C^*-Algebras and the Gelfand-Naimark Theorems

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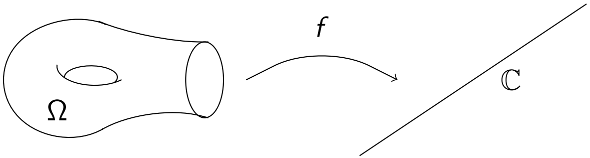
“A useful way of thinking of the theory of C^*-algebras is as non-commutative topology. This is justified by the correspondence between commutatve C^*-algebras and Hausdorff locally compact topological spaces given by the Gelfand representation. On the other hand the von Neumann algebras are a class of C^*-algebras whose studies can be thought as non-commutative measure theory.”

Banach Algebras

Definition

- An algebra A is a $\mathbb{C}$-vector space together with a bilinear map $A \times A \longrightarrow A$ which is associative. We will assume that there is a multiplicative identity 1. (Unital algebra)
- A norm $\|\cdot\|$ on an algebra A is said to be submultiplicative if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. In this case $(A, \|\cdot\|)$ is called a normed algebra.
- A Banach algebra A is a normed algebra which in complete with respect to the norm, i.e. every Cauchy sequence converges in A.

Example 1: Commutative Example



Let Ω be a Hausdorff compact topological space. Consider the space

$$C(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \mid f \text{ continuous}\}$$

with point-wise multiplication

$$(fg)(\omega) = f(\omega)g(\omega) \quad \forall \omega \in \Omega$$

and with norm the sup-norm

$$\|f\| = \sup_{\omega \in \Omega} |f(\omega)|$$

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Example 2 :Non-Commutative Example

Let $(X, \|\cdot\|)$ be a Banach space space. Consider the space

$$\mathcal{B}(X) = \{T : X \longrightarrow X \mid T \text{ bounded}\}$$

with “multiplication" given by composition. Then $\mathcal{B}(X)$ is a Banach algebra with respect to the operator norm:

If $T \in \mathcal{B}(X)$ define

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.$$

Gelfand-Naimark Theorems

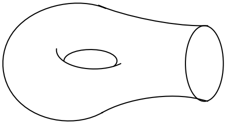
“Example 1 and Example 2 are the generic C^*-algebras."

We are going to study the commutative case

Given a C^*-algebra A how do we construct a Hausdorff compact topological space $\Omega(A)$ such that

$$A \cong C(\Omega(A))\,?$$

$$A \xrightarrow{\ ?\ } \Omega(A) =$$



Ideals

Definition

A left (respectively, right) ideal in an algebra A is a vector subspace J such that $a \in A$ and $b \in J$ implies that $ab \in J$ (respectively, $ba \in J$). An ideal in A is a vector subspace that is simultaneously a left and a right ideal.

Theorem

Let J be an ideal in a Banach algebra A. If J is proper, so is its closure $\bar{J}$. If J is maximal, then its closed.

The spectrum

Let A be a unital Banach algebra.

Definition

We define the spectrum of an element $a \in A$ to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \notin \mathrm{Inv}(A)\}$$

Example

Let $A = C(\Omega)$ where Ω is a compact Hausdorff topological space. Then $\sigma(f) = f(\Omega)$ for all $f \in A$.

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Some properties of the spectrum

- Gelfand Theorem: $\sigma(a) \neq \emptyset$ for all $a \in A$.
- The spectrum of $a \in A$ is a compact subset of $\mathbb{C}$. Moreover, it is a subset of the disc of radius $\|a\|$ and centered in the origin.
- If we define the spectral radius of an element $a \in A$ by

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$$

then

$$r(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}$$

Characters

Definition

A character τ on a commutative algebra A is a non-zero homomorphism between algebras $\tau : A \longrightarrow \mathbb{C}$. Let $\Omega(A)$ denote the space of all characters on A.

Remark

The space of characters $\Omega(A)$ is a subspace of the dual space A'.

Theorem

- If $\tau \in \Omega(A)$ then $\|\tau\| = 1$.
- The set $\Omega(A)$ is non-empty and the map $\tau \longmapsto \ker(\tau)$ defines a bijection from $\Omega(A)$ onto the set of all maximal ideals of A.
- $\sigma(a) = \{\tau(a) \mid \tau \in \Omega(A)\}$.

Topology of $\Omega(A)$

- The space $\Omega(A)$ is contained in the closed unit ball B of A'. Endow $\Omega(A)$ with the relative weak* topology.
- Weak Topology: A sequence $(\chi_n)_n \subseteq A'$ converges to an element $\chi \in A'$ is the weak* topology if $\chi_n(a) \longrightarrow \chi(a)$ for all $a \in A$.

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Theorem

The space $\Omega(A)$ is a compact Hausdorff topological space with respect to the weak* topology induced by A'.

Proof

- $\Omega(A)$ is weak* closed in the unit ball B of A'.
- B is weak* compact (Banach-Alaoglu Theorem).

The Gelfand Representation

If $a \in A$ define a function $\hat{a} : \Omega(A) \longrightarrow \mathbb{C}$ by $\hat{a}(\tau) = \tau(a)$. We call $\hat{a}$ the Gelfand transform of a.

Remark

Note that the topology on $\Omega(A)$ is the weakest (smallest) topology making all this functions continuous.

