How to write a translator to Dedukti

The case of Agda

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How to write a translator to Dedukti

Previous talks. How to define theories and write proofs in Dedukti (eg. the theory $\mathcal{U}$).

This talk. How to write an automatic translator from a proof assistant to Dedukti:

- General principles on writing such a translator
- Specific case of the Agda2Dedukti translator

From Agda to Dedukti

1. Principles on translating from a proof assistant to Dedukti
2. What is Agda?
3. Encoding Agda in Dedukti
4. Implementation of Agda2Dedukti
5. Inductive types and dependent pattern matching
6. Universe polymorphism
7. Eta equality & irrelevance
8. Conclusion

How to translate from a proof assistant to Dedukti

Step 0. Find/define a system $\mathcal{O}$ corresponding to the proof assistant's logic (not easy!)

Step 1. Define a Dedukti theory $D[\mathcal{O}] = (\Sigma, \mathcal{R})$ representing the object logic in Dedukti.

Step 2. Define a translation $[\![-]\!] : \Lambda_{\mathcal{O}} \to \Lambda_{DK}$. The pair $(D[\mathcal{O}], [\![-]\!])$ is an encoding of $\mathcal{O}$.

Step 3. Implement the translating function, making use of the APIs and other tools offered by the proof assistant.

Not all encodings are created equal

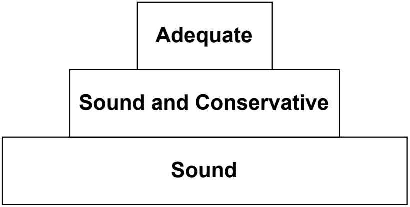
- An encoding is sound if:

$$\vdash_{\mathcal{O}} M : A \quad \text{implies} \quad \vdash_{D[\mathcal{O}]} [\![M]\!] : El\ [\![A]\!]$$

- An encoding is conservative if:

$$\vdash_{D[\mathcal{O}]} M : El\ [\![A]\!] \quad \text{implies} \quad \exists N,\ \vdash_{\mathcal{O}} N : A$$

- An encoding is adequate if for each type A:
  $[\![-]\!]$ is a *compositional bijection* between A and $El\ [\![A]\!]$



Nor are all proof assistants equal

The difficulty of encoding (the core language of) a proof assistant depends on its features:

Dependent types are in Coq, Agda, Lean, …

Inductive types are in most proof assistants.[1]

Universe polymorphism is in Coq, Agda, Lean, …

Impredicativity is in all proof assistants, except Agda and Epigram.

Eta-equality & irrelevance are present in different shapes in different proof assistants.

[1] Most type-theoretic proof assistants also support inductive families.

Neither are their implementations

The difficulty of writing a translator also depends on the *implementation* of the proof assistant:

- In systems based on Curry-Howard (Coq/Agda/Matita), proof terms are already *in the internal syntax*, so are easier to translate.
- In LCF-like assistants (Isabelle/HOL), there are no proof terms, so we need to *reconstruct* them from proof derivations.
- In other systems (PVS), proofs derivations are not even internally available.[2]
- ...

[2] See Gabriel's talk for a solution.

From Agda to Dedukti

What is Agda?

Agda is a dependently typed programming language and proof assistant based on Martin-Löf type theory.

It has indexed datatypes, dependent pattern matching, and explicit universe polymorphism.

Its type checker identifies terms up to β-equality and η-equality for functions and records, and supports definitional proof irrelevance.

Data types in Agda

$$
\begin{array}{l}
\mathsf{data}\ \_\uplus\_\ (A\ B : \mathsf{Set}) : \mathsf{Set}\ \mathsf{where} \\
\quad \mathsf{left}\ \ : A \to A \uplus B \\
\quad \mathsf{right} : B \to A \uplus B
\end{array}
$$

$$
\begin{array}{l}
\mathsf{data}\ \_\leq\_ : \mathbb{N} \to \mathbb{N} \to \mathsf{Set}\ \mathsf{where} \\
\quad \leq\text{-}\mathsf{zero} : \forall\ \{n\} \qquad\qquad\qquad\ \to \mathsf{zero} \quad \leq n \\
\quad \leq\text{-}\mathsf{suc}\ \ : \forall\ \{m\ n\} \to m \leq n \to \mathsf{suc}\ m \leq \mathsf{suc}\ n
\end{array}
$$

