MATH 521, WEEK 7:
Open Covers, Compact Sets

1 Insufficiency of Open and Closed Sets

Consider the question of *embedding* metric spaces inside of one another. That is to say, suppose that we have a metric spaces (X, d) and a subset $Y \subseteq X$, and decide to consider (Y, d) as its own metric space. Given a common metric, how do the open and closed sets of (X, d) and (Y, d) compare to one another? Are we guaranteed to have the same topology, or can it be different?

We first of all formally define the following.

Definition 1.1. *Suppose (X, d) is a metric space and $Y \subseteq X$. We will say that a subset $S \subseteq Y$ has property P* ***relative to*** *Y if S has property P in the embedded metric space (Y, d).*

In particular, for a given metric space (X, d), we will talk about sets $S \subseteq X$ which are *open* or *closed* relative to an embedded metric space (Y, d) where $Y \subseteq X$. We will see, however, that there is another classification of set which will be more desirable for embedded metric spaces.

Consider the following example.

Example: Consider the metric spaces $(\mathbb{R}, d)$ and $(\mathbb{Q}, d)$ where $d(x, y) = |x - y|$ and (clearly) $\mathbb{Q} \subset \mathbb{R}$. Consider the set

$$S = \left\{ x \in \mathbb{Q} \mid -\sqrt{2} < x < \sqrt{2} \right\}.$$

This set is *both* open and closed relative to the topology of $(\mathbb{Q}, d)$. To see this, note that, for every $x \in S$, we may pick an $r > 0$ sufficiently small so $B_r(x) \cap S$ contains only rational points in the set S, so that $B_r(x) \subseteq S$. It follows that every $x \in S$ is an interior point so that S is open relative to $(\mathbb{Q}, d)$.

Now consider the limit points. We have (from the previous argument) that every $x \in S$ is a limit point; however, we must also check $x \in S^c$. We have that every $x \in S^c$ (i.e. $x \in \mathbb{Q}$ such that $x < \sqrt{2}$ or $x > \sqrt{2}$) we may pick a sufficiently small $r > 0$ so that $B_r(x) \subseteq S^c$. It follows that no such point may be a limit point of S, so that $S' = S$. It follows that S is closed relative to $(\mathbb{Q}, d)$.

Now consider S relative to the topology of $(\mathbb{R}, d)$. This set is *neither* open nor closed. To see this, note that every ball $B_r(x)$ for $x \in S$ contains irrational numbers (which we must now consider!) and that every irrational number between $-\sqrt{2}$ and $\sqrt{2}$ is a limit point of S but not contained in S. It follows that S is neither open nor closed relative to the topology of $(\mathbb{R}, d)$.

We make the following crucial observations:

1. *A set $S \subseteq Y \subseteq X$ may be open relative to (Y, d) but not (X, d).*

2. *A set $S \subseteq Y \subseteq X$ may be closed relative to (Y, d) but not (X, d).*

We should wonder if there are types of sets which are guaranteed to have the same topological properties in both (X, d) and (Y, d). The answer is a definitive *yes*, although we will have to do a little work laying some groundwork first. We will see shortly that the types of sets we are looking for are *compact sets*.

2 Open Covers and Compact Sets

The following topological notions which will factor significantly when we consider the convergence of sequences and continuous functions in the next few weeks. It is recommended (although certainly not necessary!) that attention be paid to the *consequences* of a set S being compact, even if the definition appears obtuse at first glance. It is often easier to grasp what follows from compactness than the subtleties of the definition itself.

The definitions are notable topological in flavor (i.e. they are defined in terms of *open sets*) but we will relate them to more familiar notions soon.

Definition 2.1. *Suppose (X, d) is a metric space and $S \subseteq X$. We will say that the family of sets $\{S_\alpha\}_{\alpha \in A}$ is an* ***open cover*** *of S if all S_α, $\alpha \in A$, are open sets and if*

$$S \subseteq \bigcup_{\alpha \in A} S_\alpha.$$

Given an open cover $\{S_\alpha\}$ of S, we will furthermore say that the family $\{S_\beta\}_{\beta \in B}$, is a ***subcover*** *of $\{S_\alpha\}$ if $B \subseteq A$ and $\{S_\beta\}$ is an open cover of S.*

The definition really is as straight-forward as it sounds. We need the sets in the family to be open, and for them to "cover" S. A subcover is just a subfamily of the sets in the family which also covers S. For example, consider the following sets S in the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$:

1. $S = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$: For any $\epsilon > 0$, the family of intervals $\{S_\alpha\}$ where $S_\alpha = (\alpha - \epsilon, \alpha + \epsilon)$ and $\alpha \in A = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is an open cover. There are many subcovers. For example, we may choose the family $\{S_\beta\}$ where $S_\beta = (\beta - \epsilon, \beta + \epsilon)$ and $\beta \in B = \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$ (since $B \subseteq A$).