Gelfand Representation Theorem

Suppose that A is a unital Banach algebra. Then the map $A \longrightarrow C(\Omega(A))$ given by $a \longmapsto \hat{a}$ is a norm-decreasing homomorphism, i.e. $\|\hat{a}\| \leq \|a\|$ and $r(a) = \|\hat{a}\|$.

Proof

$$\|\hat{a}\| = \sup_{\tau \in \Omega(A)} |\hat{a}(\tau)| = \sup_{\tau \in \Omega(a)} |\tau(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a) \leq \|a\|.$$

Involutions and C^*-Algebras

Definition

An involution on an algebra A is a conjugate-linear map $* : A \longrightarrow A$ such that $a^{**} = a$ and $(ab)^* = b^* a^*$.

Definition

A C^*-algebra is a Banach $*$-algebra such that $\|a^* a\| = \|a\|^2$.

Examples

- $\mathbb{C}$ is a C^*-algebra under conjugation.
- Example 1: $A = \Omega(A)$ is a C^*-algebra algebra with $f^*(\omega) = \overline{f(\omega)}$.
- Example 2: The set of bounded operators $\mathcal{B}(H)$ of a Hilbert space H is a C*-algebra under taking adjoints.

Gelfand-Naimark Theorem

Let A be a C^*-algebra , then the Gelfand representation $\phi : A \longrightarrow C(\Omega(A))$ is an isometric $*$-isomorphism.

Proof

Is it easy to see that ϕ is a $*$-homomorphism. No note that

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \|\phi(a^*a)\| = r(a^*a) \overset{!}{=} \|a^*a\| = \|a\|^2.$$

Therefore ϕ is an isometry (and hence injective). The set $\phi(A)$ is a closed $*$-subalgebra of $C(A)$ separating the points of $\Omega(A)$ and having the property that for any $\tau \in \Omega(A)$ there is an element $a \in A$ such that $\phi(a)(\tau) \neq 0$. The Stone-Weierstrass theorem implies that $\phi(A) = C(A)$.

Example 1 Revised

Let Ω be a Hausdorff compact topological space.

- $A = C(\Omega)$ is a C^*-algebra.
- The space of characters $\Omega(A)$ is a Hausdorff compact topological space.
- By the Gelfand-Naimark theorem the map $A \longrightarrow C(\Omega(A))$ is a C^*-algebra isomorphism, i.e. A is the space of complex-valued continuous functions over $\Omega(A)$.

Question

What is the relation between Ω and $\Omega(A)$? Topology?

Example 1 Revised

- To each point $\omega \in \Omega$ we associate a character in $A = C(\Omega)$

$$
\begin{aligned}
\chi :\ \Omega &\longrightarrow \Omega(A) \\
\omega &\longmapsto \chi_\omega : A \longrightarrow \mathbb{C} \\
&\qquad\qquad f \longmapsto f(\omega)
\end{aligned}
$$

- Note that $C(\Omega)$ separates points in Ω (Urysohn's lemma): if $\omega_1 \neq \omega_2$ then $\chi_{\omega_1} \neq \chi_{\omega_2}$. Therefore Ω can be embedded in $\Omega(C(\Omega))$.
- It can be shown using a compactess argument that χ is onto, i.e. $\Omega = \Omega(A)$ (As sets!).
- In spite that the left hand side carries the given topology of Ω and the right hand side carries the weak* topology relative to $C(\Omega)'$, these topologies coincide (compactess ragument again) so that

$$\Omega \cong \Omega(C(\Omega))$$

as topological spaces.

Functorial Relations

C^*-alg. to Top. Space

$$
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi_B} \\
C(\Omega(A)) & \xrightarrow{\chi\Omega\psi} & C(\Omega(B))
\end{array}
$$

Top. Space to C^*-alg.

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle \chi_X} & & \downarrow{\scriptstyle \chi_Y} \\
\Omega(C(X)) & \xrightarrow{\Omega\chi f} & \Omega(C(Y))
\end{array}
$$

Gelfand-Naimark (commutative) Theorem

Can be thought a s the construction of two contravariant functors from the category of (locally) compact Hausdorff spaces to the category of (non-unital) C^*-algebras.

Topology

- locally compact space
- compact space
- compactification
- continuous proper map
- homeomorphism
- open subset
- closed subset
- metrizable
- Baire measure

Algebra

- C^*-algebra
- unital C^*-algebra
- unitization
- $*$-homomorphism
- automorphism
- ideal
- quotient algebra
- separable
- positive linear functional

Non-Commutative Version

Definition

A representation of a C^*-algebra is a pair (H, ϕ) where H is a Hilbert space and $\phi : A \longrightarrow \mathcal{B}(H)$ is a $*$-homomorphism. We say that (H, ϕ) is faithful if ϕ is injective.

Gelfand-Neimark Theorem

If A is a C^*-algebra, then it has a faithful representation.

Mmmmm... Question

- Let $C_b(\mathbb{R})$ be defined as

$$C_b(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{C} \mid f \text{ continuous and bounded}\}$$

  with point-wise multiplication as with the sup norm and involution given by the complex conjugation.
- The algebra $A = C_b(\mathbb{R})$ is a unital commutative C^*-algebra. Thus, by the Gelfand-Naimark theorem there exists a Hausdorff compact topological space Ω such that $C_b(\mathbb{R}) \cong C(\Omega)$. What is the relation between $\mathbb{R}$ and Ω?
- If we generalize this example to a general topological space X, i.e.

$$C_b(X) \cong C(\Omega) \quad ???$$

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