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The Mathematical Proficiency Promoted by Mathematical Modelling

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Abstract: This study aims to investigate the mathematical proficiency promoted by mathematical modelling tasks that require students to get involved in the processes of developing mathematical models, instead of just using known or given models. The research methodology is grounded on design-based research, and the classroom design framework is supported by complexity science underpinnings. The research intervention consists of high-school students, from a grade 11 mathematics course, aiming to solve four different modelling tasks in four distinct moments. Data was collected during the intervention from students' written mathematical work and audio and video recordings, and from recall interviews after the intervention. Data analysis was conducted based on a model of mathematical proficiency and assisted by interpretive diagrams created for this research purpose. This research study offers insight into mathematics teaching by portraying how mathematical modelling tasks can be integrated into mathematics classes to promote students' mathematical proficiency. The study discusses observed expressions and behaviours in students' development of mathematical proficiency and suggests a relationship between mathematical modelling processes and the promotion of mathematical proficiency. The study also reveals that students develop mathematical proficiency, even when they do not come to full resolutions of modelling tasks, which emphasizes the relevance of learning processes, and not only of the products of these processes.

Keywords: C*lassroom-based research; Complexity science; Design-based research; High-school level; Mathematical modelling; Mathematical proficiency*.

Allowing for New Possibilities

Students develop their sense of what it means to "do mathematics" from their actual experiences with mathematics, and their primary opportunities to experience mathematics as a discipline are seated in the classroom activities in which they engage.
(Henningsen & Stein, 1997, p. 525)

Allowing for new possibilities in mathematics teaching can be challenging. Teachers carry their students' learning encounters for life. Britzman (2003) points out that teachers are shaped based on the school experiences they had as learners. The teaching practices, beliefs, and attitudes that were part of a learner's school experiences are intrinsically part of the teacher that same learner might become. Detaching from or deconstructing these previous experiences can be a long and hard process. Even though teachers might try to contextualize mathematics, use different strategies and engage students in their classes, mathematics teaching can still be focused on traditional instructional strategies, and founded on procedural knowledge. Silver et al. (2009) affirm that mathematics instruction is mainly based on low-level activities that do not help in promoting mathematical understanding. Strategies that promote mathematical thinking and reasoning can demand time, and outcomes are not immediate. If teachers do not allow for changes to happen, they may stick to familiar practices that might not do much more than having students reproducing steps based on the teacher's guidance. Teachers' commitment to a tight curriculum and time limitations are also complicating factors. Teachers may keep teaching in procedural ways that are perceived to be less time-consuming and more successful in terms of covering the curriculum. That may be the reason why mathematics is understood by many as a matter of memorizing procedures. Because of allegedly time and curriculum constraints, the essence of mathematics might not be worked out in class and students might not understand why they are learning mathematics or why they benefit from studying mathematics. Procedural ways of teaching are unsuccessful in terms of promoting thoughtful inquiry and reasoning abilities, which are fundamental features of the mathematics essence. Although mathematics education is a result of different scenarios and demands around the world, teachers need to be cautious not to narrow down students' learning processes and miss the opportunity to work on what matters.

It can be hard to figure out new possibilities that focus on mathematical understanding and challenge traditional entrenched ways of teaching and learning. This article portrays a doctoral research that analyzes one of these possibilities (Corrêa, 2019; Dias Corrêa, 2017). It explores a classroom complex environment, in which students' development of mathematical proficiency is investigated when students engage in mathematical modelling tasks. The investigated classroom is intentionally acknowledged as a complex environment: it is centred on students' work, allows for students' interactions, enables non-linear and adaptive knowledge building, and supports a learning process in which individual and group explorations are merged in insightful ways. Modelling tasks are chosen because they can potentially engage students and trigger their interest in mathematics, given that these approaches are usually related to likely-to-happen contextualized scenarios. Lamon (2003) states that "children are intensely motivated when they are immersed in the problem" (p. 447). Besides, mathematical modelling tasks have the features of high-level tasks (Silver et al. 2009), that is, cognitively demanding tasks that promote students' mathematical thinking and understanding. Within this scenario, students' mathematical proficiency is examined based on Kilpatrick, Swafford and Findell's proficiency model (2001). The research questions are: *How is mathematical proficiency observed and expressed in the actions and interactions of students when engaged in mathematical modelling approaches? What are the affordances on students' mathematical proficiency through modelling approaches?*

Mathematical Modelling at the School Level

To acknowledge the novelty of the present research, it is necessary to look at the work that has been done in terms of mathematical modelling at the school level. This research literature review encompasses research around the world up to 2016. Meyer et al.(2011), Almeida et al.(2013), Blum and Borromeo Ferri (2016), are a few examples of work that helps in opening up possibilities for modelling instruction at the elementary and the secondary school level by presenting teachers with practical examples. Bleiler-Baxter et al. (2016), Gann et al. (2016) and Cavey and Champion's (2016) work discuss different ways of scaffolding students involved in modelling tasks at the secondary level. These studies are helpful in what concerns teachers' adjustment to the use of modelling approaches in class. In Australia, Brown and Edwards' (2011) research claims that students' use of prior knowledge, integration between reality and mathematics, and high-order thinking are related to the promotion of deep understanding through mathematical modelling. Bahmaei (2011), Ikeda and Stephens (2010), Kawasaki et al. (2012) and Bonotto

(2010) are examples of research in Iran, Japan and Italy that discuss modelling experiences, relevance, and implementation issues.

In a different perspective, Frejd (2012) investigates how teachers' conceptions in a Swedish school impact the use of modelling approaches in high-school classes. Frejd affirms that teachers, in general, do not consider modelling as a priority and they question some modelling approaches as related to mathematics. Blum and Borromeo Ferri (2009) assert that "the gap between the goals of the educational debate and everyday school practice is that modelling is difficult both for students and for teachers" (p. 45). In recent work, Blum and Borromeo Ferri (2016) suggest that modelling should have a main role in pre- and in-service teacher education. In tune with that, Siller and Kuntze (2011) emphasize the need for introducing modelling in teacher professional developments. Doerr and English (Doerr, 2006; English & Doerr, 2004) suggest that teachers build on their own knowledge through interacting with students during modelling processes and develop a diversity of approaches in response to students' modelling work.

Although research on school mathematics and mathematical modelling can be found, to this literature review extent, no research investigates mathematical proficiency in the context of mathematical modelling. Lesh et al. (2013) assert that "most of [the] studies investigate the development of ideas – not the success of treatments and interventions" (p. 280). In this sense, apart from drawing attention to an example of a modelling task, this research presents an intervention to implement modelling tasks in class and a possible way of unpacking students' mathematics work on these tasks. From students' work analysis, the research dives into the accomplishment of the intervention by investigating how mathematical proficiency is promoted by mathematical modelling. Zbiek and Conner's (2006) work is closer to the current research perspective, given that their work analyzes the mathematical learning that occurs while mathematical modelling tasks are implemented with prospective secondary mathematics teachers. The primary differences between the current study and Zbiek and Conner's study are the research participants and the data analysis. In their case, participants are mathematics teachers and the analysis is based on modelling sub-processes, while in the current study, participants are high-school students and the analysis is based on mathematical proficiency. Considering that mathematical proficiency is a desired outcome of school mathematics education, it is relevant to investigate ways that mathematical modelling can nurture that proficiency.

