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The Evolution of Research on Teaching Mathematics

International Perspectives in the Digital Era

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The Evolution of Research on Teaching Mathematics

International Perspectives in the Digital Era

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Contents

The Evolution of Research on Teaching Mathematics: International Perspectives in the Digital Era: Introduction

Agida G. Manizade , **Nils Buchholtz** , **and Kim Beswick**

1   Introduction

Mathematics teaching is subject to cultural and temporal conditions. Not only do school and societal conditions shift, and with them the composition of the student body, but also curricular regulations and new mathematical and pedagogical insights determine the content to be taught and the approach to learning used in mathematics classes. To reflect on mathematics teaching in a changing world, there is a need for continuous scientific research into this process of teaching mathematics. Results of this research also have a retrospective impact on mathematics teacher education insofar as the conditions of education need to be continuously adapted to the professional requirements of teachers in practice. Research on teaching mathematics thus bears a great responsibility and is a constantly evolving field of research for scholars around the globe.

This book comes at the time when the world is facing an ongoing global pandemic and experiencing violence and unrest due to active war. This publication symbolizes

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a professional commitment and international collaboration par excellence apropos teaching mathematics. The editors from three different continents and researchers who represent sixteen institutions and eight countries worked constructively and collaboratively with utmost respect for each other, with intentions to reflect on existing research knowledge and to create new knowledge that can be shared and used by other educators and researchers across the world.

In preparation for this book, our international group of researchers shared current issues related to the evolution of research on teaching mathematics. We examined the present state of research on mathematics teaching and discussed the theoretical and methodological challenges associated with it, including issues related to conceptualization, instrumentation, and design. Additionally, we explored the likely direction of future research developments. In our literature review and discussions on this project, it became evident that studies on teaching frequently establish direct relationships between units of analysis that, at first glance, cannot be assumed to be directly related in a chain of effects. There are examples of studies presented in this book that directly relate teacher competencies to student achievements using empirical measurement models in a causal or relational way. Without criticizing these studies across the board, however, it seems reasonable to consider moderating or intermediate variables in this chain of effects (Baron & Kenny, 1986), such as the initiated student learning activities observable by teachers in the classroom, aspects of instructional quality (e.g., classroom management or cognitive activation), or corresponding student variables such as attention and cooperation in class or students' prior knowledge (e.g., Fig. 1).

Although there are researchers who do indeed study mediating variables (e.g., Blömeke et al., 2022), it became clear to us that there is a lack of a systematic scientific overview of the complete chain of effects between teacher characteristics, activities, and students' learning processes. Overviews of precisely these aspects of research on teaching and respective studies are scarce, which inspired this book.
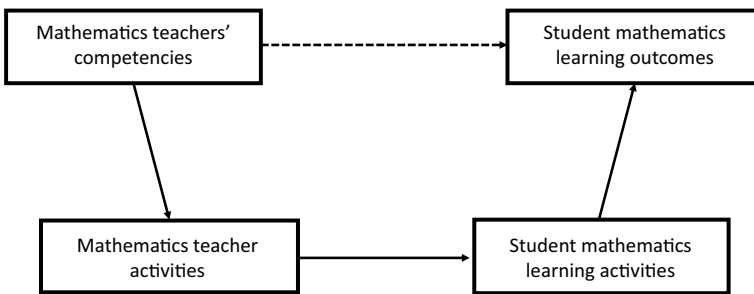
Fig. 1 Example of a chain of effect in teaching

2　The Purpose of the Book

Research that aims to relate teachers' observable actions with students' gains in achievement is referred to as *process–product research.* The term was first used by Donald M. Medley and Harold E. Mitzel (Hunt et al., 2010; Medley & Mitzel, 1963). *Presage-process–product research* then also considered other important variables, namely all the preceding and mediating variables that influence the actions of teachers in the classroom, such as teachers' professional training, knowledge, competencies, skills, personality traits, and teachers' abilities to plan a lesson or assess students. The framework for this book was based on a 1987 seminal work called "Evolution of research on teaching" by Medley (1987), in which he discussed literature on the development of research on teaching for thirty years prior to that publication vis-à-vis the presage-process–product standpoint. In it, he described a set of essential variables of research on teaching as given in Fig. 2, which he labelled online variables - "ones which lie along a direct line of influence of the teacher on pupil learning" (p. 105) and offline variables, "ones which affect pupil learning but are not under the direct control of the teacher." (ibid.).

Updating this framework is timely and, since it has not been described for mathematics teaching in particular, the framework was adapted and applied in the context of mathematics teaching and mathematics teacher education, as presented in Fig. 3 (Manizade et al., 2019). In the past twenty to thirty years, research on teaching has evolved further, and researchers have used a wide range of conceptual and theoretical frameworks in an effort to advance knowledge in presage-process–product research in mathematics education (e.g., Blömeke et al., 2016; Buchholtz, 2017; Liljedahl, 2016; Manizade & Martinovic, 2018). For this reason, the terms of the variables used by Medley (1987) have been adapted to the current research discourse. Although the
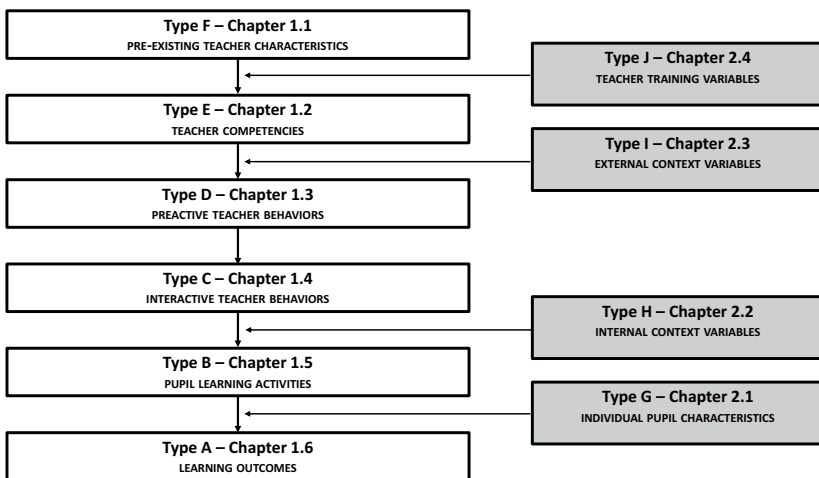
Fig. 2 Representation of Medley's 1987 framework mapped to the book's chapters

field of research on teaching mathematics has considerably advanced during the past twenty to thirty years, we find that the main units of analysis in the current research studies have remained the same: thus, Medley's framework is still valuable as it gives an orientation to all possible variables that become apparent qua the chain of effects from teacher behavior to student achievements. Moreover, the abiding challenges associated with the conceptualization, instrumentation, operationalization, and research design that Medley described are still complex, despite recent advances in technology and research methodology in the digital era.

One of the aims of the book is to update and situate Medley's framework within mathematics education research of the last three decades. Societal and educational realities have changed significantly since Medley wrote his seminal paper. Therefore, based on current research, additional variables must be considered in the chain of effects. Another goal is to provide researchers, who are scientifically concerned with more than one main unit of analysis—as described in Fig. 3—with current knowledge and methods apropos of the respective variables in the overview chapters. Each chapter of the book is based on reviews of research conducted over the past twenty to thirty years and written by leading experts in the respective fields. The chapters therefore also address cultural and technological aspects of the research on the respective variables.

Main Units of Analysis of Research on Teaching Mathematics

Within Cultural and Epistemological Contexts

Type F:
Pre-existing mathematics teacher characteristics
a mathematics teacher's beliefs and aptitude for teaching, characteristics needed to acquire professional competencies during training

Type J:
Mathematics teacher training and experiences
designed to increase mathematics teachers' range of competencies

Type E:
Mathematics teachers' competencies, knowledge, and skills
to function effectively in mathematics teaching situations

Type D:
Pre- and post-active mathematics teacher activities
such as planning, assessment, reflection, and other out-of-class activities of mathematics teaching

Type I:
External context variables
support systems: materials, technology, facilities, supervision, administrative support, community and parental support

Type C:
Interactive mathematics teacher activities
activities of the mathematics teacher while in the presence of students

Type H:
Internal context variables
characteristics of students or groups of students which affect response to mathematics teacher behaviors and actions

Type B:
Student mathematics learning activities
occur in the mathematics classroom. The types of student experiences that will result in desired learning outcomes

Type G:
Individual student characteristics, abilities, and personal qualities
which determine outcomes of any specific learning experience

Type A:
Student mathematics learning outcomes
measured after the teaching is over

Within a Digital Context

Fig. 3 Updated framework of research on teaching mathematics

Additionally, in his original work, Medley focused on discussion surrounding *good teaching* and the complexity of defining such a term in research (Medley, 1987). In the past twenty to thirty years, myriad of new theoretical perspectives on teaching mathematics have emerged in the field. These perspectives assume that a wide range of mathematics learning goals based on theoretical frameworks are enacted by teachers in the classroom (Manizade et al., this volume). Depending on these goals, the definition of *good teaching* and what is valued in the mathematics classroom can have an array of meanings (Manizade et al., this volume). These include reproducing the perfect sequence of steps when solving a mathematical problem, engaging students in productive struggle and productive failure, developing mathematical constructs through collaborative discourse, and addressing students' lived cultural experiences as mathematical experiences, to name a few. The updated framework, therefore, considers the epistemological contexts of research on teaching mathematics with respect to main units of analysis, in addition to considering the cultural and digital contexts that also affect all units of analyses of research presented in the framework (Fig. 3).

3 Book Structure

The book is comprised of two parts. In part one, we examine research in mathematics education with focus on units of analysis that Medley called *online variables* (Medley, 1987). In contrast to current use, the term *online* has a distinct and different meaning in Medley's work. *Online variables* are units of analysis of research that can be under the control of mathematics teachers. They included research on mathematics teaching and teacher education that examined: pre-existing mathematics teacher characteristics (Type F); mathematics teacher competencies, knowledge, and skills (Type E); pre-post-active mathematics teacher activities (Type D); interactive mathematics teacher activities (Type C); student mathematics learning activities (Type B); and student mathematics learning outcomes (Type A) (Fig. 3).

In part two, we examine mathematics education research with main units of analysis that are not under the direct control of teachers. These include *offline research variables* (Medley, 1987) such as individual student characteristics, abilities, and personal qualities (Type G); internal context variables (Type H); external context variables (Type I); and mathematics teacher training and experiences (Type J). A detailed discussion of both parts of the book is presented later in this chapter. Because the offline (Types J, I H, and G) research foci that are not under the direct control of mathematics teachers are so broad, our authors selected a subset of research variables within each type to discuss in their respective chapters included in part two of the book. We understand the importance of each research focus and unit of analysis and acknowledge that a larger publication would be needed to include all their components.

In the following section, we give an overview of the individual units of analysis of research on teaching mathematics, as well as the chapters of the book.

4   Part 1: Online Variables

4.1   Pre-Existing Mathematics Teacher Characteristics

Pre-existing teacher characteristics include abilities, knowledge, and attitudes that a candidate for admission to a teacher preparation program possesses on entry, as well as a candidate's aptitude for teaching. In order for teachers to learn the necessary competencies for teaching in teacher education processes, they must possess appropriate entry-level prerequisites that sustain competency development.

Mathematics teacher competencies include, for example, cognitive abilities such as prior mathematical and pedagogical knowledge at the point of study entry, attitudes toward mathematics as a subject or toward the learning and teaching mathematics, as well as motivational and volitional variables such as enthusiasm for the subject of mathematics and personality traits and identity aspects such as one's own understanding of one's role, self-regulation and self-concept, and ability to reflect and collaborate with students and with colleagues. More recent research also counts emotional aspects such as personal well-being or stress resilience among personal factors that play a role in competence acquisition at entry level. It should be noted here that all the influencing variables themselves also change in the context of teacher education. That is, in line with Medley, the changeability of personality structures is assumed.

In Chap. 2, Olive Chapman compiles findings on these main research units of analyses based on extensive literature reviews spanning over more than twenty years. With respect to the prior mathematical knowledge of pre-service teachers, Chapman focuses on studies in the content area of fractions, whole number operations, geometry and algebraic thinking and problem-posing. Many of the current studies demonstrate, in part, large gaps in knowledge related to conceptual understanding of elementary mathematical concepts and operations, which pose an ongoing challenge to teacher education. For the area of prior mathematics-related pedagogical knowledge, Chapman focuses on studies examining skills in observing instruction and noticing and analyzing student work and thinking and evaluating tasks. Here, too, the systematic review revealed weaknesses among beginning pre-service teachers who, for example, can generate few pedagogical decisions from observations of instruction or fail to recognize the potential of mathematics tasks. In the area of attitudes, Chapman adds to existing findings with those related to attitudes toward technology use and mathematical processes or specific mathematics areas such as algebra.

Overall, Chapman notes a shift in studies over the past twenty years away from focusing on single "hard" categories, such as high school graduation or mathematics grades, to examining content aspects of prior knowledge and learning conditions including those influenced by culture and technology at the beginning of the teacher education program. Finally, Chap. 2 also addresses methodological challenges and future directions for Type F research, including different survey formats, designs, and methods of research analysis.

4.2 Mathematics Teachers' Competencies, Knowledge and Skills

Medley described Type E teacher competencies as knowledges, skills, and values that a teacher possesses. Without going into detail about what exactly is meant by competencies, knowledges, or skills, he describes these as the "tools" of teaching in an instrumental, functional sense. They are the prerequisites for successful and competent teacher action in various situations. This assumes that the prerequisites for teaching can be precisely specified for a given situation - as is done in later research, for example, through requirements analysis by observing teachers. Interestingly, Medley also included values in these prerequisites and thus included affective characteristics of teachers among the competencies. A conceptual understanding of competency can be discerned here, the scope of which was recognized in the early 2000s in the educational psychology discussion on the conceptual understanding of competencies and was more widely received. In contrast to Type F, however, Medley saw this online variable less as the personality characteristics of teachers. He understood teacher competencies as a measurable outcome of teacher education and experiences - in contrast to Type F, pre-existing mathematics teacher characteristics. Teacher competencies thus always remain a potential trait in the exclusive research of Type E, since the (measurable) performance of these competencies only takes place in the actual preparation and implementation of teaching (Type D and C).

In Chap. 3, Nils Buchholtz, Björn Schwarz, and Gabriele Kaiser describe the development of mathematics education research on teacher competencies in the last 30 years, especially the research on teacher knowledge and affective variables such as beliefs or self-regulatory skills. For the subject of mathematics, normative requirements have always been formulated for teachers in terms of their content knowledge. However, the researchers see the starting point of research on Type E in psychological cognition research, which has strongly influenced research on mathematics teaching and teacher education. At its starting point, research on Type E was thus still closely aligned with Medley's description. However, Buchholtz, Schwarz and Kaiser describe how Lee Shulman's work in particular inspired, developed, and advanced the research. A broad research field of qualitative and quantitative studies on teacher cognitions developed, resulting in a plurality of different conceptualizations of teacher knowledge that refer to different knowledge bases (mainly: content knowledge, pedagogical content knowledge, pedagogical knowledge). Teacher competencies are thus conceived in research as a multidimensional construct, the complexity of which poses major challenges to research in terms of its measurability. Different ways of measurement (especially through knowledge tests) have been used in research. Overall, the plurality in a research field is perceived as a strength, especially since it is broadly based internationally. In recent years, research on teacher competencies has started to focus more on the situational performance of competencies, which has already extended the focus from Type E to Types

D and C. The reason for this development has been on the one hand methodological developments through video-based competence measurement, and on the other hand the increasing conviction that teacher competencies can only be examined to a limited extent outside of the situational context of practical teaching. That is, an isolated consideration of Type E is less insightful. To this end, the chapter provides an overview of current research on situational-based mathematics teacher competency measurement and the relationships among teacher competencies, instructional quality, and student outcomes.

4.3   Pre- and Post-Active Mathematics Teacher Activities

In his original work, Medley referred to the online variable, Type D, as *preactive teacher behaviors.* These included such activities as "planning, evaluation, and other out-of-class activities of teaching, the things a teacher does to promote pupil learning while no pupils are present". These are practices that demonstrate how teachers' professional competencies knowledge and skills (Type E) affect the quality of their classroom interactions with students (Type C), and therefore, indicate how successfully the teacher can meet their goals for teaching.

In their Chap. 4, Agida Manizade, Alex Moore, and Kim Beswick named this variable *pre- and post- active* because several of the Type D activities (e.g., lesson, and unit planning) are performed prior to teaching, while others (e.g., reflection, and assessment) are conducted after lessons have been taught. Manizade, Moore, and Beswick focused on lesson planning, assessment, and reflection as the key actions that teachers perform when students are not present in the classroom. These "pre- and post-" actions are the most direct ways through which teachers shape observable teaching work, as mediated by their goals for teaching. These goals are representations of teachers' epistemological commitments apropos teaching mathematics, whether those commitments be consciously espoused or unconsciously reproduced due to constraints within which they work. The researchers surveyed the literature on lesson planning, assessment, and reflection according to eight epistemological paradigms that are known in the field of mathematics teaching, namely Situated Learning Theory, Behaviorism, Cognitive Learning Theory, Social Constructivism, Structuralism, Problem Solving, Culturally Relevant Pedagogy, and Project- and Problem-Based Learning. They place other perspectives on learning theory, which are derivatives of these prevailing paradigms, within this overarching frame. They detail each perspective, providing a definition, goals for teaching, pros and cons of each theoretical perspective, and examples from the literature on teaching mathematics. The chapter revealed that some of the theoretical perspectives are well-reported in the literature whilst others have not received the same amount of attention from researchers. The researchers recognized that the chapter focused on the western cultural context and more research is needed in a variety of cultural settings, considering each of the settings affects every unit of analysis in research on mathematics teaching and teacher education (Fig. 3). The researchers posited that, amidst cultural

contexts and the technological advent of the digital era of mathematics education, researchers must engage more explicitly with the theoretical perspectives identified as underserved and must themselves reckon with their own epistemological commitments more intentionally when engaging and reporting on studies regarding Type D.

4.4 Interactive Mathematics Teacher Activities

Medley (1987) described interactive teacher behaviors as "the behaviors of the teacher while in the presence of students" (p. 105). He explained that these behaviors are typically what are referred to as teaching and are the means through which teachers influence students. They are directly observable actions through which teachers translate their pre-post-active behaviors (i.e., planning and other out-of-class activities, Type D) into learning experiences for students. They are the bridge between teachers' plans to promote student learning (Type D) and the things that students do that result in their learning (Type B).

In Chap. 5 Kim Beswick, Felicity Rawlings-Sanaei, and Laura Tuohilampi discuss the research literature related to the activities that mathematics teachers engage in when they are with students. Importantly in the digital era teachers can be with students without being physically with them. Teachers' interactive behaviors in online or virtual contexts remain under-researched but have attracted increased attention in recent years in which the pandemic forced the closure of schools for periods of weeks or months in many countries, necessitating a move to online interaction.

The authors structure their chapter in two main parts. The first surveys what we know about normative teaching practices; the things that typically happen in mathematics classrooms whether physical or virtual. They rely primarily on large scale studies, principally the *Trends in International Mathematics and Science Study* (TIMSS) and the *Programme for International Student Assessment* surveys (PISA). These studies rely on teacher self-reports as well as student reports of the activity that occurs in their mathematics classrooms. TIMSS video studies provided more direct access to teacher behaviors but have still relied on teachers to indicate the extent to which the video-recorded lessons were typical of their practice. The second part of Chap. 5 deals with teachers' interactive behaviors documented by researchers interested in promoting or supporting teachers to implement particular behaviors or to adopt in some way an approach to mathematics teaching that the researchers believe will be beneficial. Beswick et al. describe the interactive behaviors reported in these studies as atypical because they represent approximations of changed behaviors that align with the researchers' perspective.

4.5   Student Mathematics Learning Activities

The variable Type B was described by Medley as student learning activities. By this, he meant all types of student experiences within the classroom that result in the learning outcomes desired by the teacher. These student activities and behaviors always take place under teaching objectives, in that they are directed or oriented by the teacher and therefore a direct result of an interactive teacher's behavior (Type C). For a direct influence of teaching activities on student learning to be assumed, Medley presupposed that all learning is based on learners' activity. That is, student activity can be used as an indicator of learning processes. Most particularly, therefore, it is important that any activity is perceived as purposeful.

Maria Timmerman addresses this purpose of students' learning activities from a mathematical perspective in Chap. 6, presenting different ways in which students' learning activities could be understood and seen as productive or purposeful for learning mathematics. She illustrates that effective and equitable experiences of students are related to how mathematics learning has been defined over recent decades, in different countries internationally and also under different educational premises, whereby respective curricula can provide an orienting framework.

Timmerman notes a shift in mathematics education research towards student thinking over the last 30 years, where the focus is no longer exclusively on student behavior. This has been driven by developments of new epistemological perspectives on mathematics teaching, and the development of new curricular objectives, including but not limited to problem-solving, or project-based learning activities. Additionally, process-oriented goals, in contrast to the teaching of pure factual knowledge as well as the competence orientation, have fundamentally changed student learning activities by broadening the horizon of what over the years is considered as a learning activity in mathematics.

Regarding the development of the theoretical perspective on student learning activities, Timmerman describes different conceptualizations of student learning in mathematics, including the theory of progressive coordination of actions and the development of cognitive schemata, the research model of learning through activity, and research on student engagement, which plays a particularly important role in problem solving processes. Timmerman also focuses on how in the context of such activities, the affective learning conditions of the students, such as productive dispositions or student perseverance, which can positively influence student learning activities (e.g., when students are struggling or failing and can use this for learning processes). This also brings the individual prerequisites of students (Type H) more into focus when examining the effectiveness of learning activities.

4.6   Student Mathematics Learning Outcomes

Medley identified student learning outcomes as the first online variable (Type A), which he associated with each type of "changes in pupils" (p. 105) that can be measured after teaching has been completed. He referred to the outcome of a completed learning process, which at that time was primarily measured in the form of achievement gains on standardized tests. In this sense, he called it a "production of learning outcomes" (p.105) as a result of teaching with the attention given to progress towards teaching goals that could be detected through close observation. Learning outcomes are seen as the ultimate goal and the measurability criterion of teaching effectiveness. There are, however, challenges associated with the measurability of this criterion, that specifically relate to different theoretical frameworks and approaches used for teaching mathematics. These challenges, therefore, continue to be a part of the mathematics education research discourse.

In Chap. 7, Jelena Radišić presents an overview of the challenge of describing mathematical understanding and knowledge as a measurable learning outcome, addressing different conceptualizations of mathematical competence, literacy, or proficiency. Making something as vague as mathematical understanding measurable based of certain criteria remains a challenge of mathematics education research to this day. Various mathematical activities, such as problem-solving, modelling, reasoning, and proving have continuously found their way into mathematics education curricula internationally over the last 30 years and still elude measurability of mastery. For this reason, teaching effectiveness that is measured according to students' acquisition of these skills, is challenging. Jelena Radišić's research perspective is based on international large-scale assessment studies (ILSAs), which have been developed internationally since the late 1980s for comparative educational monitoring and which still today systematically collect and compare learning outcomes on the basis of high scientific standards. Since the studies are almost exclusively methodologically quantitative and use big data by collecting a large number of variables on many cases, they now allow the simultaneous statistical correlation of multiple variables and consideration of different contextual conditions in the tradition of presage-process–product research. Whereas Medley's assessment of "good teaching" with respect to Type A tended to be general in its maximization of learning outcomes, today's Type A research takes a more nuanced view in measuring effectiveness of learning for students with individual learning needs.

The fact that specific methodological problems arise with the measurement of student outcomes is addressed in the chapter, as is the growing influence that technology has on learning and therefore on our understanding of learning outcomes. Finally, Radišić takes a new perspective on research on Type A by describing affective variables such as student motivation and self-belief as learning outcomes in their own right. Affective variables remain underrepresented in research on teaching.

5 Part 2: Offline Variables

5.1 Individual Student Characteristics, Abilities and Personal Qualities

In Medley's model, individual student characteristics (Type G), that is abilities and other personal qualities of students, mediate between student learning activities (Type B) and student learning outcomes (Type A). This mediating offline variable is explained by the observation that students do not show the same outcome even under identical learning conditions. Learning processes in the classroom depend to a large extent on individual students' cognitive and affective preconditions, which can be shaped by family, social, cultural identity-forming experiences, and physical conditions.

Education is increasingly characterized by high levels of student diversity in many countries due to migration movements and cultural and transnational multiple attributions. Individual student characteristics can, therefore, include variables such as race, gender, or socio-economic background. The language requirements of students today are diversified to a greater extent than in Medley's time. In many countries, students with special educational needs are included in mainstream education, so that learning processes are also influenced by students' physical or social-emotional development and how they can overcome learning difficulties or learning disabilities. Mathematics education research also takes up emotional and physical characteristics such as resilience, mathematics anxiety, or students well-being as psychological variables influencing the individual learning process.

In Chap. 8, Rhonda Faragher describes central aspects of Type G in an overview and focuses on the subset of Type G, namely learners with intellectual disabilities, learning difficulties, and learned difficulties. She starts by describing two significant developments in the last decades: the recognition of streaming (tracking) as harmful; and the recognition of inclusive education as beneficial. These have changed the nature of mathematics classrooms substantially. Faragher first describes different approaches of mathematics education, neuro-psychological research, and general pedagogical research on special needs education to understand learning difficulties and learning disabilities of students and to make them accessible for research. She then presents different approaches that have developed in recent years to address the impact of these learning difficulties and learning disabilities on student achievement in the classroom and to provide equal opportunities for all students. The researcher claims that in doing so, teachers can adapt instruction in ways such as by the use of Universal Design for Learning (UDL), using digital tools that make instructional content more accessible to students, or adapting curriculum and learning activities to students' achievement levels and prior knowledge. Faragher uses case studies of achieving equity for students with Down syndrome to illustrate the latter throughout the chapter. Faragher argues that with the increasing acceptance and implementation of inclusive learning in the classroom, in research the Type G offline variable is ultimately not only a mediator between Type B and Type A, but as the direction

of future research, this offline variable must also play a role in other research variables, for example when teachers' lesson-planning is analyzed or appropriate support structures are created in schools.

5.2   Internal Context Variables

Internal context variables (Type H) affect individual or group student responses to any teacher actions in the classroom. They mediate between the interactive teacher behaviors (Type C) and the learning activities (Type B), thus influencing the way students respond to the teacher in social interaction and behave during initiated or mediated learning activities. By its nature, the Type H variable is close in content to the Type G variable, as psychosocial factors of student diversity are both evident at the individual level of learning processes and express their collective expression in the responses of students or groups of students to the teacher's teaching activities. This may include, for example, students' work behavior, motivation, self-efficacy, or self-regulation. Recent mathematics education research has also focused on the social and emotional experience of students and their well-being in the classroom. The offline variable, Type G, addresses intrapersonal cognitive preconditions and processing, as well as affective attitudes of the students, and thus primarily focuses on individual appropriation processes of the students against the background of diversity, the variable Type H. Additionally, this main unit of research analysis focuses on social and interpersonal factors of the students' diversity, which become particularly important in the interaction between student and teacher and leads to different observable actions of the students in the classroom.

Megan Che and Even Baker, in Chap. 9, follow this broader perspective on context variables by focusing on identity-creating aspects of individual student personality in their description of the Type H variable. The central thesis of their chapter is that the identity of students is not only based on individual elements, but also on collective elements and the learning context, i.e., the mathematical experiences of the students as doers of mathematics, which consequently requires a situated consideration of identity-forming aspects and internal context variables both in research and in teaching within external contexts. In their description of the future direction of research on student internal context, Megan Che and Evan Baker call for further consideration of research approaches based on critical theory and postmodern perspectives on educational contexts. The researchers claim that these perspectives can provide additional insight into "understandings of students' mathematical identities and internal social contexts in a variety of technological mathematical learning environments, including gaming environments, online mathematics classrooms, and social media environments" (Che & Baker, this volume) without dismissing the importance of students' access to the technology. Additionally, they discuss another future research focus, "online communities and the potential to inhabit yet another identity as a virtual being in virtual worlds." (Che & Baker, this volume).

5.3   External Context Variables

External context variables stand for the support system within which teachers act and thus exploit and develop the potential of their competencies for professional practice. Medley understood this as the material, the facilities, the supervision, and administrative support provided by the school or the community of practitioners. Since these offline variables are mediating factors between teacher competencies and pre-post-active teacher activities, external context variables mainly influence how teachers carry out activities such as lesson planning, evaluation, and reflection depending on contingently given formal and material structures in the global educational system or the local school. Medley illustrated this dependency by highlighting that teachers with the same, or even assumedly identical competency profiles would act differently in differently supported instructional settings.

What does the support mean within the school context in the sense of mathematics educational research on Type I? If we look at research on textbooks and curricula, for example, culturally shaped task and examination cultures and national educational standards come into view, and form the normative guidelines for teachers' work in formulating learning goals and planning lessons. For the practical implementation of these guidelines, lack of free access to teaching materials and books is too often an obstacle. The collegial support of mathematics teachers at school can also be counted as part of this support system. The opportunities for further training through involvement in informal or national teacher associations, access to professional development (PD) and local feedback structures at school, for example through the principal, parents, or peers, are part of the support system described.

In Chap. 10, Birgit Pepin and Ghislaine Gueudet consider an offline variable of the technological support of teaching. This new variable, which Medley could not yet include among the external context variables at the end of the 1980s, has continuously shaped the schoolwork of teachers within the last 30 years. In their chapter, Pepin and Gueudet shed light on the educational policy preconditions and anchors for the use of digital resources and educational technologies, as well as research on the willingness and preconditions for teachers to use or not use technology and digital resources in the classroom, or on the reasons why they do not. Overall, they note, the role of the teacher is changing toward supporting the learning process as students become more self-regulated learners in their engagement with digital learning tools. The integration of programming into mathematics instruction, which has been increasingly promoted over many years, also requires new knowledge on the part of the teacher. Research on the quality criteria of digital resources is also receiving attention, for example, on the development of electronic curriculum materials, electronic textbooks, and dynamic mathematics tasks that, in terms of student learning of mathematics, require teachers not only to integrate these materials into the classroom, but also to design their instruction around them.

5.4   Mathematics Teacher Training and Experiences

The duration and quality of teacher training can differ qualitatively and quantitatively across teachers, as Medley described in the Type J offline variable. Different teacher training factors are the influential variables that mediate teachers' personal characteristics (Type F) and learned competencies (Type E). This means, for example, the extent to which teachers can develop their personal potentials in the context of training processes and translate them into learned competencies and skills is influenced by aspects of their training. Medley (1987) understood this as the experiences during teacher training designed to increase the "teacher's repertoire of competencies" (p. 106). Thus, indirectly, the abilities and mediation approaches of teacher educators, coaches and trainers come into view, as well as engagement in teacher PD.

In the field of mathematics education research, there have long been many approaches to assessing the quality of teacher education and training and to evaluating the influence of corresponding variables on the development of teacher competencies by means of empirical studies. International studies have considered, for example, the duration of teacher training, the quality of the courses offered, and the number of courses attended during training. The form of teacher training (e.g., how courses are structured or which seminars and courses are effective in teacher training to acquire mathematical knowledge for teaching) can also be analyzed and assessed from the perspective of cultural and national educational policy influences or normative values of "good" teaching. The importance of continuous professional development for teachers has increased over recent decades. As a result, respective corresponding variables are considered, such as engagement and participation in teacher PD. Recent mathematics education research also focuses on incorporating variables such as duration, structure, and quality of PD as well as effectiveness of PD assessment measures.

In Chap. 11, Joyce Peters-Dasdemir, Lars Holzäpfel, Bärbel Barzel and Timo Leuders, describe a special unit of analysis assigned to Type J—the qualification of teacher educators or adult educators providing PD. This unit of analysis refers to the qualification of facilitators of PD in mathematics, which is an area that has been insufficiently researched and that Medley did not consider. The teaching profession is characterized by experiential and lifelong learning and continuous professional development has gained traction in educational studies. This development has led to scientific research on the quality of PD. The chapter's central idea here in terms of advancing research on teaching and Medley's framework is to extend the chain of effects upward to include the corresponding effectiveness of those engaged in teacher education. To this end, Peters-Dasdemir et al. developed a competency framework model that can be used to describe the necessary professional profile of facilitators. Based on the results of overview studies on the criteria of effective teacher training, development, and based on systematic findings in adult education, the model includes aspects of the role of trainers as facilitators, their content and field-specific knowledge, professional values, and beliefs. In addition, their role identity, professional

self-monitoring skills, and social competencies. The PD facilitators need to have fundamental professional knowledge and skills of the school subject that go beyond the knowledge of teachers (e.g., regarding curricular standards or current relevant empirical research findings).