2. $S = \mathbb{R}$: The family of open intervals $\{S_\alpha\}$ where $S_\alpha = (\alpha - 1, \alpha + 1)$, $\alpha \in \mathbb{Z}$ is an open cover of $\mathbb{R}$ which contains no non-trivial subcovers (i.e. we always have $\{S_\beta\} = \{S_\alpha\}$, otherwise the family does not cover S).

3. $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$: For any $\epsilon > 0$, the family of intervals $\{S_n\}$ where $S_\alpha = (\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$ and $n \in \mathbb{N}$ is an open cover. Notice that, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $n > N$ implies $0 < \frac{1}{n} < \epsilon$. It follows that, for any $N^* > N$, we have that S_{N^*} contains $\frac{1}{n}$ for all $n > N$. It follows that the *finite* subfamily

$$\{S_1, S_2, \ldots, S_N, S_{N^*}\}$$

is a subcover of $\{S_n\}$.

Open covers are not that interesting in and of themselves. As demonstrated by the above examples, it is rather simple to find an open cover of a set. What is not so trivial—but turns out to be intimately related to many important topological properties of metric spaces—is when we can say things about *every* open cover of a set. We have the following definition.

Definition 2.2. *Suppose (X, d) is a metric space and $S \subseteq X$. We will say that S is a* ***compact set*** *if every open cover $\{S_\alpha\}$ of S has a finite subcover.*

Note here that, by a *finite* subcover, we mean that the number of sets in the family comprising the subcover is finite not that the sets themselves has a finite number of elements. More explicitly, we have that there is a finite set of indices $\alpha_1, \ldots, \alpha_N$ so that

$$S \subseteq \bigcup_{n=1}^{N} S_{\alpha_n}$$

where $\{S_{\alpha_n}\} \subseteq \{S_\alpha\}$.

Before worrying too much about the details (and wondering how this might be useful), let's reconsider our previous examples.

Example 1: Consider the set $S = [0,1]$. For a fixed $\epsilon > 0$, consider the open cover $\{S_\alpha\}$ where $S_\alpha = (\alpha - \epsilon, \alpha + \epsilon)$ for $\alpha \in [0,1]$. Show that S has a finite subcover from the family $\{S_\alpha\}$.

Solution: While the family S_α is an uncountably infinite family of sets, there is a lot of overlap between them. A fixed $x \in [0,1]$ is "covered" by every set in the S_α where $\alpha \in (x - \alpha, x + \alpha)$. We want to remove as much redundancy as we can.

What we are going to do is, instead of taking including all sets S_α in a continuous range, we will take discrete *jumps*. If we jump by an appropriate amount, we should not exclude "covering" the elements of the interval. It should not take much convincing to agree that the family

$$\{S_0, S_\epsilon, S_{2\epsilon}, \ldots, S_{N\epsilon}\}$$

where N is the largest $N \in \mathbb{N}$ so that $N\epsilon < 1$ also covers $S = [0,1]$. (This is, of course, not the only choice!) It is clear that every set in this family is in the family $\{S_\alpha\}$ and that this set is finite, so that we have found a finite subcover, and we are done.

Note: This result does not prove that the set is compact—since this is just ***one*** *open cover and compactness requires this to hold for* ***all*** *open covers—but it is a start toward understanding how open covers work! In fact, this set is compact, but we are not quite ready to prove this yet.*

Example 2: Show that $\mathbb{R}$ has an open cover which does not have a finite subcover, and therefore that $\mathbb{R}$ is not compact.