Research Methods

Setting Up the Classroom

An important part of this research study was the environment in which students engaged in modelling tasks. Classrooms have the potential to be complex systems. Nevertheless, the school system imposes a structure and some conditions on classrooms that end up inhibiting the complex nature of classrooms. In the case of mathematics classrooms, Ricks (2009) asserts that students' difficulties in mathematics learning are precisely because of the lack of acknowledgment of mathematics classes as complex systems. In view of this, this research recognizes that the complex nature of a mathematics classroom must be respected, so that classroom intrinsic conditions are not suppressed. Hence, complexity science is used as a theoretical framework for this research classroom design; that is, as a framework to set up the classroom environment in which the modelling intervention is held. In the proposed classroom design, students are encouraged to collectively investigate the given tasks and to explore the mathematics behind them. This research analysis investigates the individual outcomes derived from the collective work among students, given that individuals are nested in a collective learning system. The goal is to set up the classroom as a non-linear, spontaneous and self-organizing environment, and ultimately reduce mathematics learning issues commonly witnessed in the more instrumental and teacher-centred classes.

According to Davis and Simmt (2003), five conditions – namely *internal diversity*, *redundancy*, *decentralized control*, *organized randomness*, and *neighbour interaction* – are necessary to create, promote, and sustain a complex environment within mathematics classes. *Internal diversity* refers to the necessity of having students from different perspectives and with different backgrounds to generate possibilities for diverse contributions to the class. *Redundancy* is when students have common knowledge, experiences and expectations so that interactions among students are more likely to happen. *Decentralized control* speaks to the necessity of having the teacher stepping aside at various points and for varying lengths of time to leave students free to lead activities, thinking processes and the development of mathematical knowledge in class. *Organized randomness* allows students to manage their own work. It grants students the opportunity to organize their thoughts and reasoning in ways that make more sense to them, supporting their processes of understanding. Finally, *neighbour interactions* is related to the essential notion of having students interacting with each other, the teacher, the mathematics, and other useful thoughts or resources. These interactions allow students to share insights and ideas that might help in building on their mathematical understanding. This research endorses that these five conditions can facilitate the emergence of mathematical proficiency and the production of mathematical knowledge. Ricks (2009) defends that, except for redundancy, all the other conditions are not present in conservative mathematics instruction. This would explain why students struggle when studying mathematics: because they are immersed in a complex system, in which not all necessary conditions to support it are present.

In addition to that, Henningsen and Stein (1997) suggest that different factors support engagement in cognitively high-level mathematical thinking. The authors highlight that these factors are not necessarily related to the mathematical task itself, but they can be related to the environment in which the task is being implemented. The task might not be enough to engage students in mathematical thinking. A proper supportive environment is necessary. Consistent with this finding, Henningsen and Stein assume that students' failures in school are not due to the lack of students' capability, but to the lack of opportunities to engage in adequate learning experiences instead. In this sense, a complexity setting seems to be an appropriate option for generating this supportive atmosphere, as it respects and preserves the authenticity of a mathematics classroom learning system. The proposed framework for the design of this research classroom setting consists of carrying out mathematical modelling tasks in an environment that sustains Davis and Simmt’s (2003) five complex conditions. The pentagon in Figure 1 represents this environment.

Outlining the Participants

An in-service teacher and university graduate student were invited and accepted to host the research in their classroom. Among the different classes the teacher was teaching, the one selected was the one that accommodated the researcher's schedule. The researched group was composed of a high-school class enrolled in a 4-month grade 11 Alberta (Canada) mathematics course (Mathematics 20-1). The course started in February and ended in early June. The group was an International Baccalaureate (IB) class composed of 27 students. Although all of them participated in the modelling tasks, data analysis was based on the work of 12 students who fully consented data collection and fully participated in the tasks. The research was based on four tasks presented in different moments. All 27 students were divided into groups to work on each of the tasks. The 12 participants, who allowed video recordings, were tentatively concentrated into three groups. For task number one, the classroom teacher was more directive when forming the groups. For the other three tasks, students were free to organize themselves into groups, as long as they (preferably) changed previous group configurations.

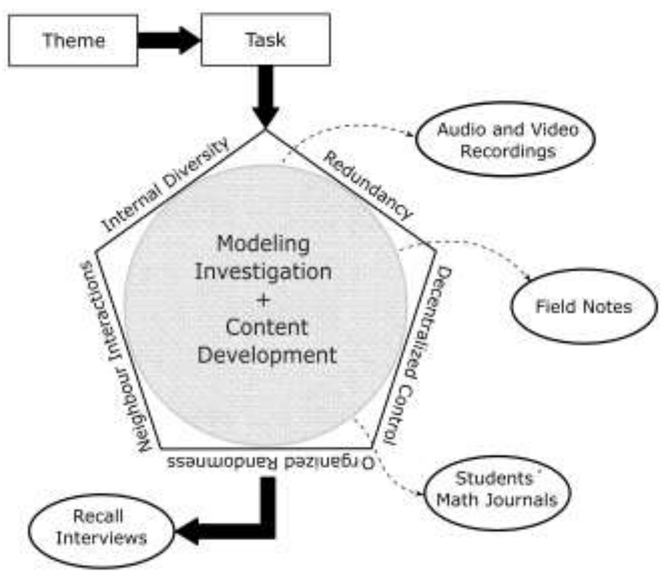
Designing the Intervention

This research presents a dual purpose: the design of a complex learning environment, and the development of a theory that claims that mathematical modelling is of benefit to the development of students' mathematical proficiency. The research considers the implementation, analysis and improvement of a planned intervention. It produces a supportive theory that aids teachers and educational designers in understanding how and why mathematical modelling approaches should be employed in mathematics classes. It is implemented in an ordinary mathematical class that presents common learning issues and challenges. Finally, the research addresses important outcomes to the mathematics teaching community. Based on these characteristics, according to the Design-Based Research Collective (2003), the present research can be categorized as design-based research. Indeed, Zawojewski (2013) points out that the nature of mathematical modelling research is related to design-based research; because when investigating modelling processes, the researcher is aware of the potential changes in students' modelling processes, given that this is part of the nature of modelling.