Pattern matching in Agda

```
_<_ : ℕ → ℕ → Set
m < n = m ≤ suc n

compare : (m n : ℕ) → (m ≤ n) ⊎ (n < m)
compare zero    n       = left  ≤-zero
compare (suc m) zero    = right ≤-zero
compare (suc m) (suc n) with compare m n
... | left  m≤n         = left  (≤-suc m≤n)
... | right n<m         = right (≤-suc n<m)
```

Agda as a PTS

At its core, Agda is a pure type system with sorts Set ℓ where ℓ is a universe level.

$$\mathsf{U} : (\ell : \mathsf{Level}) \to \mathsf{Set}\ (\mathsf{lsuc}\ \ell)$$
$$\mathsf{U}\ \ell = \mathsf{Set}\ \ell$$

$$\begin{aligned}\mathsf{prod} :\ & (\ell_1\ \ell_2 : \mathsf{Level}) \\ & (A : \mathsf{Set}\ \ell_1)\ (B : A \to \mathsf{Set}\ \ell_2) \\ \to\ & \mathsf{Set}\ (\ell_1 \sqcup \ell_2)\end{aligned}$$
$$\mathsf{prod}\ \_\ \_\ A\ B = (x : A) \to B\ x$$

From Agda to Dedukti

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$
Def. symbol	$[\![\mathsf{f}]\!]$	$=$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$
Lambda	$[\![\lambda x \to u]\!]$	$=$
Application	$[\![u\ v]\!]$	$=$
Pi type	$[\![(x : A) \to B]\!]$	$=$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$

Encoding Agda terms in Dedukti

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Tarski- vs. Russell-style universes[3]

Agda uses Russell-style universes: Elements are *types* themselves.

$$\frac{A : \mathsf{Set}_l}{A \ \ \textsc{type}}$$

In Dedukti, if $A : \mathsf{Set}$, we cannot have $a : A$. Thus, Dedukti uses a form of Tarski-style universes: Elements are *codes* that can be *interpreted* as types.

$$\frac{c : \mathtt{U}\ (\mathtt{set}\ l)}{\mathtt{El}\ (\mathtt{set}\ l)\ c \ \ \textsc{type}}$$

[3] https://www.cs.rhul.ac.uk/home/zhaohui/universes.pdf

Encoding Agda's PTS in Dedukti

```
Sort : Type.
set  : Lvl -> Sort.

U : (s : Sort) -> Type.
def El : (s : Sort) -> (a : U s) -> Type.

def axiom : Sort -> Sort.
[i] axiom (set i) --> set (s i).

def rule : Sort -> Sort -> Sort.
[i, j] rule (set i) (set j) --> set (max i j).
```

(We postpone the definition of `Lvl` until later,
for now you can assume `lvl` $= \mathbb{N}$.)

Encoding pi types

- Add a constant $\texttt{prod}$ for encoding the pi type:

$$\frac{A : \texttt{U}\ s_A \quad B : \texttt{El}\ s_A\ A \to \texttt{U}\ s_B}{\texttt{prod}\ s_A\ s_B\ A\ B : \texttt{U}\ (\texttt{rule}\ s_A\ s_B)}$$

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- Identify elements of $\texttt{prod}$ with the *metatheoretic arrow type*:

$$\begin{aligned}&\texttt{El}\ \_\ (\texttt{prod}\ s_A\ s_B\ A\ B)\\ &\quad = (x : \texttt{El}\ s_A\ A) \to El\ s_B\ (B\ x)\end{aligned}$$

Encoding pi types in Dedukti

```
prod : (s_A : Sort) ->
       (s_B : Sort) ->
       (A : U s_A) ->
       (B : (El s_A A -> U s_B)) ->
       U (rule s_A s_B).

[s_A, s_B, A, B]
      El _ (prod s_A s_B A B)
  --> (x : El s_A A) -> El s_B (B x).
```

Reconstructing sorts

For translating pi types, we need access to the sort of the domain and codomain.

Luckily, Agda's type checker already annotates each type A with its sort $s(A)$.