Solution: It is enough in this case to take the already mentioned open cover of $\mathbb{R}$, that is, $\{S_\alpha\}$ where $S_\alpha = (\alpha - 1, \alpha + 1)$, $\alpha \in \mathbb{Z}$. This is clearly an infinite cover (otherwise, we would already be done!). Now imagine removing any individual set S_{α^*} from S_α. That is to say, we remove the interval $(\alpha^* - 1, \alpha^* + 1)$ from the covering set. We notice, however, $\alpha^* \in \mathbb{R}$ is no longer covered by the remaining sets. It follows we may not remove any sets in the family $\{S_\alpha\}$ while maintaining the property that the family covers $\mathbb{R}$. It follows that not every open cover has a finite subcover, and therefore that $\mathbb{R}$ is *not compact.*

Example 3: Show that the set

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

is not compact.

Solution: This should seem like a strange demand for this example. After all, we showed previous that this set had an open cover $\{S_\alpha\}$ which had a finite cover $\{S_1, \ldots, S_N, S_{N^*}\}$. *This is not, however, enough to show a set is compact!* It is possible that one subcover may contain finite subcovers while another may not. In this case, we need to find an open cover $\{S_\alpha\}$ which *does not* have a finite subcover.

From first principles, this is a question we would not know how to start; however, we will use a trick which will become commonplace for our understanding of compact sets. We know that each element of S is an isolated point. This is because, for every $n \in \mathbb{N}$, there is an $r_n > 0$ so that

$$\frac{1}{n+1} < \frac{1}{n} - r_n < \frac{1}{n} < \frac{1}{n} + r_n < \frac{1}{n-1}. \tag{1}$$

(Explicitly, any $0 < r_n < \frac{1}{n(n+1)}$ will do.)

Now consider the family $\{B_{r_n}(\frac{1}{n})\}$ for $n \in \mathbb{N}$. That is to say, take the particular family of open balls centered at each point $x = 1/n$ with the values of r_n satisfying (1). It is clear that this family covers S so that it is an open cover of S. Consider now the question of whether a *finite* subcover exists. (Note that the set itself is countable, but infinite, since it is indexed by $\mathbb{N}$.) Each point $\frac{1}{n} \in S$ is covered by *exactly one* element in cover $\{B_{r_n}(\frac{1}{n})\}$. It follows that removing any single set from the family produces a family which is no longer a cover. The family therefore does not have a finite subcover, which means (by definition) that S is not compact.

The method constructed here may seem ad hoc, but notice what we have done: we have constructed small open balls around *isolated* points in such a say that each ball covers precisely one point. It is easy to establish that any infinite set of isolated points (or set *containing* and infinite set of isolated points) may not be compact.

What is not quite obvious yet is, when embedding metric spaces inside of one another, the property we are interested in is *compactness* of sets. We have the following result.

Theorem 2.1 (Theorem 2.33 in Rudin)**.** *Suppose* (X, d) *and is a metric spaces and* $Y \subseteq X$. *Then a set* $S \subseteq Y$ *is compact relative to* (Y, d) *if and only if it is compact in* (X, d).

In other words, when we are talking about *compact* sets, we do not have to worry about the embedding or our metric spaces as we did with open and closed sets. It is a very desirable topological property to have (for this, and many other reasons)!

Proof. We show first that every open cover of S having a finite subcover in (Y, d) implies that every open cover of S has a finite subcover in (X, d).

Suppose $S \subseteq Y$ is compact in (Y, d) and let $\{X_\alpha\}$, $\alpha \in A$, be an open cover of S is X. We claim that the family $\{Y_\alpha\}$ with $Y_\alpha = X_\alpha \cap Y$, $\alpha \in A$, is open in (Y, d). Indeed, for every $y \in Y_\alpha$, we have that there is an $r > 0$ so that $B_r(y) \subseteq X_\alpha$ (since $y \in X_\alpha$ as a consequence of $y \in Y_\alpha$). It follows that $(B_r(y) \cap Y) \subseteq (X_\alpha \cap Y) = Y_\alpha$ so that x is an interior point of Y_α. Since $\{Y_\alpha\}$, $\alpha \in A$, covers S and S is compact in (Y, d), it follows that there is a finite subcover

$$S \subseteq \{Y_{\alpha_n}\}_{n=1,\dots,N} \subseteq \{X_{\alpha_n}\}_{n=1,\dots,N}.$$

It follows that S is compact in (X, d).