Biembengut and Hein (2002) (cited in Zorzan 2007) claim that the implementation of mathematical modelling in class is based on five different stages: 1) diagnosis of students' interest; 2) selection of the mathematical model or theme; 3) development of the content to be studied; 4) students' orientation towards the modelling process; and 5) assessment of the whole process. The present research intervention (Figure 1) was based on these implementation stages. A poll was done with students so that they could contribute to the choice of themes for the modelling tasks (Stage 1). The four most preferred themes were selected, and the researcher formulated four tasks, one for each theme, to be used in four different interventions (Stage 2). Next, the content to be studied should be developed (Stage 3) and students should move towards the modelling process (Stage 4). In this research, students were supposed to work on the content to be studied (Stage 3) and undergo the modelling process (Stage 4) simultaneously. This means students were not taught part of the necessary content knowledge to model the task beforehand. Students were invited to investigate the modelling task, work on it, maybe struggle due to the lack of necessary mathematical content knowledge, and then come up with the new mathematical content knowledge or learn from the teacher during the modelling process. The teacher and the researcher were available to scaffold these processes. Having students learning or producing the necessary knowledge during modelling tasks was one of the differences of this research in relation to other previous studies. Contrast this approach with Bracke and Geiger's (2011) research, in which they inquire about the viability of the use of modelling tasks regularly during a whole grade 9 mathematics course. The authors report that "students were directed to use methods which have been discussed in the lessons before the start of the respective task" (p. 532). This is not an uncommon practice, in particular, when there is a concern about fulfilling curriculum demands. In open modelling activities, students are at risk of not using the desired curriculum mathematical content, and teachers might be more worried about covering specific content than about doing modelling activities.

Figure 1

Research Intervention



During and after tasks were completed, the intervention was assessed so that the next intervention could be improved if needed. Although the core structure of the intervention remained the same, the proposed tasks and the approaches the teacher used during the implementation of each task were the results of these reflective assessments. This assessment refers to Stage 5 of Biembengut and Hein's (2002) (cited in Zorzan 2007) implementation stages. The intervention was repeated for each of the four tasks. The first intervention was completed at the end of February, the second at mid-March, the third at the end of April, and the fourth at the beginning of June.

Creating the Modelling Tasks

Before diving into the creation of the research modelling tasks, it is important to elucidate what constitutes mathematical modelling according to this study. Cirillo et al. (2016) explain that an agreed-upon definition for mathematical modelling cannot be found; instead, there are descriptions, definitions or assumptions made by single authors. Dym (2004) defines a *mathematical model* as "a representation in mathematical terms of the behaviour of real devices and objects" (p. 4). Blum and Borromeo Ferri (2009) define *mathematical modelling* as "the process of translating between the real world and mathematics in both directions" (p. 45). For this research study, *mathematical modelling* encompasses *developing a mathematical model*, that is, students create a mathematical representation that translates the situation they are analyzing into mathematics. The care in distinguishing between *using* a model and *developing* a model was grounded in the literature. Cavey and Champion (2016) assert that "most textbook 'modelling' problems engage students in *using* a given model, not in *developing* their own model or thinking about how to make improvements" (p. 132, italics added). Stillman (2001) states that teachers are encouraged to assess students' abilities in using models, rather than their abilities in developing models. Similarly, Zbiek and Conner (2006) assert that "many modelling tasks in our schooling context are fundamentally applied problems in disguise and are presented to use existing mathematical knowledge rather than to evoke new mathematical knowledge" (p. 100).

Cirillo et al. (2016) describe mathematical modelling based on the following five features:

> a) the connection with ordinary situations; b) the ill-defined nature; c) the necessity of a creative modeler who is able to make assumptions, choices and decisions; d) the recursive behaviour given that the modeler needs to constantly confront the model and the phenomena to validate the model; and e) the non-unique or non-strict nature since the modeler can choose from multiple paths and get to different solutions (p. 144).

In addition to these features, Bassanezi (1994) brings up an important aspect of mathematical modelling. He asserts that working with mathematical models in the teaching and learning processes is not only about expanding students' knowledge, it is overall a matter of structuring the way students think and act. Therefore, mathematical modelling endorses students as agents of change (Viecili, 2006).

With this perspective of mathematical modelling in mind, the next step is to understand how the research modelling tasks were created. Blum and Borromeo Ferri (2016) present six criteria that are essential for the creation of modelling tasks. The criteria are: 1) focus on the necessity of creating a task that truly addresses *genuine situation contexts*; 2) create an *open task*, in which a single correct answer is not the only possible solution for the task; 3) make the task *complex enough*, so that the task is not straightforwardly solved without thought-provoking students; 4) create a *challenging task* so that it is problematic enough to trigger students' high-level thinking; 5) create a *cognitively accessible* task that invites students to work within their zone of proximal development (Vygotsky 1978); and 6) ensure the task considers *all modelling cycle stages* (constructing, simplifying/structuring, mathematizing, working mathematically, interpreting, validating, presenting) in order to have students working on all modelling competencies. The tasks created for this study followed these six criteria.

Nonetheless, there was an extra criterion to follow in the task creations; the tasks were to promote the study of a not-already-taught mathematical concept or procedure, which was anticipated to be used in the process of solving the task. This additional criterion emphasizes modelling as a "vehicle" and not as a "content". Galbraith (2011) explains that when modelling is approached as a "vehicle" the focus is on using modelling to learn or to enhance the learning of mathematical concepts and contents; while when modelling is approached as a "content" the focus is on learning how to model. The necessity of learning new content knowledge during the modelling process posed an additional challenge for the researcher in the creation of the tasks. Yet, it was an intentional approach, because this sort of encounter with new knowledge during an investigation – in which the new knowledge is useful and necessary – presumably enables mathematics appreciation and meaningful and longer-lasting learning.

Based on the above criteria, research tasks reflected likely-to-be-experienced situations and were designed to develop reasoning skills and mathematical knowledge, by offering students thought-provoking, challenging, and high-level thinking tasks (as defined by Silver et al.. 2009). Four different tasks were elaborated for this research purpose; each research intervention was based on one of them. Tasks aimed to address four different content areas from the Mathematics 20-1 program of studies while respecting the teacher's planning for the course. The first task was about quadratic functions in a profit context (Figure 2), the second task was about the cosine law in a flight simulator context, the third task was about rational equations in a medley relay context, and the fourth task was about arithmetic and geometric sequences in a linear and binary search context.

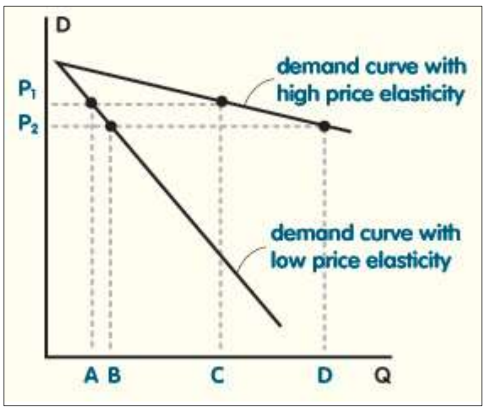
The class teacher suggested changes to the tasks when appropriate and approved all of them before implementation. The first task was implemented during three consecutive 80-minute classes, while each of the other three tasks was implemented during two consecutive 80-minute classes. Although students were encouraged to work collaboratively in teams, at the end of the process they could find different and individual solutions for the same task. Aligned with this research methodology, after one or more tasks were applied and based on students' prolonged struggle, whatever was deemed enabling or constraining in promoting students' mathematical modelling was considered to modify or adapt the following task and the following intervention. Stillman (2001) asserts that task scaffolding is directly related to the number of decisions students are to face and deal with. The lower the task scaffolding the higher the number of decisions to be made. The higher the task scaffolding, the lower the number of decisions to be made. The main challenge in the elaboration and implementation of the tasks was the decision about how much data was just the "right amount" for learners to work through the task relatively autonomously from the teacher and researcher.