5.5 Research Methods, Techniques, and Tools for Research on Teaching in the Digital Era

Following the description of the ten online and offline variables, Medley (1987) pointed out methodological issues to be considered in research on teaching. These methodological issues can refer to all stages of the research process in relation to the variables, their conceptualization, their instrumentation in empirical studies, the design of studies to investigate them, and the quality of the analysis of the data collected in studies. In relation to the conceptualization of the variables in research, Medley noted that the critical definition of effectiveness, that is, of "good teaching," varies intersubjectively, so all variables can potentially be affected by researcher bias. Challenges are also posed by the instrumentation of studies, that is, how the variables under study are operationalized in studies. Here, the evolution of research on teaching has led to increasingly better refinement of methods, which is taken up by all the authors in this volume. Medley further identified challenges of a more methodological nature in how studies examining the different variables must be specifically designed and what forms of data collection must take place. Finally, statistical data analyses and interpretation of results also pose challenges to researchers, but Medley recognized an ongoing elaboration of statistical analysis procedures. With increased sophistication of technological tools access to powerful statistical procedures has improved. Due to the fact that in the 1980s, the primary research methods accepted in the education community were first and foremost quantitative, Medley's work focused on quantitative methods of analysis. However, his concerns related to conceptualization, instrumentation, and design in research on teaching are still valid and relevant today, even with new technological and methodological developments and a wide range of modern qualitative and mixed methods used in mathematics education research.

Chandra Orrill, Zarina Gearty and Kun Wang in Chap. 12, provide information about methodological developments in mathematics education research and how it is positioned in the twenty-first century. They note that in addition to the quantitative research that Medley had in mind, qualitative research methods continued to be developed steadily in the 1980s and have led to profound insights in the research on teaching. Since overcoming of what has been characterized as trench warfare between quantitative and qualitative methodologies, a growing number of mixed-methods studies have also been observed with respect to the main units of analysis of research described by Medley. Looking specifically at quantitative research, Orrill et al. consider the item response theory (IRT) as an influential psychometric

model which has significantly contributed to the further development of method-ology in mathematics education research on teaching – especially, when it comes to the measurement of effectiveness. However, the researchers also present method-ological advancements related to study design. For example, they describe teaching experiments, design-based-research, and cultural historical activity theory as new developments of design frameworks that meet the specific demands and needs of mathematics education research. Orrill et al. also separately address technological developments in research (e.g., eye-tracking, DGS and 360° video capture), and how these have led to both new insights and further development of methods in research.

6   Conclusion

Through the process of writing this book, we updated the original framework consid-ering current research on teaching mathematics (Fig. 3). In addition to presenting new connections between main units of analyses of research, we acknowledge that each research variable must be considered within its cultural context and changes from one culture to another. The book focused on a western cultural perspective. Additionally, epistemological contexts are major factors in considering every unit of analysis of research on teaching mathematics. Depending on researchers' conceptual framework, the ideas surrounding Medley's "good teaching" change as the goals of teaching are directly tied to epistemological stances. Ultimately, new developments in technology change the way we can define (e.g., students' digital identities), eval-uate (e.g., new instruments/measures of teachers' knowledge), and connect (e.g., modern research tools, methods, and techniques) main units of analysis described in framework presented in Fig. 3.

Finally, in Medley's original work, he warned against using variables that were far removed from one another within one study. New research methods and techniques described in Chap. 12 show that there are ways to consider multiple units of analyses, as well as the ones that are not adjacent to each other within the framework (Fig. 3). However, even with new technologies and advances, we found through writing this book that units of analyses (Types A though E) further removed from each other have less predictive value in contrast to those variables within the framework that are closer to each other. Although researchers considered and studied mediating variables between those that they intended to measure and report, it became clear to us was that there is a lack of a systematic scientific overview of the complete chain between the units of analysis described in Medley's original framework. Our intention was to provide such an overview and to offer scholars potential directions for research related to each unit of analysis as presented in the chapters of this book. This was the inspiration for our project, and we hope the chapters broaden the readers' horizons just as our views were expanded through collaboration with this international team of scholars.

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Mathematics Teachers' Influence on Students' Learning: Online Variables

Pre-existing Mathematics Teacher Characteristics

Olive Chapman

1 Introduction

This chapter deals with one of the essential *online variables* of research on teaching (i.e., pre-existing teacher characteristics) promoted by Donald M. Medley in his seminal work on the evolution of research on teaching (Medley, 1987). Medley developed a framework of variables that research in teaching from a presage-process–product perspective must be concerned with to effectively contribute to the understanding and improvement of teaching. This framework provided the theoretical basis for framing this book on "evolution of research on teaching mathematics" (Manizade, Buchholtz, & Beswick, Chap. 1, this volume). As Manizade et al. explained,

> Medley's framework is still valuable as it gives an orientation to all possible variables that become apparent qua the chain of effects from teacher behavior to student achievements. Moreover, the abiding challenges associated with the conceptualization, instrumentation, operationalization, and research design that Medley described are still complex, despite recent advances in technology and research methodology in the digital era. (p. 5)

However, for this book, Manizade et al. updated the framework to take into consideration cultural and epistemological contexts and digital contexts and to situate it within research on teaching mathematics (see Fig. "Updated framework of research on teaching mathematics", Manizade et al., this volume).

Medley's (1987) framework includes six types of essential "online variables", that is, "ones which lie along a direct line of influence of the teacher on pupil learning" (p. 105). Medley labelled and sequenced these variables from Type F to Type A. This chapter deals with the Type F variable that is at the beginning of this direct

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line. Figures two and three in the introductory chapter illustrate these variables as presented by Medley for research on teaching and adapted by Manizade et al. for research on teaching mathematics (Manizade, Buchholtz, & Beswick, Chap. 1, this volume).

According to Medley (1987):

> Pre-existing teacher characteristics include abilities, knowledge, and attitudes that a candi-
> date for admission to a teacher preparation program possesses on entry; they make up a
> candidate's aptitude for teaching. Part of it consists of the characteristics a teacher needs
> in order to acquire those competencies that training and experience can provide; part of it
> consists of those competencies that a teacher must possess on entry. (p. 105)

In relating it to mathematics teaching, Manizade et al. (this volume, p. 6) defined the Type F variable as "a mathematics teacher's beliefs and aptitude for teaching, characteristics needed to acquire professional competencies during training." This definition was adapted in this chapter to explore research of pre-existing mathematics teacher characteristics [PMTC] that prospective teachers possess on entry into a teacher education program or mathematics teacher education [MTE] as a necessary stage in understanding the mathematics teacher and mathematics teaching.

In addition, Medley's four factors regarding methodological issues that research on teaching must deal with were adapted in this chapter to discuss the evolution of research on PMTC. These factors, discussed later, are conceptualization, instrumen-tation, design, and analysis. Medley explained that "evolution of research on teaching depends on advances made in how each has been dealt with" (p. 106).

In general, the chapter provides an overview of research that addressed PMTC of prospective teachers of mathematics [PTs] through a systematic review and synthesis of relevant published empirical studies for the period 2000 to 2020. It begins with an overview of the scope of the literature review, followed by an overview of the types and nature of PMTC covered in the studies reviewed, then a discussion of the evolution of the research on PMTC and suggestions regarding future evolution of research on PMTC.

2 Scope of Literature Review to Determine Studies of PMTC

Given the large body of literature on PTs, it was decided to focus only on high profile peer-reviewed international journals (Williams & Leatham, 2017) that likely included studies on PTs' PMTC. They included: Educational Studies in Mathe-matics (ESM), Journal for Research in Mathematics Education (JRME), Journal of Mathematics Teacher Education (JMTE), Journal of Mathematical Behavior (JMB), International Journal of Science and Mathematics Education (IJSME), Mathematical Thinking and Learning (MTL), and ZDM—Mathematics Education. The author and a trained research assistant conducted a search of these journals for the period 2000–2020 using various combinations of keywords that included: prospective teachers;

future teachers; teacher candidates; preservice teachers; student teachers; characteristics; competencies; abilities; knowledge; attitudes; beliefs; identity; and recruitment. Based on our review of titles and abstracts, we prepared a list of articles with potentially relevant studies. We examined these articles to determine whether participants were at the beginning of their MTE. This process produced very few studies. We then decided to focus on studies that addressed PTs' characteristics at the beginning of a course or prior to an intervention during a course or in situations that reflected the nature of their background knowledge or ability (e.g., interpreting students' work or evaluating tasks), which seemed to be a more promising approach to obtain studies of PMTC. The assumption was that these studies would suggest characteristics the PTs held prior to entering MTE if these characteristics were directly related to their school experiences with mathematics (e.g., mathematics curriculum content and pedagogy).

We obtained a large list of these studies by examining the methodology section of articles in our list. We then examined these studies to determine if the findings provided information that was clearly related to PMTC to identify those studies to exclude. Many intervention studies highlighted the changes resulting from the intervention but not the initial characteristics of the PTs and were removed from the list. Studies at the beginning of a course that investigated characteristics that were related to prior mathematics or mathematics education courses in the program were also removed from the list. In keeping with the theme of this book, all studies not situated in a Western context were also later removed. This process resulted in a list of 51 studies from the above-noted journals, to which were added a few studies from other journals based on citations of relevant studies in articles on this list. These studies were situated mainly in the USA, with some from different regions internationally. To highlight this, in reporting the studies, the countries are noted for those that were not situated in the USA.

For each article on the final list, the author and research assistant identified and recorded the PTs' characteristics explicitly investigated based on the aim of the study. On examining the characteristics, we determined that they generally involved PTs' mathematics knowledge, pedagogical knowledge, or beliefs, which became initial categories used to group the characteristics. The content of these categories consisted of, for each study, the type of characteristics, the aim of the study related to the characteristics and key findings regarding the nature of the characteristics. Further examination of the content of each of the three categories and cross-checking of findings for agreement between the researcher and research assistant, resulted in sub-groups of characteristics consisting of different types of mathematics concepts and skills, different types of pedagogical knowledge and ability, and different types of beliefs or conceptions, respectively. This process also validated that all the characteristics were appropriately accounted for and could be represented by three broad categories: *pre-existing mathematical knowledge and skills*, pre-existing *mathematics-related pedagogical knowledge and ability*, and *pre-existing mathematics-related beliefs*. These final categories, described in the next section, provided a landscape of PMTC related to the Type F variable that were researched in the period 2000–2020. We also documented examples of research tools, design and analysis that formed the basis of discussion of the evolution of research on the PMTC.

3   Landscape of PMTC Researched in 2000–2020

The studies reviewed provided a landscape of several PMTC researched in 2000–2020 regarding what PTs knew or were able to do on entering MTE. These PMTC, grouped in three categories, are presented in this section in terms of the aims of the studies, with examples of key findings of the studies related to the PTs' aptitude for teaching mathematics. The goal is to provide an overview of PMTC for the three categories: *pre-existing mathematical knowledge and skills, pre-existing mathematics-related pedagogical knowledge and ability,* and *pre-existing mathematics-related beliefs.*

3.1   Pre-existing Mathematical Knowledge and Skills

This category consists of studies that investigated PTs' mathematical knowledge and skills connected to school mathematics in the period 2000–2020. Collectively, these studies included primary, elementary, middle, and secondary school PTs and their knowledge of different content areas (i.e., fractions, whole number operations, geometry, algebra) and skills (i.e., problem posing). They addressed one category of PMTC that is central to teaching mathematics and important for PTs to have on entering teacher education. The following overview of these studies is organized by each content area and skill to highlight the extent to which they were addressed in terms of the aims of the studies and nature of the PTs' PMTC and in reversed chronological order to indicate distribution in the period beginning with most recent studies.

3.1.1   Fractions

These studies on fractions focused mostly on elementary school PTs and addressed their knowledge of fractions in a variety of ways. During the second 10 years of the period: Lee and Lee (2020) investigated elementary school PTs' exploration of model breaking points in fractions that included the area model of fraction addition. Most of the PTs represented fraction addition well with simple fractions but had difficulty representing fraction addition with improper fractions or fractions with unlike and relatively large denominators and tended to use algorithm-based thinking. The area models drawn by several of the PTs revealed various misconceptions. Lovin et al. (2018) investigated elementary and middle school PTs' understanding of fractions as they were starting their first required mathematics course and found that they relied on procedural knowledge. Most of them had constructed the lower-level fraction schemes and operations but less than half had constructed the more sophisticated ones. Baeka et al. (2017) investigated elementary and middle school PTs' pictorial strategies for a multistep fraction task in a multiplicative context. They found that

many of the PTs were able to construct valid pictorial strategies that were widely diverse regarding how they made sense of an unknown referent whole of a fraction in multiple steps, how they represented the wholes in their drawings, in which order they did multiple steps, and the type of model they used (area or set). Whitacre and Nickerson (2016) investigated elementary school PTs' fraction knowledge at the beginning and end of their first mathematics content course. In the beginning, the PTs used predominantly standard strategies with weak performance and flexibility in comparing fractions. Lin et al. (2013) explored an intervention for enhancing elementary school PTs' fraction knowledge and found that, prior to the intervention, the PTs held procedural understanding of basic fractional ideas and basic fractional operations, including equivalent fractions and addition, subtraction, multiplication, and division of fractions. Finally, Osana and Royea (2011) explored an intervention centered on problem solving to support Canadian elementary school PTs' learning of fractions. The PTs were initially challenged to generate word problems for number sentences involving fractions, construct meaningful solutions to fraction problems, and represent those solutions symbolically.

Regarding the first 10 years of the period: Newton (2008) studied elementary PTs enrolled in a course on elementary school mathematics to obtain a comprehensive understanding of their fraction knowledge. Findings at the beginning of the course indicated that they had limited and fragmented knowledge of fractions. For example, they misapplied fraction algorithms, attended to superficial conditions when choosing a solution method, and demonstrated little flexibility in solving problems. Although they remembered many procedures, such as cross-multiplying and finding a common denominator, they were using them in inappropriate ways. Their most common error was to keep the denominator the same when it was not appropriate to do so. Tirosh (2000) investigated fraction division and found that in a class of Israeli elementary PTs, most of them knew how to divide fractions but could not explain why the procedure worked.

3.1.2 Whole Number Operations

This group of studies addressed elementary school PTs' knowledge of addition, subtraction, multiplication, and division of whole numbers. Norton (2019) examined Australian primary school PTs' mathematics knowledge at the beginning and end of their education course. Findings indicated that the PTs had low levels of knowledge of whole numbers at the beginning of the course. The most challenging whole-number computation for them was division by a double-digit divisor. Kaasila et al. (2010) investigated Finnish elementary PTs' conceptual understanding, adaptive reasoning, and procedural fluency based on a non-standard division problem and concluded that division seemed not to be fully understood. Less than half of the PTs were able to produce complete or mainly correct solutions. The main reasons for their issues in understanding the task consisted of staying on the integer level, inability to handle the remainder, difficulties in understanding the relationships between different operations, and insufficient reasoning strategies. Thanheiser (2010) examined PTs'

responses to standard addition and subtraction place-value tasks and found that, at the beginning of their MTE, the PTs were often able to perform but not explain algorithms. For example, they had incorrect views of regrouped digits that included: interpreting all regrouped digits consistently as having the same value (all as 1 or all as 10); treating the value of the digits as dependent on the context (addition or subtraction); interpreting the digits consistently within but not across contexts (i.e. all as 10 in addition but all as 1 in subtraction); and interpreting the digits inconsistently depending on the task (i.e. the same digit was interpreted in multiple ways).

Thanheiser (2009) also reported on the PTs' knowledge of multidigit whole numbers in the context of standard algorithms for addition and subtraction prior to their first mathematics course in their MTE. Most of the PTs did not have a deep understanding of numbers and struggled relating the values of the digits in a number to one another. They did not provide mathematical explanations of the algorithms. They referred to the digit in the tens place as ones rather than in terms of the reference unit tens or the appropriate groups of ones. While some drew on a conception that enabled them to explain the algorithm in at least one way, few exhibited an understanding of numbers that enabled them to explain the algorithm flexibly, including why the digits in any column can be treated as ones and why we can treat any pair of adjacent digits as if they were ones and tens.

3.1.3   Geometry

These two studies addressed different aspects of elementary and middle school PTs' knowledge of geometry concepts. Miller (2018) analyzed PTs' definitions of types of quadrilateral based on a survey of elementary school PTs who, since high school, had not yet studied geometry in their MTE. Findings included that the majority of the PTs' definitions contained necessary attributes, but not sufficient or minimal attributes. The PTs were most comfortable with squares, followed by parallelograms, then rectangles, trapezoids, rhombi, and finally kites. They did not include hierarchical relationships as a means of defining one shape in terms of another and often created definitions that were aligned with emergent concept images of the shape types with only typical examples. Yanik (2011) investigated middle school PTs' knowledge of rigid geometric translations and found that the PTs had difficulties recognizing, describing, executing, and representing geometric translations. They viewed geometric translations mainly as physical motions based on their previous experiences, that is, as rotational motion, translational motion, and mapping. They interpreted the vector that defines translations as a force, a line of symmetry, a direction indicator, and a displacement. Many of them knew that a vector has a magnitude and a direction but did not conclude that vectors define translations.

3.1.4 Algebraic Concepts and Thinking

This group of studies addressed elementary and middle school PTs' knowledge of algebraic concepts and their ability to think algebraically. Hohensee (2017) examined the insights and challenges elementary school PTs experienced when exploring early algebraic reasoning. Findings indicated that they were challenged conceptually to identify the relationships contained in algebraic expressions, to distinguish between unknowns and variables, to bracket their knowledge of formal algebra, and to represent subtraction from unknowns or variables. You and Quinn (2010) investigated elementary and middle school PTs' knowledge of linear functions and found that they were stronger on procedural than conceptual knowledge of linear functions. They were weak in representation flexibility, for example, ability to transfer flexibly: (i) between visual and algebraic representations to recognize relevant properties of algebraic and visual representations and to make connections among them when treating functions as an entity; (ii) from functions to a word problem situation; and (iii) from word problem situations to various forms of functions. Richardson et al. (2009) studied how pattern-finding tasks promoted elementary school PTs' learning of how to generalize and justify algebraic rules from an emergent perspective to support their teaching of early algebra concepts. They found that most of the PTs, in their only mathematics methods course, initially focused on numerical data in tables and had difficulty providing a valid justification for their generalizations. Nearly all of the PTs generalized explicit rules using symbolic notation but had trouble with justifications early in the experiment. Pomerantsev and Korosteleva (2003) investigated the typical mistakes elementary and middle school PTs made as they progressed through their courses. They found that the PTs had difficulties recognizing structures of algebraic expressions at the introductory level of the courses.

3.1.5 Problem Posing

This group of studies addressed elementary and lower secondary school PTs' problem posing knowledge or ability. Crespo and Sinclair (2008) investigated elementary school PTs' problem-posing practices prior to planned interventions. They found that a majority of the problems the PTs posed consisted of *assignment* problems as opposed to the more complex *relational* or *conditional* problems for one task and *factual* problems (involving the recall of names and properties, the identification of properties, the application of measurement formulae, or the counting of shapes) for another task. The purpose was mainly to *elicit information.* Problem structure included clarity (problems not confusing, misleading, or under- and over-stated) and simplicity (numbers or shapes common and uncomplicated and right answers). Rizvi (2004) investigated Australian lower secondary school PTs' ability to pose word problems for mathematical expressions involving division before an instructional intervention. She found that none of the PTs was able to pose word problems for the expressions where the divisors were fractions. They posed only sharing type word problems for the expressions where the divisor was a whole number. While many

were aware of the repeated subtraction, no participants posed any word problem based on the repeated subtraction model for any division expression. Crespo (2003) investigated elementary school PTs' beginning approaches to posing problems and found that they consisted of: making problems easy to solve (e.g., the narrow mathematical scope of the original version of the problem and the work of students); posing familiar problems (e.g., quick-translation story problems or computational exercises); and posing problems blindly (i.e., unawareness of the mathematical potential and scope of problem).

3.1.6   Summary

The overview of studies in this section on *pre-existing mathematical knowledge and skills* offers insights of the nature of the PTs' content knowledge at the point of entry into a teacher education program. The studies investigated the PTs' knowledge of different content areas (i.e., fractions, whole number operations, geometry, algebra) and their problem-posing skills. There was more attention on elementary than secondary PTs and on fractions than the other areas. Those studies dealing with fractions focused on meaning of fractions, arithmetic operations with fractions, strategies for solving fraction tasks and models of representing fractions. They indicated that the PTs' fraction knowledge contained many misconceptions and was generally limited, fragmented, weak, low level, and procedural. Studies dealing with whole numbers focused on the arithmetic operations (addition, subtraction, multiplication, division). They indicated that the PTs did not have deep understanding of these procedures. Studies dealing with geometry focused on two-dimensional shapes and rigid motions. They indicated that the PTs' had superficial knowledge or difficulties in dealing with these concepts. Studies dealing with algebraic concepts addressed algebraic expressions, linear functions, and algebraic rules. They indicated that the PTs had weak knowledge of the concepts, were challenged conceptually, and had difficulties with the concepts. Problem posing received the least attention with these studies focusing on posing word problems. The studies indicated that the PTs' problem-posing ability was limited to posing problems of low level of cognitive demand. Overall, the studies highlighted that the PTs' pre-existing knowledge of mathematical content was plagued with difficulties and low conceptual understanding of specific mathematics concepts that are central to school mathematics curricula and their future teaching.

3.2   Pre-existing Mathematics-Related Pedagogical Knowledge and Ability

This second category consists of studies that investigated PTs' mathematics-related pedagogical knowledge and ability in the period 2000–2020. These studies collectively included early childhood and primary, elementary, middle, and secondary school PTs and their pedagogical ability (e.g., to notice, observe, analyze, and/or interpret teaching situations). They addressed another category of PMTC that is important for a teacher to function effectively in mathematics teaching situations and that PTs should have on entering MTE. The following overview of these studies is organized based on their foci on the PTs' knowledge or ability involving (i) observing and analyzing teaching, (ii) noticing and interpreting students' work or thinking, and (iii) evaluating tasks, to highlight the extent to which each was addressed in terms of the aims of the studies and the nature of the PTs' PMTC. The studies are presented in reversed chronological order to indicate distribution in the period beginning with the most recent studies.

3.2.1   Observing and Analyzing Teaching

This group of studies addressed elementary, middle, and secondary school PTs' ability to observe and/or analyze teaching by engaging the PTs in exploring videos of mathematics lessons. Star and Strickland (2008) investigated the impact of video viewing as a means to improve secondary school PTs' ability to be observers of classroom practice. Their findings of the pre-assessment indicated that the PTs generally did not enter teaching methods courses with well-developed observation skills. They were astute observers of classroom management regarding what the teacher did to maintain control in the classroom and what students did that might influence the teacher's ability to maintain control. They were also reasonably attentive to the actions of the teacher to support the lesson objectives, such as her use of notes, her presentation of the material, how she structured the group work, and her assignment of homework. However, their ability to notice other aspects of the classroom was not as strong. They did not attend to features of the classroom environment and/or did not feel that such features needed their attention. They were weak in observing the mathematical content, for example, questions about the representation of the mathematics, the examples used, and the problems posed. They did not notice subtleties in the ways that the teacher helped students think about content. In general, the PTs were very attentive to issues of classroom management but mostly unaware of static features of the classroom environment and the subtleties of classroom communication and mathematical content. Stockero (2008) investigated the use of a video-case curriculum in a middle school mathematics methods course for PTs to develop a reflective stance to enable them to analyze classroom interactions. Findings early in the course indicated that the PTs' level of reflection or observation was at the two lowest levels of reflection; that is, describing and explaining levels. Their reflection focused on

describing and explaining what they observed in the videos and did not demonstrate the higher levels of reflection consisting of theorizing, confronting, and restructuring. They also tended to analyze classroom events based on affective measures instead of pedagogical and mathematical reasons for instructional decisions.

Morris (2006) provided the only study that focused on this observing/analyzing PTs' ability when the PTs entered their MTE compared to others that considered it prior to an intervention in a course later in their MTE. She investigated the "learning-from-practice skills" that elementary and middle school PTs possessed by requiring them to analyze a videorecorded mathematics lesson regarding the effects on student learning, to support their analysis with evidence, and to use their analysis to revise the lesson. She found that many of the PTs could carry out a cause-effect type of analysis of the relationships between specific instructional strategies and student learning and could use this analysis to make productive revisions to the instruction. But their ability to collect evidence that supported their analysis was less developed. Their analysis of the effects of instruction on the students' learning was dependent on the video-task conditions. For example, when the task instructions indicated that the lesson was not successful, the PTs attended to both teacher and students and could make some elementary claims about how teaching and learning might be connected, but specific types of deficiencies in their evidence-gathering were apparent including the ability to collect evidence that supported conjectures about the effects of instruction. When the condition allowed the PTs to decide whether the lesson was successful and which instructional activities worked well or not, most of them focused primarily on the teacher, implying that students learn what the teacher explains. For example, they saw a teacher giving explanations and children giving correct responses, concluded that the children understood the teacher's explanations, and made minimal revisions to the lesson. In general, the PTs' support of hypotheses about student learning involved: no references to students' responses, referring to students' responses that were marginally related to the claims, attributing a wide range of understandings to students based on little or no objective evidence, and failing to refer to students' responses that provided the most access to students' thinking.

3.2.2 Noticing and Interpreting Students' Work and Thinking

This group of studies collectively addressed early childhood and primary, elementary, and secondary school PTs' knowledge of, and ability to notice and interpret, students' mathematical work and thinking. Regarding the second 10 years of the period: Shin (2020) examined secondary school PTs' noticing of students' reasoning about mean and variability. Findings indicated that the PTs had difficulties noticing students' reasoning about variability. None of the PTs explicitly interpreted the students' limited understanding of variability when comparing data sets with unequal sample sizes. Some showed no evidence of differentiating between students' different levels of reasoning. Superfine et al. (2019) investigated different facilitation moves to support the elementary school PTs in noticing children's mathematical thinking and found that they generally did not discuss their noticing at a high-level and there were

few instances where they provided evidence for their noticing. Sánchez-Matamoros et al. (2019) examined the relationships between how secondary school PTs in Spain attended to the mathematical elements in students' solutions and interpreted students' understandings for the derivative of a function at a given point. Their findings indicated that the PTs had different levels of pre-existing ability consisting of those who provided general comments about students' learning, who found it difficult to recognize characteristics of the students' understanding, and who had difficulties in using mathematical elements in students' solutions to recognize differences among students' understanding. Callejo and Zapatera (2017) investigated Spanish primary school PTs' noticing, describing, and interpreting of students' mathematical thinking in their solution to a pattern generalization task. They found that the PTs were able to name various mathematical elements to describe the students' answers but did not always use them to interpret the understanding of pattern generalization of each student. Some PTs could not recognize the understanding of the students.

In addition, Simpson and Haltiwanger (2017) investigated how secondary school PTs made sense of students' mathematical thinking of an algebra and function mathematics problem, when professional noticing was not a formal part of their MTE. They found that the PTs exhibited a lack of rigorous evidence when interpreting what the students may or may not have understood. The PTs discussed only what the students understood in terms of the written work. They did not consider misconceptions or errors in the students' mathematical thinking. Sánchez-Matamoros et al. (2015) examined the ability of secondary school PTs in Spain to notice students' understanding of the derivative concept in the beginning and end of a "training module". At the beginning, the PTs' noticing was limited to describing students' answers in the graphical and analytical modes of representation but without identifying the relevant mathematical elements and interpreting the students' understanding by making general comments related to "the good or bad understanding of the student." Lastly, Son (2013) examined the secondary and elementary school PTs' interpretations of and responses to a student's error(s) involving finding a missing length in similar rectangles through a teaching scenario task. Findings indicated that although the student's errors came from conceptual aspects of similarity, a majority of the PTs identified the errors as stemming from procedural aspects of similarity and consequently drew on procedural knowledge as a way to guide the students.

Regarding the first 10 years of the period: Harkness and Thomas (2008) investigated early childhood PTs' mathematical understanding of a student's invented multiplication algorithm and found that a majority of the PTs relied on procedural and memorized explanations rather than using mathematical properties to describe the validity of the algorithm. Generally, their responses demonstrated a procedural or memorized understanding of the invented algorithm. Crespo's (2000) study on how elementary school PTs in Canada interpreted their students' work indicated that their interpretations were initially from a limited focus on the correctness of the students' solutions and not meaning.

3.2.3   Evaluating Tasks

This group of studies addressed elementary and middle school PTs' knowledge of, and ability to evaluate, features of mathematical tasks to support students' learning. Magiera et al. (2013) explored middle school PTs' ability to recognize opportunities to engage students in algebraic thinking. They found that the PTs demonstrated limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students, recognizing only some features in the analyzed tasks. Stephens (2006) examined elementary school PTs' awareness of equivalence and relational thinking to assess their initial preparedness to engage students in these aspects of early algebraic reasoning. She found that the PTs collectively demonstrated an awareness of relational thinking in identifying opportunities offered by the tasks to engage students in this thinking. But in proposing difficulties students might have with selected tasks, few of them demonstrated an understanding that many students have misconceptions about the meaning of the equal sign. Osana et al. (2006) examined the nature of elementary school PTs' evaluations of elementary mathematics problems using a model designed to discriminate among tasks according to their cognitive complexity. Results demonstrated that, overall, the PTs had more difficulty accurately classifying problems considered to represent high levels of cognitive complexity compared to less complex problems. They were influenced by the surface characteristics of task length and tended to label short problems as less cognitively demanding and long problems as more so.

3.2.4   Summary

The overview of studies in this section on *pre-existing mathematics-related pedagogical knowledge and ability* offers insights of the nature of the PTs' knowledge and ability related to teaching and learning mathematics at the point of entering a teacher education program. The studies addressed the PTs' knowledge of how to observe and analyze teaching, notice and interpret students' work or thinking, and evaluate tasks. Those dealing with observing and analyzing classroom behaviours of teachers and students indicated mostly weaknesses in the PTs' ability to observe and make appropriate conclusions or suggestions regarding instruction. For example, they were strong in observing classroom management but weak in noticing other aspects of the classroom and demonstrated lowest levels of analysis of classroom interactions. Those studies dealing with noticing and interpreting students' mathematical thinking and work indicated that the PTs demonstrated several areas of difficulties in noticing or recognizing students' reasoning, providing rigorous evidence to support their noticing, and discussing noticing at a high level. Studies dealing with evaluating or interpreting features of tasks to support students' learning indicated that the PTs had limited ability to recognize potential of tasks or difficulties students could experience with a task or classifying a problem of high level of cognitive complexity. Overall, the studies highlighted that there were much more weaknesses

than strengths in the PTs' pre-existing ability to notice, analyze, and interpret mathematics classroom behaviors of teachers and students and students' mathematical work.

3.3 Pre-existing Mathematics-Related Beliefs

This third and final category consists of studies that investigated PTs' mathematics-related beliefs in the period 2000–2020. The studies collectively included primary, elementary, middle, and secondary school PTs and their content and pedagogical beliefs (conceptions and perceptions). They addressed another category of PMTC that are important to how teachers conceptualize and enact their teaching of mathematics and PTs would have on entering MTE. The following overview of these studies is organized based on their foci on the PTs' beliefs about: (i) nature of mathematics, (ii) teaching and learning mathematics, (iii) use of technology, (iv) mathematical processes, and (v) mathematics concept, to highlight the extent to which they were addressed in terms of the aims of the studies and the nature of the PTs' PMTC. The studies are presented in reversed chronological order to indicate distribution in the period beginning with the most recent studies.