Now suppose S is compact in (X, d) and let $\{Y_\alpha\}$, $\alpha \in A$, be an open cover of S in Y. We claim that, for each Y_α, there is a set $X_\alpha \subseteq X$ which is open relative to (X, d) such that $Y_\alpha = X_\alpha \cap Y$. Because Y_α is open, we have that, for every $y \in Y_\alpha$ there is an $r_y > 0$ so that $B_{r_y}(y) \subset Y_\alpha$. We define the set X_α to be

$$X_\alpha = \bigcup_{y \in Y_\alpha} B_{r_y}(y)$$

and note that this set is open by an earlier Theorem (infinite union of open sets is open!). We therefore have that $\{X_\alpha\}$, $\alpha \in A$, is an open cover of S. Since S is compact in (X, d) we have that there is a finite subcover $\{X_{\alpha_i}\}_{i=1,\dots,N}$. It follows from $Y_\alpha = X_\alpha \cap Y$ and $S \subseteq Y$ that

$$S \subseteq \{Y_{\alpha_n}\}_{n=1,\dots,N} \subseteq \{X_{\alpha_n}\}_{n=1,\dots,N}.$$

It follows that S is compact in (Y, d), and we are done. □

Example: $S = \left\{x \in \mathbb{Q} \mid -\sqrt{2} < x < \sqrt{2}\right\}$ is not compact relative to either $(\mathbb{Q}, d)$ or $(\mathbb{R}, d)$. (Consider the open cover $\{S_n\}$, $n \in \mathbb{N}$, where $S_n = \{x \in X \mid -\sqrt{2} + \frac{1}{n} < x < \sqrt{2} - \frac{1}{n}\}$, and $X = \mathbb{Q}$ or $X = \mathbb{R}$.)

3 General Properties of Compact Sets

This is all well and good, but it does not really give us any insight into what compact sets really are, or why they are useful. Consider the following properties.

Definition 3.1. *Let (X, d) be a metric space and $S \subseteq X$. We will say that S is **bounded** if there is an $x \in X$ and an $r > 0$ so that $y \in B_r(x)$ for all $y \in S$. Otherwise, S is said to be **unbounded**.*

That is to say, a set is bounded if there is a maximal distance between any points in the set. Note that in $\mathbb{R}^n$ we will typically take the "center" (i.e. the $x \in X$) to be $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$.

The following result contains the most well-known consequences of compactness.

Theorem 3.1 (Theorems 2.34 and 2.37 of Rudin)**.** *Let (X, d) be a metric space. Suppose $S \subseteq X$ is a compact set. Then:*

(a) S is bounded;

(b) S is closed; and

(c) every infinite subset $K \subseteq S$ has a limit point in S.

Proof of Theorem 3.1(a). Suppose $S \subseteq X$ is a compact set. Take $x \in X$ to be fixed and consider the sets $B_n(x)$ where $n \in \mathbb{N}$. It is clear that the family $\{B_n(x)\}$ is an open cover of S, since it is an open cover of X. Because S is compact, there exists a finite subcover $\{B_{n_1}(x), B_{n_2}(x), \ldots, B_{n_N}(x)\}$. We can see that $B_n(x) \subseteq B_{\bar{n}}(x)$ for all $n \in \mathbb{N}$ where $\bar{n} = \max\{n_1, n_2, \ldots, n_N\}$. This implies that S is bounded, and we are done. □

Proof of Theorem 3.1(b). Suppose $S \subseteq X$ is a compact set. We show that S^c is open (i.e. every point $p \in S^c$ is an interior point of S^c).

Suppose that $p \in S^c$ and consider the open cover $\{S_q\}$, $q \in S$, of S given by

$$S_q = B_{r_q}(q)$$

where $r_q = \frac{1}{2}d(q, p)$. This clearly covers S, does not contain p, and consists only of open sets. Since S is compact, it follows that there is a finite subset $\{q_1, \ldots, q_N\}$ so that $\{S_{q_N}\} = \{B_{r_{q_n}}(q_n)\}$, $n = 1, \ldots, N$, also covers S. Because this set is finite, we can pick

$$r_p = \min_{n \in \{1, \ldots, N\}} \{r_{q_n}\} > 0$$

and consider $B_{r_p}(p)$. By construction, this ball does not intersect any $\{B_{r_{q_n}}\}$ and therefore does not contain any points in S. It follows that $B_{r_p}(p) \subseteq S^c$ so that p is an interior point of S^c. Since the choice of $p \in S^c$ was arbitrary, it follows that every point of S^c is an interior point, so that S^c is open. It follows that S is closed. □

Proof of Theorem 3.1(c). Suppose S is compact and $K \subseteq S$ is an infinite subset. Suppose there does not exist a $q \in K' \cap S$. That is to say, there is not limit point of K inside S. This implies that, for every $q \in S$, there is an $r(q) > 0$ (which depends on the choice of q) such that $B_{r(q)}(q) \cap K$ does not contain any points of K except q (if $q \in K$). Furthermore, this is true for *every* $q \in S$, so that the family $\{B_{r(q)}(q)\}$, $q \in S$, is an open cover of S, and also of K as a result.