Figure 2

Task 1 – Quadratic Functions

Price elasticity tells how much of an impact a change in price will have on the consumers' willingness to buy that item. If the price rises, the law of demand states that the quantity demanded of that item will decrease. Price elasticity of demand tells you how much the quantity demanded decreases. Elastic demand means that the consumers of that good or service are highly sensitive to changes in price. Usually, a good which is not a necessity or has numerous substitutes has elastic demand. Inelastic demand means that the consumers of that good are not highly sensitive to price changes. If the price of an inelastic good, say [bread], rises by 10 percent, maybe sales will only decrease by 1 percent. Consumers will still buy that good, typically because it is essential or has no substitutes. (Tuck, n.d.)

In the graph below, D stands for price and Q stands for quantity demanded.



(Elasticity of demand, n.d.)

Total revenue is calculated as the quantity of a good sold multiplied by its price. It is a measure of how much money a company makes from selling its product, before any costs are considered. Obviously, the goal of a company is to maximize profits, and one way to do this is by increasing total revenue. The company can increase its total revenue by selling more items or by raising the price. (Tuck, n.d.)

Suppose you are responsible for the accountancy of a bookstore. You were requested to do a price analysis for two products: a textbook and a reading book. In this analysis, you should consider cost, revenue and profit. The table below presents some data about variation in quantity sold when prices vary. Other than that, it might be helpful to know that there is a fixed cost of CAD$ 618 to produce either book. Based on your analysis, what is the best option to price each product in order to maximize the store profit?

Textbook			Reading Book		
Unit Price	*Quantity Sold*	*Cost/Unit*	*Unit Price*	*Quantity Sold*	*Cost/Unit*
CAD$ 26	102	CAD$ 18	CAD$ 24	112	CAD$ 18
CAD$ 30	100		CAD$ 30	100	
CAD$ 34	98		CAD$ 34	92	
CAD$ 40	95		CAD$ 38	84	

When students were working on the tasks, in the same way as in task elaboration, the strategy was not to give unnecessary support for them to start with. However, mathematical modelling was not the kind of activity that the class and the teacher were used to engage in, so this was taken into consideration. In the first researcher's visit to the class (to explain the research and invite students to participate) an example of a modelling task was given to students so that they could understand what was expected from them. Besides, the researcher was in class during all interventions and helped the teacher with whatever was necessary during the modelling intervention implementation. Stender and Kaiser (2016) suggest that, when modelling, students should get adequate help, not too much and not too little. As a rule, after students were given enough time to struggle, investigate and elaborate on the task, they would get an appropriate piece of advice or information, just enough to scaffold their meaning-making and allow them to move forward. If a group did not require scaffolding, they would not get it and would have the opportunity to face difficulties and challenges on their own. Vorhölter et al. (2014) highlight that scaffolding practice cannot be based on the teacher's immediate thoughts about students' work. It needs to be based on the diagnosis of students' understanding instead. Kaiser and Stender (2013) propose this diagnosis be done by asking students what their state of work is. The intention of the scaffolding in this study was not to induce students to choose a path, neither to get in the way of their thinking processes by anticipating steps they were supposedly able to achieve. The aim was to allow students to fully experience the modelling processes, promote high-level mathematics reasoning, and foster confidence.

Collecting Data

As illustrated in Figure 1, four data collection methods were used. During each intervention, *audio and video recordings* of students' activities, and researcher's *field notes* about students' discussions were gathered. Then, at the end of each intervention, students' *mathematics journals* were collected. Lastly, after each intervention was completed, students were invited to participate in *recall interviews* to further discuss what they have experienced in class.

Video and audio recordings allowed for recurrent examination. These recordings were used to create transcripts, which were of fundamental relevance for the analysis of students' mathematical proficiency. While student group discussions about the task were being audio and video recorded, the researcher observed students and took field notes based on students' work, questions and conjectures about tasks. These field notes were to complement perceptions and comments gathered through recordings. During activities, students were encouraged to report their discussion processes and investigation outcomes by writing notes in their individual journals. Students were asked and remembered to record variables, assumptions, strategies, solutions, thoughts, changes in reasoning, and doubts in their journals. These written materials helped to portray students' development of mathematical proficiency. Journals were contrasted with audio and video recordings, providing adequate warrants for the conclusions made concerning students' mathematical proficiency. These documents were also used in recall interviews as a way of reminding students of what they did during the tasks.

Recall interviews were *stimulated recall interviews* (Anderson et al., 2009), in which students were asked to recall situations that happened in class and discuss them. The researcher conducted the interviews. It was extremely important to provide students with a comfortable and respectful environment, not to create expectations about what students were to say, not to induce students by researcher thoughts, and not to drive students to fulfill researcher expectations. A relevant aspect of stimulated recall interviews is that if students feel comfortable enough to speak freely, they can drive the interviews and clarify their thoughts. This strategy is likely to draw the researcher's attention to relevant issues that might not have been noticed before and could enrich research findings. Nine out of the 12 invited students accepted the invitation to participate in recall interviews.

The interview questions were meant to unpack students' work and investigate students' mathematical proficiency. Some general questions were elaborated for discussion, and some specific questions were posed based on students' individual work during the modelling tasks. The combination of asked questions depended on the task, the student, and the course of the interview. Because stimulated recall interviews are based on post-reflections about what students meant or thought, they might not reflect students' activities and opinions with exactness. In this sense, it is essential to consider other methods of data collection, such as the ones described before. Audio and video recordings, field notes and mathematics journals reflect students' activities in the actual intervention setting; therefore, students' ideas and thoughts might be more accurate when gathered by these methods. In analyzing and overlapping the data collected from the four different methods, findings supplement each other, bring up nuances, give insights about students' thinking, and, as a result, provide a richer picture of the situation.

Analyzing the Data

Analyzing students' mathematical proficiency involves comprehending in what ways students understand and process mathematics. Three theoretical frameworks were considered for this study analysis: Pirie and Kieren's (1994) model, Tall's (2013) model, and Kilpatrick et al.'s (2001) model. Due to the challenges in pointing out students' proficiency, Kilpatrick et al.'s (2001) mathematical proficiency model was chosen to operationalize this process through some indicators. The use of this model offers some advantages to this classroom-based research when compared to the other analyzed models: it includes features of mathematical understanding that are present in the literature; it relates students' mathematical performance to teachers' daily practice; it thoroughly describes students' mathematical performance in terms of concepts, procedures, strategies, and reasoning; and it considers student-related aspects.