Examples. $s(\mathbb{N}) = \mathsf{Set}$, $s(\mathsf{Set}) = \mathsf{Set}_1$, $s(\mathsf{Set}_1 \to \mathsf{Set}) = \mathsf{Set}_2$

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
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Application	$[\![u\ v]\!]$	$=$	$[\![u]\!]\ \ [\![v]\!]$
Pi type	$[\![(x : A) \to B]\!]$	$=$	???
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Pi type	$[\![(x : A) \to B]\!]$	$=$	$\mathtt{prod}\ \lvert s(A)\rvert\ \lvert s(B)\rvert\ [\![A]\!]\ (x \Rightarrow [\![B]\!])$
	where $\lvert \mathsf{Set}\ \ell \rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

(We will see how to translate levels later.)

Encoding universe types

- Add a constant $\mathtt{u}$ for encoding the Set type:

$$\frac{s : \mathtt{Sort}}{\mathtt{u}\ s : \mathtt{U}\ (\mathtt{axiom}\ s)}$$

Encoding universe types

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$$\frac{s : \mathtt{Sort}}{\mathtt{u}\ s : \mathtt{U}\ (\mathtt{axiom}\ s)}$$

- Identify elements of $\mathtt{u}\ s$ with the ones of $\mathtt{U}\ s$:

$$\mathtt{El}\ \_\ (\mathtt{u}\ s) = \mathtt{U}\ s$$

In Dedukti:

```
u : (s : Sort) -> U (axiom s).
[i] El _ (u s) --> U s.
```

Encoding Agda terms in Dedukti

Variable	$[\![x]\!]$	$=$	x
Def. symbol	$[\![\mathsf{f}]\!]$	$=$	$\mathtt{f}$
Constructor	$[\![\mathsf{D.c}]\!]$	$=$	$\mathtt{D\_\_c}$
Lambda	$[\![\lambda x \to u]\!]$	$=$	$x \Rightarrow [\![u]\!]$
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	where $\lvert\mathsf{Set}\ \ell\rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	???

(We will see how to translate levels later.)

Encoding Agda terms in Dedukti

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	where $\lvert \mathsf{Set}\ \ell \rvert$	$=$	$\mathsf{set}\ [\![\ell]\!]$
Universe	$[\![\mathsf{Set}\ \ell]\!]$	$=$	$\mathtt{u}\ (\mathsf{set}\ [\![\ell]\!])$

(We will see how to translate levels later.)

Encoding Agda definitions in Dedukti

Data types (no parameters or indices)

$$\left[\!\!\left[\begin{array}{l} \mathsf{data}\ \mathsf{D} : U\ \mathsf{where} \\ \quad \mathsf{c} : A \end{array} \right]\!\!\right] = \begin{array}{l} \mathtt{D} : \mathtt{El}\ |s(U)|\ \llbracket U \rrbracket\,. \\ \mathtt{D\_\_c} : \mathtt{El}\ |U|\ \llbracket A \rrbracket\,. \end{array}$$

Function definitions (no pattern matching)

$$\left[\!\!\left[\begin{array}{l} \mathsf{f} : A \\ \mathsf{f}\ x = v \end{array} \right]\!\!\right] = \begin{array}{l} \mathtt{def\ f} : \mathtt{El}\ |s(A)|\ \llbracket A \rrbracket\,. \\ \mathtt{[x]\ f\ x\ \text{-->}}\ \llbracket v \rrbracket\,. \end{array}$$

From Agda to Dedukti

Implementation of Agda2Dedukti

Agda2Dedukti is implemented as an Agda backend.

This allows us to reuse parts of Agda's implementation:

- Internal syntax representation
- Type checking monad `TCM`

Structure of the Agda typechecker

.agda file

lexer & parser $\Downarrow$

Concrete syntax

scope checker $\Downarrow$

Abstract syntax

type checker $\Downarrow$

Internal syntax

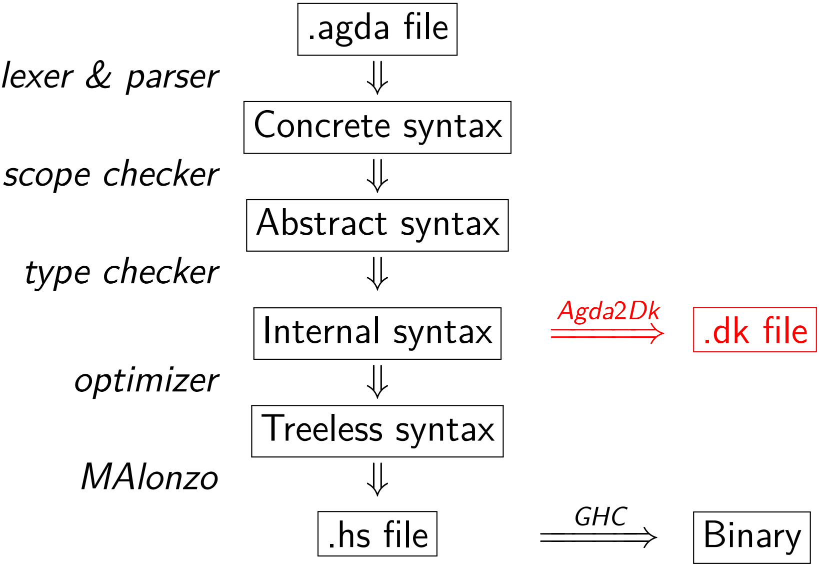
optimizer $\Downarrow$

Treeless syntax

MAlonzo $\Downarrow$

.hs file $\xRightarrow{\textit{GHC}}$ Binary

Structure of the Agda typechecker



Agda's internal syntax[4]

```
data Term
  = Var Int Elims                  -- x u v ..
  | Lam ArgInfo (Abs Term)         -- λ x → v
  | Lit Literal                    -- 42, 'a', ...
  | Def QName Elims                -- f u v ..
  | Con ConHead ConInfo Elims      -- c u v ..
  | Pi (Dom Type) (Abs Type)       -- (x : A) → B
  | Sort Sort                      -- Set, Set₁, Prop, ...
  | Level Level                    -- lzero, ...
  | MetaV MetaId Elims             -- _X_235
  | DontCare Term
  | Dummy String Elims
```

[4] Code from `Agda.Syntax.Internal`

Agda's TCM monad

Agda's typechecker uses a type-checking monad **TCM**:

```
type TCM a
getConstInfo :: QName -> TCM Definition
getBuiltin :: String -> TCM Term
getContext :: TCM Context
addContext :: (Name, Dom Type) -> TCM a -> TCM a
checkInternal :: Term -> Type -> TCM ()
reconstructParameters :: Type -> Term -> TCM Term
...
```

Putting it all together

$$\text{example} : (1 \leq 2) \uplus (2 < 1)$$
$$\text{example} = \text{left}\ (\leq\text{-suc}\ \leq\text{-zero})$$

```
{|!_⊎___left|}
  ({|!_≤_|}
    (Nat__suc Nat__zero)
    (Nat__suc (Nat__suc Nat__zero)))
  ({|!_<_|}
    (Nat__suc (Nat__suc Nat__zero))
    (Nat__suc Nat__zero))
  ({|!_≤___≤-suc|}
    Nat__zero
    (Nat__suc Nat__zero)
    ({|!_≤___≤-zero|} (Nat__suc Nat__zero)))
```

From Agda to Dedukti

Translating datatypes and constructors to constants

Data types and their constructors do not reduce, so we translate them to constants in Dedukti.

Example. _≤_ is translated to:

```
{|!_≤_|} : El (set (s 0)) (prod (set 0) (set (s 0))
  Nat (_0 => (prod (set 0) (set (s 0))
    Nat (_0 => (u (set 0)))))).

{|!_≤___≤-zero|} : El (set 0) (prod (set 0) (set 0)
  Nat (n => ({|!_≤_|} Nat__zero n))).

{|!_≤___≤-suc|} : El (set 0) (prod (set 0) (set 0) Nat
  (m => (prod (set 0) (set 0)
    Nat (n => (prod (set 0) (set 0)
      ({|!_≤_|} m n)
      (_0 => ({|!_≤_|} (Nat__suc m) (Nat__suc n)))))))).
```

Reconstruction of data parameters

Constructors in Agda do *not* store their parameters.

Reconstructing parameters requires a type-directed traversal of the syntax.

We can reuse Agda's `reconstructParameters`, which does exactly this!

