{array}{ccc} Te & \xrightarrow{Tf} & Te' \\ & \searrow p & \swarrow p' \\ & & b \end{array}$$

Dually, for $b \in C_0$ and $S: D \rightarrow C$, **comma category** $(b \downarrow S)$ of **objects** S -**under** b .

Proposition: For adjunction $(F: \mathbf{X} \rightarrow \mathbf{A}, U: \mathbf{A} \rightarrow \mathbf{X}, \eta, \varepsilon)$, unit $\eta_X: X \rightarrow UFX$ is an initial object of $(X \downarrow U)$ and counit $\varepsilon_A: FUA \rightarrow A$ is a terminal object of $(F \downarrow A)$.

Proof.

$$\begin{array}{ccc} & X & \\ \eta_X \swarrow & & \searrow p \\ UFX & \dashrightarrow & UA \\ & U\varphi_{X,A}^{-1} & \end{array} \quad \text{and} \quad \begin{array}{ccc} FX & \dashrightarrow & FUA \\ & \searrow p & \swarrow \varepsilon_A \\ & & A \end{array} \quad \square$$

Cor: Given $\mathbf{A} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \mathbf{X}$, unit and counit uniquely determined.