Kilpatrick et al. (2001) use the term mathematical proficiency to refer to aspects deemed as necessary to successfully learn mathematics. Their notion of mathematical proficiency states that students need to accomplish five different strands to achieve mathematical proficiency, namely: *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition*. The authors understand that these five strands are interconnected as a complex whole and all of them influence students' mathematical proficiency. *Conceptual understanding* enables students to retrieve and apply mathematical content more easily because they can make sense of mathematics as a whole and not as isolated parts. *Procedural fluency* involves knowing and understanding which procedure to use, when, how and why to use it. *Strategic competence* requires students to work on problem formulation, problem representation, and problem-solving. *Adaptive reasoning* is the ability to logically relate mathematical concepts and situations, to adapt thoughts, conjectures and approaches. Finally, *productive disposition* is defined as students' positive perception about the worthiness and usefulness of mathematics, and the belief in their own ability to make sense of, learn and do mathematics. Based on these strands, a list of indicators was created to identify each strand in students' mathematical work (Table 1). Data collected was coded according to these indicators.

Table 1

Data Analysis.Indicators for each of the Strands of Mathematical Proficiency

Conceptual Understanding	Connect mathematical content.
	Retrieve mathematical content.
	Understand mathematical content.
Strategic Competence	Build a strategy to understand the problem.
	Build a strategy to represent the problem.
	Build a strategy to solve the problem.
Procedural Fluency	Choose a right procedure.
	Choose a right moment to apply the procedure.
	Perform the procedure correctly.
	Understand the procedure.
Adaptive Reasoning	Logically relate contents.
	Logically relate situations.
	Logically relate content and situation.
	Transfer content between situations.
Productive Disposition	Perceive mathematics as worthwhile.
	Believe in his/her ability to learn mathematics.
	Believe in his/her ability to do mathematics.

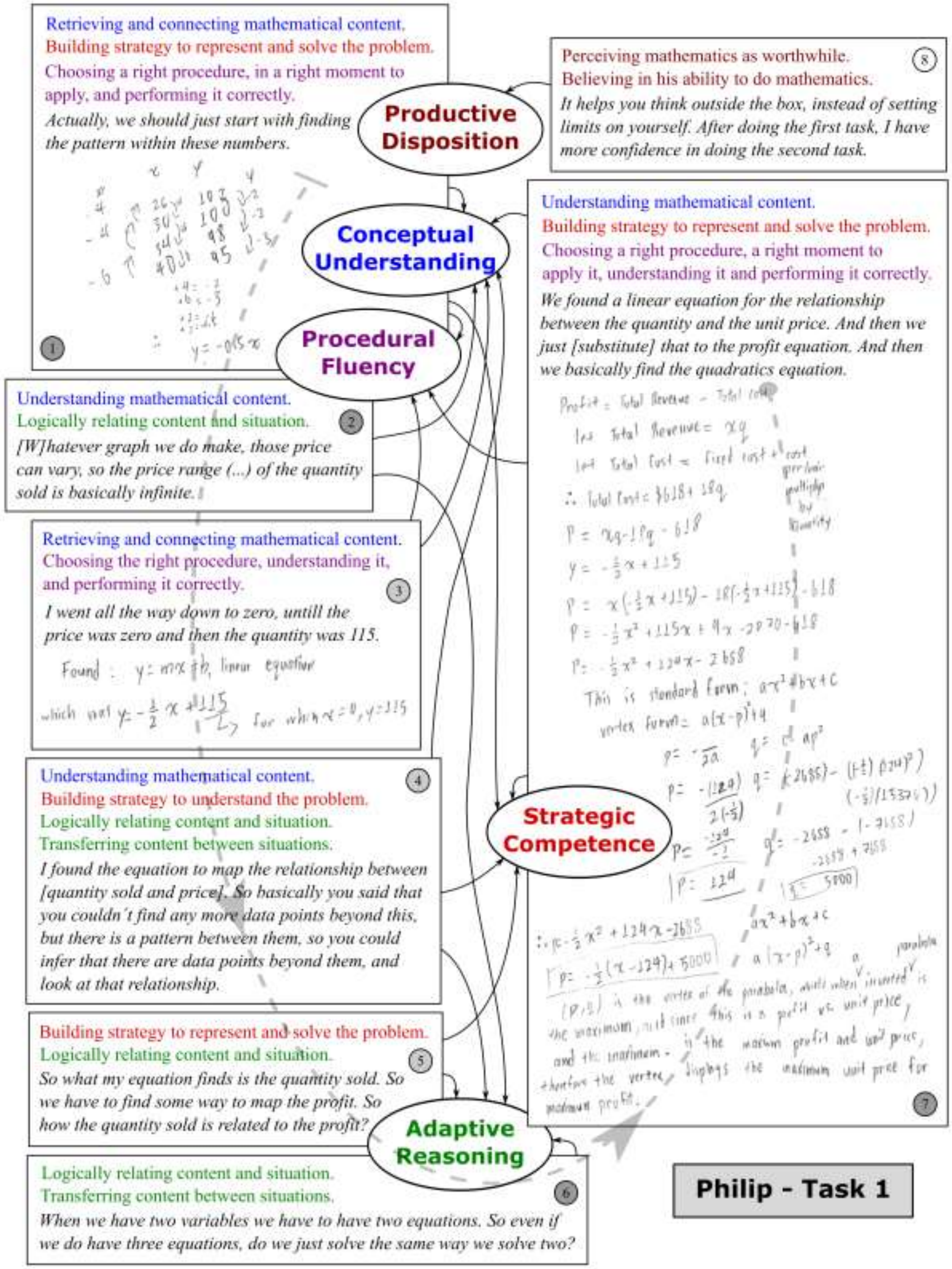
To organize and investigate the collected and coded data, interpretive diagrams illustrating individual student's thinking path were elaborated by the researcher. Although students were working in groups, diagrams were based on students' individual work. These diagrams portray fragments collected from students' written materials, audio and video transcripts, and interview transcripts. Researcher field notes are not used in the diagrams' creation. An example of an interpretive diagram is shown in Figure 3. Fragments from student's work are shown inside rectangles within the diagram and are mapped according to the student's journey through the task from introduction to completion. Italics are used to represent the student's verbatim quotations. Kilpatrick et al.'s (2001) proficiency strands are illustrated on the diagram within bubbles. Fragments were categorized based on the indicators presented in Table 1. To illustrate the interpretation of a fragment as a specific strand or strands of mathematical proficiency, black arrows connect them. The chronological order that fragments were collected is indicated by a grey ellipse-shaped arrow. The beginning of the process is represented by a dash at one extreme of the arrow, and the end of the process is represented by a dot at the other extreme of the arrow. If the grey arrow does not overlap fragments, it means these fragments are from recall interviews. To analyze the data across students and across the strands of mathematical proficiency, students' work needed to be made comparable. Therefore, a decision was made to divide their work into phases. Phases 1, 2 and 3 refer to the beginning, the middle and the end of a student's work respectively. Depending on the student's work, these phases slightly varied. Although these phases speak to three different moments in time during the modelling investigation process, they do not intend to characterize the modelling process as a three-stage process. Fragments of Phases 1 and 3 are numbered in dark gray. Fragments of Phase 2 are numbered in light gray. Fragments of interviews are numbered in white.