Now, because S is compact, it follows that there is a finite subset of $q \in S$ (denoted hereafter as $\{q_1, \ldots, q_N\}$) so that

$$\left\{B_{r(q_n)}(q_n)\right\}_{n=1,\ldots,N}$$

is an open cover of S, and therefore is also an open cover of K. We have already established, however, that the only points in K which can be "covered" by any individual set $B_{r(q)}(q) \cap K$ is q itself. Since K is an infinite set while the family $\{B_{r(q_n)}(q_n)\}_{n=1,\ldots,N}$ may cover at most a finite number of elements from K, the family does not cover K. Since this is a contradiction, the result follows. □

We also have the following result about *families* of compact sets.

Theorem 3.2 (Theorem 2.36 in Rudin). *Let (X, d) be a metric space. Suppose $\{S_\alpha\}$, $\alpha \in A$, is a family of non-empty compact subsets and that, for every finite subfamily $\{S_{\alpha_n}\}$, $n = 1, \ldots, N$, of $\{S_\alpha\}$ we have*

$$\bigcap_{n=1}^{N} S_{\alpha_n} \neq \emptyset.$$

Then

$$\bigcap_{\alpha \in A} S_\alpha \neq \emptyset.$$

Recall that we have already seen an example of an open family where this was *not* true. Specifically, for the family $\{S_n\}$ where $S_n = \left\{x \in \mathbb{R} \mid 0 < x < \frac{1}{n}\right\}$, $n \in \mathbb{N}$, we found that every finite subfamily had a non-empty intersection while the infinite intersection was empty. Similarly, we can show this property does not necessarily hold for *closed* families. For example, consider the family with elements $S_n = \{x \in \mathbb{R} \mid x \geq n\}$, $n \in \mathbb{N}$.

We now show that this (somewhat counter-intuitive) situation cannot occur for families of *compact* sets.

Proof. Suppose otherwise. That is to say, suppose that $\{S_\alpha\}$, $\alpha \in A$, is a family of compact sets, that every finite subfamily $\{S_{\alpha_n}\}$, $n = 1, \ldots, N$, has a nonempty intersection, but that the intersection of the whole family is empty.

If the intersection of the whole family is empty, it follows by definition that, for any $\alpha^* \in A$, we have

$$S_{\alpha^*} \cap S = \emptyset \tag{2}$$

where

$$S = \left(\bigcap_{\alpha \in A, \alpha \neq \alpha^*} S_\alpha \right).$$

Now consider the set

$$S^c = \left(\bigcap_{\alpha \in A, \alpha \neq \alpha^*} S_\alpha \right)^c = \left(\bigcup_{\alpha \in A, \alpha \neq \alpha^*} S_\alpha^c \right)$$

where the form of S^c follows from DeMorgan's laws. Note importantly that each S_α is compact, which implies that it is closed, so that S_α^c is *open* for every $\alpha \in A$.

It follows from (2) and the fact that S_{α^*} is non-empty that S^c is a cover of S_{α^*}. That is to say, we have

$$S_{\alpha^*} \subseteq \left(\bigcup_{\alpha \in A, \alpha \neq \alpha^*} S_\alpha^c \right).$$

We know, however, that each S_α^c is open so that $\{S_\alpha^c\}$, $\alpha \in A$, $\alpha \neq \alpha^*$, is an open cover of S_α^*. Since S_α^* is compact, it follows that there is a finite subcover $\{S_{\alpha_n}^c\}$, $n = 1, \ldots, N$, $\alpha_n \neq \alpha^*$. That is to say, we have

$$S_{\alpha^*} \subseteq \left(\bigcup_{n=1}^{N} S_{\alpha_n}^c \right) = \left(\bigcap_{n=1}^{N} S_{\alpha_n} \right)^c$$

so that

$$S_{\alpha^*} \cap S_{\alpha_1} \cap \cdots \cap S_{\alpha_N} = \emptyset.$$

This, however, contradicts the assumption that every finite intersection within the family is non-empty, so that the result is shown. □

4 Sufficient Conditions for Compactness

So far, we have considered what follows *if* a set is compact, but we have given little consideration to directly showing that any given set *is* compact. That is to say, we have not considered sufficient conditions for compactness.