3.3.1 Nature of Mathematics

This group of studies addressed primary, elementary, middle, and secondary school PTs' beliefs of the nature of mathematics. Weldeana and Abraham (2014) investigated an intervention to change beliefs of middle school PTs. They found that before the intervention a majority of the PTs did not hold progressive beliefs related to the nature of mathematics. For example, they believed that for every problem of mathematics, there is one unique approach leading to its solution. Shilling-Traina and Stylianides (2013) investigated changes in the beliefs about mathematics held by elementary school PTs in a mathematics course and found that their initial beliefs largely reflected instrumentalist and Platonist views. Conner et al. (2011) investigated secondary school mathematics PTs' beliefs about mathematics. Findings indicated that their initial views of mathematics were primarily Platonist and instrumentalist. Some of the most prevalent descriptors of mathematics across participants were mathematics is logical and less subjective than other disciplines, and mathematics is unambiguous in the sense that, while multiple solution paths are possible, each problem has a single, correct answer. Finally, Bolden et al. (2010) investigated primary school PTs in the UK early in their education course at the beginning of the program and found that the PTs held narrow, absolutist views of mathematics as a subject. Most conceived mathematics as a subject of a set body of knowledge that offered little or no room for freedom of expression, imagination, and independence. Most also believed that mathematics was not a creative subject, and it was difficult to encourage creativity in mathematics.

3.3.2 Teaching and Learning Mathematics

This group of studies addressed primary, elementary, middle, and secondary school PTs' beliefs or conceptions about teaching and learning mathematics, including qualities of teachers and learners, mathematical behaviour and creativity, and doing and understanding mathematics. Regarding the second 10 years of the period: Stohlmann et al., (2014/2015) investigated changing elementary school PTs' beliefs about mathematical knowledge. At the beginning of the course, the majority of the PTs showed little or no evidence of the belief that conceptual understanding of mathematics is more powerful or generative than remembering mathematical procedures and they appeared to be focused on understanding mathematics in terms of procedural fluency. The majority of them also showed weak or no evidence of beliefs that: (i) One's knowledge of how to apply mathematical procedures does not necessarily go with the understanding of the underlying concepts. (ii) Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures. (iii) If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn procedures first, they are less likely ever to learn the concepts. For (i), very few PTs showed evidence or strong evidence of the belief that if a child knows procedures, they may not understand the underlying concepts. Weldeana and Abraham (2014) investigated an intervention to change beliefs of middle school PTs. They found that before the intervention a majority of the PTs did not hold progressive beliefs related to the way mathematics is learned, taught, and practiced. The PTs began with a strong belief that mathematics can be learned and understood through memorization of facts and formulae. They held many traditional beliefs related to knowing in mathematics (e.g., step-by-step procedure; getting the right answers quickly; retrieving information quickly; and figuring out formulae and equations to solve problems immediately).

In addition, Conner et al. (2011) investigated secondary school PTs' beliefs about mathematics teaching. They found that the PTs' initial beliefs of characteristics of effective mathematics teachers included: having positive affective characteristics (a good teacher is nice, patient, friendly) and mathematical knowledge, attending to student needs, and facilitating student participation. They also initially held the belief that students should participate in class, asking questions and working together, but sometimes they described a teacher centered view of student participation that included direct instruction as the primary method for teaching new content. Bolden et al. (2010) investigated conceptions of creativity of primary school PTs in the UK early in their education course at the beginning of their MTE. Findings indicated that the meaning of creativity in primary school mathematics was not well understood by the PTs based on their conception of it. Their conceptions were narrow, predominantly associated with the use of resources and technology, and tied to the idea of "teaching creatively" rather than "teaching for creativity". They viewed creativity in terms of the types of resources used and the way in which they were used to teach mathematical topics and the way in which real-life examples were used to explore mathematical concepts.

For the first 10 years of the period: Ambrose (2004) investigated an intervention to build elementary school PTs' beliefs. Findings indicated that initially the PTs held beliefs that teaching involves explaining things to children, which most of them continued to hold following the intervention. They initially equated doing mathematics with using memorized procedures. Their actions indicated a teaching-as-telling belief along with a belief about mathematics learning as the acquisition of standard symbolic procedures. Szydlik et al. (2003) explored elementary school PTs' beliefs about the nature of mathematical behavior both at the beginning and end of the education course. At the beginning, the majority of the PTs believed that, as learners, they could not "figure out" mathematics for themselves. They could not imagine being asked to do a problem significantly different from those in the text-book or having a teacher who did not first show them how to do similar problems. They believed that they must memorize formulas, procedures, or template problems in order to work on new problems.

3.3.3 Use of Technology

These two studies addressed middle and secondary school PTs' beliefs about the use of technology. Wachira et al. (2008) assessed middle school PTs' beliefs about the appropriate use of technology in mathematics teaching and learning prior to taking the methods course. They found the PTs' beliefs to be limited to the use of technology as computational tools and for checking the accuracy of these computations. The PTs' conceptions indicated a lack of understanding of technology as powerful tools to help students gain knowledge, skills, deeper understanding and appreciation of mathematics. They did not provide specific ways on how technology could be used to promote learning. None indicated that technology could be used to explore patterns, discover more about mathematics concepts or investigate mathematical relationships, which suggested that they lacked understanding of how technology could be used appropriately to develop concepts. Leatham (2007) investigated secondary school PTs' beliefs about teaching mathematics with technology later in their program after taking an education course on technology and found that their beliefs about the nature of technology in the classroom were about the availability of technology, the purposeful use of technology, and the importance of teacher knowledge of technology.

3.3.4 Mathematical Processes

These two studies addressed elementary school PTs' beliefs (conceptions, views) of two mathematical processes: problem solving and representing mathematical concepts or situations. Son and Lee (2020) examined elementary PTs' problem-solving conceptions and performances. They found that a majority of the PTs held conceptions of problem solving as a means to a solution, that is, they expressed a skill-based or means-to-an-end view by focusing on solutions or procedural steps.

Their conceptions of problem solving were related to their performances. Dreher et al. (2016) investigated views about using multiple representations held by British and German elementary school PTs at the beginning of their first year of their MTE. They found that the PTs showed little awareness for the role of representations for mathematical understanding. They viewed the role of multiple representations for understanding mathematics as less important than other non-discipline reasons for using multiple representations. They mostly were not able to recognize the learning potential of tasks focusing on conversions of representations, in comparison with tasks including rather unhelpful pictorial representations, to which they tended to assign a higher learning potential.

3.3.5 Algebra

One study addressed PTs' conceptions of algebra. Stephens (2008) examined conceptions of algebra held by elementary school PTs enrolled in their only course addressing the teaching of mathematics. Findings suggested that their conceptions of algebra as subject matter were narrow. Most of them equated algebra with the manipulation of symbols. Very few identified other forms of reasoning, in particular, relational thinking, with algebra. Several made comments implying that student strategies that demonstrated traditional symbol manipulation might be valued more than those that demonstrated relational thinking, suggesting that what was viewed as algebra is what will be valued in the classroom. Tasks were often judged to be algebra or non-algebra problems by the presence or absence of a variable or letter, and students were often judged to have used or not used algebra based on how closely their work matched the symbol-manipulation model.

3.3.6 Summary

The overview of studies in this section on *pre-existing mathematics-related beliefs* offers insights of the nature of the PTs' pre-existing beliefs or conceptions related to mathematics and mathematics pedagogy at the point of entering a teacher education program. The studies addressed the PTs' beliefs about the nature of mathematics, teaching and learning mathematics, use of technology, mathematical processes, and mathematics concepts. Those dealing with beliefs about the nature of mathematics indicated that the PTs held beliefs of a Platonist or absolutist perspective of mathematics. Those dealing with beliefs about teaching and learning mathematics indicated that the PTs' beliefs mostly reflected a traditional or 'teacher-centered' perspective of teaching and learning mathematics. Those dealing with beliefs about mathematics concepts and processes indicated that the PTs held narrow conceptions of algebra, problem solving, and multiple representations. Those dealing with technology indicated that the PTs lacked understanding of use of technology to support students' learning and to develop concepts and held beliefs that could significantly limit their use of technology in teaching mathematics. Overall, the studies highlighted that

the PTs' pre-existing beliefs were mostly inappropriate to support contemporary perspectives of reform-based mathematics education.

3.4 Summary of Landscape of PMTC

Table 1 provides a summary of the PMTC that were addressed by studies investigating PTs' characteristics in the period 2000–2020. The table includes the three categories of PMTC researched and the types of PMTC researched for each category. These PMTC could directly impact PTs' learning during initial teacher education and their teaching as future teachers. They are further discussed in the sections that follow concerning how research has evolved in PMTC.

4 Evolution of Research on PMTC

The preceding section outlined studies relevant to Medley's (1987) Type F variable that provided a landscape of the types of PMTC they addressed. These studies formed the basis to consider the evolution of research on PMTC in the period 2000–2020, based on what was done (i.e., the scope of research) and how it was done (i.e., methodological factors) in establishing and advancing research in this area of mathematics education. The scope of research involves the extent to which PMTC were studied. The methodological factors involve those that Medley proposed are necessary to consider the evolution of research on teaching, adapted to address research on PTs' PMTC. These factors are conceptualization (e.g., of good teaching), instrumentation

Table 1 Landscape of PMTC of PTs Researcher in 2000–2020

Categories of PMTC researched	Types of PMTC researched
Pre-existing mathematical knowledge and skills	Fractions Whole number operations Geometry Algebra Problem posing
Pre-existing mathematics-related pedagogical knowledge and ability	Observing and analyzing teaching Noticing and interpreting students' work or thinking Evaluating tasks
Pre-existing mathematics-related beliefs	Nature of mathematics Teaching and learning mathematics Use of technology Mathematical processes Algebra

(e.g., valid tools), design, and analysis. This section is organized in terms of the *scope of research on PMTC* and *methodological factors*.

4.1 Scope of Research on PMTC

The scope of research suggests an evolution of research on PMTC in terms of the extent of the types of PMTC researched, the extent to which PMTC were addressed by the studies, and the extent to which the studies were framed at the point of entry into MTE.

4.1.1 Types of PMTC Researched

The studies suggested a shift from researching teacher candidates' characteristics, such as level of school mathematics they completed, mathematics courses they completed, their overall grade point average (GPA), and their mathematics GPA, which were not considered in any of the studies for 2000–2020. They also suggested growth in research on PMTC in terms of different types of PTs' characteristics that were investigated. They addressed specific aspects of nine types of PMTC associated with three categories of characteristics (Table 1) the PTs held on entering MTE. These characteristics included PTs' knowledge, skills, and beliefs that were connected to what they would have learned, directly or indirectly, as students in school mathematics classrooms. For example, prior to entering teacher education, PTs would have developed knowledge of mathematical:

- Content—directly, based on what was taught.
- Processes—directly, based on what they engaged in.
- Learning—indirectly, based on how they were engaged and their personal orientation.
- Teaching—indirectly, based on how they were taught and assessed.
- Technological tools—directly or indirectly, based on how they were used in their learning.
- Beliefs—indirectly, based on what was taught and how it was taught.

While the studies touched on all of these areas of PTs' learning, aspects of them were not explored enough or at all to add depth to the body of research on PMTC. For example, little or no consideration was given to secondary school mathematics concepts, mathematical problem solving and thinking skills, technological tools, and assessment of learning. The studies also did not address contextual variables that impacted the characteristics, in particular, cultural context and technological context (discussed later). Thus, while the studies offered insights of some important PMTC held by PTs on entering MTE, in advancing the field of research on the mathematics teacher and teaching, they were limited in types and number of PMTC covered in a recent 20-year period.

4.1.2 Attention to PMTC by Studies

The extent to which the studies attended to the PMTC researched, based on number of studies, suggested relative levels of evolution of each of the three categories and types of PMTC (Table 1) for the period 2000–2020. For example, the *pre-existing mathematical content and skills* category received the most attention, suggesting higher interest in content-related characteristics, in particular, knowledge of fractions that represented almost half of the studies in the category, with most of them occurring in the second half of the period. The *pre-existing mathematics-related pedagogical knowledge and ability* and *the pre-existing mathematics-related beliefs* categories received the same level of attention, but when combined were higher by about 20% more studies than the mathematical content and skills category. This suggested that overall, research focused on PMTC regarding pedagogical ability and beliefs had grown more than for content knowledge.

In particular, in the pedagogical ability category, there was significant attention in the studies on noticing and interpreting students' work and thinking, which formed about two-thirds of the studies in this category, were the largest group of studies for the three categories, and were mostly occurring in the second half of the period. For the beliefs category, beliefs about teaching and learning received the most attention, but was third behind the interpreting students' work and the fractions PMTC. Most of the studies for these three types of PMTC were also in the second half of the period, compared to the other types of PMTC that collectively had more studies in the first half of the period. All of the studies on problem posing, use of technology, and the ability to observe and analyze teaching, and most of those on ability to evaluate tasks and knowledge of whole number operations were in the first half of the period. Hence, there was a shift in focus from the first to the second half of the period that suggested a shift in research on PMTC that may be considered a partial growth regarding some PMTC researched and a limitation regarding lack of continuation of attention to those that are of ongoing importance to support effective teaching of mathematics.

4.1.3 Studies at Point of Entry

The extent to which the studies were framed at the PTs' point of entry into MTE provided another perspective of the level of growth on research on PMTC for the period 2000–2020. While there is a large body of mathematics education research on PTs in this period, it is lacking in addressing PMTC of teacher candidates at the point of entering MTE. Only about 8% of the studies addressing PMTC in 2000–2020 focused on PTs at the beginning of their programs, which was not defined. Hence, as previously discussed, the majority of the studies were framed at the beginning of mathematics education courses, or prior to research-based instructional interventions in a mathematics education course or based on activities in mathematics for teachers or mathematics education courses that depended solely on the use of prior knowledge related to school experience. This framing suggested that, in 2000 -2020, there was

no growth of research of PTs' PMTC at the point of entry into MTE or there was an evolution in terms of a shift to, or consideration of, more practical approaches of obtaining access to PTs and their PMTC (e.g., participants in mathematics education courses at different points in a program).

Given the importance of understanding teacher candidates' PMTC, the little attention of research on them could partly be related to challenges associated with the *point of entry*, which could be messy regarding accessing information on teacher candidates for one discipline and dealing with complexities associated with different admission requirements, different academic backgrounds of candidates, and different programs. For example, in a Western cultural context, mathematics teacher candidates could enter a teacher education program directly from high school with or without a college entrance test/exam, or after receiving an undergraduate or graduate degree in mathematics or some other related degree, or while jointly working on a mathematics education degree and another related degree. They could have completed only middle or high school mathematics, or a mathematics degree, or a mathematics-related degree (e.g., physics, engineering), or some mathematics or mathematics-related courses prior to beginning a teacher education or mathematics education program. These various groups of PTs would need to be considered separately, in addition to the various groups according to different school levels for which PTs are preparing (e.g., elementary, middle, or high school) to obtain a reliable and meaningful picture of PMTC.

A related issue is what is the point of entry for candidates in mathematics education–the beginning of a general teacher education program that includes mathematics education, the beginning of a specialized mathematics education program, the beginning of mathematics education courses, and mathematics courses for teachers? All of these possibilities could lead to different versions of the nature of PMTC held by PTs. Although the studies on PMTC in 2000–2020 addressed school levels, there was an underlying assumption that the PTs in a course formed a homogeneous group in terms of their academic backgrounds, which might or might not have been the case. However, the intent of most of these studies was not explicitly to address PMTC at the point of entry into an MTE, but to obtain baseline information to evaluate their instructional approaches, which could be another way of considering how research on PMTC has evolved.

Another possible challenge for researching teacher candidates' PMTC involves the use of institutions' recruitment criteria or admission requirements as a basis of their point-of-entry PMTC, since what institutions want and what they get may not fully align. For example, as Artzt and Curcio (2008) explained, regarding recruiting high school students for secondary school mathematics education programs in the USA:

> There were several obstacles we faced in recruiting talented mathematics secondary students; it requires finding students from the intersection of three sets: those who love mathematics, those who want to become teachers, and those who are interested in attending Queens College …. Overcoming the barriers requires a multitude of recruitment strategies. (p. 249)

Schmidt et al. (2012) also raised concerns about selectivity for institutions in the USA. They noted:

> There is great variation in what secondary mathematics individuals have had before entering teacher preparation. (p. 265).
>
> Variation was especially large for the college entrance mathematics score, … revealing a very large selectivity factor across institutions. (p. 270)
>
> From a policy perspective, selectivity relates to differences in mathematics knowledge among future teachers before they began their teacher preparation—the issue of who enters teaching. This is manifested by large differences among institutions. The policy issue related to selectivity includes recruiting more mathematically able students into primary teacher preparation no matter which institution they might attend. (p. 275)

Selectivity could also be an issue within an institution where admission could be based on a combination of grades, interviews, portfolio, etc. which adds another layer of complexity in using admission requirements to determine PMTC. Research on PMTC in 2000–2020 did not address recruitment criteria or admission requirements of institutions, which could also be considered as representative of a shift in interest of the type of PMTC that seemed to be more relevant in this period.

In general, then, the lack of research on PTs' PMTC in 2000–2020, based on the journals reviewed, could be related to challenges in addressing variables associated with a point of entry into an education program or a shift in interest from considering PMTC at point of entry to indirectly addressing PMTC for PTs based on their participation in mathematics education courses regardless of where they are situated in a teacher education program.

4.2 Methodological Factors

Each of the four methodological factors adapted from Medley (1987) is discussed in this section regarding the evolution of research on the PTs' PMTC. *Conceptualization* is interpreted in terms of relationship to 'good teaching', teacher education, and technology and culture. *Instrumentation* is interpreted as procedures or tools used in collecting the data. *Design* is interpreted as what was used or done to support the data collection process and *analysis* is interpreted as the means used to extract information from the data. These interpretations are appropriate to address the information provided in the studies.

4.2.1 Conceptualization of PMTC in Relation to 'Good Teaching'

In adapting Medley's (1987) framework as a guide for research on teaching, it is important for studies to provide a conceptualization of *good teaching*. For the period 2000–2020, while good or effective teaching of mathematics was not explicitly conceptualized in the studies on PMTC, it was implied based on the theoretical bases

framing the PMTC being investigated for the three categories of mathematical knowledge and skills, pedagogical knowledge, and beliefs. These theoretical perspectives were related to reform-based perspectives of mathematics education that promoted significant shifts in school mathematics curriculum, teaching, and learning and were associated with effective teaching of mathematics. Collectively, the studies directly or indirectly considered PMTC concerning specific elements of these perspectives that include: (a) standards and principles for mathematics education (NCTM,); (b) mathematical proficiency (Kilpatrick et al., 2001); (c) mathematical thinking (Mason et al., 2010; Schoenfeld, 1992); (d) mathematics knowledge for teaching (Ball et al., 2008); (e) teaching practices (NCTM, 2014); and (f) beliefs about the nature of mathematics (Ernest, 1989) and teaching and learning mathematics (Beswick, 2012).

The studies, then, indicated an evolution in conceptualizing PMTC to reflect contemporary perspectives of teaching and learning mathematics, with implications about the nature of the PMTC required for a PT to become "the teacher who has a set of personal characteristics closest to those of the ideal teacher" (Medley, 1987, p. 106). This implication seemed to underlie most of the studies considering the nature of the PTs' PMTC mainly from a deficit perspective. The result was an evolution of research in the period to highlight what was wrong with the PMTC of the PTs on entering MTE.

Regardless of whether addressing PTs at the primary, elementary, middle, or secondary school level, the studies showed an ongoing focus on issues and limitations in their PMTC that indicated what they did not know or were not able to do at the beginning of their MTE. For example, collectively, the PTs: did not have deep conceptual or relational understanding of arithmetic and algebraic concepts; could not pose complex relational or conditional problems; were not able to think mathematically beyond a low level; were not able to observe details of the classroom environment, mathematical content of a lesson, and subtleties of classroom communication and mathematics content beyond a surface level; were not able to collect, beyond surface level, appropriate evidence to support analysis of instruction and learning; could not base their analysis of instructional decisions on pedagogical and mathematical reasons instead of affective reasons; could not reflect on a level to theorize or restructure; were not able to analyze or interpret students' solutions with depth, notice students reasoning based on meaning of student work, identify appropriate evidence for their noticing, identify conceptual aspect of errors, and classify tasks of high levels of cognitive complexity for students; and did not hold views of inquiry-based, constructivist-oriented perspectives of mathematics, teaching and learning, technology as tool to support deeper understanding and appreciation of mathematics, nature of genuine problems and problem solving in terms of their openness, conceptual use of multiple representation to support deep learning, and algebra as reasoning and relational thinking.

The deficit perspective of the PTs' PMTC also suggested limited evolution of school mathematics teaching based on reform recommendations since the different types of PMTC involved were directly related to what the PTs should have learned or experienced directly (e.g., mathematics content) and indirectly (e.g., mathematics pedagogy and beliefs) in school prior to entering teacher education. This probably

limited evolution of teaching was also suggested by the studies based on what they noted or implied about what the PTs knew or were able to do, which reflected learning from traditional, pre-reform-oriented teaching of mathematics. For example, the PTs' PMTC included: procedural or instrumental understanding of the some key school mathematics concepts; ability to pose problems of low level of cognitive complexity; ability to conduct instrumental analysis of relationship between instruction and student learning; ability to observe, describe and explain generic instructional issues, classroom management, instructional tasks, and affective factors of classroom interactions; ability to notice, interpret and describe students' work or thinking on a procedural level; ability to identify tasks of low cognitive demand and attend to surface characteristics of tasks; beliefs of mathematics as absolute, teaching and learning as teacher-centered (e.g., teaching as telling, learning as memorizing), technology as computational tool; use of representations on an instrumental level; problem solving as a means to a solution involving procedural steps; and algebra as manipulation of symbols and in terms of surface features (e.g., variable or letter).

In general, whether the PTs' PMTC were viewed from a perspective of what they knew or did not know, were able or not able to do, the studies indicated continued issues with their PMTC when the PMTC were conceptualized in relation to reform expectations for effective teaching of mathematics in the period 2000–2020. This outcome suggested that the impact of the reform movement in school mathematics had not materialized in this period and many PTs were entering teacher education with PMTC that did not align with appropriate knowledge, skills, and beliefs in relation to effective teaching. But this conclusion might not be representative of the actual situation given the limitations of the studies regarding small sample sizes and little information on those PTs with PMTC that reflected reform-based teaching at the point of entry into MTE.

4.2.2   Conceptualization of PMTC in Relation to Teacher Education

Based on the definitions of Type F variable (Medley, 1987; Manizade et al., this volume), research should also conceptualize PMTC in relation to "the characteristics a teacher candidate needs in order to acquire those competencies that formal education and experience can provide" (Medley, p. 105). Hence, as Medley indicated, Types F and E variables should be combined in research on teaching, where Type E involves mathematics teacher competencies, knowledge, and skills to function effectively in mathematics teacher education (Manizade et al.). In addition, Medley explained:

> Type FE research is the proper research to provide support for selective admission to teacher preparation, and may be called research in teacher selection. What characteristics of entering students identify teachers who will acquire the competencies they need as a result of training? (p. 111)

His perspective suggested that PMTC, in addition to being conceptualized in relation to good or effective teaching, should be conceptualized in relation to *good*

or effective teacher education, for example, the types and nature of the PMTC that are consistent with the role of MTE or are most needed to support PTs' learning in MTE.

The studies on PMTC in 2000–2020 did not address this relationship or lacked clarity about it. While they suggested that the PTs' PMTC were not adequate in relation to effective teaching, there was less clarity regarding whether or not the PMTC were adequate to support their learning in MTE. But, based on the design of many of the studies, there were underlying assumptions of the relationship between the PTs' PMTC and possible roles of MTE to help PTs develop the knowledge and competencies for effective teaching of mathematics. For example, the intervention studies with focus on fixing deficiencies in the PTs' PMTC implied PMTC were conceptualized in relation to a remedial role of the teacher education programs. In general, the studies did not suggest that PMTC were conceptualized in relation to a constructivist role of MTE in which the PMTC were viewed as resources to build on and not deficiencies to fix. For both of these roles, the quality of the PMTC on entering MTE might not be important beyond some minimum standard required to complete high school and/or to enter an education program. Overall, then, while conceptualization of PMTC in relation to teacher education was unclear or limited for research in 2000–2020, there was a shift in the underlying implication of the studies that the role of mathematics teacher education was important to researching and understanding the PMTC PTs needed on entering MTE for them to succeed in it.

4.2.3   Conceptualization of PMTC in Relation to Technology and Culture

Context is important to understanding the mathematics teacher and teaching and should be considered in the conceptualization of teachers' PMTC in research on teaching. Medley (1987) identified four categories of context-related variables, also adapted by Manizade et al., (this volume) for mathematics education research, that should be considered, but he associated these categories with practicing teachers and did not directly connect any with the Type F variable of PMTC. Thus, context-related factors that could have impacted the PMTC prior to being engaged in teacher education were not highlighted in Medley's model. However, in a digital age and a twenty-first century society, the evolution of research on PMTC in 2000–2020 should reflect the availability of technology and the cultural context in and outside of classrooms. This view means that, for research in this period, the conceptualization of PTs' PMTC should also be related to technology and culture. This was not, however, reflected in the studies, directly or indirectly, based on theories to support the importance of culture and technology to mathematics teaching. For example, equity and technology are two of the six principles recommended by NCTM (2000) as fundamental to high-quality mathematics education.

Regarding Technology. NCTM (2000) promoted technology as being essential in teaching and learning mathematics. In addition, NCTM (2011) explained:

> Technological tools include those that are both content specific and content neutral. In mathematics education, content-specific technologies include computer algebra systems; dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications. These technologies support students in exploring and identifying mathematical concepts and relationships. Content-neutral technologies include communication and collaboration tools and Web-based digital media, and these technologies increase students' access to information, ideas, and interactions that can support and enhance sense making, which is central to the process of taking ownership of knowledge. (NCTM, 2011)

Despite this range of tools and importance of technology, there was no study in the last ten years and only two studies in the early 2000s that considered technology in relation to PMTC. There was, therefore, a lack of information regarding the influence of technology on PTs' PMTC based on technology in general or the different types of technology the PTs would have encountered in or out of the classroom. The two studies on technology focused on PTs' beliefs about it in a general sense (without consideration of specific types), but neither considered the relationship between the use of technology in learning and the PMTC.

Of the two studies, only one (Wachira et al., 2008) focused explicitly on pre-existing beliefs at the beginning of a semester based on students' responses to two prompts: (a) to indicate their experiences with instructional technology use in mathematics, and (b) to provide compelling arguments for the use of technology in mathematics learning. Both studies highlighted the limitations of the PTs' beliefs, which suggested that their exposure to technology was not enriching to their PMTC. But these studies occurred in the early 2000s when access to technology was not as available in schools as later in the period. They may not, therefore, be representative of the significant changes and access to technology in western cultural contexts and the impact on teacher candidates' learning and thinking on entering MTE. Overall, there was a lack of growth in conceptualizing PMTC in relation to technology.

Regarding Culture. The actions of teaching and learning exist in cultures that vary greatly from society to society, from school to school, and even from classroom to classroom in the same school. Thus, culture could be a problematic variable regarding its meaning in researching teaching and a basis of conceptualizing PMTC. It is considered here in relation to the classroom. The culture of a mathematics classroom determines and is determined by the type of learning that takes place, affects the types of experiences students engage in, and could interact with students' personal cultural (e.g., home or societal culture) experiences in positive or negative ways. Thus, in the period 2000–2020, there has been the promotion of culturally responsive teaching in general (Gay, 2010; Taylor & Sobel, 2011) and specific to mathematics education (Greer et al., 2009; Presmeg, 2007) and of equity as a principle in school mathematics education (NCTM, 2000) to meaningfully address diverse student population based on various cultural heritage and social backgrounds of students in the western cultural context.

Culture then should be of importance in considering the evolution of research on PMTC in relation to its impact on PTs' experience and learning of mathematics that are connected to the nature of their PMTC on entering MTE. These PMTC include the mathematical identity PTs developed based on the classroom culture and the personal cultural-based resources they bring to MTE with implications for the type of teacher they will become. None of the studies attended to PMTC in relation to culture. This lack of conceptualization of PMTC in relation to culture suggested a significant deficiency in the evolution of research on beginning PTs. There seemed to be a lack of a humanistic perspective in framing the studies that made culture irrelevant in considering the PMTC. There was also a lack of focus on affective factors that are directly associated with culture. Regardless of whether the PTs were from culturally homogenous classrooms with homogenous cultural backgrounds, culture still mattered regarding their identity as a teacher and the nature of their PMTC. Thus, overall, there was a lack of growth in conceptualizing PMTC in relation to culture.

4.2.4   Instrumentation in Researching PMTC

Instrumentation is the second factor Medley (1987) indicated as important in considering the evolution of research on teaching, interpreted here as procedures or tools used in collecting the data. In earlier studies, Medley, for example, noted that instrumentation focused on surveys consisting of closed response questionnaires or brief written response items about teaching. For the period 2000–2010, while the studies on PMTC continued to use surveys, they also used a variety of tools for data collection. This evolution of instrumentation mirrored the evolution in the conceptualization of PMTC in relation to contemporary perspectives of effective mathematics teaching. Four categories of instruments, discussed in turn in the following paragraphs, were used in the studies to determine the PTs' PMTC at the beginning of a program or course or prior to an intervention to support their learning.

Questionnaires. Some studies used only questionnaires, for example: motivation questionnaire (Newton, 2009); beliefs questionnaire (Dreher et al., 2016; Weldeana & Abraham, 2014); questionnaire to analyze student work (Simpson & Haltiwanger, 2017); diagnostic questionnaire (Tirosh, 2000); content knowledge questionnaire (Lee & Lee, 2020); questionnaire on concept maps and definitions (Miller, 2018); and questionnaire with true/false, multiple choice, and short answer questions (Star & Strickland, 2008). Other studies used open-ended questionnaires with semi-structured interviews, for example: regarding conceptions and beliefs of mathematics, problem solving or creativity (Bolden et al., 2009; Conner et al., 2011; Son & Lee, 2020; Szydlik et al., 2003) and mathematics backgrounds and teaching interests (Stephens, 2008).

Interviews. Some studies used only semi-structured interviews based on participants solving mathematical tasks (Stephens, 2006; Thanheiser, 2009, 2010; Yanik, 2011). In addition to interviews being combined with questionnaires, other studies combined

interviews with written responses to analyse students' written work (Magiera et al., 2013; Shin, 2020) and to prompts on beliefs (Shilling-Traina & Stylianides, 2013).

Written Responses. Some studies used only written responses, including: journals on mathematics problem posing (Crespo, 2003); mathematical autobiographies (Harkness et al., 2007; Wachira et al., 2008); responses to interpreting students' solutions to mathematical tasks (Callejo & Zapatera, 2017; Sánchez-Matamoros et al., 2015, 2019); responses to incorrect students' solutions to the same problem solved by PTs (Son, 2013); responses on reflecting on a student's invented algorithm (Harkness & Thomas, 2008); responses to written standard place-value-operation tasks (addition and subtraction) (Thanheiser, 2010); and responses to the analysis of video recorded mathematics lessons (Morris, 2006; Star & Strickland, 2008) and analysis of a mathematics video curriculum (Stockero, 2008).

Mathematics Tests and Tasks. Some studies used content knowledge tests on rational numbers and computations (Lovin et al., 2018); fractions (Lin et al., 2013; Osana & Royea, 2011); whole number operations (Kaasila et al., 2010; Norton, 2019); linear functions (You & Quinn, 2010); and algebraic language (Pomerantsev & Korosteleva, 2003). One study combined a number sense test with interviews (Whitacre & Nickerson, 2016). A few studies used students' work on mathematical tasks involving solving algebra tasks (Hohensee, 2017); posing mathematics problems (Crespo & Sinclair, 2008); solving pattern-finding tasks (Richardson et al., 2009; and sorting mathematics problems (Osana et al., 2006).

The different ways of collecting data outlined above were used throughout the period. There were about the same number of studies that used questionnaires, interviews, written responses, and tests and tasks, alone or in different combinations. Overall, the growth in instrumentation consisted of very little use of only interviews and an increased use of combinations of questionnaires and interviews, open written responses, and tests or tasks which have the potential to produce more valid data regarding the types of PMTC that were studied.

4.2.5 Design and Analysis in Researching PMTC

Design and analysis are the last two factors Medley (1987) indicated were important in considering the evolution of research on teaching. However, based on Medley's perspective, they were problematic to address for the studies on PTMC in 2000–2020, most of which were not specifically designed to address PTs' PMTC on entering MTE, but had broader goals. Thus, the design was considered in terms of what the studies used to support the data collection process at the beginning of a course or prior to an intervention and analysis as the means used to obtain information from the data.