Four diagrams (representing the work of four different students) were created for each of the four tasks. In total, 16 interpretive diagrams were created and analyzed. The goal was to have diagrams representing all 12 students, all four tasks, and all 12 discussion groups (three groups for each of the four tasks). All 12 students were considered in the data analysis regardless of their interest, background or readiness in mathematics. Unfortunately, diagrams did not include the data from three out of the 12 students, either because students were not verbally contributing during the task period, either because they did not have enough records in their journals or both. All the other nine students were contemplated once or twice in the diagrams created. All four tasks and students' work from 11 of the 12 discussion groups were represented in the interpretive diagrams. Students' work was chosen based on the quantity of students' written, and verbal contributions, and also based on the quality of audio and video recordings. Some recordings presented too much noise in the background and, although software resources were used to try lessening the noise effect, that did not turn out successfully. The use of individual microphones would have helped in this matter.

Although the interpretive diagrams were created based solely on students' written and verbal communications, and the mathematical proficiency indicators used to code the data were meant to be clear and well-defined, the diagrams reveal the researcher's interpretation of students' mathematical work. This interpretation accounted for overlaps between the different indicators, given that one single fragment from students' work could inherently reveal more than one strand of mathematical proficiency. Besides, a detailed narrative explaining the interpretation of each diagram was written, and the field notes were used in some of the narratives to account for some of the researcher's observations and perspectives. While readers may agree with the researcher's interpretations of students' work, it may also be the case that this is not true, which would speak to a limitation of the research.

Figure 3

An Example of an Interpretive Diagram



Research Outcomes

Mathematical Proficiency Expressions

To answer this study's first research question – how mathematical proficiency is observed and expressed when students are engaged in mathematical modelling – it was necessary to look at students' actions and interactions in terms of the five strands of mathematical proficiency. Fragments from interpretive diagrams were analyzed to identify specific ways of expressing individual strands of mathematical proficiency; that is what actions in students' work characterized each strand. Students' expressions of conceptual understanding were mainly afforded by their discussions and explanations in their groups, and on a smaller scale, these expressions were afforded by the teacher or by self-talk. Students expressed conceptual understanding when: retrieving a concept; explaining a concept they were willing to use; explaining how or why they were willing to use a determined concept; describing a situation related to a specific concept; explaining what they were doing; explaining how or why they got to a determined stage; describing a situation under analysis and making a conceptual conclusion; describing a procedural step that was justified based on the concept behind it; representing a situation by using mathematical terms and concepts; representing a situation through a mathematical illustration and mathematical concepts.

As for strategic competence, most evidence was taken from the students' verbal discourse, but there was also written evidence. Some ways of expressing students' strategic competence were: describing a strategy; explaining the usefulness of a strategy; describing the approach to solve/model the task; explaining what needs to be done; explaining why following a determined plan; making conjectures about a possible plan; making questions to better organize thoughts and understand a task; representing a strategy by using mathematical terms; representing a strategy through a table; representing a strategy through a mathematical illustration. Procedural fluency is the easiest strand of mathematical proficiency to be identified in students' work. This is because – different from other strands – students usually make notes of the procedures they use, given that they typically need to write down their mathematical procedures to work on them. If some students are not prompted to write down their thoughts, conjectures and reasoning, they will probably only write down procedures. Some ways of expressing procedural fluency were: writing and working on a mathematical equation (or any other procedure) that represented a problem; organizing data in a table or in another format and finding patterns or relations; writing a mathematical algorithm that represented a problem; describing a procedure; explaining a procedure.

In terms of adaptive reasoning, it is worth noticing that part of the evidence was gathered during students' reflections in stimulated recall interviews. This fact confirms that students do not always reveal their thoughts while engaged in classroom mathematical tasks. Sometimes students need to be asked or prompted to reflect on what they are doing so that their reasoning is disclosed and elucidated. Some ways of expressing adaptive reasoning were when students: realized a change in a task situation and conjectured about new possibilities; explained why or how a determined content was used to approach the given situation; explained why or how a content interfered in the way a situation unfolded; compared different contents in relation to one same feature; conjectured about how different situations related to each other;

explained how or why transferring one content from one situation to another; inquired about the use of certain content in a certain situation. Finally, productive disposition was expressed when students: valued figuring out the task on their own, instead of being told what to do; identified what needed to be done and worked towards it; took their own decisions; described what to be done; showed belief in their ability to infer; showed belief in their capacity to understand; approached the hardest first; trusted that at some point they would get to an answer; shared and used their ideas within the group; showed enjoyment when getting to an end; inferred what they would get when following a different path; valued the fact that the task did not limit them; acknowledged they were more confident; pointed up to the relevance of mathematical modelling tasks done in class; underlined the usefulness of associating tasks with contextualized situations.

These five lists of how strands of mathematical proficiency can be expressed in students' work are not intended to be exhaustive lists. Their main purpose is to show the richness of actions that can reveal mathematical proficiency along with students' work, indicating options that can be encouraged or even required in students' mathematical written work, and pointing out new possibilities when assessing students' mathematical work.

Mathematical Proficiency Behaviours

From the analysis of students' interpretive diagrams, an extra research question emerged: What mathematical proficiency behaviours can be observed in students' modelling processes? All 16 interpretive diagrams were analyzed in four different ways: individually; across students; across tasks; and across strands of mathematical proficiency.

The first analysis approach, based on individual student's work on single tasks, brought up four remarks. The first one highlights that mathematical modelling can promote the development of students' mathematical proficiency, which is a sought-after goal in students' mathematics education. The 16 diagrams presented at least one fragment of each of the proficiency strands; which means that students were somehow working on mathematical skills related to concepts, strategies, procedures, reasoning, and attitude towards mathematics while engaged in the modelling tasks. This outcome indicates modelling as a prospective approach to mathematics teaching and learning. The second remark is that, when working on mathematical modelling, students work on the strands of mathematical proficiency even when they do not complete the task requirements. This outcome can be confirmed through students' diagrams. Some of the students did not finish their tasks, mainly because they needed some extra time; but students' diagrams still portray all five strands of mathematical proficiency. This outcome decreases the pressure on students to get to a final answer, as if this was a condition for them to learn mathematics. It highligths the development of mathematical proficiency throughout the processes of learning. The third remark refers to the difficulty in gathering fragments related to adaptive reasoning and productive disposition. Students did not commonly or openly express these strands, and when they did it was usually by verbal communication. Adaptive reasoning was expressed when students explained their thoughts or options to their peers, or when they were thinking out loud as a way of understanding or clarifying their ideas. It was not usual for students to write down their reasoning. In general, they wrote down the procedures derived from their reasoning processes. Therefore, many times their reasoning was implicit in their procedures. This implicit reasoning might be clear by analyzing students' procedures; however, changes in reasoning might be almost unobservable, in particular, when students erase or do not register their attempts to solve the task. As for productive disposition, this strand was mainly observed as student excitement and confidence about what they were doing, or when they verbalized their contentment in learning some useful knowledge or in exploring some likely-to-happen situations. Some students wrote down positive comments about their accomplishments, but this was not standard. Opportunities to identify productive disposition were scarce. The last remark observed in students' individual work refers to the use of procedural fluency as a way of analyzing the task, instead of just a way of solving the task. Procedural fluency is typically expected when students are implementing procedures to solve the task. This individual analysis drew attention to the fact that students could use procedural examples to illustrate or represent what was being described in the task. By doing that, students were able to express what they were supposed to visualize and analyze, which made their comprehension process more tangible and easier.