This is not surprising. In general, it is easier to prove a set is *not* compact than to prove it *is* compact. In fact, in general metric spaces (X, d), it can be very challenging to prove any given set is compact. Nevertheless, we will now turn our attention to a case where we can make a positive claim about compactness. (Notice importantly, that the example is drawn from our familiar space of the real numbers. Other spaces are more difficult.)

Theorem 4.1 (Roughly Theorem 2.40 in Rudin)**.** *Consider the metric space* $(\mathbb{R}, d)$ *where* $d(x, y) = |x - y|$*. Then any closed and bounded interval* $I = [a, b] \subset \mathbb{R}$ *is a compact set.*

Proof. We want to prove that every open cover $\{S_\alpha\}$ of $I = [a, b]$ has a finite subcover. We suppose otherwise. That is to say, we suppose there is an open cover $\{S_\alpha\}$ of I which *does not* have a finite subcover.

We start by making the following observation: for any $c = \frac{b-a}{2}$, at least one of the intervals $[a, c]$ and $[c, b]$ does not have a finite subcover. (otherwise, both $[a, c]$ and $[c, b]$ would have a finite subcover, so that $[a, b]$ would as well.) We will label whichever of $[a, c]$ and $[c, b]$ does *not* have a finite subcover I_1.

We then repeat this process, successively dividing intervals, each time choosing a new interval I_n, $n \in \mathbb{N}$, which does not have a finite subcover. We have that:

1. $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$;

2. I_n is not covered by any finite subfamily of $\{S_\alpha\}$ for any $n \in \mathbb{N}$; and

3. $d(x, y) = |x - y| \leq (b - a)2^{-n}$ for all $x, y \in I_n$.

We now claim that there is an $x^* \in I_n$ *for all* $n \in \mathbb{N}$. Notice that this is equivalent to the claim that

$$x^* \in \bigcap_{n \in \mathbb{N}} I_n.$$

To prove this, denote each interval $I_n = [a_n, b_n]$ and let $I^- = \{a, a_1, a_2, a_3, \ldots\}$ and $I^+ = \{b, b_1, b_2, b_3, \ldots\}$. That is to say, let I^- be the set of lower endpoints and I^+ be the set of upper endpoints. We notice that I^- has an upper bound, and because $\mathbb{R}$ has the least upper bound property, we may conclude that $\sup(I^-)$ exists in $\mathbb{R}$. Similarly, I^+ has a lower bound, so from the same property we have that $\inf(I^+)$ exists in $\mathbb{R}$. We clearly have that $\sup(I^-) \leq \inf(I^+)$ (otherwise, there would be an I_n which is empty) so that there is an $x^* \in \mathbb{R}$ (in particular, $\sup(I^-) \leq x^* \leq \inf(I^+)$) so that $x^* \in I_n$ for all $n \in \mathbb{N}$.

Consider what this means. By the definition of the open cover, we have that there is an α so that $x \in S_\alpha$. The set S_α is open, so that there is an $r > 0$ so that $x \in B_r(x^*)$ implies $x \in S_\alpha$. However, we can clearly take an N sufficiently large so that $(b-a)2^{-N} < r$. Since every $x^* \in I_n$ for all $n \in \mathbb{N}$, it follows that $I_N \subset B_r(x^*) \subseteq S_\alpha$. However, this implies that I_N is covered by a finite subset of $\{S_\alpha\}$ (in particular, a single set). This contradicts the construction of the sets I_n, $n \in \mathbb{N}$. It follows the assumption that $[a, b]$ is not compact was in error, so that $[0, 1]$ must in fact be compact. □

Note: It is important to recognize how this proof fails for intervals in $\mathbb{R}$ which are not either bounded or closed. We will consider this case separately:

1. Suppose the interval $I \subseteq \mathbb{R}$ is closed but not bounded (e.g. we have $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$). While we can subdivide the interval, we have not guarantee that the sequence of intervals I_n which does not possess a finite subcover may be selected so that their lengths decrease (as required by point 3. above). The result does not hold (and can in fact be contradicted).