Design. There was an evolution in the design of research in the period in terms of the use of school students' mathematical work and videos of mathematics teaching to engage the PTs in situations to apply their PMTC. Students' work included: actual

solutions to various mathematical tasks (Callejo & Zapatera, 2017; Magiera et al., 2013; Sánchez-Matamoros et al., 2015, 2019; Simpson & Haltiwanger, 2017); hypothetical written solutions (Shin, 2020); incorrect solutions to the same problem solved by PTs (Son, 2013), and a student's invented algorithm (Harkness & Thomas, 2008). Videos included videotaped mathematics lessons (Morris, 2006; Star & Strickland, 2008) and a video-case curriculum (Stockero, 2008).

There was also evolution in terms of significant variations in the design of instrumentation (e.g., questionnaires, written responses, tests and tasks) to match the different types of PMTC and in terms of the combination of interviews with other instruments to obtain reliable data. One area of limitation involved studies not being designed solely for researching PMTC at the beginning of an education program, which could have resulted in aspects of the PMTC not being identical to the PTs' PMTC on entering the program. A few studies were designed at the beginning of courses, while most were designed as intervention studies with a pre-post-intervention design. The intent of the intervention studies was more about promoting the intervention as a way of impacting change and less about the nature of the PMTC. Thus, they tended to provide little information on the pre-intervention characteristics, with the emphasis being on the post-intervention. The design also tended to use convenient samples of PTs enrolled in specific courses and small sample sizes regardless of the nature of the instrumentation. Thus, the studies did not necessarily provide a representative picture of PMTC within an institution or a region, even though they offered useful insights about the PMTC.

Analysis. The analysis approaches used in the studies depended on the instrumentation and thus showed an evolution of approaches consisting of both quantitative and qualitative strategies. These approaches varied within and across categories of instruments depending on the design of the instrument. For example, some questionnaires used the Likert scale (e.g., Dreher et al., 2016; Newton, 2009; Szydlik et al., 2003; Weldeana & Abraham, 2014) while others used open-ended items with a rubric or scale for scoring or categories to compile and rank frequencies (e.g., Conner et al., 2011; Lee & Lee, 2020; Miller, 2018; Son & Lee, 2020). Interviews by themselves were semi-structured and based on PTs solving mathematics tasks (Stephens, 2006; Thanheiser, 2009, 2010; Yanik, 2011); when combined with questionnaires they were semi-structured and based on following up on questionnaire items or ideas (Ambrose, 2004; Conner et al., 2011; Son & Lee, 2020; Stephens, 2008; Szydlik et al., 2003). Interviews as well as open written response tasks (Crespo, 2003, Harkness et al., 2007; Sánchez-Matamoros et al., 2015; 2019; Son, 2013; Thanheiser, 2010; deCallejo & Zapatera, 2017; Harkness & Thomas, 2008; Morris, 2006; Star & Strickland, 2008; Stockero, 2008) were generally analyzed through coding to produce themes or categories. Tests, which dealt with mathematics content, used scoring schemes that indicated the level of correctness or error in participants' responses (e.g., Lin et al., 2013; Lovin et al., 2018; Norton, 2019; Osana & Royea, 2011; You & Quinn, 2010). While Medley (1987) suggested more use of technology in analysis, this was not

reflected in the studies because of the shift to more qualitative approaches or quantitative approaches with small sample sizes that did not necessarily require complex statistical analysis.

4.3 Summary of Evolution of PMTC Research

Overall, consideration of the scope and the methodological factors of the studies on PMTC for the period 2000–2020 indicated that there were both growth and limitations or gaps in research on PTs' PMTC at the point of entry in MTE. There was evolution in the scope of research in terms of the extent of the types of PMTC researched and the extent to which PMTC were addressed by the studies. For example, while the mathematical knowledge and skills category of PMTC received the most attention in the studies, suggesting ongoing interest in content-related characteristics, a significant shift was the pedagogical skills category regarding studies on noticing and interpreting students' work and thinking that was the largest group of studies for the three categories of PMTC (Table 1).

There was also evolution of aspects of methodology based on Medley's (1987) four factors of the conceptualization, instrumentation, design, and analysis. For example, conceptualization of PMTC evolved to reflect contemporary perspectives of teaching and learning mathematics**.** There was a shift in instrumentation from a focus on surveys with large samples in early studies to a variety of tools, used alone or in different combinations. There was growth in design regarding the use of school children's work and of videos on teaching as bases of obtaining pedagogical-related data. The analysis also shifted from mainly statistical approaches to including qualitative approaches particularly for interviews and written responses. Limitations and gaps in the evolution of the research on PMTC are addressed in the following section in relation to what needs to be considered in future research.

5 Future Evolution of Research on PMTC

Future evolution of research on PMTC refers to what should be considered to move the field forward in this area. Medley (1987) indicated that the methodological factors of conceptualization, instrumentation, design, and analysis should also be used in considering the future evolution of research on teaching, which includes research on PMTC. Consistent with the preceding section, the scope of research is also relevant. The preceding discussion of evolution on research on PMTC in the period 2000–2020 indicated that there were limitations or lack of attention regarding scope and methodological factors, which suggest areas that need attention to support future evolution of research on PMTC. The following summary highlights the key areas that future research should consider.

Regarding scope, much more attention is needed on research of PMTC for PTs at the point of entry into a teacher education program as opposed to other points during the program, which could be affected by confounding factors associated with their experience in the program in general. There also needs to be more scope and depth of the types of PMTC researched within and beyond the categories of mathematical content and skills, pedagogical knowledge, and beliefs. There are other aspects of PTs' mathematical abilities, knowledge, and attitudes, as well as aptitude for teaching that are important to understand PTs and their PMTC at point of entry MTE. One area the studies in 2000–2020 were particularly lacking in addressing, that needs future consideration, was affective factors such as PTs' attitudes and what they value. For example, do they value collaboration, know how to connect and form collaborative groups, have the skills needed to create an environment of working with others? As Blanton (2002) also asked, do they value discourse as an active process in which students use the collective knowledge of a group to build understanding (i.e., dialogic discourse)? What is their level of competence to reflect and to be curious? In addition, as Strutchens et al. (2017) also suggested, there is a need to consider the various identities that PTs have prior to their participation in preservice education.

Another area not attended to, but that is of significance in the context of the current digital age and twenty-first century society and in need of future attention, is the impact of technology and culture on the PTs' PMTC at the point of entry into MTE. Both are important to the nature of PTs' PMTC and culture, in particular, to their developing mathematical identity. Finally, regarding beliefs, the scope was limited to types of beliefs but future attention could also be given to PTs' ability to reflect on them.

Regarding conceptualization, more attention is needed to conceptualize PMTC in relation to teacher education, for example, regarding the types and nature of PTs' PMTC on entering teacher education that are consistent with the role of the teacher education program or are most needed to support the PTs' learning in the program. In addition, PMTC should be conceptualized in relation to technology and culture regarding specific characteristics of the latter that influence the nature of the PMTC.

Regarding instrumentation, design, and analysis, the future evolution will depend on the scope and characterization of future studies on PMTC. Some considerations are: designing studies with the sole aim of exploring PMTC at point of entry to MTE, which may also require different or "better instruments" (Medley, 1987) and analysis; designing studies that are more humanistic in focusing on what PTs' know and can do based on their PMTC, which could be more practical when using convenient samples and qualitative instruments; and the use of more rigorous mixed methods research design with more rigorous statistical analysis and use of technology.

To conclude, overall, the studies suggested that there have been significant changes in research on teaching with a focus on the Type F variable regarding PMTC of candidates MTE. But ongoing work is necessary for this area of research given its importance to understanding the selection of teacher candidates, the mathematics teacher, teaching of mathematics, and teacher education. Since PMTC at the point of entry are important starting points of PTs' formal education to become a teacher,

then more attention is needed to understand these PMTC and how to work with them in mathematics teacher education.

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The Evolution of Research on Mathematics Teachers' Competencies, Knowledge and Skills

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1 Introduction

What characterizes competent mathematics teachers, what types of knowledge do they need in order to be able to teach successfully, and what skills do they draw upon for successful teaching? These questions have long concerned mathematics education research, teacher education and educational policy. The NCTM standards (2000), for example, refer specifically to teacher knowledge as a ground to start from, stating that "[t]eachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (p. 17). Teachers need not only sufficient disciplinary mathematical knowledge and knowledge of the school subject (Bromme, 1994). As Shulman (1986, 1987) argues, teachers need a specialized knowledge base for teaching that is different from pure mathematical knowledge and that differs from other professions, thus coining the term of pedagogical content knowledge. However, to determine what teachers should know, what other aspects constitute teacher competencies, and to specify how teachers acquire these in teacher education, as well as how teachers use their skills and act competently in practical situations in teaching is not an easy task. Teacher competencies are also related to underlying beliefs about the role of teachers

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and the teaching profession, which are culturally shaped. All these entities are subject to change over time and are continually evolving. For example, further challenges arise over time as new demands emerge in the context of teachers' professional practice—such as the increasing integration of technology and digitization, which also require thinking about additional necessary teacher competencies.

In the framework of presage-process–product research underlying this volume (see Chap. 1), Medley (1987) names the facet of teacher competence (Type E) as a central variable within the research on teaching, which he understands as the "knowledges [note: plural!], skills, and values which a teacher possesses" (p. 105) and which he considers being "the tools of teaching" (ibid.). Teacher competence has thus an impact on student learning (the outcome of teaching), as it enables teachers to teach successfully and competently in classroom situations. However, it becomes clear that in order to be able to assess this effect, additional mediating variables should be taken into account as good as possible (see Chap. 2). For example, pre- and post-active teacher activities (Type D), such as planning, assessment, reflection and out-of-class activities of mathematics teaching (see Chap. 3) and interactive mathematics teacher activities (Type C), that take place when in the presence of the students (see Chap. 4). Yet, teacher competencies play a central role in the quality of instruction.[1] Characterized by a cognitivist and individualist perspective, what most research on teacher competence today seem to agree on is that teachers' professional knowledge is central within teacher competence and is considered an essential component of the job-specific prerequisites for successful classroom action. It represents an important cognitive resource for interpreting classroom situations and generating informed decisions for actions needed for successful and competent teaching (Baumert & Kunter, 2006; Gitomer & Zisk, 2015; Guerriero, 2017).

Since the beginning of presage-process–product research, and based on theoretical reflections on a subject-specific characterization of teacher cognitions in teaching, which were initiated in the U.S. in the late 1980s, the question of the theoretical conceptualization and empirically examination of teachers' professional knowledge has become increasingly important (e.g. Carpenter & Fennema, 1992; Carpenter et al., 1988, 1989; Fennema et al., 1996; Neubrand, 2018; Petrou & Goulding, 2011; Rowland, 2014). The research initially sought to identify and isolate more general variables of successful teaching, but has since taken somewhat different forms. Presage-process–product research meanwhile is transitioned into the content-dependent, more situation-specific study of teachers' professional knowledge and its implications for the quality of mathematics instruction. In recent years, a new branch of research on the theoretical description and empirical measurement of professional knowledge of mathematics teachers has become firmly established in the international mathematics education research discipline (a.o. Ball et al., 2008; Baumert et al., 2010; Buchholtz et al., 2014; Carrillo-Yañez et al., 2018; Davis & Simmt, 2006; Even & Ball, 2009; Hill et al., 2004, 2008a, 2008b; Kaiser et al., 2014, 2017;

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[1] For further student-related variables as well as external and internal context variables that play a role in the relation of teacher competence and student outcome see Chapter 1 and the other Chapters in this volume.

Krauss et al., 2008; Kunter et al., 2013; Lindmeier, 2011; Manizade & Martinovic, 2016; Manizade & Mason, 2011; Rowland & Ruthven, 2011; Scheiner et al., 2019). However, this work already builds on research approaches that have developed over the past 30 years, as we will show in this chapter.

On the theoretical level, following the seminal work of Shulman (1986, 1987), different dimensions of knowledge are often distinguished in the study of teachers' professional knowledge, depending on assumed aspects of content, referring to the so-called domain specificity. This classification of teachers' professional knowledge has also been used in large-scale international comparative studies of the effectiveness of teacher education programs, such as TEDS-M 2008 (Blömeke et al., 2014a, 2014b; Tatto et al., 2012) and its predecessor study MT21 (Schmidt et al., 2007, 2011). Despite the abundance of studies in this area, however, there is still no agreement on a unified theoretical conceptualization because different conceptualizations are based on different domains attributed to teachers' professional knowledge, differ in their theoretical assumptions, and also have different grain-size of the knowledge elements considered (Even, et al., 2017; Neubrand, 2018).

However, the complexity of the construct of professional knowledge in contemporary research on Type E has not only increased as a result of different theoretical conceptualizations, but also because of the question of the extent to which it is situationally available in school practice as a cognitive prerequisite 'in the head of a teacher' in the form of requirements-related knowledge and skills. When such knowledge is operationalized and measured context-independently for empirical studies (for example in psychometric scalable knowledge tests), research to date showed mixed results as to whether or not it is possible to separate different knowledge domains empirically (Bednarz & Proulx, 2009; Buchholtz et al., 2014; Charalambous et al., 2019; Depaepe et al., 2013). Current discourses within research on teaching, however, put up for discussing the extent to which a context-independent investigation of teachers' professional knowledge seems to be useful at all. Thus, on various occasions, the importance of approaches that allow for a more situation-specific measurement of teachers' cognitive processes in teaching has been pointed out to strengthen the contextual study of teacher competence (e.g., Kaiser et al., 2015; Shavelson, 2010). Since then, scholarly advancements in the last decade have consisted in the differentiation of the current conceptualizations for teaching mathematics according to the theoretically-sound and empirically-based integration of action-oriented knowledge facets (Blömeke et al., 2015; Kaiser et al., 2017; Neubrand, 2018). Among other things, this has led to current mathematics education research approaches to the study of teacher competencies, such as the Knowledge Quartet (Rowland, 2008a; b), Lindmeier's action-based competence approach (Lindmeier, 2011; Lindmeier et al., 2020), and the German TEDS research program (Kaiser & König, 2019). These research approaches focus more on the situational manifestation of professional knowledge and its relation to perceived instructional quality (Even et al., 2017). At the same time, however, the call for a situation-embedded study of knowledge is countered by the fact that the more contextually knowledge is analyzed in studies, the even more difficult it becomes to empirically

distinguish knowledge from other factors such as teacher personality or affective variables (Even et al., 2017).

Newer challenges in the description and study of teachers' professional knowledge are also posed by the ever-changing demands of professional practice, which have increased significantly since the late 1980s so there is a constant need to rethink what specialized teacher competencies are needed for successful teaching. Medley (1987) identifies this as a distinct branch of research in teacher competence (Type E and Type D research, p. 111), which is normatively oriented and includes both preactive teacher behaviors like planning or evaluating as well as situational aspects of competence. For example, current topics in research on teaching include the study of teachers' diagnostic skills. These are becoming increasingly important because of the need to deal with an ever-increasing linguistic and cultural diversity of students in the classroom due to transnationalization processes and multiple cultural attributions. Furthermore, novel challenges concerning competencies in the use of technology and digital media in mathematics teaching and dealing with the challenges of digitization play a role (e.g., Mishra & Koehler, 2006) as well as skills and attitudes for achieving equity and educational justice in mathematics classrooms (Schoenfeld et al., 2019).

This chapter provides an overview of the most important developments in the field of describing professional competencies of mathematics teachers, especially taking up the perspective of the development of research over time since Medley's (1987) reflections. However, this overview chapter does not follow the criteria of a systematic review; rather, we provide a narrative review (Snyder, 2019) to give as good and comprehensive as possible an overview on the progress of the research in the field. As a result, however, the perspective is inevitably subjective, and not all work is included. First, we will discuss the development of research on teacher competencies, knowledge and skills over time, before discussing various facets of teacher competencies and teacher professional knowledge separately in Sect. 2. Section 3 deals with the different conceptualization and operationalization of teacher competencies in key studies and research programs and the further development of research towards the consideration of situational aspects. We conclude the chapter with an outlook on the further development of Type E research and a summary reflection.

2 Evolution of Research on Teacher Competencies, Knowledge and Skills

Research on teachers' competencies, knowledge and skills has been influenced by different research directions over time. To be able to chronologically situate the developments and to describe the further developments in terms of thematic content, it is necessary to reflect on the underlying paradigms of research on teaching. Research on the teaching profession has undergone several paradigm shifts since the 1960s, changing the underlying theories and the research approaches used. In the process, existing paradigms were critically examined for weaknesses and further developed

so that today's research on teacher competencies, knowledge and skills is based on different paradigmatic approaches which have complementary strengths and set different accents.

The so-called personality paradigm or traits paradigm, which prevailed until about the 1960s, attempted to attribute the pedagogical effectiveness of teachers' actions to measured personality traits (e.g., patience or emotional stability). However, the paradigm had its weaknesses in that it was unable to explain how these characteristics impact different classroom situations (Bromme, 2001). Since its research has produced few or only trivial results on the relationship between teacher action and learning success, the paradigm is not considered very fruitful today.

Originating in teacher effectiveness research, Medley's reflections on directly detectable relationships between different variables in the chain of effects (Medley, 1987) on the outcomes of teaching can be assigned to the presage-process–product paradigm, which took over from the 1960s when research on teaching became more systematic and empirical. This research paradigm questions what effects certain characteristics of teachers have on the desired learning outcomes of their students, assuming stable behavior (Floden, 2001).

In the past, researchers following this paradigm deliberately did not examine teachers' cognitions, but rather behavioral features that are easy to control and observe, e.g., the number and level of questions asked, the waiting time after questions, or the frequency of feedback on students' responses (e.g., Gage & Needels, 1989). An assumption of many studies was that effective teaching practices were domain-general, and researchers could look across teaching in different domains and make generalizations about what teaching expertise looked like overall (Russ et al., 2011). The assumption of the paradigm, that a teacher's behavior exerted a direct influence on student's learning experienced significant criticism in later years, in part because the focus in observing teachers in some studies tended to be only on isolated surface characteristics and did not look at more complex structures of instructional quality (e.g., deep structures rather than surface structures) or the combination of multiple variables, including those of Types E and F[2] (Bromme, 2001). It further became clear that the impact of specific teacher actions depended on the context and the learner much more than assumed and findings on teacher behavior were not as transferable to the realities of different classrooms as one had hoped for (Weinert et al., 1989). However, back in the 1980's, researchers only had access to different (less advanced) research tools (e.g., compared to today's multilevel structure equation modelling), and considered different evidence in their work. One outcome of the criticism was the programmatic remodelling of the paradigm, basically in the expert paradigm (Ornstein, 1995). But presage-process–product research nevertheless continued to evolve. An important aspect of presage-process–product research that continues to shape research on the teaching profession today is the holistic approach that seeks to make connections between teacher behavior and student

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[2] This criticism does not undermine the overall framework developed by Medley in general, as it is open to the conceptualization of the variables studied and also takes into account corresponding contextual variables.

learning in particular ways. It thus identifies relevant variables for successful teaching as shown in recent meta-analyses on the effectiveness of teaching (Hattie, 2009; Seidel & Shavelson, 2007). The presage-process–product paradigm thus continues to influence research on instructional quality today and has established its standards.

Based on findings from cognitive psychology research, since the mid-1980s the individual cognitions of the teacher had become the focus of interest in research on the teaching profession. This approach was initially promising in that it was hoped that an understanding of the teacher's thinking would provide insight into why teachers behaved in certain ways in the classroom. Again, however, the focus was in the beginning on cross-domain, rather than initially subject-specific, approaches (Russ et al., 2011). This changed mainly due to the growing influence of the research program "Knowledge Growth in Teaching" by Lee Shulman (Shulman, 1986, 1987) and the work of his research group at Stanford University. Shulman pointed out the importance of subject matter in the study of teacher knowledge. In his famous Presidential Address at the 1985 annual meeting of the American Educational Research Association and the article published in 1986, Shulman cautioned against teacher effectiveness evaluations at the time that focused purely on generic teacher behaviors (such as orientation to simple rules like appropriate waiting times on student responses). He proposed a classification of teachers' professional knowledge that accounted for subject-specific viewpoints and, he saw subject matter knowledge as central to the pedagogical preparation and accessibility of subject content in the classroom. The most important consideration for the research on teacher cognitions at that time was his postulation of pedagogical content knowledge (PCK), which differs from the knowledge required by other professions, such as mathematicians. PCK is specifically oriented towards teaching and includes knowledge about different student cognitions and teaching approaches. "Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Shulman himself did not aim for the development of a catalog of corresponding knowledge content but specified his idea of PCK in his article published the following year, "Knowledge and Teaching: Foundations of the New Reform" (Shulman, 1987) as a "specific amalgam" of knowledge about subject content and pedagogy, which focuses on subject representations and concepts of understanding as well as misconceptions. Shulman (1987, p. 8) distinguishes various forms of knowledge in his typology of professional knowledge:

- Content knowledge,
- General pedagogical knowledge (strategies of classroom management and organization),
- Curricular knowledge (including materials that serve as "tools of the trade" for teachers),

- Pedagogical content knowledge, a special "amalgam" of subject content and pedagogy that is found exclusively among teachers and forms the basis of their professional understanding,
- Knowledge of learners and their characteristics,
- Knowledge of educational contexts (e.g., about working of groups, administration and funding of school districts, or the character of communities and cultures),
- Knowledge of educational goals and values and their philosophical and historical grounds.

Later on, pedagogical knowledge, content knowledge, and pedagogical content knowledge had a great impact in terms of the theoretical design of research studies on teachers' professional knowledge.

According to Shulman, teachers must transform subject content into pedagogical forms such as examples, illustrations, and classroom tasks that make the content accessible to learners. This transformation of subject matter into pedagogically effective forms of learning is understood as the central intellectual task of the teacher and has become the defining characteristic of pedagogical content knowledge (Deng, 2007a, 2007b). Thus, for Shulman, PCK means the integration of subject matter knowledge and pedagogical knowledge that enables teachers to translate subject matter knowledge into pedagogically effective forms of presentation that match learners' abilities and interests. Shulman's work, however, did not go uncriticized and the criticism led to further developments in research on teacher knowledge. Among other things, it was noted that Shulman had a static understanding of knowledge as something that could be acquired and applied regardless of the complexity of the instructional context, and that the idea of "transforming" or "translating" subject matter into pedagogical forms amounted to a routine, mechanistic transmission of a fixed canon of knowledge. Shulman's critics objected that mathematical knowledge itself could also be assumed to be multidimensional and dynamic in nature, from which it follows that teachers' knowledge is characterized by its "interactive and dynamic nature" (Fennema & Franke, 1992, p. 162). Other scholars adopted this dynamic view of knowledge, essentially viewing it as physically and socially situated in the act of teaching in a particular context (Bednarz & Proulx, 2009; Döhrmann et al., 2018; Meredith, 1995).

This situatedness of teachers' cognitions was taken up by the so-called expert paradigm. The presage-process–product research at that time looked more for the general abilities and skills of teachers and was less concerned with the question of whether these individual bundles of behaviors could actually be found in a person in reality. The expert paradigm focused on the successful teacher "as a whole" (Bromme, 2001; Schön, 1983), and the focus henceforth was on teachers' knowledge and skills. Central to this development was the work of Berliner (2001), for example, in which he calls the teacher an 'expert teacher' and speaks of 'teaching expertise'. According to the expert paradigm, teachers are called experts because they can successfully manage a very specialized, complex task such as school teaching. Expertise is manifested, for example, in the immediacy of action expected of a responsible teacher in his or her teaching and the resulting time pressure of acting as well as in acting under

information deficit concerning the current situation, the complexity and dynamics of which are continuously changing due to the students' behavior. In this context, teachers draw on specific knowledge and skills, which can be technically described within the research approach through detailed analyses of requirements—such as those derived from psychology (e.g., Bromme, 1992, 2008).

A recent further development of the expert paradigm has been the approach of professional competence of teachers for about twenty years (Kunter et al., 2013). In this approach, teachers' knowledge and skills are not only identified using requirement analyses in terms of the expertise paradigm but are furthermore complemented by the examination of personality traits such as motivation and self-regulation. The concept of competence was introduced into the discussion by Franz Emmanuel Weinert (1999, 2001) about twenty years ago as part of an influential review of different definitions of competence in a report prepared for the OECD. In describing professional action competence, Weinert states:

"The theoretical construct of action competence comprehensively combines those intellectual abilities, content-specific knowledge, cognitive skills, domain-specific strategies, routines and subroutines, motivational tendencies, volitional control systems, personal value orientations, and social behaviors into a complex system. Together, this system specifies the prerequisites required to fulfill the demands of a particular professional position, social role, or personal project" (Weinert, 1999, p. 10). In summary, competence can thus be defined as "the ability to successfully meet complex demands on a particular context through the mobilization of psychosocial prerequisites (including both cognitive and noncognitive aspects)" (Rychen & Salganik, 2003, p. 43).

A feature of this definition of competence is that it is first understood as context-based. Second, in addition to purely cognitive components, it includes affective components such as volitional, motivational and social readiness to apply the competence in situations. It should also be noted that there is a distinction between competence as a general overarching concept, and the distinction between individual competencies if individual content-related competence facets are meant. According to this understanding, the professional competence of mathematics teachers consists of subject-related and subject-overlapping cognitive dispositions—teachers' professional knowledge (cf. also Baumert & Kunter, 2006)—as well as additional affective personality traits like beliefs, motivation or values (Hannula et al., 2019) specifically for the subject mathematics. These form the basis for mastery of specific situations that arise in professional demands.

Today's research on teacher competencies, knowledge, and skills invokes the different approaches of these paradigms. These are perceived as complementary so that the boundaries between the different paradigms often fade. For example, the current approach to professional competence combines the systematic analysis of teachers' characteristics and abilities of the presage-process–product paradigm with the approach of researching teacher cognitions and the approach of looking at certain characteristics of teachers' personality, such as motivation and values. Consequently, Medley's variables of Type E are still valid as the main units of analyses in research

studies, even with today's advances in research on mathematics teaching and teacher education.

3　Components of Teachers' Professional Competencies

Taking into account Shulman's (1986, 1987) reflections on the professional requirements of teachers, which we will discuss in more detail in the next section, the professional competencies of mathematics teachers and its components.

3.1　Content Knowledge

Teachers need knowledge of relevant facts, concepts, and their relations oriented to the subject body of knowledge, as well as subject-specific procedures for generating knowledge and justifying it. This means that teachers of mathematics must be proficient in mathematics, which can be expressed, for example, by the "five strands" of mathematical proficiency by Kilpatrick et al. (2001), which are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The deeper understanding of reasoning also implies that argumentation and proving is part of the professional knowledge of teachers so that they are able "to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions" (Shulman, 1986, p. 9). Neubrand et al. (2009) address the connections of teachers' content knowledge to more general mathematical skills such as explaining, communicating, and even modeling, and include insights into the history and epistemology of mathematics among the content knowledge of mathematics teachers. Somewhat later than Shulman, Bromme (1994)—a representative of the expert paradigm—also formulated on this basis the central insight that when describing teachers' content knowledge, a distinction should be made between the knowledge of the discipline and that of the school subject, since the school subject has a "life of its own" (p. 74), with its own body of knowledge and epistemologies. In mathematics education research, this distinction by Bromme contributed to the identification of professional knowledge of school mathematics (Deng, 2007a, 2007b), or elementary mathematics from a higher standpoint in relevant studies (Buchholtz et al., 2013) going back to approaches by Felix Klein (1908/2016). Dreher et al. (2018), for example, conceptualized this type of knowledge as so-called school related content knowledge (SRCK).

3.2 *Pedagogical Content Knowledge*

Although Shulman identified two components that are central to PCK, namely knowledge of instructional strategies and representations, and knowledge of students' (mis)conceptions, he did not specify PCK for mathematics. To describe the subject-specific PCK for mathematics, it is not sufficient to focus only on mathematical content, which would neglect cognitive and social preconditions of the learning processes of students. In terms of content, mathematical pedagogical content knowledge presupposes an understanding of subject knowledge, but central to this is knowledge of the potential of school subject matter for learning processes (curricula and syllabi, learning goals and principles), knowledge of subject-related student cognitions (student ideas and errors, learning prerequisites), and knowledge of subject-specific instructional strategies (representations, subject-related diagnostics, performance measurement, and subject-related explanatory and mediation strategies). Subsequently, Shulman's model has been refined more and more, also in response to criticism (for an overview, see the systematic review on PCK by Depaepe et al., 2013). Grossman (1990) for example distinguished four components that are central to teachers' PCK: (1) knowledge of students' understanding, (2) knowledge of curriculum, (3) knowledge of instructional strategies, and (4) knowledge of purposes for teaching. Depaepe et al. (2013) even distinguish a total of eight different facets based on their systematic review: (1) knowledge of students' (mis)conceptions and difficulties, (2) knowledge of instructional strategies, (3) knowledge of mathematical tasks and cognitive demands, (4) knowledge of educational ends, (5) knowledge of curriculum and media, (6) context knowledge, (7) content knowledge, and (8) pedagogical knowledge. A relevant extension of Shulman's understanding of PCK was undertaken in the U.S. in the late 2000s in the Learning Mathematics for Teaching (LMT) project, amongst others, through the formulation of the construct mathematical knowledge for teaching (MKT) or content knowledge for teaching mathematics (CKTM) by Ball and colleagues (e.g., Ball et al., 2008; Hill et al., 2004, 2005, 2008a, 2008b), which we will discuss in more detail below.

If one takes a closer look at the relationship between mathematics and pedagogy within the construct of PCK, however, some aspects can be identified that are more strongly influenced by the subject, while other aspects are more clearly related to pedagogy (see also Chick et al., 2006). With respect to a normative description of the content of the PCK, the perspectives of referring to the scientific discipline of mathematics education (i.e., mathematics, psychology, educational science, general didactics), which have been discussed since the 1970s and which continue to shape the mathematics education discourse today, provide orientation (Buchholtz et al., 2014). By more subject-related pedagogical content knowledge we can therefore understand primarily mathematical aspects of teaching and learning mathematics. This includes, for example, knowledge about subject-specific approaches to teaching, basic ideas, and mental representations of mathematical content, e.g., fractions,

percentages, or the concept of derivation, and being able to identify critical mathematical components within concepts that are fundamental for understanding; knowledge about the interconnectedness and interdependence of mathematical concepts (to establish connections between the different subject areas of mathematics education and their mathematical backgrounds, connections to other subjects in the sense of interdisciplinary learning, and connections between mathematical concepts and the real world (Freudenthal, 1991)); knowledge about fundamental mathematical ideas and mathematical activities (e.g., abstraction or algorithmic thinking); knowledge of students' subject-specific preconcepts and barriers to understanding, as well as levels of conceptual rigour and formalization (important in analysing and interpreting student solutions and student questions); knowledge of the role of everyday language and mathematical language in concept formation; knowledge of subject-motivated approaches to mathematical content (e.g., different approaches to the concept of probability; justifications for number range extensions); knowledge about subject-matter-based diagnostics of student solutions and errors (e.g., student misconceptions; appropriateness of student solutions); as well as knowledge about different types of tasks (important for using tasks as a starting point for learning processes).

Under more teaching-related pedagogical content knowledge in mathematics, we can locate perspectives beyond mathematical subject knowledge, which focus more on educational-psychological areas, but which are constitutive for mathematics education. These include knowledge about concepts of mathematical education (e.g., theoretical concepts of mathematical thinking and general competencies such as modeling, problem-solving, and reasoning); knowledge about dealing with different forms of heterogeneity in mathematics education (e.g., the use of different teaching goals in mathematics education, differentiation, and individualization); knowledge about dyscalculia, giftedness, and special education support (important for developing support plans for dyscalculic and gifted learners or inclusive learning groups, taking into account specific learning requirements); knowledge about forms and concepts for teaching and learning mathematics in schools (e.g., genetic learning, discovery learning, dialogical learning, extracurricular learning); knowledge about educational standards, curricula, and textbooks for the subject of mathematics; and knowledge about aims and forms of assessment in mathematics education (formative and summative).

The different requirements for PCK make clear that this knowledge is closely connected to content knowledge because the teacher consciously must choose between all the possible representations the subject provides for teaching (Neubrand et al., 2009). This may be one of the reasons for which there are still mixed findings of the empirical separation of these different knowledge facets (Charalambous et al., 2019; Depaepe et al., 2013), depending on respective measures. However, it is also clear from these lists that there are overlaps with general pedagogical knowledge—which we describe in the next section, for example in the area of assessment and in the area of dealing with heterogeneity, and that subject-specific curricular aspects also play a role (Grossman, 1990), which Shulman (1987) had rather assigned to general curricular knowledge.