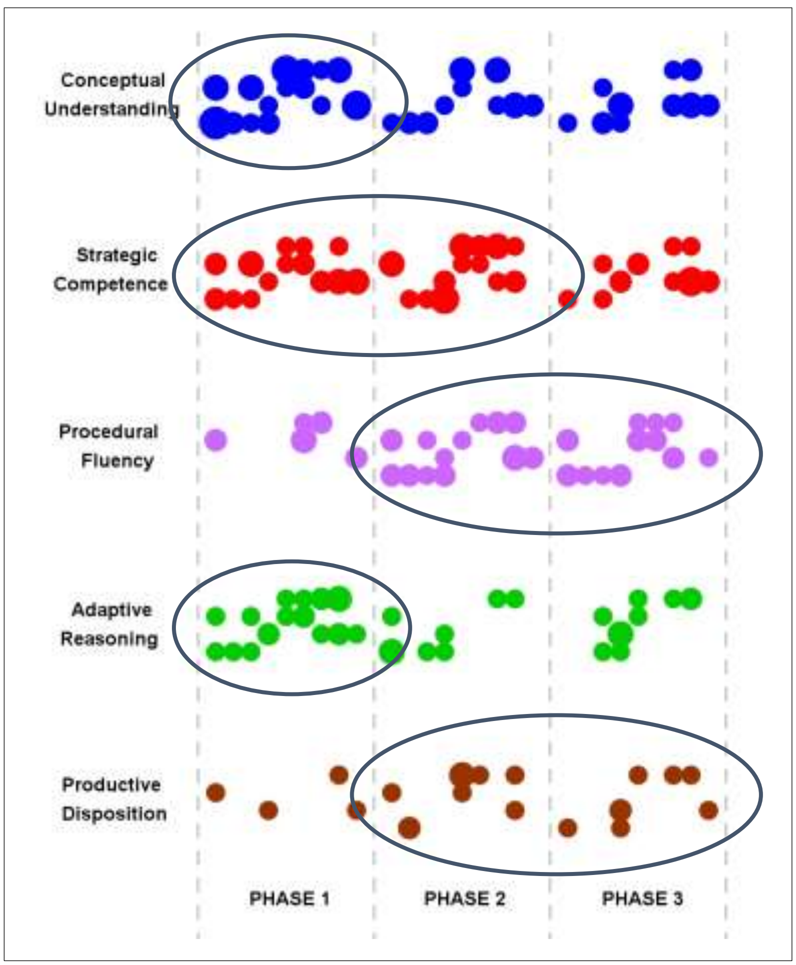
The second analysis approach was by crossing data over students, that is, by looking at one same task when done by different students. All four diagrams referring to the same task – from four different students – were investigated and mapped in search of common aspects or behaviours. The goal was to figure out if the task was accountable for prompting specific thoughts or raising specific strategies that would result in a mathematical proficiency pattern. The same procedure was repeated for tasks one, two, three and four. However, no patterns or tendencies triggered by the tasks were found. The third analysis approach was done by crossing data over tasks, that is, by looking at one same student when doing different tasks. In this case, two different tasks done by the same student were analyzed and mapped looking for similar ways of conducting the tasks. The idea was to identify if the student's mindset could possibly influence the resolution of different tasks. This behaviour could be observed if similar thoughts, approaches or strategies were consistently used in different tasks, due to the student's particular ways of doing mathematics. If that was the case, peculiar mathematical proficiency patterns could be noticed. Nevertheless, that was not the case, which means no patterns or tendencies seemed to be generated by students' individualities.

The final analysis approach was by crossing data over the strands of mathematical proficiency, that is, by looking at the strands along time for all students and all tasks. A distribution of the strands of mathematical proficiency, throughout task phases, is portrayed in the graph shown in Figure 4. Task phases refer to Phases 1, 2 and 3 identified in each diagram. Each dot in the graph is obtained by coordinates, in which x corresponds to a number composed by [Phase Number. Student Number] and y corresponds to a number composed by [Strand Number. Task Number]. Productive disposition is considered as strand 0, adaptive reasoning as strand 1, procedural fluency as strand 2, strategic competence as strand 3, and conceptual understanding as strand 4. Each student and each task were assigned a number. If a dot has coordinates (1.7, 2.4), it refers to student number 7, doing task number 4, during phase number 1, and working on strand number 2. In the graph, strands are weighted, which means that if the same strand occurs more than once within the same phase, same task and same student, the respective dot will be increasingly bigger according to the number of occurrences.

As Figure 4 shows, the different strands are present in all stages of students' work. However, there is a larger concentration of conceptual understanding and adaptive reasoning at the initial phases, attesting to students' greater need to retrieve previous concepts, reason and ponder different options, understand the task and establish a working plan. Strategic competence was more intense in the initial and middle phases, indicating students' attempts to come up with strategies to solve the task. Procedural fluency was mainly concentrated in the middle and final phases, suggesting students were doing the necessary procedural work to carry out the planned strategies. Finally, productive disposition was expressed more often in the middle and final phases, after students have done some work and have experienced reassuring results. Students' mathematical proficiency may unfold along time following this tendency. Yet, this study does not suggest that this is a fixed pattern for students' proficiency growth while modelling.