2. Suppose the interval $I \subseteq \mathbb{R}$ is bounded but not closed (e.g. we have $(a, b]$, $[a, b)$, (a, b)). While we are able to subdivide intervals into a sequence I_n for which the lengths decrease, we cannot guarantee there is a point $x^* \in I_n$ for every subinterval I_n without a finite subcover. For instance, if we have $I = (a, b]$, we could have $I_1 = (a, b_1]$, $I_2 = (a, b_2]$, etc., where $b > b_1 > b_2 > \cdots$ and clearly do not have that a is in every (or any!) subinterval. The result again does not hold.

We now have some sense about what it takes to conclude that a set $S \subseteq X$ is compact. The picture is not pretty. This was a lot of work! It would take ages to confirm any given set is compact by this method.

Fortunately, for our traditional Euclidean spaces $\mathbb{R}^n$, the previous argument can be generalized to give very precise necessarily and sufficient conditions for compactness. We have the following.

Theorem 4.2 (Heine-Borel Theorem, Theorem 2.41 in Rudin)**.** *Consider the Euclidean space* $(\mathbb{R}^n, d_2)$ *and* $S \subseteq \mathbb{R}^n$*. Then* S *is compact if and only if it is closed and bounded.*

Sketch of proof. We will not go through the details, except to note that the methodology of Theorem 4.1 can be employed. Rather than consider a single coordinate, however, we must consider *multiple* coordinates. Analogously to carving the interval I into small intervals, we can carve box in $\mathbb{R}^n$ into smaller boxes, and since any closed and bounded subset of $\mathbb{R}^n$ must fit into one of these boxes, we may use the fact that a closed subset of a compact set is compact (Corollary of Theorem 2.35 in Rudin) to prove that the set is compact.

The other direction in the if and only if statement follows from Theorem 3.1. □

Note: It is fair at this point to wonder how a set could possible be closed and bounded and fail to be compact. This seems to be where everything is headed. In fact, in more general spaces than $\mathbb{R}^n$, and with more pathological metrics than d_2, it is possible for this to fail. That is to say, it is possible for a set to be closed and bounded without being compact. Consider the following examples:

1. Consider the metric space $(\mathbb{R}^\infty, d)$ where

$$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R}, i \in \mathbb{N}\}$$

and

$$d(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i|$$

for $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. This is very similar to the Euclidean spaces we have considered with the notable exception that there are an *infinite* number of coordinates. A consequence modify our d_∞ metric to calculate the supremum rather than the maximum. (We are guaranteed this supremum exists in real numbers by the least-upper-bound property of the reals.)

Now consider the subset $S = \{\mathbf{x} \in \mathbb{R}^\infty \mid d(\mathbf{0}, \mathbf{x}) \leq \frac{1}{2}\}$ where $\mathbf{0} = (0, 0, \ldots)$. It is clear this set is bounded (by definition!) and we can check that it is closed (for any $\mathbf{x} \in S^c$, we can take a ball of radius $r_x = \sup_{i \in \mathbb{N}} \frac{|x_i| - 1}{2}$). Consider the open cover $\{S_n\}$, $n \in \mathbb{N}$, where

$$S_n = \{x \in \mathbb{R}^\infty \mid |x_i| < 1 \text{ for } i \leq n, x_i < \frac{1}{2} \text{ for } i > n\}.$$

(It can be checked with a little work that these sets are open in the topology of $(\mathbb{R}^\infty, d)$.) This family clearly covers S; however, no finite subcover of $\{S_n\}$ does. It follows that, in general metric spaces (X, d), it does not follow that every bounded and closed $S \subset X$ is compact.

2. Consider also the metric space $(\mathbb{R}^n, d)$ with the *discrete* metric

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{for } \mathbf{x} = \mathbf{y} \\ 1, & \text{for } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

   Consider an arbitrary infinite subset $S \subseteq \mathbb{R}^n$. It can be easily checked that every subset is both open and closed relative to $(\mathbb{R}^n, d)$, so that S is closed. It can also be easily seen that S is bounded since $d(\mathbf{0}, \mathbf{x}) \leq 1$ for all $\mathbf{x} \in \mathbb{R}^n$.

   Consider the open cover $\{S_x\}$, $x \in S$, where $S_x = \{x\}$. Since every subset of $\mathbb{R}^n$ is open (relative to the discrete metric d), we have that this is an open cover of S. We may not, however, select a finite subcover since S is an infinite set, and $x \in S$ is only covered by S_x. We have therefore found another metric space for have a bounded and closed subset $S \subseteq X$ is not compact.