3.3 General Pedagogical Knowledge

What kind of general pedagogical knowledge a mathematics teacher should possess is not an easy question. As König et al. (2011) indicate, the shape of general pedagogy is strongly influenced by cultural perspectives on the objectives of schooling and on the role of teachers (Hopmann & Riquarts, 1995). However, König et al. (2011), identify, based on a literature review, two core tasks: instruction and classroom management. "Less agreement exists as to what extent and what kind of knowledge about counseling and nurturing students' social and moral development or knowledge about school management should also be included in the area of general pedagogy" (König et al., 2011, p. 189). When it comes to knowledge about effective instruction, theories of learning, an understanding of the various educational philosophies, and general knowledge about learners (Grossmann & Richert, 1988) should be added to teachers' GPK along with knowledge about effective classroom management. By combining research on the quality of instruction and general didactics based on task analyses, König and colleagues were able to develop a framework for mathematics teachers' GPK consisting of four different dimensions of pedagogical knowledge. Thus GPK in the model of König et al. (2011) comprises knowledge about structures (structuring of learning objectives, lesson planning and structuring the lesson process, lesson evaluation), knowledge about motivation, and classroom management (achievement motivation; strategies to motivate single students or the whole group, strategies to prevent and counteract interferences, effective use of allocated time and routines). Furthermore knowledge about adaptivity (strategies of differentiation, use of a wide range of teaching methods) and knowledge about assessment (assessment types and functions, evaluation criteria, teacher expectation effects).

3.4 Beliefs

Research on teacher action assumes that the application of professional knowledge in action situations presupposes corresponding subjective beliefs (Felbrich et al., 2014; Schmotz et al., 2010). This relation makes the connection between Medleys Type E and Type F (Chapter 1.1 on pre-existing mathematics teacher characteristics) clear since pre-service teachers already have initial beliefs about teaching and learning and about mathematics at the beginning of their studies, which also influences the acquisition of professional knowledge (Blömeke et al., 2014a, 2014b; Buchholtz, 2017). Beliefs are thought to serve an orienting and action-guiding function for applying learned knowledge (Schmotz et al., 2010; Schoenfeld, 1998; Thompson, 1992). However, despite intensive research on teachers' beliefs, especially in the context of pedagogical-psychological oriented approaches, no precise and selective definition of the concept of beliefs can be discerned so far (Leder, 2019; Törner, 2002). Philipp (2007) defines beliefs as "the lenses through which one looks when interpreting the world" (p. 258). Richardson (1996) proposes a domain-unspecific

definition of beliefs that are based on a broader understanding. She understands beliefs to be "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). This refers to a person's epistemological stands towards an object, which includes affective attitudes and the readiness to act (cf. Grigutsch et al., 1998) and which, in contrast to knowledge, are dependent on the degree of individual agreement (Beswick, 2005, 2007). Still beliefs are seen by many researchers as largely cognitive in nature (Beswick, 2018). So far, however, it has not been sufficiently clarified to what extend beliefs contain cognitive components, and which components can be identified. With regard to long-term development of beliefs, however, it can be assumed according to the current state of research that they are relatively stable with respect to restructuring, and to a certain extent can act as psychological "filters" and/or "barriers" (Reusser et al., 2011). On the other hand, however, beliefs can change in teachers' professional development (Eichler & Erens, 2015; Swars et al., 2009). For mathematics teachers, despite the vagueness of the term, there is a broad consensus on the differentiation of profession-related beliefs (Ernest, 1989). Among others, it is assumed that beliefs can be domain-specific (Eichler & Erens, 2015; Törner, 2002) or even situation-specific (Kuntze, 2011; Schoenfeld, 2010). With respect to epistemological beliefs about the structure of mathematics, according to Grigutsch et al. (1998), the emphasis on the formal aspect of mathematics (formalism aspect) or an orientation towards procedures and calculation schemes (schema orientation) can be brought to the fore with respect to static views. With respect to dynamic views, the application aspect and the processual character of mathematics are mostly emphasized (cf. Grigutsch et al., 1998). In addition, beliefs about the acquisition of mathematical knowledge or the teaching and learning of mathematics (Handal, 2003; Kuntze, 2011; Staub & Stern, 2002) represent another significant dimension of epistemological beliefs. Here, transmission-oriented beliefs, in which students are viewed as passive recipients of knowledge, are often distinguished from constructivist-influenced beliefs that endorse the principles of constructive learning (Staub & Stern, 2002). Although the question of how teacher beliefs influence student achievement is far from conclusive, it is likely that dynamic beliefs about mathematics and constructivist teaching–learning beliefs are more strongly related to an emphasis on processual, iterative mathematics in instructional settings (Reusser et al., 2011).

3.5 Motivation and Self-regulation Skills

Motivational research in psychology counts motivation as a personal trait which refers to the individually varying personal characteristics that constitute the reasons for and the persistence of human behaviour (Kunter, 2013; Pintrich, 2003; Rheinberg, 2006). It serves as an important predictor of how successful people can handle situational demands that occur in teaching. Thus, motivation and self-regulation are vital for teachers to succeed in their profession in the long term (Alexander, 2008; Kunter et al., 2013; Woolfolk Hoy, 2008). The beginnings of research on teacher

motivation in the 1970s were still in the study of why people decide to become teachers (Lortie, 1975). Within presage-process–product research, the motivational orientation would likely be described as a characteristic of beginning teacher candidates (Type F, Chapter 1.1) and connections would be sought between career choice motivation at career entry and teachers' learning outcomes (with the goal of selective admission to the teaching career). Medley (1987) describes this as Type FE research (research in teacher selection, p. 111). However, because affective personality traits have (re)entered the professional competence research, contemporary research on professional teacher competencies examines differences in motivation and self-efficacy between practicing teachers, such as in the form of intrinsic motivation and enthusiasm for the subject of mathematics and for teaching, and further, what influence these forms of enthusiasm have on teaching quality and, if applicable, student achievement (Kunter, 2013). By this, the research goes far beyond Type F and Type E research. The description of the manageable psychological construct of self-efficacy by Bandura (1997) in the late 1990s also contributed significantly to this development. Self-regulatory skills are now also part of many studies of professional teacher competence, as the teaching profession is believed to have implications for teacher health and well-being due to its high demands. In order to meet the demanding challenges over extended periods of time, teachers need to develop self-regulation skills in order to maintain their occupational commitment over time and to preclude unfavorable motivational and emotional outcomes (Kunter et al., 2013).

4   Different Conceptualizations of Teacher Knowledge

As knowledge is considered a major component of teacher competencies, we will focus on recent conceptualizations of mathematics teacher knowledge in the following. Worldwide, many conceptualizations of professional knowledge are based on Shulman's fundamental description, such as in the U.S. the Learning Mathematics for Teaching project by the research group around Deborah Ball (LMT; cf. Hill et al., 2008a, 2008b), the study on Mathematics Knowledge in Teaching (Rowland & Ruthven, 2011) in the U.K., as well as different frameworks in Australia (Beswick & Chick, 2020; Chick et al., 2006). In Germany, the COACTIV study builds on this work (Kunter et al., 2013) but also frameworks developed by other researchers (Buchholtz et al., 2013; Dreher et al., 2016, 2018). International comparative studies such as MT21 (Schmidt et al., 2007, 2011) or the Teacher Education and Development Study in Mathematics (TEDS-M; Blömeke et al., 2014a, 2014b; Tatto, et al., 2012), also built on this work and investigated teachers' professional knowledge at the end of their education with a framework based on Shulman. A more systematic overview of the description of professional knowledge by teachers can be found, for example, in the ICMI study by Even and Ball (2009), in the Handbook by Wood et al. (2008), or in various different publications such as by Cochran-Smith and Zeichner (2005), Rowland (2014), Neubrand (2018) or Manizade and Orrill (2020). In the following,

we describe some of these key frameworks that have been more widely received internationally.

4.1 Mathematical Knowledge for Teaching (MKT)

A model that has been widely acknowledged and applied internationally is the Michigan group's Mathematical Knowledge for Teaching. This approach to describing and measuring teachers' professional knowledge consists of developing a practice-based theory of the mathematical resources entailed by the work of teaching on the basis of the knowledge facets identified by Shulman. To this end, extensive observational categories were derived from mathematics tasks and observation of primary teachers' practical work with students. Thus, rather than normatively specifying Shulman's classification in technical terms, the project took, as its starting point, a requirements analysis that first identified three key responsibilities of teachers. The requirements were "(1) [t]o provide effective opportunities to learn substantial mathematics and treat the mathematics with intellectual integrity (Bruner, 1960); (2) to be able to hear student thinking, take it seriously, and make it an integral part of the instruction; and (3) to be committed to the learning of every student, and further to the learning of the class as an intellectual community" (Ball & Bass, 2009, p. 26). The goal of the project was initially to empirically study instruction to characterize the mathematical knowledge necessary "to carry out the work of teaching mathematics" (Hill et al., 2005, p. 373; Ball & Bass, 2003). In the process, knowledge facets were also specified in more detail (Ball et al., 2005, 2008), resulting in the development of a model of professional knowledge (the MKT model). MKT covers three categories that relate to teachers' subject matter knowledge: (1) common content knowledge (CCK, i.e., mathematical knowledge and skills used in settings other than teaching), which describes knowledge held in common with professionals in other mathematically intensive fields; (2) specialized content knowledge (SCK, i.e., mathematical knowledge and skills that are unique to the teaching of mathematics); and (3) horizon content knowledge (HCK, i.e., an awareness of how distinct mathematical topics are related to each other), which Bass and Ball (2009) described as an "elementary perspective on advanced knowledge that equips teachers with a broader and also more particular vision and orientation for their work" (Bass & Ball, 2009, p. 34). In contrast, there are three categories that can be considered constituent of teachers' PCK: (4) knowledge of content and students (KCS, i.e., knowledge about students' mathematical thinking or typical student errors); (5) knowledge of content and teaching (KCT, i.e., knowledge to introduce a new concept or method); and (6) knowledge of content and curriculum (i.e., knowledge on educational goals, standards, and grade levels where particular topics are typically taught) (Ball et al., 2008). Later on, the project developed measures of MKT (Hill et al., 2004) and used teachers' scores as a predictor of students' mathematics achievement. They found that "teachers' mathematical knowledge was significantly related to student achievement gains in both first and third grades […]" (Hill et al., 2005, p. 371).

Internationally, the model gained much recognition and was transferred or applied in many other countries including Ireland, Norway and Indonesia (Blömeke & Delanay, 2012; Delanay et al., 2008; Fauskanger, 2015; Ng et al., 2012). However, although widely used the model has also been criticized as the empirical differentiation of the dimensions has not been shown sufficiently and it is not clear whether the model can be transferred to the secondary level (Speer et al., 2015). Furthermore, its operationalization for the empirical measurement of teachers' knowledge and the use of multiple-choice operationalization items in a respective instrument have been criticized because this operationalization might underestimate the complexity of some of the knowledge facets (especially those involving students learning and thinking) (Manizade & Mason, 2011).

4.2   The Knowledge Quartet

Tim Rowland and his colleagues in the United Kingdom took a perspective away from the empirical testing of teachers' knowledge that is present in the Michigan project and other projects. They analyzed videotaped data from classroom observations and proposed a framework for describing the knowledge the teacher enacts in the classroom. The aim of their project, which became known as "Knowledge Quartet", was to make visible and describe the professional knowledge and beliefs acquired during training in classroom teaching situations in which this knowledge becomes visible (Rowland, 2008a, 2008b). Their theoretical framework for the observation, analysis and development of mathematics teaching has been developed in the context of primary education, although approaches to transfer to the secondary level exist (Rowland et al., 2011). The approach of the study followed methods similar to grounded theory research. The identified theoretical model consisted of four categories: (1) foundation, which describes the teachers' knowledge base; (2) transformation, which includes situations in which knowledge about chosen representations, examples, analogies, explanations, etc. is revealed—a category that takes up the ideas of PCK; (3) connection, which describes situations in which students' misconceptions are revealed, and the teacher knows about what is 'hard' or 'easy' to grasp for the students; and finally, (4) contingency, which refers to unexpected, unplanned moments, i.e. students' unexpected responses and questions (Rowland et al., 2005). The framework is now used in several countries by collaborating colleagues (including Norway, U.K., the U.S., Ireland, Turkey, Italy, Cyprus and Australia). However, qualitative reconstructive studies with a rather smaller sample size dominate the study of teacher knowledge here (e.g., Maher et al., 2022; Petrou, 2009).

4.3 Modelling Teachers' Knowledge in Relation to Teaching Practice

Researcher groups from Australia, Canada and the U.S. developed frameworks for empirical research on teachers' knowledge which especially account for the blurriness of Shulmans knowledge domains when it comes to teaching practice. The work of the Michigan group was criticized for that "the precise way in which they conceive of knowledge and how aspects of such a conception beyond 'facts that are known' is incorporated in their model is not clear" (Beswick et al., 2012, p.133). Furthermore, teachers "do not always employ the same sort of knowledge in apparently equivalent situations, and they draw upon a range of types of knowledge concerning many of their everyday tasks, moving among them seamlessly and flexibly" (ibid., p.154). In the work of the Australian researchers Beswick and colleagues, therefore, the conception of knowledge also includes teachers' beliefs and confidence as central components in corresponding frameworks, thus also taking into account affective competence characteristics in particular, which were thought to be more intertwined with knowledge facets here than in other frameworks because they have such a major impact on teachers' actions in practice (Beswick & Chick, 2020; Beswick et al., 2012). To investigate the professional knowledge of Tasmanian middle school teachers in mathematics, a profile framework was developed with eight different facets. Specifically, the framework refers to teachers' knowledge and readiness: (1) to nominate how they would improve middle school students' mathematical understandings and how mathematics might be used to enhance students' learning more broadly; (2) to outline a plan for teaching a mathematics concept that they considered important; and (3) to rate their confidence about developing their students' understanding of a range of middle school mathematics topics, and their ability to make connections between mathematics and other curriculum areas. Furthermore (4) to use of mathematics in everyday life; (5) their beliefs on mathematics teaching and learning; (6) and to anticipate appropriate and inappropriate responses that their students might give to mathematics problems and to describe how they could use each of the items in their classroom. The framework furthermore contains teachers' background variables and their perceived professional learning needs (Beswick et al., 2012). The model developed and operationalized for an empirical study thus acts as counter to highly analytic models such as MKT. In order to provide evidence-based insights into how Australian teacher education prepares mathematics teachers for their professional requirements, empirical studies examined the teacher knowledge of primary and secondary mathematics teacher education students in MCK and PCK using Rasch-scaled knowledge tests (Beswick & Goos, 2012; Goos, 2013). In particular, the studies found close empirical relationships between the two knowledge facets. Chick and her colleagues on the other hand developed a framework for analysing primary teachers' PCK for teaching decimals (Chick et al., 2006). Their framework shows especially the blurriness between content knowledge and pedagogical knowledge. It entails three categories with a large number of sub-categories in which pedagogy and content are thought intertwined and set in a mutual context.

Their PCK framework contains the knowledge of teaching strategies, knowledge of students' thinking, knowledge of representations, knowledge of the cognitive demand of tasks, knowledge of explanations, as well as resources and the curriculum. Furthermore, a category "content knowledge in a pedagogical context," covers a profound understanding of fundamental content, knowledge to deconstruct content to its key components, an awareness of mathematical structure and connections, as well as procedural knowledge when for example solving problems or using an algorithm. The third category of the framework is "pedagogical knowledge in a content context." It contains sub-categories of knowledge of the goals of learning, assessment practices, and classroom techniques that are needed for example when students need to work in groups. (Beswick & Chick, 2020).

The Canadian framework "Mathematics-for-Teaching" (Davis & Simmt, 2006) considers the complex structure of professional knowledge dynamically and distinguishes in knowledge acquisition "between the relatively stable aspects of mathematical knowledge itself and the somewhat more volatile qualities" (Davis & Simmt, 2006, p. 297). The model distinguishes relatively stable aspects of knowledge e.g. about curriculum structures or mathematics and dynamic aspects of "knowing", e.g. classroom collectivity or a subjective understanding to attend to both explicit and tacit aspects of teachers' mathematical knowledge. Other researchers describe the professional knowledge of mathematics teachers as situated within a specific mathematical content.

Important in this context are the works of Manizade and Martinovic on professional-situated knowledge in geometry (Manizade & Martinovic, 2016, 2018; Manizade & Mason, 2011) in the U.S. and Canada, respectively, which are characterized by the fact that Shulman's CK and PCK are situated and considered and scrutinized for very specific mathematical topics commonly taught in secondary mathematics, such as the area of trapezoids (see also e.g., rational numbers, Depaepe et al., 2015). The researchers highlight the importance of the development of measures of professionally-situated knowledge. They focus on developing valid and reliable measurements of mathematics teachers' situational manifestation of PCK and CK within specific geometry contexts. In their work, Manizade and Martinovic (2016) describe the following five dimensions of such knowledge, including: (1) geometry knowledge; (2) knowledge of student challenges and conceptions; (3) ability to ask diagnostic questions; (4) knowledge of applicable instructional strategies and tools; and (5) ability to provide geometric extensions with respect to a specific topic in geometry. Martinovic and Manizade (2017, 2018) describe the development of instruments—which they referred to as probes—for assessing teachers' knowledge for teaching geometry. Unlike assessing mathematics teacher competence on a more generic level, they argue the benefits of developing assessment instruments within a well-defined and narrow topic in mathematics, and of combining different measures to ensure the validity of the assessed construct.

4.4 Teachers' Knowledge About the Integration of Technology in the Classroom

With the increase of the integration of technologies and digital tools in the teaching of mathematics, necessary new developments emerged for conceptualizations of teacher knowledge. Based on the premise that technology integration efforts should be creatively designed or structured for particular subject matter ideas in specific classroom contexts, Mishra and Koehler (2006) developed the TPACK framework based on Shulman's description of PCK to describe the teacher knowledge needed when integrating technology in teaching. The TPACK framework was also later revised and adapted (Koehler & Mishra, 2008, 2009). The framework includes seven categories of knowledge: Technological knowledge (TK) includes the technical knowledge of using emerging media, including digital media, such as programs, devices, or hardware. It also includes pedagogical knowledge (PK), content knowledge (CK), and four other categories defined by the intersections of these knowledge categories. These facets embrace the technological content knowledge (TCK), which is the knowledge of how technology and subject knowledge affect each other. From the perspective of mathematics education, this includes knowledge about technical possibilities for representing mathematics, for example, through dynamic geometry programs, pedagogical content knowledge (PCK), and technological pedagogical knowledge (TPK), which is knowledge about how the use of technologies affects general teaching and learning processes. In the intersection of all knowledge areas lies the so-called technological pedagogical content knowledge (TPACK), which describes a combination of subject-specific PCK with knowledge about the use of technology for learning. TPACK also takes into account the relationship between teachers' decisions and the contextual factors of teaching, such as class size, environment, resources, and culture (Koehler & Mishra, 2009). The TPACK framework was specifically designed to enable research on the knowledge teachers need to effectively integrate technology into their teaching in a particular content area. Mathematics educational research has increasingly adopted the rather generic framework in recent years to describe mathematics-specific requirements of each knowledge facet and to explore how these develop, for example, for the area of curriculum development or in terms of describing instructional practices (Niess et al., 2009). Furthermore, the framework has been applied to observe mathematics teachers' practices in using technology in teaching and to describe them at the level of the knowledge facets involved (Muir et al., 2016; Patahuddin et al., 2016).

4.5 COACTIV

Also based on the approaches of the Michigan group and the work of Shulman, a study with representative samples of German secondary school teachers developed in the mid-2000s to investigate teachers' professional knowledge and its empirical relation

to student achievement. The key factor was the facilitation of a national extension of the 2003 PISA sample, in which individual and grade-level aggregated student performance from the PISA study could be extended longitudinally and related to teacher characteristics of about 300 teachers teaching in these grades. The COACTIV research program (Baumert et al., 2010; Kunter et al., 2013) aimed to investigate the professional competencies of practicing mathematics teachers, including making statements about the relationship to student achievement. Standardized achievement tests of teacher professional knowledge were used (Krauss et al., 2008). The framework for teacher knowledge developed by COACTIV is based on content knowledge, but identifies three different facets of subject-specific knowledge: first, knowledge of student conceptions and prior knowledge (e.g., knowledge about typical student errors or the difficulty of mathematical tasks); and secondly, knowledge of subject-specific instructional strategies (for example, knowledge about representations and making content "accessible"). An innovative feature of the COACTIV theoretical framework was that subject-specific knowledge was operationalized in part through knowledge about task quality and the cognitive potential of the tasks used in the classroom. In this context, a corresponding classification of tasks used placed particular emphasis on the content-specific cognitive activation of mathematical tasks (Neubrand et al., 2013). This classification allowed "the recognition, for example, of how conceptual thinking is incorporated in a lesson, how teachers select the tasks, and if that selection influences the learning progress of the students" (Neubrand, 2018, p. 606). The research program investigated the competence of practicing German mathematics teachers differentiated in the areas of content knowledge and pedagogical content knowledge. Among other things, COACTIV found that systematic differences in performance existed between teachers for higher track secondary level in content knowledge, some of which could be attributed to differences in teacher education characteristics. A central finding of the study was also that the content knowledge of teachers was a necessary prerequisite for the acquisition of pedagogical content knowledge, but that ultimately the pedagogical content knowledge of a teacher had a greater explanatory power for predicting student performance than their content knowledge (cf. Kunter et al., 2013)—which did not mean, however, that content knowledge was less important in teacher education.

4.6   TEDS-M

The studies from the TEDS-M research program focus on different aspects of professional competencies, each with a different emphasis. While earlier international comparative studies such as TEDS-M 2008 (Blömeke et al., 2014a, 2014b; Tatto et al., 2012) or its predecessor study MT21 (Schmidt et al., 2007, 2011) focused mainly on knowledge-related (dispositional) aspects and knowledge at the end of teacher education, subsequent studies of the TEDS-M research program in Germany included in addition situational aspects of professional competence and thus also focus to a greater extent on the competencies of practicing teachers. Particular attention in the

following first is given to the results of the TEDS-M 2008 study, which was commissioned by the International Association for the Evaluation of Educational Achievement (IEA) and examined the teacher professional knowledge of prospective primary and secondary mathematics teachers in 16 participating countries. With regard to the underlying framework, the TEDS-M 2008 study and its predecessor study MT21 refer to the different knowledge facets of Shulman (1986, 1987) and differentiate PCK two-dimensionally, namely along with different requirements for teachers (Döhrmann et al., 2012). Within the theoretical framework between teaching-related demands like "Mathematics Curricular Knowledge" and "Knowledge of Planning for Mathematics Teaching," as well as learning process-related demands like "Enacting Mathematics for Teaching and Learning" are distinguished (Tatto et al., 2012, p. 131). Curricular and instructional planning requirements include the selection of subject-specific teaching content for students, as well as its justification, simplification, and preparation using various representations. This therefore includes knowledge of mathematics curricula, assessment methods, and teaching methods. Interaction-related requirements, which reflect the teacher's activities during the lesson, intend to include the classification of student answers against the background of cognitive levels, possible errors, and error patterns. These are therefore analytical and diagnostic skills that prospective teachers should possess. An overview of international research findings is provided by Tatto et al. (2012). Furthermore, Blömeke and Delanay (2012) describe the current state of research from TEDS-M 2008 in a review article from the perspective of similarities and differences between TEDS-M 2008 and the Learning Mathematics for Teaching study (LMT; Hill et al., 2008a, 2008b). Meanwhile, several complementary and in-depth national analyses have emerged from TEDS-M 2008 and MT21, looking in detail at specific issues in participating countries (2014a, 2014b; Blömeke et al., 2009a, 2009b). Furthermore, within the TEDS-M research program TEDS-LT followed as a new study, expanding the concepts of TEDS-M 2008 for a German sample to both a longitudinal design and more subjects, as German and English were included besides mathematics (Blömeke et al., 2011, 2013).

5 Recent Extensions in the Concept of Mathematics Teacher Competence

Despite the blurry lines between CK and PCK, like Kaiser and König (2019) note, several studies to date have provided evidence that the knowledge facets as proposed by Shulman (1987) can be theoretically and empirically differentiated and separated (e.g., Blömeke et al., 2016; Krauss et al., 2008), provided that appropriate instruments, topics and sampling are used. Fundamental to this were scientific studies that examined the structure of professional knowledge in particular. Regarding the correlations between the specific facets, it turned out that, "as Shulman (1987) with his "amalgam" hypothesis on the nature of PCK suggested PCK is related to both CK and GPK, whereas CK and GPK are more distant to each other" (Kaiser & König,

2019. p. 603f.). For example, in the COACTIV study, a strong correlation between CK and PCK was found (0.61) (Baumert et al., 2010). Important scientific developments about the professional competence of teachers can be located especially in the last five to ten years. Since teachers access different forms of their professional knowledge in different instructional contexts—so the assumption—it seems reasonable to focus not only on the structure but especially on its application in different teaching situations when examining professional knowledge (Even et al., 2017; Kaiser et al., 2015; Rowland, 2008b). Thus, as a new guiding question in research on mathematics teacher competencies, knowledge and skills, if we follow up on Medley's Type E, it was added how content knowledge, pedagogical content knowledge, and general pedagogical knowledge can be surveyed in connection with teaching practice using suitable instruments, which led in particular to the investigation of situation -specific skills, in other studies referred to as professional noticing (Sherin et al., 2011; Van Es & Sherin, 2008).

5.1 Situational Aspects of Mathematics Teachers' Professional Competencies

When situational aspects of teachers' professional competencies are addressed in the context of empirical studies, the main aim is to survey competencies as closely as possible to real situations from everyday teaching. With their conceptualization of competence as a continuum, Blömeke et al. (2015) aimed to overcome an opposition that had increasingly emerged between different approaches to understanding competence. On one hand, there existed the analytical approach of dispositional aspects of competence, which formed essentially the basis of cognitively oriented empirical studies from educational research mainly using paper-and-pencil tests. According to this approach, one starts from analytically separable areas of competence (e.g., the knowledge facets) which can then be measured and considered in terms of their structural relationships. The goal here is to promote specific competencies as a resource for behavior in specific situations. As we described, competence here includes both cognitive and affective-motivational domains. The analytical approach was now opposed by a holistic approach in the research tradition from organizational psychology, which focused on the observation of behavior and performance in an appropriate real-life context. Competence then influences this behavior, whereby competence is still understood as a collection of diverse cognitive and affective-motivational components that constantly change—depending on the situation and requirements. The idea of Blömeke and her colleagues was to combine both approaches in a common continuous model. Specifically, they assume that the behavior of, for example, a teacher in concrete situations is influenced by his or her competence (in the sense of the holistic approach). However, competence is then not understood as a constantly changing collection of different components, but as a

fixed sum of clearly describable individual components (in the sense of the analytical approach).

The starting point of the new model of competence as a continuum is the disposition of a teacher, which is characterized by cognitive (CK, PCK, GPK) and affective-motivational areas (a.o. beliefs). These cognitive and affective-motivational dispositions are complemented by situation-specific skills, which are also referred to as professional noticing (here, the teachers' noticing discussion plays a role, in particular, see Sherin et al., 2011). That is, in a specific situation, a teacher first perceives the situation, interprets what is perceived, and makes appropriate decisions. The teacher does this influenced by the situation at hand, but of course also by their basic disposition. Based on the teacher's perception, interpretation, and decision, their actual actions in the situation then emerge. It is therefore said that professional noticing consisting of the areas of perception, interpretation, and decision-making plays a mediating or transforming role between disposition and actual action which is an observable performance. While pure surveys with tests represent a proven possibility for the investigation of competence in the sense of the analytical approach (for example with instruments of MKT or TEDS-M), it is immediately clear that situational aspects are difficult to assess in this way, because the reality of teaching can only be represented in test items to a limited extent. An alternative way of assessing competence in a situation-related manner is the use of video vignettes or dynamic geometry software as a stimulus for answering test items. Subsequently, many recent studies investigating situational teacher competence built on the use of video vignettes (e.g., Bruckmaier et al., 2016; Kaiser & König, 2019; Kersting, 2008; Kersting et al., 2010; Knievel et al., 2015; Seidel & Stürmer, 2014). Martinovic and Manizade (2020) for instance used interactive dynamic instruments (that incorporate dynamic software such as GeoGebra) to mimic the classroom simulations and a variety of student responses to a given mathematics problem question. This way, they evaluated teachers' professionally situated knowledge (PCK and CK) based on teachers' responses to the questions that follow up a dynamic simulation.

5.2 Further Developments of the Studies of the TEDS-M Research Program

The further developments of the TEDS-M research program in Germany, which aim at investigating the competence development of mathematics teachers in the first years of their professional activity, are also based on this approach. Central to this is the outlined understanding of competence as a continuum. In addition, expertise research (Berliner, 2001) with its basic distinction between experts and novices forms a central pillar of the conceptual framework for further developments. Specifically, the different areas of teacher knowledge from TEDS-M were conceptually supplemented by the situation-specific skills of professional noticing, which were surveyed with video vignettes aimed at eliciting different aspects of expertise. The TEDS-M

Follow up study (TEDS-FU) for example measured perception, interpretation, and decision-making as facets of professional noticing of in-service mathematics teachers (Kaiser et al., 2015); The relation of knowledge and noticing concerning GPK was evaluated by König et al. (2014), differentiating mathematics teachers' pedagogical competence into knowledge and noticing facets. Kaiser and König (2019, p. 605) also report structural connections in this context, with a connection between dispositional and situational facets of professional competence being particularly evident in interpreting classroom perception: "Whereas teacher knowledge and interpretation skills are moderately related to each other (0.37), perception is only loosely related to interpretation (0.17) and knowledge (0.13)."

5.3 Relationships of Teacher Competencies to Instructional Quality and Student Achievement

The results presented so far give us clues about the relationship between teachers' knowledge and their skills. What needs to be questioned, however, is why appropriate skills were considered valuable components of teacher competence in the first place. One obvious answer is that skills in the area of professional noticing help with the design of instruction and are linked to this assumption that ultimately student achievement can also be improved by good instruction. Specifically, some studies in recent years have surveyed the direct relationship between teacher skills and instructional quality (Hill et al., 2008a, 2008b; Santagata & Lee, 2021). In the TEDS-Instruct study and the TEDS-Validate study, for example, two observers each assessed lower secondary mathematics teaching on different criteria using a comprehensive rating manual that focused on four facets of teaching quality, namely efficient classroom management, constructive support, the potential for cognitive activation, and content-related structuring (for details Schlesinger et al., 2018). At the same time, results from the subject-related competence facets were available for the participating teachers, which were collected using TEDS-M and TEDS-FU instruments (Blömeke et al., 2020). Thereby, efficient classroom management did not correlate significantly with the subject-related competence facets. The remaining three quality dimensions correlate significantly positively with teachers' professional noticing of mathematics teaching, but not consistently with subject-related knowledge facets (Jentsch et al., 2021). As TEDS-Validate and TEDS-Instruct furthermore had access to the results of students' achievement tests, the studies especially offer the opportunity to fully observe the linkage between teachers' competences, instructional quality, and students' achievements. Results revealed that with regard to the dimensions of instructional quality cognitive activation was found as a predictor for students' progress in achievement. In addition, general pedagogical knowledge and situation-specific classroom management expertise (CME) serve as predictors for instructional quality (GPK for all three dimensions, CME only for cognitive activation). Furthermore, there is a direct effect of teachers' professional competence

on students' achievement but without mediation by the instructional quality (König et al., 2021). Also, other studies investigated the relationships between professional competence, teaching quality, and student achievement (cf. Kaiser & König, 2019, p. 606). In the COACTIV study, a strong positive effect of PCK on student learning progress was found to be mediated by the quality of instruction. In particular, the dimensions of cognitive activation and individual learning support played a crucial role. For CK, however, the mediation model applied only to a very limited extent. Despite the high correlation with PCK, teachers' CK had lower predictive power for students' learning progress (Baumert et al., 2010). Similarly, Hill et al. (2005) and Hill and Chin (2018) furthermore showed that teachers' knowledge and their instructional quality were significantly related to students' outcomes.

6 Concluding Remarks

In the present chapter, we provided an overview of important lines of development and the evolution of mathematics education research on professional teacher competencies, knowledge and skills. Research has evolved from the process–product paradigm and has been developed especially in the period of 30 years after Medley's (1987) reflections. The starting point in this process were basic theoretical reflections on teachers' professional knowledge, which were strongly influenced by cognitivism. Subsequently, an independent branch of research in mathematics education developed, which dealt with the professional competence of teachers, thus broadening the focus by not only taking single cognitive aspects into account. As in Medley's time, the starting point to this shift in the research was the intention to measure and describe what makes a good teacher and how to improve student achievement in mathematics. From the critique of the studies in the following years, research evolved further towards the inclusion of more situation-specific teacher competencies, examining connections and effects between the different variables within the chain of effects, namely teachers' competence, instructional quality, and student achievements.