Figure 4

Graph of Weighted Strands Over Investigation Phases



Mathematical Proficiency in the Context of Mathematical Modelling

This study's final research question speaks to the affordances on students' mathematical proficiency when modelling is explored in high-school classes. To answer this question, it is necessary to describe how the strands of mathematical proficiency function in the context of mathematical modelling, that is, how mathematical modelling supports the development of these strands by demanding students to engage in activities that are not simply based on the use of procedures. Blum and Borromeo Ferri (2009) suggest seven different stages in a modelling cycle, namely: understanding the task; simplifying/structuring; mathematizing; working mathematically; interpreting; validating; and presenting. Thinking in terms of these different stages and supported by Palharini and Almeida's (2015) thoughts about investigating mathematical thinking processes through modelling, students' activities were explored to bridge the modelling processes and the strands of mathematical proficiency. A set of 13 modelling processes that reflect the activities students engaged in when working on the proposed mathematical tasks was determined. Based on the list of indicators used for data analysis (Table 1), these processes and respective activities were then correlated with the promoted strands of mathematical proficiency. The result of this correspondence is illustrated in Table 2

Table 2

Mathematical Modelling. Processes and Promoted Strands of Mathematical Proficiency

Mathematical Modelling Processes	Students' Activities	Promoted Strands of Mathematical Proficiency
Engagement	Investigating likely-to-happen situations.	*Productive Disposition*
Motivation	Realizing mathematics usefulness.	*Productive Disposition*
Positive Attitude	Realizing the ability to do and learn mathematics, and to solve problems.	*Productive Disposition*
Investigation	Exploring a task and its goals.	*All five strands.*
Conceptual Analysis	Relating a task to mathematical concepts.	*Conceptual Understanding*
Content Analysis	Relating a task to mathematical contents.	*Conceptual Understanding*
Strategy Building	Figuring out strategies to solve a task.	*Strategic Competence*
Decision Making	Choosing appropriate options to solve a task.	*Strategic Competence*
Choice of Procedures	Determining proper procedures to solve a task.	*Procedural Fluency*
Use of Procedures	Implementing mathematical procedures.	*Procedural Fluency*
Knowledge Production	Producing or researching the necessary knowledge to solve a task.	*All five strands.*
Mathematical Reasoning	Finding out logical relations.	*Adaptive Reasoning*
Explanation	Explaining or justifying decisions or solutions.	*Adaptive Reasoning*

Although listed in a certain order, these processes do not adhere to a determined order. Modelling is not linear and, as a result, these processes can coexist in diverse ways. As suggested by Lamon (2003), when presented with a modelling problem that addressed a likely-to-happen situation and portrayed the usefulness of mathematics, students' behaviours reflected engagement and motivation. Also, when students perceived that they were capable of doing and learning mathematics they showed a positive attitude towards mathematics. As per Kilpatrick et al.'s (2001) work, both these situations promote the development of productive disposition. During the modelling tasks, students would connect to prior knowledge in search of content or concepts that would relate to the task they were investigating. These processes were called conceptual and content analysis, speaking to the development of conceptual understanding (Kilpatrick et al.). Students also needed to analyze diverse situations and figure out a plan to approach them. Students had to search for strategies and determine which strategy would be more appropriate; focusing on strategy building and decision making and, as a result, promoting strategic competence (Kilpatrick et al.). Besides, students were expected to investigate situations that were not directly related to specific learned content, they needed to consider different procedures, identify an appropriate one, and implement it, addressing the development of procedural fluency (Kilpatrick et al.). Finally, when students were engaged in activities that required them to seek mathematical relations or to explain and justify their work, they were working on the processes of mathematical reasoning and explanation, which promotes adaptive reasoning (Kilpatrick et al.). Due to the extent of the so-called investigation and knowledge production processes, both were considered to fully promote the strands of mathematical proficiency.

Final Thoughts

The benefits of bringing modelling into mathematics classes are mostly acknowledged, and modelling is becoming more common and more appealing. Research around this topic is increasing, and new understandings and possibilities are emerging to enrich mathematics teaching and learning processes. Modelling experiences can be of benefit for teachers and students (Biembengut, 2009; Blum & Borromeo Ferri, 2009; Doerr, 2006; English & Doerr, 2004; Viecili, 2006,), and teachers should consider saving class time to work on them. As the Mathematical Modelling Handbook (Teachers College Columbia University, 2012) highlights, "modelling cannot be set aside or employed only when spare time arises" (p. vi). However, although there is adequate research about this topic – gathering several successful examples – there is not enough knowledge to lead to a consensus about how modelling should be implemented in class (Vorhölter et al., 2014). Indeed, the practical incorporation of mathematical modelling in classes still poses a big challenge. As Kaiser and Stender (2013) affirm:

> It is especially an open question, how complex authentic modelling problems put forward by the realistic or applied perspective on modelling can be integrated into mathematics education, what kind of learning environment is necessary, whether a change in the role of the teacher to a coach or mentor of the students is needed (p. 279).

This study intends to yield some insight into what kind of environment is necessary when doing modelling activities in high-school classes, and insight into the integration of mathematical modelling tasks that aim to promote mathematical proficiency. The first concern when investigating this scenario was related to the design of the classroom in which students are immersed. It is important to recognize a mathematics class as a complex system, and respect and support the complex features inherent to it (Davis & Simmt, 2003; Ricks, 2009). Maintaining this supportive environment was essential to obtain this research reported outcomes. The next research concern was observing the five strands of mathematical proficiency (Kilpatrick et al., 2001) in students' mathematical work while modelling in this complex classroom setting. Palharini and Almeida (2015) argue that modelling tasks allow mathematical thinking processes to be revealed and explored, which in turn facilitates the creation of proper learning approaches that will foster these processes. By unfolding students' mathematical proficiency, teachers can better access and grasp students' thoughts to enhance their teaching and learning strategies. This study differs from previous ones primarily in the two different aspects described above, which are: 1) the classroom design, which is set respecting complex system features; and 2) the research analysis, which focuses on high-school students' mathematical proficiency.

The integration of mathematical modelling tasks in class can pose some challenges; in particular, when students are expected to produce knowledge. Diverse reasons can be mentioned by teachers to avoid it. Three of them might be of more influence when teachers give up, they are: the difficulty in creating mathematical modelling tasks; the difficulty in fitting modelling tasks within curriculum outcomes; and the long time required implementing modelling tasks. As mentioned in the Mathematical Modelling Handbook (Teachers College Columbia University, 2012), "[t]he integrated nature of mathematical modelling, and in turn the number of curricular standards covered when working through a modelling activity, make modelling activities a very efficient use of class time" (p. vi). That is to say that the time required for the implementation and the connection with the curriculum are not exactly constraints, once modelling successfully integrates content in a timely manner. This same handbook presents 26 different modelling modules that can be used in mathematics classes, not to mention other publications indicated earlier in this paper with the same goal. The difficulty in creating modelling tasks and/or the lack of instructional materials are not as problematic as they were when tasks were not readily available. Moreover, the benefits modelling has to offer to mathematics teaching and learning processes (e.g. Blum & Borromeo Ferri, 2009; Biembengut, 2009; Viecili, 2006) outweigh the abovementioned challenges.

This research addresses mathematical proficiency expressions and behaviours when students are engaged in mathematical modelling tasks in a high-school class. It also speaks to the affordances on students' mathematical proficiency while modelling. Previous research did not analyze high-school students' engagement in mathematical modelling tasks in terms of their mathematical proficiency. This research suggests that mathematical modelling has the potential of promoting and fostering students' mathematical proficiency, even when tasks assigned are not fully completed. Considering that mathematics teachers and educators can benefit from teaching approaches that favour students' solid learning of mathematics, it is of crucial importance to draw attention to classroom-based experiences that work towards teaching for mathematical proficiency.

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