Filling implicit arguments & reconstructing parameters

$$\text{left } (\leq\text{-suc } \leq\text{-zero}) : (1 \leq 2) \uplus (2 < 1)$$

Filling implicit arguments & reconstructing parameters

Agda's type checker infers implicit arguments during type checking.

$$\mathsf{left}\ (\leq\text{-}\mathsf{suc}\ \leq\text{-}\mathsf{zero}) : (1 \leq 2) \uplus (2 < 1)$$
$$\Downarrow$$
$$\mathsf{left}\ (\leq\text{-}\mathsf{suc}\ \{\mathsf{m} = 0\}\ \{\mathsf{n} = 1\}\ (\leq\text{-}\mathsf{zero}\ \{\mathsf{n} = 1\}))$$

Filling implicit arguments & reconstructing parameters

Agda's type checker infers implicit arguments during type checking.

Agda2Dk makes all implicit arguments explicit and reconstructs constructor parameters.

$$\text{left } (\leq\text{-suc } \leq\text{-zero}) : (1 \leq 2) \uplus (2 < 1)$$
$$\Downarrow$$
$$\text{left } (\leq\text{-suc } \{m = 0\}\ \{n = 1\}\ (\leq\text{-zero } \{n = 1\}))$$
$$\Downarrow$$
$$\text{left } (1 \leq 2)\ (2 < 1)\ (\leq\text{-suc } 0\ 1\ (\leq\text{-zero } 1))$$

Translating clauses to rewrite rules

Functions in Agda are defined by a set of clauses, so we translate them to a constant + a set of rewrite rules.

Example. compare is translated to:

```
def compare : El (set 0) (prod (set 0) (set 0)
  Nat (m => (prod (set 0) (set 0)
    Nat (n => ({|!_⊎_|} ({|!_≤_|} m n) ({|!_<_|} n m)))))).
[n] compare Nat__zero n -->
  {|!_⊎___left|} ({|!_≤_|} Nat__zero n)
    ({|!_<_|} n Nat__zero) ({|!_≤___≤-zero|} n).
[m] compare (Nat__suc m) Nat__zero -->
  {|!_⊎___right|} ({|!_≤_|} (Nat__suc m) Nat__zero)
    ({|!_<_|} Nat__zero (Nat__suc m))
    ({|!_≤___≤-zero|} (Nat__suc (Nat__suc m))).
[m, n] compare (Nat__suc m) (Nat__suc n) -->
  {|!with-66|} m n (compare m n).
```

Drawbacks of generating rewrite rules

Generating a new rewrite rule for each clause means that we are extending the theory with each definition.

Moreover, checking correctness (completeness & termination) of rewrite rules is very hard.

Ongoing work: Instead, we can translate definitions by pattern matching to eliminators.[5]

```
def compare := Nat__ind...
```

[5] Ask Thiago for details!

From Agda to Dedukti

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Universe polymorphism allows definitions that can be used at multiple universe levels:

```
data List {i} (A : Set i) : Set i where
  [] : List A
  _::_ : A → List A → List A

map : {i j : Level} → {A : Set i} → {B : Set j}
  → (f : A → B) → List A → List B
map f [] = []
map f (x :: l) = f x :: map f l
```

Other forms of universe polymorphism

Universe polymorphism in Agda is very different from universe polymorphism in Coq:

	Coq	Agda
Typical ambiguity		
Cumulativity ($Set_i \subseteq Set_{i+1}$)		
Definitions carry constraints		

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In this talk we only see the encoding of Agda's universe polymorphism.

For Coq's version, see Gaspard Ferey's PhD thesis.

Universe polymorphism in Dedukti

Idea. Generalize the encoding of the arrow type:

```
setOmega  : Sort.

forall : (l : (Lvl -> Sort)) ->
         ((i : Lvl) -> U (l i)) -> U setOmega.

[l, t] El _ (forall l t) -->
          (i : Lvl) -> El (l i) (t i).
```

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```

We extend the translation function with:

Level quantification	$[\![(i : Level) \to A]\!]$	$=$	$\texttt{forall}\ (i \Rightarrow [\![s(A)]\!])$ $(i \Rightarrow [\![A]\!])$
Level application	$[\![M\ l]\!]$	$=$	$[\![M]\!]\ [\![l]\!]$
Level abstraction	$[\![\lambda i.M]\!]$	$=$	$i \Rightarrow [\![M]\!]$

Back to List

Now the constant `List` can be given the type:

```
El setOmega
   (forall (i => set (suc i))
           (i => prod (set (suc i))
                      (set (suc i))
                      (u (set i))
                      (_ => u (set i))))
```