What have these developments in common? The developments represent decisive improvements with regard to the systematic inclusion of personality characteristics of teachers as well as the contextual conditions in which teacher competencies come into play. It is clear that different conceptualizations of teacher competencies still take into account, to varying degrees, the same variables that Medley (1987) had already considered, although in the meantime a stronger emphasis on the subject-specific characteristics of mathematics has also been taken into account.

However, new conceptual challenges arose as a result of further developments. Thus, after many years, as we describe, currently a large variety of frameworks on teacher competencies, knowledge and skills is available internationally, each describing teacher competence differently and thus setting different emphases. Conceptualizations are based on different domains attributed to teachers' professional knowledge, differ in their theoretical assumptions, and also have different

grain-size of the knowledge elements considered (Even et al., 2017; Neubrand, 2018). On the one hand, the boundaries of what is understood by teacher competencies in certain domains are pragmatically determined from theoretical considerations or in the context of empirical studies, but on the other hand, Delphi methods, or the Grounded Theory Approach, for example, could also be used to develop content-valid conceptualizations (Manizade & Mason, 2011; Martinovic & Manizade, 2017). Either way, however, the conceptualizations of teacher competencies, knowledge, and skills for research purposes remain normative—and thus dependent on cultural traditions, epistemologies, and values. We expect the field to evolve further with great progress in the next years.

While we often assume that mathematics education is culture-neutral, research indicates that the way in which we express ourselves and view mathematics is in fact highly cultural (Leung et al., 2006). Although many of these different frameworks are used in several countries to assess teacher competencies, the cultural dependency of the frameworks should not be overlooked (Blömeke & Delanay, 2012), so that a transfer to other educational systems is by no means trivial and should require validation studies (e.g. Yang et al., 2018). In the future, therefore, it can be assumed that the cultural sensitivity of research on teacher competencies will be more critically scrutinized. International research on teacher competencies can nevertheless benefit from this polyphony, although it suffers from it at the same time. The multiplicity and diversity of frameworks need not be seen as confusion but can be seen as richness—if one takes a comparative perspective, however, it seems profitable when frameworks and conceptualizations are synthesized and compared based on their similarities and differences.

What is clear from our overview, however, is that after more than three decades of developing research on teacher competencies, knowledge, and skills, there are still methodological challenges to empirical measurement. Certainly, current tools of measuring allow us to capture teacher competencies more accurately than in the past. Methodological advances such as multilevel structural equation modelling (Teo & Khine, 2009) allow for the consideration of numerous relevant (background) variables and differences between individuals, classes, and schools when examining relationships between teacher competencies and student outcomes. These analyses can be used to identify interactions between teachers' characteristics, personal and affective traits, and various other factors, all those that are related to teacher competencies. Nevertheless, even today we do not have the means to realize Medley's vision of taking into account the interrelationships of all variables in studies, and often only proxies can be used for variables to be measured so that even in the future the validity of measurement instruments, in particular, will have to be critically analyzed. However, these methodological advances should still not lead research on teacher competencies to neglect mediating variables in the chain of connections between teacher competencies and student outcomes. Teaching activities in instructional quality as a mediating variable, and thus the situatedness of teacher competencies has played and probably will play an increasingly central role as a site for observing and measuring competencies, especially in recent years.

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The Research on Mathematics Teaching and Planning: Theoretical Perspectives and Implications of Teachers' Pre-post Classroom Activities

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1 Introduction

In this chapter, we focus on research in mathematics teaching such as mathematics teacher planning, assessment, and other teacher-related activities when students are not present ("pre- and post-active"; Type D). These are the types of activities that mathematics teachers do to promote student learning while no students are present; in other words, this chapter focuses on the invisible part of teaching mathematics. These activities are important means by which the teacher exercises control over their teaching and are also the main way that a teacher's professional knowledge, competencies, skills, and beliefs (Type E; see Fig. 3 in Chap. 1, this volume) impact the process of teaching mathematics. Type E is necessary but not sufficient for producing quality student–teacher interactions in the mathematics classroom (e.g., Sullivan et al., 2009). Enactment of the teaching practice interactively with students is a direct result of teachers' pre- and post-active (Type D) actions. Type D, therefore, determines how well the teacher performs the main interactive function of teaching mathematics (Type C) and how successfully the teacher accomplishes the purpose of

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teaching. Our perspective is premised on the belief that theory drives teaching practice, and, in this chapter, we demonstrate how that connection functions in relation to teacher's pre- and post-active actions (Type D).

The chapter includes a discussion of how teachers relate the problems they encounter in the practice of teaching mathematics to their professional knowledge, competencies, skills, and beliefs in deploying their available resources and their own abilities. Teachers' subjective decision-making is based on the Type E elements that they possess, and the configuration of a teacher's Type E elements "stabilizes [the teacher's] world" (Žižek, 2012, p. 367, as cited in Brown, 2016, p. 86) through their subjective, decision-making process. This decision-making process could be based, for instance, on the most recent professional development presentation that they attended and ideas that they bought into, or on a conference presentation that their principal attended and practices subsequently imposed on the teachers. In either case, and even without having attended a recent conference, teachers' choices and objectified beliefs are a product of the constraints within which they are working (Ingram & Clay, 2000). Thus, their decision-making process produces observable pre-post classroom actions that are the focus of this chapter. We also address the ways in which cultural and digital contexts affect Type D. Additionally, we discuss the theoretical and methodological challenges associated with conceptualization of the Type D domain, instrumentation, and research design.

1.1 Statement of the Problem

It is important to study Type D because teacher planning, which includes introducing key ideas, selecting associated tasks, and creating assessments to measure student understanding, has a great effect on the learning opportunities for students (Akyuz et al., 2013; Sullivan et al., 2009). It provides for targeted understanding of the lesson content, managing classroom transitions, and allows for a focus on classroom processes (Clark & Yinger, 1987). This applies not only to daily lesson plans, but also to unit plans that cover a range of related topics (Roche et al., 2014). McAlpine et al. (2006) specifically called for more research on the ways in which teachers think and the connection between teachers' thinking and its influence on their teaching actions. This connection is hypothesized to be particularly useful in studying the relationship between teachers' "theories-in-use" and teachers' thinking (Kane et al., 2002; McAlpine et al., 2006). Sullivan et al. (2009) called for more professional development to improve teachers' abilities to take a mathematical task and convert it into a "meaningful learning experience" (p. 85), noting that Type E was necessary but not sufficient for this conversion. Thus, regardless of a teacher's perspective on teaching mathematics as described by the theoretical framework that we present in this chapter (see Fig. 1), research on improving Type D is important because it contains the potential for improving the quality of mathematics instruction (Lewis et al., 2013).
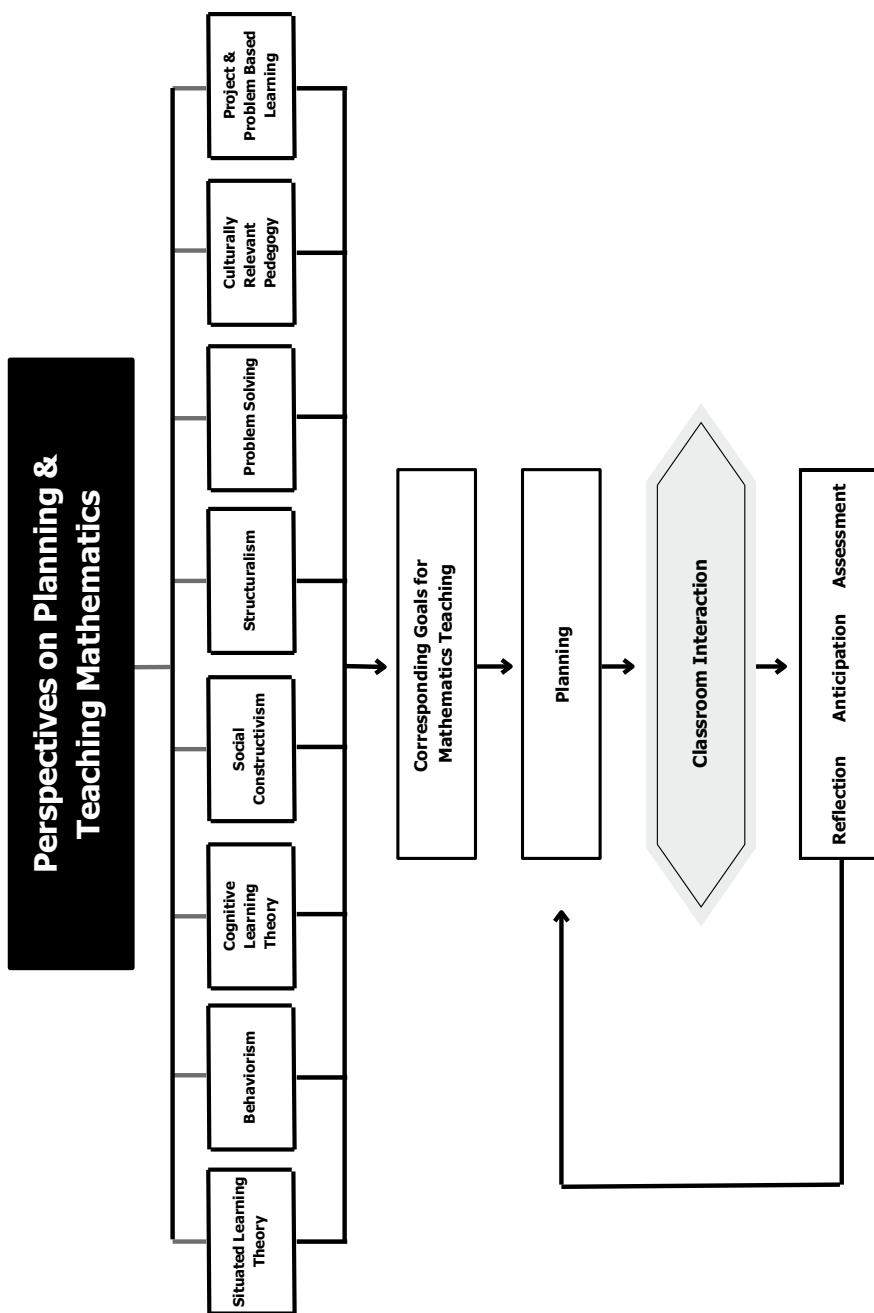
Fig. 1 Epistemological perspectives on teachers' Type D

Depending on the importance the teacher assigns to planning, the outcomes of planning manifest in different ways through the observable and enacted lesson. Zazkis et al. (2009) compared Japanese teachers' planning—which was focused on the process of student learning and discovery of concepts—and American teachers' planning—which was focused on specific content outcomes. They found that the ways in which teachers talked about the professional act of planning varied greatly between the two groups and differences aligned with the observable outcome of the written lesson plan and its enactment. If teachers think of planning as a high-level professional task that is an important part of the act of teaching, then the produced written plan and its enactment lead to different types of teaching than is the case for a teacher who does not think of planning as having a central role in their practice (Zazkis et al., 2009). Designing lesson plans that incorporate teachers' goals and are focused on "students' anticipated learning" (Akyuz et al., 2013, p. 94) has been the focus of key reform-based documents, including *Adding It Up* (Kilpatrick et al., 2001). Further, Hiebert et al. (2003) emphasized the importance of developing teachers' Type D: "[T]eachers need to design lessons with clear goals in mind, monitor their implementation, collect feedback, and interpret the feedback in order to revise and improve future practice" (p. 206).

Regardless of the knowledge, competencies, skills, and beliefs that teachers develop in their teacher preparation programs, they often go back to the way they were taught when faced with the challenges of the everyday classroom: "People learn to teach, in part, by growing up in a culture—by serving as passive apprentices for 12 years or more when they themselves were students. When they face the real challenges of the classroom, they often abandon new practices and revert to the teaching methods their teachers used" (Hiebert et al., 2003, p. 201). It becomes extremely important therefore to develop teachers' abilities to plan quality mathematics lessons with specific goals in mind, and to use student data to make decisions about subsequent planning and instruction. As McAlpine et al. (2006) suggested, we need to develop a "language" (p. 129) for talking about teachers' Type D activity to fully realize it as a domain of mathematics education research. To develop this necessary "language" for Type D research, we propose the conceptual framework in Fig. 1 for discussing literature related to Type D. We subscribe to Akyuz et al.'s (2013) definition of Type D, that cyclically relates preparation (pre-active) to reflection, anticipation, assessment, and revision (post-active). In their model relating these variables, reflection, anticipation, and assessment interrelate laterally with each other, all of which then inform revision. Revision, then, cycles back to preparation as the pre-active variable.

The chapter can be thought of as broken into three main themes. In the first theme, we situate our current work: we discuss the connection between Type D and Type E (knowledge, competencies, skills, and beliefs), followed by a discussion of the goals of Type D broadly. The second theme constitutes the bulk of the chapter: we discuss each of the epistemological perspectives in Fig. 1, including a definition, goals of teaching, and examples from the literature. The third theme provides commentary on the first two themes: we discuss pros and cons of each perspective, followed by a commentary on the relationship of each perspective to cultural contexts, and we

conclude with a brief discussion of the implications of each perspective on the task of lesson planning. We close by noting implications for future directions of research as a result of the intervention we offer in the present chapter.

2 The Connection of Type D to Knowledge, Competencies, Skills, and Beliefs

Teachers' knowledge, competencies, skills, and beliefs (Type E) are connected to the way they plan their mathematics instruction and influences their decision-making, evidenced in the ways they implement their lessons. In some cases, this connection is conscious, and in other cases, it is unconscious. There are various factors that can affect teachers' lesson planning and instruction, including their beliefs about the nature of mathematics, such as whether they hold an *instrumental*, *Platonist*, or *problem-solving* view (Beswick, 2005; Ernest, 1989); learning theories; and the pedagogical practices and approaches in which they have been trained. Most importantly, teachers can use combinations of these factors to produce and implement a lesson, and to assess students.

Kilpatrick et al. (2001), for example, discussed the connection between Types C, D, E, and F—what they call *teaching for mathematical proficiency*—with the following components:

(1) conceptual understanding of the core knowledge required in the practice of teaching;
(2) fluency in carrying out basic instructional routines;
(3) strategic competence in planning effective instruction and solving problems that arise during instruction;
(4) adaptive reasoning in justifying and explaining one's instructional practices and in reflecting on those practices so as to improve them; and
(5) productive disposition towards mathematics, teaching, learning, and the improvement of practice.
(Kilpatrick et al., 2001, p. 380)

We align components (1) and (2) with Type E; component (3) with Types D and C; component (4) with Type D; and component (5) with Type F.

While theories may influence teachers consciously and unconsciously, typically once they are with their students, they operate (and make pedagogical—and specifically, planning—*decisions and choices*) in a way that strives for synchronicity and harmony between their previous experiences of successes in teaching; their knowledge, competencies, and skills; their unique mixes of students and content for each of their course preparations; and the institutional constraints within which they are working (cf. Ingram & Clay, 2000). While a theory of learning may influence these decisions and choices, it wouldn't necessarily have to "inform" them *per se*. For example, if a teacher has attended a "project-based learning" workshop, they may be motivated to try some of the techniques or lessons they were exposed to, but they are

likely to do so as an adaptation of their existing lesson-planning practices. Regardless of the reasons behind specific instances of decision-making that are evident in their lesson planning, assessment, or instruction, we focus on the teachers' objectified beliefs—that is, observable behaviors—that ostensibly have been chosen because they stabilize the teacher's world. The focus of this chapter is not, then, on the "why" of the decision-making processes, but rather on "how" it manifests in the observable research components of Type D.

In science education, researchers (e.g., Carlson et al., 2019) have reconceptualized teachers' knowledge to include *Enacted Pedagogical Content Knowledge* (ePCK) and *Personal Pedagogical Content Knowledge* (pPCK), both of which include knowledge associated with Type D, namely, planning and reflection. These researchers defined ePCK as knowledge for planning and reflection that is situated within the school, classroom, and individual students' interactional contexts (cf. Ingram & Clay, 2000) with the teacher and the teacher's subject matter knowledge and discipline-related skills. Personal PCK includes the PCK influences that have occurred over the teacher's life, experiences, and interactions with other professionals (e.g., fellow teachers, researchers, coursework, professional development, reading journal articles) that have accumulated to shape and inform their ePCK. In other words, pPCK builds over time and experience to increase the sophistication with which they deploy their knowledge in thinking about, planning, and reflecting on their lessons. Additionally, ePCK is a subset of pPCK, meaning that the enacted—viz. observable—knowledge of a teacher is contained within their set of personal PCK, indicating that observable Type D can be conceived as objectification of personal knowledge and epistemological commitments. Thus, utilizing epistemological frameworks facilitates insight when studying teachers' Type D. When conceptualized in this way, the teacher's pPCK is a privately held knowledge that is unique to the teacher, whereas ePCK is the mode of the teacher's knowledge with which the students most directly interact. The connection between ePCK and pPCK reveals that teachers' epistemological commitments, knowledge about how students learn mathematics, and corresponding knowledge about how best to teach mathematics interrelate both at a micro-level (e.g., planning a particular lesson) and a macro-level (e.g., shaped over their lives, experiences, and professional formation). It is pertinent, therefore, for mathematics education researchers to consider how the objectifications of this knowledge—as Type D observable behaviors in the form of decisions and choices—shape their teaching. As we explore in this chapter, some epistemological commitments have been explored apropos of Type D whilst others warrant further investigation by researchers.

3 Goals of Pre- and Post-active Teacher Activities

The purpose of this chapter is to focus on the decisions and choices teachers make while students are not present. Ultimately, these decisions and choices—regardless of any espoused epistemology or pedagogy—directly impact their lesson plans,

which become the basis for what they subsequently enact in the classroom. Tricoglus (2007) explored planning and collaboration amongst mathematics teachers, revealing that teachers' development of tasks and lesson plans involves cyclical thinking as they become more knowledgeable about them. The study highlighted three types of thinking that teachers engaged in when planning: deliberative thinking, which is the considered thought that generates ideas and future plans; interpretive thinking, which is the part in the process where decisions are made and problems are managed; and metacognition, which is evaluative, reflective thinking. These three types of thinking are the basis for teachers' decisions and choices about the tasks and lesson plans they create, and are informed by their perspectives on teaching and learning mathematics. As teachers think about their beliefs and knowledge in this cyclical process, they formulate goals for their teaching as well as actions they intend to take to reach them (Aguirre & Speer, 2000; Akyuz et al., 2013; Schoenfeld, 1998).

In Fig. 1, we provide a framework emergent from the literature presenting eight categories that researchers have used when describing lesson planning and mathematics instruction. These eight categories are not strictly types of learning theories but rather the perspectives that researchers have characterized teachers as *appearing to be enacting*, through their decisions and choice-making, to represent observable and objectified beliefs, knowledge, competencies, and skills (Type E). The aim of Type D is to articulate "the learning goals for the lesson, and the hypotheses that link planned instructional activities with expected learning outcomes" (Hiebert et al., 2003, pp. 207–208). Importantly, the focus of the teaching can be either on the goal of the lesson, or on the activities included in the lesson, regardless of the perspective the teacher adopts. When the focus is on the goal of the lesson, the teacher is likely to consider common student challenges with respect to the mathematical concepts taught, typical questions or difficulties students might experience based on their developmental levels, as well as ways to address those challenges and difficulties (West & Staub, 2003). However, if the focus is on the activity itself, then it is less likely that teachers will think of students' conceptual development of the mathematics. Instead, the focus lands on the "how to": lessons can become more prescribed and rigid, which potentially allows for missed opportunities when teachable moments arise (e.g., Akyuz et al., 2013).

Regardless of teachers' theoretical frameworks or beliefs about teaching mathematics or the nature of mathematics, all teachers operate within institutional, economic, cultural, familial, and logistical constraints (Ingram & Clay, 2000). Thus, the goals adopted by teachers for their teaching can be diverse, encompassing achieving high standardized test scores, developing interest in mathematics in their students, preparing students for futures of entering the workforce or college, meeting curricular demands of the administration, communicating ideas about mathematics, enabling students to solve a given problem multiple ways or to see the connection between different mathematical ideas, and describing patterns students might see in the world using mathematics. These are different performance or learning goals, and while not mutually exclusive, they cannot all be goals for a single lesson. The teacher will consciously and unconsciously use goals to formulate the actions they take when planning a lesson or assessing student work. For example, if a teacher's

goal is for the students to be able to talk to each other about the mathematics, then the lesson planning decisions will be very different from what they would be if the goal was for students to be able to replicate the process presented by the teacher in instruction.

Despite the many challenges, the teacher must know how to create and carry out their plan and, most importantly, how to be flexible with their plan in changing classroom circumstances (Akyuz et al., 2013). A teacher will be faced with goals from other sources such as department chairs, parents, principals, school boards, state departments of education, co-workers, and so forth. Each of these sources has an interest in what the teacher is doing in the classroom. The goals produced by these parties and subsequently imposed on the teacher, whether directly or indirectly, exist in addition to the teacher's own goals. The best-case scenario is that the teacher's goals and the goals of these other sources are in harmony. However, if that is not the case, and the teacher has the job of synthesizing disparate goals from various sources, the result will be that the teacher's goals—as objectified beliefs, observed through their lesson planning process—become a product of stabilizing the reality of their world and being.

4 Theoretical Perspectives for Teaching Mathematics that Are Present in Type D

In this section, we discuss and define each of the perspectives in Fig. 1, the goals for teaching associated with each, and provide some examples from the literature. The literature surveyed was selected for the representativeness each article provided for illustrating each of the epistemological perspectives in Fig. 1 apropos of Type D. An EBSCOhost search of the electronic library system of a major research university in the United States with keywords related to Type D was performed. We operated with three inclusion criteria: (1) Western context (US, Europe, Australia, New Zealand, etc.) for either the author or study setting; (2) articles written between 2000 and the present; and (3) articles must be about mathematics education specifically. Literature by a Western author performing a study in a non-Western context was also excluded. We then manually accessed each article to screen to relevance. Articles in which the primary focus was not on Type D were excluded. On reviewing the included literature, we noticed the emergence of eight epistemological themes with respect to the way the articles were situating and discussing Type D. We organized the literature into these eight themes, as seen in Fig. 1.

In each of the examples discussed, some researchers explicitly cited a theoretical perspective from Fig. 1, and in others, our characterization of a study as having employed a particular epistemological perspective is based on our interpretation of the researchers' work. The examples given are in no way exhaustive but rather serve to demonstrate each particular perspective in recent literature. While we acknowledge that cultural context is of great importance in understanding Type D, we only

considered research from the Western context in the survey of literature that follows. A more complete discussion of the cultural context of the eight epistemological perspectives is included later in the chapter.

4.1 Situated Learning Theory as Perspective

4.1.1 Definition

Situated learning theory (SLT), originally developed in the work of Lave and Wenger (Lave & Wenger, 1991; Wenger, 1998), represented a departure from other learning theories in existence in the 1990s by situating knowledge in the community rather than in individuals. The tonality shift here from individuals to communities is important to distinguish situated learning theory from other sociocultural theories of learning: epistemology itself is exclusively located in the community, entirely "decentered" (Lave & Wenger, 1991, p. 86) from the individual, and exists as a historical function of time including the future. This perspective differs from social constructivism because situated learning theory does not posit any individual possession of knowledge. Instead, knowledge is experienced and participated in rather than possessing deterministic ontology, because at any point in time it is not possible to capture the community's learning as such. An individual person's role in this epistemology exists in their emergent formation of an identity as a member of that community.

A useful metaphor for conceptualizing SLT is apprenticeship (Lave & Wenger, 1991). The apprentice first must initiate a connection to one of the community members, an act that expresses the aspiring apprentice's desire to participate in the community. The apprentice then trains under the tutelage of one of the experts in the community. The community discourse proceeds through interactions between *oldtimers* and *newcomers* (Lave & Wenger, 1991). Oldtimers are established members of the community who have participated in the community and its activities for a long time. Newcomers are novices and just beginning their apprenticeship journey into the community. Through this process, the apprentice—over time—becomes an expert and an oldtimer in their own right, thus bringing in new apprentices and repeating the cycle. If a student—a newcomer—desires to become a member of a community, they must initiate an apprenticeship with an expert, and spend the substantial amount of time required performing the labor and activities of the community to become an expert theirself. If one were to ask either the oldtimer or the newcomer *when*—exactly—they had learned to become a central member of the community or an expert, their answer would involve their entire career, perhaps with one or two critical moments when their status as an expert was validated by other member(s) of the community. Traditionally, university mathematics departments function in this way, where students are immersed in the practices of the mathematics community under the apprenticeship of their professors (experts). However, if a mathematics teacher does not think like a mathematician, they are unlikely to train their newcomers to think as mathematicians. In this case, instead of raising up

newcomers as experts (mathematics doers as people who are able to perform "doing mathematics" tasks; see Stein et al., 2000, p. 16), they will raise people who can act mathematically by producing a memorized sequence of procedures, a devolvement that more closely resembles behaviorism.

4.1.2   Goals of Teaching

For the teacher taking the perspective of SLT, the goal of their teaching is to build and operate a model of apprenticeship and legitimate peripheral participation (LPP)—the novice participation of newcomers who are not yet central to the community nor close to its experts—in the classroom. "[T]he important point concerning learning [in SLT] is one of access to practice as resource for learning, rather than to instruction. Issues of motivation, identity, and language deserve further discussion" (Lave & Wenger, 1991, p. 80). The teacher acts as the expert—the oldtimer—who is responsible for enabling the students—the newcomers—to engage in LPP into the community of mathematics. Of course, the teacher cannot expect that all their students will become professional mathematicians in the future, but that is not the purpose of the students' LPP. Rather, the students' LPP in the community of mathematics represents their experience of mathematics "learning"—that is, the experience of the peripheral participation is homologous to the content being learned. In other words, the content of knowledge-learning is conceived as the process of becoming a certain type of person in the mathematics classroom. There is a critical bifurcation that must be at the forefront of the teacher's mind, and that will determine whether the students' LPP is indeed located within the landscape of the community of mathematics, or if it is located within the (nominal) community of public-school actors, wherein the focus is more closely aligned with following a perfect sequence of steps as with the behaviorist approach. If the former has been achieved in the teacher's mindset, then the teacher is prepared to enact teaching from the SLT perspective. In practice, this may involve modeling for students the correct mathematical language (Morgan et al., 2014; cf. Pimm, 2014) and discourse patterns, how to solve problems and practice communicating their thinking (Sfard, 2008), and the rigor needed to perform mathematical labor in the normative cadence and standards of the community (Herbel-Eisenmann et al., 2015).

The SLT perspective is different from the behaviorist perspective in that the latter is concerned with student outcomes on assessments. In the latter, knowledge outcomes are outcomes in and for themselves, rather than being related to a process of becoming a certain type of person who has a sense of belonging in the mathematics community. In SLT, the product of LPP is that students develop a mathematical identity that affords them membership in the mathematics community—even if at a very surface level—because they have been trained to act and think like mathematicians, so in situations where mathematics is at the fore, their mathematical identity acts as a membership card for that situation. When using technology in the mathematics classroom from the SLT approach, teachers use educational technology to propose and provide opportunities to solve complex problems that students would not be able

to solve without technology, instead of using technology as a black-box where mathematical reasoning is not required in order for learners and teachers to produce a correct answer (Gueudet & Pepin, this volume; Goos et al., 2000; Leung & Bolite-Frank, 2015; Straesser, 2002).

4.1.3 Examples from the Literature

Sfard (2008) has contributed greatly to the development of SLT as a perspective in mathematics education. One of the key notions developed by Sfard was that of commognition, which is a neologism of "communication" and "cognition." The notion of commognition emphasizes that "cognitive processes and interpersonal communication are … different manifestations of basically the same phenomenon" (Sfard, 2008, p. 83), thus showing that the psychological aspect of other perspectives in Fig. 1 are homologous to the acts of LPP in the SLT perspective. Her theoretical work has led other scholars such as Krummheuer (2011) to develop empirical representations of what it looks like to see LPP occurring in a mathematics classroom between teachers and students. Pertinent for this chapter is the way in which teachers plan their interaction with students. The SLT teacher will have planned for things like "eavesdropping" during active learning student activities, to qualify the students' participation in the activity and help students maintain the desired direction of their LPP, presumably towards the "doing mathematics" of the mathematical topic of the planned lesson. The SLT teacher would anticipate during the planning process key observable behaviors of the students, qualified through the characteristics of "doing mathematics" for the mathematical topic of the lesson. This is similar to the "key questions" that are typically taught to future teachers as a lesson planning tactic (Atkin & Karplus, 1962). The work of Sfard, Krummheuer, and others elucidate the dangers of assuming that cognition and participation are separate phenomena: psychological phenomena do not occur without the environment, culture, history, and materiality of people in community with each other. Thus, in SLT, the "psychology"—if it can indeed be called that—of mathematics teaching and learning is precisely homologous to the participation of teachers and students in the LPP dyad of oldtimers and newcomers.

Dawkins and Weber (2017) described the process of teaching students how to prove in an undergraduate mathematics class. They seemed to herald the time-honored mathematical practice of developing a proof argument as an enculturation mechanism into the enlightened learning state that is ostensibly desired for all students of mathematics. While Dawkins and Weber's article deals with undergraduate mathematics, their critique directly translates to K-12 mathematics by aligning with the "doing mathematics" (Stein et al., 2000, p. 16) level of activity as described in curriculum standards such as those from the National Council of Teachers of Mathematics. In teacher preparation programs across the US (from which context we are writing the present chapter), pre-service teachers are trained to design and implement tasks that point towards an end-goal of being at the "doing mathematics" level. Indeed, the obsession with "doing mathematics" led Baldino and Cabral

(2021) to question the qualifications applied by the teachers to the students who are ostensibly attempting to "do mathematics" in the classroom. Is the students' effort (i.e., labor) good enough to be considered "doing mathematics?" This qualification is for the teacher to decide, and in thus deciding, excludes knowledge from students who are not laboring in the intended way, that is, in the way that a mathematician would be. This judgement differential crystallizes the hidden dimension of the teacher's exercise of power in the SLT learning environment: that it is not clear who the "mathematicians" are, nor is it clear what their practice might be *or not be*.

4.2 Behaviorism as Perspective

4.2.1 Definition

Behaviorism as an approach for teaching mathematics originated in classical psychology (e.g., Bloom, 1956; Gagne et al., 1993; Skinner, 1938; Thorndike, 1898, 1905). In mathematics classrooms, the goal of the behaviorist is to elicit a desired response when a given stimulus is presented and to make undesired responses less likely based on consequences (see Freudenthal, 1978). This relationship between consequences and behaviors is called *conditioning*. Many behaviorists take the stance that students are "born as blank slates," and thus without mathematics, and that by learning the desired behaviors, acquire mathematical knowledge. The learning of mathematics is thus largely the result of the classroom environment being structured with the behavior–consequence doublet at its fore. In the current educational environment, behaviorism is still widely used if we look at the software and programs being utilized by teachers and schools, where students are given games to play that are focused on efficient achievement of low-level skills rather than engaging in conceptual understanding of higher-order tasks (e.g., Reflex Math software). Another example is when technology is used as a *master* (Geiger, 2005; Goos et al., 2000; Martinovic & Manizade, 2014), meaning that the technology knows the mathematics and the student believes or takes-for-granted any output produced by the software—lacking knowledge, competencies, and skills to engage his or her mathematical thinking to evaluate the outcome produced by the technology.

4.2.2 Goals of Teaching

The goal of the behaviorist approach is to create a perfect sequence of steps when teaching a mathematical topic that can be taught to a student procedurally with an expectation that the student will be able to repeat this sequence to produce the desired outcome. The main goal is to master the procedure and produce a desired outcome rather than examine the idea, construct new meaning, or make connections to other ideas within mathematics.

Psychology has historically described two types of behaviorism—classical conditioning and operant conditioning (Ormrod, 2020). The operant paradigm is more widely used in mathematics education settings, such as by pairing a reward with a desired behavior such as correctly answering mathematics problems. The goal of operant conditioning is to change current behavior towards the desired behavior by incentivizing with rewards. In the case of mathematics education, the desired behavior becomes the ability to correctly replicate procedures on tasks, and contrasts with a student's own mathematical thinking because the student is being rewarded for replicating the steps as prescribed by the teacher rather for exercising creative agency over their own mathematics. The result of this, in the classroom, is that mathematics procedures are reproduced in students with high efficacy and efficiency; it is easy for the teacher to see that her students are performing the mathematics in the desired way and that the entire class is making progress. This is useful in training students to be prepared for standardized tests. Thus, the goals of the behaviorist teacher are to create a classroom environment that focuses on the desired behaviors as propagated by the teacher, along with a culture of expecting rewards for those desired behaviors. In this way, the behaviorist teacher's goals are more focused on conditioning the behavior (e.g., reproducing a procedure) rather than understanding the mathematical concepts involved in the procedure.

4.2.3   Examples from the Literature

Kilpatrick et al. (2001) described vignettes of teachers' planning and subsequent lesson enactment that follow the behaviorist perspective. In one example, Mr. Angelo (pseudonym) planned a lesson on multiplication by selecting only examples that would make it "likely that all students [would] be able to produce correct answers" (p. 329), as long as they memorized the presented rule. Despite these researchers' elucidation of this early in the twenty-first century, more recent research has continued to detect the same behaviorist phenomena in mathematics classrooms.

Amador and Lamberg (2013) found that veteran teachers were guided by a behaviorist orientation towards their lesson planning, whereas novice teachers were not—the latter tended towards cognitive learning theory instead, which will be discussed in the next section. For the veteran teachers, lesson planning was guided by what Amador and Lamberg characterized as a *testing trajectory*, borrowing the term and structural diagram from Simon's (1995) work on Hypothetical Learning Trajectories (HLTs). In the testing trajectory, lesson plans were reverse engineered to produce a desired behavior in the students: that they could read, understand, and answer test questions correctly. In this type of lesson planning, the veteran teachers objectified four types of professional knowledge: (1) their knowledge about the test content and structure; (2) their beliefs about how to best prepare for a test; (3) their knowledge about how students achieve apropos of specific mathematics content, i.e., anticipating student misconceptions; and (4) their knowledge about how classroom activities and representations directly support test preparation and align with test questions, i.e.,

as opposed to concept development. Interestingly, one of the veteran teachers in the study also described her beliefs about the nature of mathematics as being inherently procedural.

Also straddling the line between the behaviorist and cognitive learning theory perspectives, Chizhik and Chizhik (2016) issued a call to the field to reject behaviorism in favor of cognitive learning theory—the latter will be discussed in the next section. Chizhik and Chizhik embraced the cognitivism of Vygotsky (1986) and the cognitive learning theorists as a retort to the behaviorist legacy that has overshadowed teacher preparation curricula on lesson planning since Tyler's (1949) foundational text on curriculum and instruction. At the time of publication of Tyler's work, the education field was primarily influenced by behavioral psychology, a focus that has since shifted—at least in other areas of educational research—to cognitive psychology. Chizhik and Chizhik (2016) argued that research on lesson planning has not similarly updated to the cognitive perspective, remaining "stuck" in the behavioral psychology of 70 years ago.

Technology-influenced Return to Behaviorism. The advent of technology in mathematics classrooms presents the danger of a return to the behaviorist approach for teaching mathematics, by focusing on rote memorization and practice of skills that are not based on student cognition of the mathematics presented through the technology. When used in this way, some technology functions as a novel way to keep students occupied with activity during class and makes it easy for teachers to monitor student completion of work—i.e., students successfully performing a desired behavior such as answering questions correctly—rather than students' development of the mathematical concepts. The operant reward is the two-fold novelty of the use of technology in and for itself, as well as the novelty of "winning" games or completing the puzzle correctly. In this way, the mathematics is secondary to the game's or puzzle's architecture, and thus the reward is not mathematical in nature.

During the COVID pandemic, school systems across the world were forced to transition to a virtual learning environment to protect the health and safety of students. They utilized various educational platforms and software (e.g., Mathletics, Reflex Math, and Sumdog). These Online Mathematics Instructional Program (OMPI) platforms use the behaviorist approach for teaching mathematics to motivate students (Darragh, 2021; Jablonka, 2017). As a result, opportunities for collaboration, problem solving, and using contextual mathematics were no longer present but were replaced by the development of superficial mathematical skills (Darragh, 2021).

We do not suggest that all technology is inherently slanted towards the behaviorist approach for teaching mathematics, but we do suggest that technology can be and often is used in that way (Parkhurst et al., 2010). In the institutionalized context of schooling, adopting any educational technology or platform is tied to teachers' goals for teaching mathematics and their goals for students' learning. Widely used dynamic platforms and software such as GeoGebra or Desmos can be pedagogically utilized in a way that is explicitly counter to the return to behaviorism (Edwards and Jones, 2006; Hohenwarter et al., 2008; Verhoef et al., 2015). The ways teachers use technologies present evidence of the perspective for teaching mathematics they

enact, as described in the framework we present in this chapter. More research is warranted to explore gaps between the potential of technology and the actual uses of technology in schools, a need that is compounded by the fast-paced evolution of technologies that are available and adopted by schools (Moore, 2020).

4.3 Cognitive Learning Theory as Perspective

4.3.1 Definition

Psychology is also the origin of cognitive learning theory (CLT) as a perspective informing what happens in mathematics classrooms, although innovations such as those of Piaget (1970a, 1970b) changed the perspective from a focus on behaviors to a focus on cognitive development. This perspective includes radical constructivism, which conceptualizes knowledge as the product of cognitive processes that construct or form individualized understandings of concepts. Cognitive learning theory strictly focuses on matching learning opportunities in mathematics with learners' natural cognitive abilities and processes. The major departure of cognitive learning theory from behaviorism was encapsulated in the development of the notion of *mental representations and associations*. Mental representations and associations describe the ways in which students "build up their picture of the world piece by piece" (von Glasersfeld & Steffe, 1991, p. 92); thus, knowledge is not conceived of as a "commodity that can be transferred from a teacher to a learner" (p. 93). Mental representations and associations are not necessarily in exact correspondence with observable behaviors; that is, observable behavior does not necessarily capture the entirety of what a student knows. The student has mental representations and associations, what Piaget (1970a, 1970b) called structures, that may or may not be reflected in their behavior. Behaviors are separate from these structures. Thus, a pedagogical approach that merely focuses on behaviors falls short of accurately designing for and assessing the goals and products of teaching.

4.3.2 Goals of Teaching

Cognitive learning theory focuses on students' development in thinking and provides learning opportunities that match the progression of this development. Unlike the behaviorist perspective where the focus is on the outcome of learning, cognitive learning theory focuses on enhancing the process of conceptual development through learning experiences. The goal of the teacher is to take an active role in helping students to make connections among their ideas, thus progressing from a simple conception of a topic towards a more complex one.

Freudenthal (1973, 1991) and Gravemeijer (2004) described the process of guided reinvention, wherein the teacher pre-actively conducts "a thought experiment to envision a learning route the class might invent itself" (Stephan et al., 2014, p. 39) to

mimic the evolution of mathematical concepts over decades or centuries. As a result, the teacher intentionally creates opportunities for students to explore and develop their own meanings of mathematics and mathematical concepts over the course of planned lesson units. The planned lesson units, which are teachers' Type D, act as the guide for students' reinvention of concepts and personal construction of meaning about those concepts: "[T]he learning route is designed so that the concepts emerge as students engage in the instructional sequence. It is in this sense that we say that students 'reinvent' mathematics" (Stephan et al., 2014, p. 39). In addition, the CLT teacher can only plan for what students *might* do, not what they will do—and for that matter, what students' constructed mathematical concepts and meanings *might* be as a result of their teaching.

Guided reinvention is pedagogically operationalized through work of radical constructivists Simon and colleagues (e.g., 1995, 2018). They developed the Learning Through Activity (LTA; Simon et al., 2018) framework based on the assumption that teachers can promote abstraction with engineered sequences of tasks. The framework extends from the teacher's pre-active engineering and sequencing of tasks all the way through students' abstraction of concepts. The engineering and sequencing portion of the framework is Type D; the task actively presented to students is Type C; the reflective abstraction is Type B; and the concept as the product of students' abstraction is Type A. Thus, the CLT teacher's Type D consists chiefly of task engineering and sequencing for the intended conceptual development trajectory: "If a concept is a result of reflective abstraction, that is an abstraction derived from activity, then it should be possible to engineer a sequence of tasks that elicits appropriate activity that promotes abstraction from that activity" (Simon et al., 2018, p. 103).

4.3.3   Examples from the Literature

Stephan et al. (2014) presented a cogent description of guided reinvention planning and teaching through a case study of 7th grade teachers who designed an instructional sequence of tasks intended to guide students in reinventing the rules of positive and negative integers, and integer operations. The task sequence started with a realistic context of financial transactions (e.g., net worth, assets, and debts) symbolized by integers and signs. The sequence progressed towards a purely abstract symbolization of signed integers and their operations, so that students through this process would explore and reinvent the meaning of these concepts for themselves.

Amador and Lamberg (2013) studied the ways in which teachers designed learning trajectories and planned corresponding lessons within the institutional constraints of the school, such as testing. They conceptualized the study using CLT in order to investigate how teachers addressed the high-stakes nature of standardized testing apropos of the daily work of teaching, and how the task of preparing students for such tests might be different for different teachers. In order to do this, they theorized an analog to the HLT called a *testing trajectory* where preparation for the testing environment was the driver for decision-making in the planning and teaching process. Four teachers were interviewed—three veteran teachers and one novice, first-year

teacher. The three veteran teachers' planning was guided by the testing trajectory whereas the novice teacher's planning was guided by conceptual development in the form of an HLT. That is, the veteran teachers worked backwards from the known parameters and format of the testing environment in order to theorize a hypothetical trajectory for preparing students to answer test questions whereas the novice teacher worked backwards from the mathematics conceptual goal in order to theorize a hypothetical learning trajectory for students' development of that concept. The direct link to this conceptual outcome was not implicated in the testing environment for the novice teacher. We identify the novice teacher as having taken the CLT perspective whereas the veteran teachers were effectively acting as behaviorists.

Chizhik and Chizhik (2016) issued a call to embrace CLT in lesson planning in tandem with a rejection of behaviorism. They note that, historically in teacher education, the texts being used in teacher preparation programs (TPPs) trace their origins to the behavioral psychology of Tyler (1949). Despite the field of education having since moved away (since the 1970s and 1980s; e.g., Vygotsky, 1986) from behaviorism in its predominant theoretical stance, TPP curricula on lesson planning have not consistently made the same update. This could potentially lead to the divide in teacher planning practices observed by Amador and Lamberg (2013). Chizhik and Chizhik (2016) argued for a reformulation of the lesson planning TPP curriculum to involve the following major components: (1) reconceptualize learning objectives by theorizing a CLT-based version of Bloom's Taxonomy; and (2) reconceptualize instruction to maximize student engagement, sharing of ideas and thinking, and (3) meaningful teacher feedback, with these three components being the major drivers towards students' success on tests.

Fernandez and Cannon (2005) conducted a case study comparison of Japanese and US teachers' lesson planning habits. They found that the two groups of teachers conceptualized the task of lesson planning in very different ways. The Japanese teachers' views of the task of lesson planning centered on conceptualizing students as active participants in the learning process and prioritized students' development of positive attitudes towards learning mathematics. The US teachers' views of the task of lesson planning, conversely, centered on conceptualizing themselves as effective teachers of the content. While the US teachers were concerned with student engagement, they grouped it under the characteristics of being an effective teacher. Thus, the difference in the two cultural paradigms of the teachers in the study meant that the purpose of their lesson planning manifested in very different ways. The US teachers planned with their own performance in mind, whereas the Japanese teachers planned with their student's cognition and affect in mind. In both cases, CLT could be implicated, but it is certainly more evident in the Japanese teachers' privileging of the students' position in the CLT paradigm. In both cases, CLT could be implicated because of the focus on trajectorial development of the mathematical content through the planned lesson.

Lewis et al. (2009) contributed a theoretical model that combined elements of CLT and SLT for the purpose of application to Lesson Study (LS), a pedagogical approach to analysis of, reflection on, and refinement of lesson plans typically done in groups of teachers. In their model, they synthesized elements of both theories

in various stages of LS, with the aim of theorizing how lesson improvement materializes from the LS process. The model connects CLT with these aspects of LS: (1) building understanding of the content area as well as students' and colleagues' thinking about it; (2) studying the standards, curriculum, and existing lesson plans to decide on building blocks for the conceptual development of the target lesson; (3) writing down lesson plan ideas that elucidate goals for student thinking—and student learning differences—to make them visible to colleagues; (4) observe a colleague teach a version of the lesson, paying attention to links between students' thinking and lesson design apropos of learning goals; and (5) noting instructional practices that should be improved to support the learning goals of the lesson. Other aspects of the model focused more on SLT, such as collaboration amongst colleagues and sharing ownership of the LS process. By theorizing the LS process through the CLT perspective, Lewis and colleagues offered a more cogent description of the work of Simon et al. (2018) with specific regard to the task of lesson study and planning.

Finally, Sullivan et al. (2013, 2015) have used CLT to research lesson planning. In their work, they connect Type D to Type C, with the distinction being that in Type D, the focus is on helping teachers identify important mathematical ideas that are fundamental for teaching a topic, whereas in Type C, they suggest that by improving teachers' knowledge of mathematics through collaborative planning, their ability to plan for a given mathematical objective will improve. In their 2013 paper, they described the ways in which teachers use the Australian national curriculum documents during planning. They found that many teachers assume agency over planning decisions when reading and interpreting curriculum documents and execute those decisions with resolve. Thus, they claimed that authors of curriculum document should construct such documents with explicit focus on inspiring teachers' decision agency in enacting the national curriculum standards. In short, Sullivan et al. (2013) argued that external actors in the school context (such as administrators, curriculum developers, etc.) should focus on unlocking teachers' planning agency potential rather than attempting to structure or restrict it, and that collaborative planning should be ritualized in schools. In this process, the teachers in the 2013 study took up agential decision-making about their planning through a process of reading the curriculum documents. Sullivan et al. (2015) closely tied this work to the CLT perspective, by focusing on the teachers' engagement with lesson planning practices that match students' cognitive process (assumedly along an HLT or similar trajectory) with the difficulty of struggle in the lesson's sequencing. In this way, teachers use CLT to reduce negative student experiences with the HLT and improve the lesson without reducing the cognitive demand—a crucial component of a successful application of CLT—of the lesson's tasks.

Sullivan et al. (2015) necessarily leads us to a discussion of the *productive struggle* and *productive failure* literature. This contrasts with the didactic perspective on teaching mathematics wherein the teacher "must produce a *recontextualization* and a *repersonalization* of the knowledge. It must become the student's knowledge" (Brousseau, 2002, p. 23, emphasis in original). The term productive struggle is defined as a necessary student learning behavior for building conceptual understanding and for promoting students' sense making (Heibert & Grouws, 2007).

Other researchers describe the goal of a CLT teacher as to *teach for the robust understanding of mathematics* by supporting students in *productive struggle* while building understandings through actively engaging in mathematical practices (Schoenfeld, 2014; Schoenfeld & TRU Project, 2016). The TRU project team describes five dimensions of powerful classrooms: (1) mathematics; (2) cognitive demand; (3) equitable access to content; (4) agency, ownership, and identity; and (5) formative assessment (Schoenfeld & TRU Project, 2016). They stress the importance of focusing on these dimensions during the lesson planning and reflection process (Type D). In addition, researchers encourage teachers to plan for students' productive failures as a necessary inseparable and cyclical portion of mathematics problem solving and learning (Kapur, 2010, 2014; Simpson & Maltese, 2017). Warshauer et al. (2021) highlight the importance of preservice mathematics teachers learning to identify strategies and practices that can be used for planning and supporting *productive struggle* in the classroom.

4.4 Social Constructivism as Perspective

4.4.1 Definition

In social constructivism (SC), which is based on the work of Vygotsky (1960), the classroom community constructs knowledge and understanding as a cultural product of students' learning experience. In the SC mathematics classroom, the shared nature of the knowledge is distinct from the cognitive learning theory perspective, because the personally held mathematical knowledge of an individual student is a reflection of the community's construction of that knowledge rather than a personal product of construction of mental representations. The social constructivism perspective was further developed in varying ways by teams of scholars, including but not limited to the work of Bishop (acculturation; 1988), Resnick (socializing; 1988), and Cobb and Yackel (emergent perspective; 1996). *Acculturation* is induction of students into a foreign or alien culture (e.g., the mathematics classroom). In the culture of the mathematics classroom, this process includes interacting with others to perform the activities of counting, locating, measuring, designing, playing, and explaining (Bishop, 1988) to develop mathematical knowledge. *Socializing* refers to social constructivism over time, where "personal habits and traits are shaped through participation in social interactions with particular demand and reward characteristics," with the goal of the student "gradually tak[ing] on the characteristics of [the teacher]" (Resnick, 1988, p. 12). In the *emergent perspective* (Cobb & Yackel, 1996), cognitive and social perspectives work in parallel, with the teacher simultaneously interpreting students' cognition (e.g., beliefs about self and others, the nature of mathematics, conceptions of mathematical ideas) and students' social interactions (e.g., classroom norms, mathematical norms, and mathematical practices).

4.4.2 Goals of Teaching

The goal of the SC teacher is to involve students in a community discourse about mathematics through teaching that is focused on classroom discussion. The discourse promoted by the teacher is guided by the relevant mathematical tasks and investigations, with the teacher being the instrument of enculturation into the mathematizing culture; the students meanwhile are the ones being acculturated (Bishop, 1988). By encouraging students to form new understandings of mathematics using their interpretations of prior mathematical knowledge, the teacher aims to empower students to contribute to the reconstruction—instead of reproduction—of new mathematical knowledge. In other words, SC teacher's goal is to provide students with opportunities to develop subjective knowledge that must be constructed and validated through and within sociocultural interactions so that the subjective knowledge may become objective knowledge of the group.

4.4.3 Examples from the Literature

Purdum-Cassidy et al. (2015) used social constructivism in their study of the way in which teachers plan for questioning (e.g., key questions on a lesson plan) in elementary mathematics classes. In their Vygotskian framing, they noted how "conceptual knowledge first occurs between learners … and then moves within the learner" (p. 81). Social constructivism thus positions the teacher's key point of access—apropos their potential for impacting student learning outcomes—as that of influencing what happens between students in the classroom. The intrapsychological impacts that occur consequently are left to each student's own psyches for the purposes of meaning-making. The teacher should thus be primarily concerned with impacting the social construction of knowledge. In their study, Purdum-Cassidy and colleagues focused on the role of questioning (their own plans for key questions) and the role of interpreting and answering students' questions during the lesson—overall what is generally called *discourse* in the mathematics education literature. In particular, they note how pre-service elementary teachers struggle to plan for and write key questions when planning a lesson. As an intervention, these researchers investigated the possibilities of children's books that have mathematical topics in helping teachers plan for mathematical questioning in their lessons. Since children's literature is discursively organized (viz. into the format of a story), the same structure can be ported over into the structuring of questions *qua* discourse. Such discursively structured questioning prompts the classroom community to socially construct knowledge—*vis-à-vis* questioning and discourse—that is then internally reified for each student.

Sullivan et al. (2015) investigated the connection between professional development and teachers' abilities to plan for scaffolding challenging mathematics tasks. In their study, they investigated how teachers exposed to challenging tasks that require student collaboration (such as inquiry tasks). Such tasks were initially uncomfortable for teachers to use, but once they had been supported by professional development,

teachers felt confident in planning for such tasks and were more likely to seek out more inquiry tasks. This finding indicates that teachers are often hesitant to engage in the SC perspective when planning for and enacting mathematics lessons, but that this hesitation can be alleviated through the use of directed training on the approach.

4.5 Structuralism as Perspective

4.5.1 Definition

The structuralist approach originates from both mathematics and psychology. Dienes (1960) emphasized the importance of children learning through the use of manipulatives (e.g., Gningue, 2016); however, classic examples of the structuralist approach can be found in every branch of mathematics. For instance, the Poincaré and Beltrami-Klein models for describing hyperbolic geometry are used to help learners develop fundamental understanding of hyperbolic space that is challenging to visualize otherwise. The focus of the structuralist perspective is on the structures and theories that underlie the mathematics presented. This perspective is often conflated with the constructivist approaches for teaching mathematics. It overlaps with radical constructivism in that there is a focus on theories of cognitive development and students' concept formation of a specific mathematical idea. However, the structuralist approach differs in that the focus is on *discovering* the structures that are introduced to students by the teacher, who is using those structures as a framework around which mathematical understanding can be developed, rather than constructing them. An example in K-12 teaching is the use of AlgeBlocks to visually demonstrate multiplication of polynomials, a mathematical process that would otherwise be only abstract and symbolic (de Walle et al., 2017). The difference, thus, is that the structuralist perspective is focused on *discovering* the existing mathematical structure of polynomial multiplication, a structure that is already there. Conversely, the constructivist perspective does not conceive of an existing structure that the student must reach through their mathematical activity, but rather, is *exploring* to construct their own concepts from scratch. The structuralist, therefore, can talk about misconceptions and misunderstandings when a student's understanding is incongruent with the relevant structure, whereas constructivists do not use that term since the student's concept is its own referent.

4.5.2 Goals of Teaching

The goal of the structuralist teacher is to guide students in discovering a mathematical structure, through the use of exemplary and sequenced tasks, each of which draws particular attention to some limit-case aspect of the structure. As already mentioned, one of the characteristics of the structuralist approach is an emphasis on the use of manipulatives. The manipulatives allow the student to experience an embodiment of

the mathematical structure during their discovery process. Additionally, there is a spiral design in the curriculum that allows for revisiting key mathematical structures in a cycle, delving into them more deeply each time.

A relevant metaphor here is the way in which a house is built: the cornerstones of the foundation must be located first, followed by the joists in the floors and beams in the walls, and it is not until this skeleton is truly discovered from behind the plaster walls (the general case examples), that the house's structure could be said to be truly discovered. Thus, the structuralist teacher must first check their own understanding of the structure against the mathematical community and literature, and then locate exemplary cases of problems that will illuminate the cornerstones, beams, and joists of the structure. These must be carefully sequenced so that the student will follow the same path in discovering the hierarchy of the structure; for example, it would make no sense to study the roof trusses without first having discovered where the studs in the walls are. This identification and sequencing of exemplary tasks then leads students to develop a more general understanding of the structure as the teacher generalizes these exemplary cases. In practice, this typically follows the arc of beginning with manipulatives, then moving to a pictorial representation of the manipulatives (i.e., drawings), then associating the drawings with abstract symbols or ideas, and finally removing the manipulatives altogether so that the abstract symbols represent the structure itself in the student's mind.

In the structuralist approach, models and manipulatives are used to help students discover mathematical structures, but none of the models are robust enough to be applicable in every case or to demonstrate every attribute of the mathematical structure's complexity. The teacher, therefore, uses manipulatives to help students discover a particular aspect or develop understanding of a particular example. These particulars can then be used to develop understanding of the mathematical structure more generally.

Mathematics education literature on multiple representations exemplify the practical use of the structuralist approach in mathematics classrooms. Researchers discuss the importance of using multiple representations and developing fluency in flexibly moving between them, such as visual, pictorial, graphical, numeric, and algebraic (e.g., Ainsworth, 2006; Deliyanni et al., 2016; Goldin, 2002; Goldin & Shteingold, 2001; Mitchell et al., 2014; Stylianou, 2010). Manipulatives can be used as a tool for making and presenting representations and can either be virtual (on computers) or physical. Planning for these types of lessons thus includes envisioning tasks with different representation in sequence from concrete to abstract, and tasks that promote the students' movement between the multiple representations' uses. Illustrations of classroom implications of the structuralist approach to Type D can be found in mathematics educators' works designed for training pre-service teachers (e.g., Beckmann, 2022; Kilpatrick et al., 2001; Van de Walle, 2017).

4.5.3 Examples from the Literature

Pierce and Stacey (2009) discussed the use of graphing calculators in a structuralist classroom. In their study, they emphasized the importance of four aspects of lesson planning for the use of technology within the structuralist approach: (1) focusing on the main goal of the lesson and thoughtfully selecting multiple representations that directly support the goal; (2) identify, for each representation, a specific purpose aligned with student engagement; (3) "establish naming protocols for variables" (p. 231) so that students can translate variables across technologies and representations easily; and (4) reducing any excessive cognitive demand so that technology does not distract or detract from the lesson goal and students' engagement with the intended mathematics. These researchers argued that technology allows the teacher to support the goals of the lesson as identified by the teacher during lesson planning. Depending on the goals for the lesson, the structuralist teacher might need to restrict the strategies that emerge during the discussion or restrict the representations being used, or plan for the reduction of distractions due to the technology (Pierce & Stacey, 2009).

In their study of the low-performing middle school mathematics classrooms, Panasuk and Todd (2005) present a conceptual framework within a structuralist approach for teaching mathematics that guided the development of the instrument titled *Lesson Plan Evaluation Rubric* (LPER) for the assessment of mathematics teachers' lesson planning process. The researchers also described a *four stages of lesson planning* (FSLP) strategy comprising: (1) planning of *objectives*, formulated in terms of students' observable behavior; (2) design of *homework*, that matches the lesson's objectives; (3) inclusion of *developmental activities* that reflect the lesson's objectives and advance students' development and learning; and (4) planning *mental mathematics* that include activities to stimulate students' prior knowledge, and prepare students for the acquisition of new concepts. The FSLP strategy focuses on the development of lessons involving multiple representations such as visual representations (diagrams, pictures, graphs, tables), verbal representations (words), and symbolic representations (variables, expressions, operations, equations) to address students' misconceptions and assess students' progress toward meeting learning objectives. Moreover, the strategy produced lessons that were comprehensive and coherent by emphasizing alignment between homework, classroom activities, and mental mathematics. The researchers claimed that to incorporate FSLP effectively, lesson plans must be flexible yet logical in their design to accommodate the distinctive needs of each student. Furthermore, this strategy encourages teachers to continuously adjust and adapt to achieve the desired learning outcomes. In addition, this strategy is compatible with Gueudet and Pepin's (this volume) concept of *coherence-in-use*, which they define as the degree to which there is coherence within teachers' (enacted) propositions to their students, after teachers have consulted various curricular materials.

Harbour et al. (2016) described a process of structuralist lesson planning, that included beginning with a diagnostic interview to determine a student's existing

understanding of a concept, and then comparing that understanding with a standards-informed intended goal for the student's understanding as a result of the lesson. The teacher then, based on this gap, takes four considerations into their lesson planning: (1) plan the lesson to utilize instructional strategies that explicitly focus on students' conceptual understanding of the topic; (2) plan explicit scaffolding and feedback opportunities into the lesson; (3) plan for student think-alouds and teacher think-alouds; and (4) plan for the use of concrete materials, visual representations, and numeric representations.

4.6 Problem Solving as Perspective

4.6.1 Definition

Problem solving as an instructional approach in mathematics classrooms is a type of teaching that focuses on developing students' problem-solving skills and abilities to persist when faced with problems with which they have no experience, rather than practicing skills that they have already previously learned (e.g., from prior instruction). The focus is on both the mathematical content and the process, with the intention to produce and interpret different approaches and strategies for solving the same problem. This approach—based on the theoretical framework developed by Polya (1945/2015) in *How to Solve It*—originated in the 1980s with the *Cockcroft Report*, *Mathematics Counts* (UK), and NCTM's *Agenda for Action*, all of which called for problem solving and investigations to be included in mathematics teaching. Schoenfeld (1983) has argued that studying problem solving requires the consideration of different and distinct domains of behavior and knowledge—knowledge resources, control, beliefs, heuristics, and practices—rather than purely relying on cognitive psychology.

4.6.2 Goals of Teaching

The goal of the teacher in a classroom that focuses on problem solving as an instructional approach is to create a *thinking classroom* (Liljedahl, 2019; Liljedahl et al., 2016) in which students are given tasks that encourage thinking. This is the kind of task or activity that does not focus on precise application of a known procedure, implementation of a taught algorithm, or the smooth execution of a formula. In other words, problem solving is a messy, non-linear, and idiosyncratic process (Liljedahl, 2020). Problem solving strategies include—but are not limited to—guess-and-check, making lists or tables, looking for patterns, working backwards, making a model, drawing a picture, and trying a simpler problem first. The goal of the teacher is to encourage students to analyze each problem for what is given and what constraints are present, to highlight the relationships between variables, and to explicate the goals of solving the problem. The teacher creates opportunities for students

to explain the meaning of the problem, as well as to ask reflective questions such as "I wonder…" and "Does this make sense?" (Common Core State Standards Initiative, 2010; Kobett & Karp, 2020; Timmerman, this volume).

4.6.3   Examples from the Literature

Liljedahl (2020) listed the practices teachers have to consider when planning for lessons in a thinking mathematics classroom. The list comprises 14 general categories of practice that all teachers adhere to in some shape or form: (1) What are the types of tasks we use; (2) How we form collaborative groups; (3) Where students work; (4) How we arrange the furniture; (5) How we answer questions; (6) When, where, and how we give tasks; (7) What homework looks like; (8) How we foster student autonomy; (9) How we use hints and extensions to further understanding; (10) How we consolidate a lesson; (11) How students take notes; (12) What we choose to evaluate; (13) How we use formative assessment; and (14) How we grade.

As a case example of this approach to Type D, Lilejdahl (2015) studied teachers' planning for problem solving in numeracy lessons. He investigated how a group of mathematics teachers engaged in lesson planning from the problem solving perspective over the course of six months, discussing their challenges with each other whilst shifting their goals for the lesson from a more traditional focus on students' knowledge to a focus on planning for students' quality of participation in the problem solving tasks. Through this shift in focus, the teachers began to *plan for* students' quality of engagement with the tasks—i.e., through the act of problem solving— rather than on designing lessons to transmit and assess a quantity of knowledge. This shift in focus was characterized by teachers' embrace of open-ended, complex problems with multiple parameters that required students to engage in thinking critically about the problem and the parameters within which they would be expected to solve the problem, in other words, the boundaries that circumscribed the problem. By focusing on the problem and its particularities, teachers' Type D assumed a different form than would have been required for more traditional, knowledge-based lesson planning. In particular, Liljedahl (2015) found that teachers who aimed at problem solving Type D focused on how to design the task with problem solving as its goal rather than students' knowledge outcomes.

Another example can be found in Zazkis et al.'s (2009) theorization of the impasse of teacher educators who teach their students to plan lessons comprehensively (i.e., with knowledge and outcome goals) thereby restricting the aims of mathematics education to those captured in curriculum documents. In the article, they juxtapose "planning teaching" (i.e., planning for knowledge and content goals) versus "teaching planning" (i.e., planning for students' engagement in problem solving). By utilizing the practice of *lesson plays*, the authors theorized how planning for problem solving incorporates consideration of the "contingencies of teaching" (John, 2006, p. 487, as cited in Zazkis et al., 2009). With a shifted focus on problem solving instead of knowledge goals, "planning for teaching" instead of "teaching how to plan" prioritizes the playfulness and student activities of problem solving. These activities are

captured in Liljedahl's (2020) list of characteristics in aligning the teachers' and students' actions as "artifacts of the [lesson] planning structure" (Zazkis et al., 2009, p. 43), such as teachers' responses to students' unexpected progression through the problem solving nature of the lesson—or in other words, alignment of teachers' planning *for the teaching of their students* as opposed to teachers' planning of the lesson itself as an artifact.

4.7 Culturally Relevant Pedagogy as Perspective

4.7.1 Definition

Culturally relevant pedagogy (CRP; Gay, 2010, 2018; Jett, 2013; Ladson-Billings, 2014) as a teaching perspective emerged out of the Realistic Mathematics Education movement (RME; viz. Freudenthal, 1991) in response to the rise of post-colonial studies in education. CRP is one way of addressing the RME heuristic that mathematics be contextualized to students' cultures. As Makonye (2020) noted, the imperative of CRP and contextualized mathematics is evidenced by the high rates of failure seen in school mathematics amongst marginalized populations, many of which are the modern product of colonialist efforts. CRP directly problematizes the Western, colonialist notion that mathematics is objective, and that it is not value-laden by a culture (viz. Bishop, 1988). The erroneous belief that mathematics *is* in fact objective leads to what is experienced in modernity as the perceived universality of mathematics, and moreover, the "truth" of mathematics. Not only is this problematic for students' learning of mathematics, but also for mathematics teachers' training, because the training experiences of mathematics teachers predicate the beliefs they will have about mathematics, and subsequently will affect the ways in which they will be conditioned to enact their training in a classroom with students. Because of this, the issue of CRP has just as many implications for teaching as it does for learning.