Which, as expected, computes to:

```
(i : Lvl) -> U (set i) -> U (set i)
```

Universe levels

Levels are given by the syntax:

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Commutativity: $a \sqcup b = b \sqcup a$

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Universe levels

Levels are given by the syntax:

$$l, l_1, l_2 ::= i \mid \mathsf{lzero} \mid \mathsf{lsuc}\ l \mid l_1 \sqcup l_2\,.$$

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Subsumption: $a \sqcup \mathsf{lsuc}^n\ a = \mathsf{lsuc}^n\ a$

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3. Decision procedure integrated in Dedukti? *We leave this to the future generations.*

Current solution: levels as sets

Idea. Every level l admits a unique canonical form

$$l = \max\{n, i_1 + m_1, ..., i_k + m_k\}$$

where $i_1, .., i_k \in FV(l)$, $n, m_1, .., m_k \in \mathbb{N}$ and $m_j \leq n$.

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But idempotence and subsumption require a non-linear rule:

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This *breaks confluence of pre-terms*, and prevents proving conservativity.

From Agda to Dedukti

Eta equality in Agda

Agda supports two kinds of eta-equality:

1. Eta for functions:

$$\frac{f : (x : A) \to B}{f = (\lambda x \to f\ x) : (x : A) \to B}$$

2. Eta for records:[6]

$$\frac{u : \Sigma\ A\ B}{u = (\mathsf{proj}_1\ u, \mathsf{proj}_2\ u) : \Sigma\ A\ B}$$

[6] Also known as surjective pairing for Σ.

Definitional singleton types

Agda supports eta for *all* record types, not just Σ!
In particular, it has eta for the unit type:

```
record ⊤ : Set where -- no fields
  constructor tt

eta-unit : (x y : ⊤) → x ≡ y
eta-unit x y = refl
```

Two distinct variables might be equal!

⇒ To check if two terms are convertible, it does not suffice to compare their normal forms.

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2. Eta-reduce everything when translating?
   This is not stable under substitution and β*:*

   $$(\lambda x.y\ x\ x)\{(\lambda\_.z)/y\} \hookrightarrow_\beta \lambda x.z\ x \hookrightarrow_\eta z$$

   but $\lambda x.y\ x\ x \not\hookrightarrow_\eta$ *and* $\lambda\_.z \not\hookrightarrow_\eta$.

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5. Annotate terms with their types to be able to match them to eta expand? e.g.
   `eta (arrow nat nat) f --> x => f x`
   We get bigger terms, and the other rules make the system non-confluent on pre-terms. Moreover, variables not translated as variables.

Encoding eta in Dedukti

The next idea. Extend Dedukti with typed-directed rewrite rules.

Take inspiration from already existing works:

- Agda's implementation of eta[7]
- Andromeda 2's extensionality rules[8]

Or maybe there are still other unexplored options?

---

[7] `https://agda.readthedocs.io/en/v2.6.2.2/language/record-types.html`

[8] A. Bauer, A. Petković, An extensible equality checking algorithm for dependent type theories

Definitional irrelevance

Agda also supports definitional proof irrelevance[9] for irrelevant functions and elements of Prop:

```
postulate
  P : Prop
  f : P → ℕ

P-irrelevant : (x y : P) → f x ≡ f y
P-irrelevant x y = refl
```

This causes very similar problems to eta for $\top$, that also requires type-directed conversion to solve.

[9] In the encoding of PVS we have a simpler form of proof irrelevance, which can be encoded in Dedukti.

From Agda to Dedukti

Summary

Many features of a dependently typed language can be encoded in Dedukti directly:

- Defined symbols are mapped to constants.
- Clauses are mapped to rewrite rules.

Other features require some more work:

- Erased constructor parameters need to be reconstructed.
- Universe levels require an equational theory.

Finally, other features we don't yet know how to encode:

- Eta-equality for record types?
- Definitional proof irrelevance?

Future work

Like most translators, Agda2Dedukti is still a work in progress.

In the future, we would like to have:

- Compilation of clauses to elimination principles,
- A conservative encoding of universe polymorphism,
- An adequate and computational encoding of Agda,[10]
- An encoding of eta-equality and irrelevance (probably requires extending Dedukti).

[10] For details, see Thiago's talk about Adequate and Computational Encodings in Dedukti, at FSCD 2022

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