CRP is based on the assumptions that cultural groups engage in behavior that is based in mathematics or mathematical elements, and that knowledge is produced through and by culture and history. Thus, mathematical thinking must be consistent with the cultural context of the students. For example, enacting CRP teaching of students in a majority Black inner-city school in the US might consist of instruction that focuses on the origin of mathematical concepts within that culture and word problems or projects about mathematics within the Black culture and history of that city.

Additionally, *ethnomathematics* has been a well-known research topic in mathematics teaching since D'Ambrosio (1985) introduced the term. (Also, for a discussion of how ethnomathematics implicates ethical responsibilities, such as through the use of technology, see D'Ambrosio, 1999). He defined ethnomathematics as "the mathematics which is *practiced* among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on" (D'Ambrosio, 1985, p. 45, our emphasis). Thus, D'Ambrosio

argued that ethnomathematics is conceptualized in contrast to the school mathematics which has the aim of maintaining and reproducing economic and social structures, "reminiscent of that given to the aristocracy when a good training in mathematics was essential for preparing the elite (as advocated by Plato), and at the same time allows this elite to assume effective management of the productive sector" (p. 45). Because of this distinction, we characterize CRP as owing its intellectual heritage to ethnomathematics but is markedly separate from it in terms of its degree of institutionalization and purpose. Ethnomathematics is not concerned with the teaching of mathematics in institutionalized settings, like schools. In fact, scholars (e.g., Pais, 2011) have revealed a growing misuse of ethnomathematical research, wherein the economic and cultural reality of the identifiable cultural group is removed from the ethnomathematics in the act of institutionalizing it.

4.7.2 Goals of Teaching

CRP teaching can be characterized by mathematics content that is explicitly situated within a contextual frame unique to the culture of the students being taught. Instead of focusing on the arbitrary nature of traditional mathematical structures, CRP bases mathematics in cultural activities and knowledge most relevant to the students. By doing so, teachers empower their students and present mathematics as intimately relevant to their students' cultural lives instead of being abstract and formalist, thus giving mathematics a useful application for all students within the cultural context. As a result, for the CRP teacher, mathematics teaching includes explication of the value-laden nature of the mathematical concepts being taught, as well as adopting a perspective that their work as a teacher is to enculturate students into the culture of mathematics—whether that culture be Western and normative or Indigenous and marginal. This can, in many cases, lead to a conflict between the culture of the CRP teacher and the culture within which the mathematics is being contextualized (Bishop, 1988). Thus, teachers must be proficient in not only their content and professionally situated knowledge of mathematics, but also in the cultural history and practices of the students whom they are teaching. As a result, traditional curriculum materials must be supplemented or redesigned to support CRP instructional efforts. In this process, the CRP teacher must be careful not to essentialize students' culture (McCarty & Lee, 2014).

4.7.3 Examples from the Literature

An example of CRP teaching can be seen in American Indian and Indigenous studies, such as those conducted by Ruef et al. (2020), who studied how mathematical concepts are represented in the Ichishkíin language of the Yakima culture. In the article, the authors weave together a complex theoretical perspective to frame their work in a comprehensive framework of theory. In their analysis, they establish the fact that mathematics is culturally situated in the western/American context which

is markedly different than for the Yakima people. One of the stories they described from the interview data (with a Yakima Elder) is that the word for "fraction" comes from the same root word in their native language for the process that happened when the White European settlers broke apart their nation by giving them the "choice" of either complete genocide or surrender of lands. The native language of the tribe uses this term for "broken" when they are talking about fractions. This shows how subtle the cultural situatedness of mathematics can be, and that researchers from dominant social groups are unable to fully grasp the concept of the differences in lived experience of other peoples' mathematics. Through the CRP perspective, implications for teachers' Type D "are not subtle" (Ruef et al., 2020, p. 316): mathematics can function as a form of White supremacy *qua* White knowledge production. Thus, the CRP teacher understands the culture of the students they teach, allowing them to plan lessons and assess student learning within that cultural frame. As Ruef et al. (2020) concluded, the work of planning for mathematics lessons and assessing student work—informed by CRP—is built around the concomitance of students' cultural language and mathematical concepts so that students and teachers are connected through the place and time in which they are engaged in mathematics learning. The Alliance of Indigenous Math Circles (www.aimathcircles.org) offers resources for teachers interested in planning CRP mathematics lessons in the Indigenous American Indian context.

Jett (2013) wrote about the context of working in pre-service mathematics teacher university courses so that the content and methods taught are of racial relevance to his students. Failure to conceptualize mathematics teacher education through a lens of cultural relevance is, for Jett, an act of fracturing the identities of pre-service mathematics teachers as they are learning to plan and implement lessons. Culturally responsive pedagogy, thus, becomes a key driver in the ways that pre-service mathematics teachers are taught to plan lessons because, as Ladson-Billings (2009) said, CRP "empowers students intellectually, socially, emotionally, and politically, by using cultural referents to impart knowledge, skills, and attitudes" (p. 20). Without CRP training in teacher education, future mathematics teachers are not equipped to plan CRP lessons of their own. An example of this can see in a study by Makonye (2020), who illustrated the consequences of teaching mathematics outside of students' cultural contexts. Makonye gave the example of interest in the banking system and how it is irrelevant or unrelatable to students in South Africa, because requesting interest for any money loaned is considered immoral. When being taught about the application of interest and related mathematical concepts, South African students face challenges because the applications are based on a culturally *irrelevant* phenomenon and thus the mathematical concept's universality fails (Makonye, 2020).

Skovsmose (2021) offers additional implications for CRP in the planning of mathematics lessons, namely that situations of crisis can serve as bases of lesson plans. As a human race, there are universally shared experiences (e.g., pandemic) that create cultural cohesion constitutive of "cultural relevance" for all students in a classroom. For example, a teacher could use mathematics to teach a lesson on a crisis or a critical situation such as COVID-19 to which all students could culturally relate; the cultural

context of COVID-19 is primary to all people because they have all experienced it. From a CRP perspective, COVID-19 is a cultural phenomenon and thus can be used to create and/or illustrate how the relevance of the mathematical activity can be presented pedagogically within a cultural group.

4.8 Project and Problem-Based Learning as Perspective

4.8.1 Definition

Project and Problem-Based Learning (PBL) is a perspective for teaching mathematics that is based on the assumption that mathematical knowledge has to be presented in the real-world contexts of students' environment. The environment provides meaning to the mathematics or contexts out of which new knowledge can be drawn. An example of this approach can be found in works that use nature as a context to teach mathematics concepts (e.g., Adam, 2003/2006; Toni, 2021). On the one hand, by exploring problems in the environment students are prompted to discover mathematical concepts. On the other hand, the abstract structures of mathematics can be imposed on an environmental or contextual situation, allowing the situation or context to be reinterpreted using mathematical concepts. The PBL teacher views mathematics as inseparable from the environment or context in which it exists or originates. In other words, every mathematical topic is presented within its context: the context itself is the source of inspiration and motivation for students' interest in the mathematics, as well as the starting point for developing new mathematical knowledge. This means that the teacher is not there to impose notions onto the child, but rather, to select the influences (also see Dewey, 1897).

Based on an extensive review on PBL literature, Merritt et al. (2017) found that different definitions existed. Most relevant for the mathematics education community are the functional/curriculum design, constructivist, and conceptual change definitions. Based on their analysis, we align the *functional/curriculum design definition* with the inquiry-based learning literature, the *constructivist definition* with the cognitive learning theory perspective discussed in the present chapter, and the *conceptual change definition* with the structuralist perspective also discussed in the present chapter. We conjecture that teachers who participate in educational experiences to learn about PBL—such as professional development events—accept PBL as a novel pedagogical tool without explicit reference to one of the three aforementioned definitions. Rather, when they return to their classroom and choose to plan lessons from their newly learned PBL perspective, they retroactively assign epistemological meaning to the lesson based on their other beliefs about mathematics teaching and learning; these beliefs then become objectified through the planning, enactment, and assessment of the PBL lesson. More research is needed in this area.

4.8.2   Goals of Teaching

The goal of PBL is for individual learners to construct mathematical concepts from the context familiar to them. The teacher uses real-world contexts as a source of inspiration, abstraction, meaning, and motivation for learning mathematics. As a result, the goal of teaching mathematics from this perspective is that students understand the mathematical concepts as intimately emergent from the context and environment itself. Thus, the student learns the structure of the environment as a mathematical structure along with learning the mathematical concepts. A characteristic of this approach apropos of pedagogy is the implementation of collaborative group projects that often utilize statistical analyses, mathematical modeling, and exploratory activities (Capraro et al., 2013; Lee, 2018).

4.8.3   Examples from the Literature

Literature on PBL related to Type D has come from the STEM Education, Engineering Education, and Science Education fields (e.g., Cheaney & Ingebritsen, 2005; Miller & Krajcik, 2019; Mills & Treagust, 2003). For example, Miller and Krajcik (2019) reported on a four-year action research project they did in their own classes in teacher education, on how best to align science teacher preparation with the goals of PBL as outlined in official curriculum documents such as the Next Generation Science Standards (NGSS; NGSS Lead State Partners, 2013). They highlighted the connection between the goals of PBL and developing students' *knowledge-in-use* (see Pellegrino & Hilton, 2012), which is the "capacity that learners need to apply knowledge to make decisions and solve problems, and to evaluate when and how to get more information when necessary" (Miller & Krajcik, 2019, p. 1). Miller and Krajcik (2019) elucidated that planning for PBL lessons means planning for the creation of a specific type of learning environment: a "sense-making and knowledge generating environment" (p. 5) that is designed to pique students' interest about natural phenomena or situations in the real world through the pursuit of questions about the world. As such, the teacher creating the PBL environment must establish driving questions to guide: (1) the lesson; (2) the development of the students' knowledge-in-use; and (3) the development of artifacts (concrete representations) as the results of the PBL investigation. Specific instances of Type D in Miller and Krajcik's (2019) process include: (1) planning for driving questions "about a problem to be solved or experience to be explained that promote wonderment about the world" (p. 6); (2) including students' participation in scientific practices in the lesson plan; (3) planning for students' exploration of the driving questions through "collaborative sensemaking activities" (p. 6) that are designed to engage "in shared knowledge building" (p. 6); (4) planning for scaffolding the development of students' knowledge-in-use through the use of discursive pedagogical tools; and (5) assessing the artifacts students produce as a result of the PBL lesson for the extent to which they "scientifically address the driving questions with increasing sophistication" (p. 6).

Conversely, the mathematics education literature on PBL focuses primarily on students' conceptual understanding of mathematics and/or beliefs about and attitudes towards mathematics as a result of having participated in a PBL lesson (e.g., Knezek & Christiansen, 2020; Merritt et al., 2017). Merritt et al. (2017) found that all the studies in mathematics and science PBL that they reviewed were concerned with students' knowledge, achievement, and affectual outcomes. The literature on PBL and its Type D implications are relevant to mathematics teaching and teacher education as well. Considering the growing interest in PBL and the larger increase in focus on STEM education, this gap in the literature needs to be addressed in order to better understand how the mathematics teacher who wishes to implement PBL lessons plans for and assesses the impacts of those lessons.

5 Pros and Cons for Each Perspective

Each aforementioned instructional approach for teaching mathematics has advantages and difficulties associated with it. Planning based on each instructional approach therefore presents unique challenges. We survey these briefly, not exhaustively, in this section.

From the Situated Learning Theory (SLT) perspective, the goal of the teacher is to develop a students' sense of belonging in the mathematics community (e.g., using correct mathematical language and discourse patterns, solving problems, and practicing communication of their thinking, and doing mathematics). The knowledge is situated in the mathematics community rather than an individual and the students are treated as a newcomer apprentices in the community where the oldtimers—mathematics teachers—engage in doing mathematics as the leaders of the mathematical community. The challenge with this approach is the time needed to develop mathematical community and the norms associated with it, including the teacher's identity as a member of the community—a practicing mathematician—and the students' identities as legitimate participants in the community's periphery. Knowledge is conceived of as identity development and belonging, either as newcomers or oldtimers, in the community. This is challenging to plan for with a diverse and/or large classroom of students and institutional constraints that may not afford it or value such educational goals. An additional challenge is that the teacher needs to make the distinction between whether the students' mathematical labor is sufficient to support their apprenticeship *contra* epistemic exclusion, and to be clear about what "doing mathematics" might look like. The SLT teacher will also need to continuously reflect on their own understanding of what it means to "do mathematics," as such an inference can be rather opaque, and thus the SLT teacher needs to consider the material labor of their students as well: "To compensate for epistemic exclusion, we seek to develop a reliable way to evaluate the *effort to understand mathematics*" (Baldino & Cabral, 2021, p. 280, emphasis in original). In our words, this effort represents students' legitimate peripheral participation and desire to become an oldtimer.

The Behaviorist classroom—where the focus of mathematics teaching is on showing students how to reproduce a perfect sequence of steps when solving a mathematical problem, and where the goal is on producing correct answers rather than understanding the meaning behind the mathematical concepts—works well when developing procedural fluency of mathematical concepts and ideas that have low cognitive demand (e.g., skill/drill activities and technology-related tasks designed to improve fluency). However, when addressing mathematical concepts with a high cognitive demand or that require critical thinking, this perspective is not appropriate for mathematics planning and teaching.

In the Cognitive Learning Theory (CLT) classroom, mathematics teachers plan lessons that allow space for learners to develop their own structures and construct their own meanings for mathematics. This requires additional time for mathematics instruction and there is a danger that a lesson can become a set of unrelated activities and lose its focus with respect to the students' construction of the lesson's mathematical aim. To avoid this, the CLT teacher plays an active role in aligning lesson plans with hypothetical learning trajectories, addressing individual student's needs throughout the construction process, and assisting learners in making connections between big mathematical ideas.

In the Social Constructivist (SC) mathematics classroom, teachers plan for mathematics learning where students develop their objective knowledge of mathematics concepts through a collaborative classroom discourse during which their subjective knowledge is examined, dissected, and confirmed by the group. Thus, any knowledge production is jointly owned by the classroom community. A challenge with SC planning is that the mathematics teacher has to be knowledgeable of various mathematical approaches for a given problem, attitudes in the mathematical community, and the processes of developing new mathematical knowledge between people.

In the Structuralist classroom, mathematics teachers plan to use various mathematical models that help students to simplify and reveal abstract mathematical structures. Each model used (e.g., AlgeBlocks, Geostrips) has limitations and therefore cannot be used as the sole source of explanation. The emphasis is on a student discovering underlying structures presented by the teacher instead of exploring and constructing their own mathematical structures (as in CLT). A difficulty with planning in the structuralist classroom is that teachers have to be knowledgeable about various appropriate models for each topic taught, the strengths and limitations of each model, and appropriate supplementary instructional tools and strategies needed for comprehensive understanding of essential mathematics.

In the Problem Solving (PS) classroom, the focus is on multiple strategies for solving a given problem and being able to make connections to other strategies or other problems, rather than the content understanding that results from it. PS has the advantage of engaging students in mathematics making and creating new knowledge through the experience of dealing with unfamiliar situations as opposed to receiving knowledge through direct instruction from a teacher. It differs from radical constructivism in that it does not focus on the concepts that are constructed by students but rather focuses on the analytic schemes, critical thinking processes, and intuitions they develop for solving posed problems. A difficulty in this approach

is that it often is misaligned with institutional goals for mathematics teaching—such as standardized testing of content knowledge—so that planning for PS lessons results in planning for student outcomes that are often not represented in curriculum documents.

The Culturally Relevant Pedagogy (CRP) approach can lead to the development of students' critical consciousness of social structures and inequities, as well as centering mathematics in the cultural practices and realities most familiar to students. This has the potential to reinvigorate students' cultures with mathematical meaning and—further—to position students' lived cultural experiences as mathematical experiences. The difficulty with CRP planning and teaching is that the teacher must be an expert not only in the mathematics content taught, but also in the culture/s of their students. Thus, curriculum materials must be tailored to provide mathematical content within the cultural context.

In the Problem- and Project-Based Learning (PBL) classroom, teachers plan mathematical activities within real-world contexts, so the meaning assigned to mathematical knowledge is contextualized in the real-world situations used to teach the mathematical concepts. A PBL lesson typically includes an open-ended problem to which students attempt to develop *a* solution. Planning for such lessons requires teachers to have comprehensive knowledge about mathematics in real-world situations, as well as the pedagogical knowledge to translate this contextualized mathematics into planned activities and assessment strategies. Another issue with this approach, similarly to CLT, is that students construct their own understanding of mathematical concepts and solutions to problems, without any guarantee that those concepts or solutions will be in complete agreement with lesson objectives, institutional and curriculum documents, or community norms. Thus, planning for a PBL lesson means incorporating additional teacher guidance (to help students in their problem-solving process) into the lesson plan.

As we have shown, each perspective for planning and teaching mathematics has certain benefits and drawbacks, and each is not appropriate for every lesson. Developing teachers' Type D means developing their ability to operationalize the different perspectives we present in this chapter as well as the ability to differentiate goals for mathematics lessons and identify the most appropriate type of epistemological commitment to support the teaching of different mathematics topics. Furthermore, developing teachers' Type D means developing self-awareness of one's *a priori* epistemological commitments about the teaching and learning of mathematics, and how these commitments may help or detract from planning and teaching a particular mathematics lesson.

6 The Epistemological Perspectives in Different Cultural Contexts

Culture is a foundationally structuring element of human experience and the social link between people. None of the eight epistemological perspectives are immune from this, because it is these perspectives that describe what teachers *actually do* when planning for and reflecting on their teaching work in the classroom. In the case of SLT, apprenticeship may take very different forms in different cultural settings, such as schooling and as apprenticeship outside of formal schooling. In the case of behaviorism, the paradigm itself is value-laden in a unique way that may socio-politically charge any attempts to stray from it; consider a cultural setting where teachers are seen as the holders of knowledge and wherein students are expected to replicate exactly what the teacher tells them to do rather than discursively engage it. In the case of CLT, there may be cultural and institutional commitments that position CLT as a proposition outside of the role of school. In the case of SC, as with SLT, there may be an issue with implementing an SC agenda where discourse between students and teachers—and between students and each other—is not culturally privileged. In the case of structuralism, manipulatives are expensive, and not even in the wealthiest of Western countries are manipulates available in every classroom of every school. In the case of PS, again, this perspective is value-laden—why solve a single problem multiple ways if the economic situation in the context calls only for one solution? In the case of CRP, cultural context is central. However, consider the situation where a wealthy, White, suburban school is attempting to deploy CRP without any input from minority populations, leading inevitably to perversion of its theoretical principles; critical educators are crucial for CRP's deployment. And finally, in the case of PBL, like structuralism, classroom materials and resources are required, as well as time and institutional flexibility regarding expectations of student outcomes within prescribed timelines and alignment with state warranties (e.g., government testing). We avow all of these cultural conditions as crucial to the employment of the eight epistemological paradigms in Fig. 1 but the purpose of this chapter was to survey the status of literature and knowledge regarding Type D in the recent Western context. The constraints listed are just some of the myriad constraints under which every teacher works (Ingram & Clay, 2000), and we leave it as an intellectual challenge to the reader to envision what implications these cultural considerations may have when considering Type D.

7 Implications for Lesson Planning

There are three additional implications related to Type D that teachers must consider when planning: the layout of the learning space, instrumentation related to lesson planning, and student assessment. We survey these briefly in this section.

7.1 Layout of the Learning Space

The layout of the room, the way the physical space is prepared, the way the board is organized, and the students' access to manipulatives and technology are all key aspects of lesson planning. For example, the Instructional Quality Assessment (IQA; Boston & Smith, 2009; Boston & Wilhelm, 2017) lesson observation rubric includes a section that requires a description of the physical classroom layout, as well as a section about the potential of the task. Liljedahl (2020) emphasized the importance of the space/physical layout in creating thinking classrooms, noting that the physical layout of the room must correlate with and support the goals of the lesson. Considering the different perspectives on planning and teaching lessons presented in this chapter, the corresponding layouts that teachers plan for each approach must be coherent with the teacher's epistemological commitments, the goals of the lesson, and their corresponding intended experiences for students.

7.2 Instrumentation Related to Lesson Planning

There are various instruments for evaluating the quality of lesson plans. These include but are not limited to the Guide to Core Issues in Mathematics Lesson Design (West & Staub, 2003) based on their Framework for Lesson Design and Analysis, the IQA Lesson Plan Rubric (Boston & Smith, 2009; Boston & Wilhelm, 2017), the Lesson Plan Evaluation Rubric (Panasuk et al., 2005), the Observation and Reflection Guide for a Mathematics Lesson (Grant et al., 2009), the Thinking Through a Lesson Protocol (Smith et al., 2008), and the 5E Lesson Plan Format (Goldston et al., 2010) which originated in science education. While these tools have been invaluable over the past two decades, lesson plan evaluation practice is lagging in research because it does not account for various the theoretical perspectives that we have described in this chapter. Indeed, Chizhik and Chizhik (2016) claimed that research on lesson planning is "stuck" in the behaviorist perspective. Furthermore, Medley (1987) argued that in order to conduct quality research, the issues of conceptualization, instrumentation, and design have to be addressed; thus, it is important to advance the design of instrumentation for lesson planning that does not discriminate against any particular theoretical perspective in which it is based. Without quality lesson planning, there cannot be quality instruction in mathematics classrooms.

7.3 Student Assessment

Analyzing student assessment is an integral part of teachers' reflection on their pre- and post-classroom activities. Research related to student assessment is discussed in depth by Radišić (this volume). The advent of educational technology and online

teaching has brought with it changes to student assessment. In virtual and physical mathematics classrooms, teachers use "technology as a servant" to serve as an assessment tool (Gueudet & Pepin, this volume; Geiger, 2005; Goos et al., 2000; Martinovic & Manizade, 2014). These considerations and more are discussed by Gueudet and Pepin (this volume). Due to space limitations, we will not expand on issues related to research on student assessment in this chapter.

8 Conclusion and Future Directions of Research

Teacher preparation and development programs provide "experiences designed to increase mathematics teachers' range of competencies" (Type J). They have been conceptualized in our adaptation of Medley's (1987) presage-process-product approach to understanding the inter-relationship of the variables that impact student learning outcomes as intervening between pre-existing mathematics teacher characteristics (Type F), and the competencies, knowledge, and skills (Type E) that teachers bring to the preactive tasks of teaching (Type D). Among Type E variables, there is arguably a dual focus in mathematics teacher education and development programs on teachers' knowledge and dispositions on the one hand, and on observable classroom competence on the other. This is evident in the oft-lamented gap between theory and practice noted by Dewey (1904) and identified by Korthagen (2017) as the central problem of teacher education and development throughout the twentieth century. According to Charalambous and Delaney (2020), practice is used in the mathematics education literature in at least four different ways: to distinguish between having an idea and enacting it; as something that is repeated with a view to improving performance; to describe the practice of teaching as having taken on the identity of a teacher; and, to describe classroom activities that are done habitually. While not excluding practices that occur in preparation for or following classroom activity, in each case the emphasis is on the teachers' actions in classrooms with students. That is, efforts to address the theory–practice gap in teacher education and development can been seen as an attempt to link Type E variables (mathematics teacher knowledge, competencies, skills, and beliefs) directly to Type C variables (interactive mathematics teacher activities that occur in the presence of students) with insufficient attention to Type D variables such as planning.

The approach taken in this chapter to the pre- and post-active actions of teaching foregrounds the importance of the teachers' theoretical perspectives in determining the goals they have for teaching and hence the kinds of activities that they plan to use in their teaching, as well as the ways in which they reflect on and self-assess their teaching. We argue that teachers' pre-active competencies such as lesson and unit planning—not simply as a technical skill or means of ensuring that novice teachers have "thought through" what they will do in their interactions with students, but as a *theoretically driven bridge* between teaching knowledge (typically characterized as theory) and practice—has the potential to address the traditionally perceived theory–practice gap. One way in which such an approach might manifest is in taking a step

back from lesson planning proformas and the like that specify such things as how to formulate the objectives of a lesson—to first reflect on the over-arching goals of one's mathematics teaching and its theoretical underpinnings. It should, thereby, be possible to trace a coherent theoretical perspective along the chain from Type E to Type D to Type C variables, influenced and constrained by Type J and Type I variables.

In conclusion, we found that some theoretical perspectives with respect to Type D have been researched well (such as CLT) while others (such as SLT and PBL) have not. With the current impetus of reform in the digital era of mathematics education, we believe these under-researched perspectives warrant further research with respect to Type D to investigate their potential for improving teachers' practice of lesson planning, assessment, and reflection. Elevating planning from a technical skill to a theoretically informed aspect of mathematics teaching would likely motivate further research on the topic as called for by Kilpatrick et al. (2001) who suggested that more research needs to be done on teacher planning, specifically, "What do teachers read when planning?", "How do they interpret and use what they read?", "And how do those uses affect their teaching?" (p. 337). The answers to each of these questions depend upon the theoretical perspective through which the teacher views the goals of their mathematics teaching. Consistent with this, our consideration of examples from the literature of each of the epistemological perspectives we identified in Fig. 1 highlight the scope of and need for mathematics education researchers to be more explicit in relation to the theoretical perspectives that underpin their own thinking and the studies on which they report.

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Interactive Mathematics Teacher Activities

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1   Introduction

The focus of this chapter is on interactive mathematics teacher activities (Type C). That is, the activities in which teachers engage in the presence of students. Researchers are often interested in innovative pedagogies aimed at enhancing the teaching and learning of mathematics. Studies, therefore, typically investigate classrooms in which teachers are participating in an intervention aimed at influencing their practice in ways deemed desirable by researchers or are attempting some kind of atypical practice that aligns with contemporary views of effective mathematics teaching. Fewer studies consider the nature of normative mathematics teaching practice. Those that do are necessarily large scale and provide less rich data than is usual for studies of atypical practice. They are, however, important for system level understanding and as a context in which to consider innovative practice.

Manizade, Moore and Beswick (this volume) describe eight epistemological perspectives that formed a framework for examining research on teacher's activities, such as lesson planning, reflecting on teaching, and assessing students, when not in the presence of students (i.e., Type D variables). Aspects of Type D that occur prior to teaching are intended to inform what happens when teachers interact directly with students (Type C). Nevertheless, we know that many factors intervene to ensure that there is rarely a direct translation from plan to practice by constraining the interactive

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activities that are feasible or desirable. They include Type I variables such as system and school policies and priorities, resources available to schools and choices about their allocation, resources of the families that schools serve, and cultural considerations; and Type E variables, specifically the skills, knowledge, dispositions, and accumulated experience that individual teachers bring to their task.

Apparent disjunctions between what teacher's say they believe about teaching (an aspect of Type E), including their epistemological perspective, and hence plan to do, (Type D), and what in fact happens in their classrooms (Type C) gave rise to studies pre-dating the focus of this review that highlighted apparent discrepancies between beliefs and practice (e.g., Frykholm, 1999; Sosniak et al., 1991). An important development in recent decades has been a growing consensus that teachers are reasonable when they state and enact their beliefs (Leatham, 2006) with more than a dozen ways in which apparent discrepancies can be reasonably explained having been documented (Liljedahl, 2008). In addition, Beswick (2003) highlighted the influence of the differing contexts in which teachers typically talk about their beliefs and then enact them. This certainly applies to the contexts in which teachers plan for interacting with students (Type D) and then implement those plans (Type C). Each of the epistemological paradigms identified by Manizade, Moore and Beswick (this volume) allow for a degree of contingency; that is, the teacher needs to respond to the ways in which students respond to teaching. Indeed contingency, defined as involving deviating from the plan, responding to student's ideas, and making use of unplanned opportunities, is a dimension of the Knowledge Quartet that was developed based on observations of teacher's practice and presented as a framework for observing mathematics teaching (Rowland & Turner, 2007). Speer (2005) argued that apparent discrepancies between teacher's espoused and enacted beliefs are likely artefacts of the research methods employed, specifically a failure to consider data from practice as well as teacher reports via surveys or interviews when attempting to infer their beliefs. Care also needs to be taken to ensure that there are shared understandings between teachers and researchers of the meanings of words and interpretations of events (Beswick, 2005; Schoenfeld, 2003) which in turn are influenced by the researcher's beliefs. In the case of large-scale studies, choices about the scales and items included reflect what the test designers assume to be desirable practices (Eriksson et al., 2019).

Consistent with this, research interest in particular interactive teaching practices has followed developments in theoretical understandings of mathematics teaching and learning and the epistemological perspectives from which practices have been examined. As noted by Manizade, Moore and Beswick (this volume), these are not always explicitly stated but can be inferred with varying degrees of confidence from reports of studies. In this chapter we review what we know about teacher behaviours in typical mathematics classrooms and discuss the range of less widespread pedagogical approaches that are evident in the literature. In both cases we make links to underpinning epistemological perspectives as described by Manizade, Moore and Beswick pointing to how these appear to have both influenced the Type C variables that have been of interest and that may relate to practices observed, although we recognise the difficulties inherent in making such connections. Large scale surveys,

for example, necessarily rely on teacher's self-reports rather than on direct observations. Desimone et al. (2005) cited research showing that although self-reports are acceptably reliable and valid measures of the content taught and the teaching strategies that are emphasised (e.g., Mullens & Gayler, 1999, as cited by Desimone et al. 2005), they may not be well-suited to measuring aspects of practice such as teacher-student interaction (e.g., Mullens & Kaspryzyk, 1999, as cited by Desimone et al., 2005). According to Eriksson et al. (2019), there was almost no connection between teacher's responses to items on an "instruction to Engage Students in Learning Scale" and student's achievement, leading them to recommend relying instead upon student reports about what happens in their classrooms. Studies that involve direct observations of teaching provide more certainty about actual classroom events but are necessarily smaller in scale and present their own challenges for researchers who seek to go beyond reporting what teachers and students do and say to make inferences about their intent and motivations.

We also examine what has been found about the impacts of technology on what happens in classrooms (real or virtual) in which teachers and students interact, and the theoretical lenses that informed the work. We conclude with reflections on aspects of classroom practice that have been less or un-scrutinized, but which warrant attention in future studies.

2   Our Approach

We conducted an organically evolving search of research articles discussing teacher's interactions with students in mathematics classrooms. We began with a search of relevant databases of the high-ranking mathematics education journals identified by Williams and Leatham (2017). The databases searched were ERIC, ProQuest Education, Informit A + Education, OECD Library, EBSCO Education Source and JSTOR. We began with a small number of relevant articles which were searched for relevant keywords (e.g., classroom environments, teacher-student interaction, teacher behaviors). Further researchers engaged in the field were identified and we focused specifically on articles that discussed teacher's actions in classrooms, and identified the different perspectives, methods, recommendations, and issues raised. We restricted the sample to publications dating from 2000 and conducted further searches by prominent authors in the field and keywords (e.g., mathematics pedagogy, mathematics teaching, classroom practice). We refer to older literature when it is important to framing more recent trends and identifying their progress over slightly longer timeframes.

The matched articles were transferred into an Excel spreadsheet in which they were categorised by title; author; date of publication; type of data (direct observation, indirect, other); type of activity (e.g., problem solving); theoretical approach; and whether it concerned existing practice or practice connected with an intervention. In addition, we found articles more broadly related to the topic, such as when a particular issue, e.g., conceptual understanding of fractions, was examined with an intervention

impacting teacher's actions. Those articles only tangentially related were excluded from the core analyses but were discussed in author meetings thus informing our discussion in this chapter. For normative practices we also referred to reports of large-scale international surveys.

The chapter is structured in two broad sections; the first describing the development of research about widely practiced teacher student interactions, and the second exploring studies that have considered teacher's behaviors with students in particular projects or in response to specific interventions. For the former, normative practices, we rely on large scale assessments of mathematics teaching and learning whereas for the atypical practices described in specific studies we refer to research reports available in the mathematics education literature.

3   Normative Mathematics Teaching Practices

In this section we survey what is known about what typically happens in mathematics classrooms. We rely primarily on the large-scale international surveys, *Trends in International Mathematics and Science Study* (TIMSS) that assess mathematics achievement at Years 4 and 8 in participating countries. The first TIMSS was conducted in 1995 but we confine our attention to those in the past two decades, beginning with TIMSS 2003. The *Programme for International Student Assessment* surveys (PISA) similarly provide insights into the classroom activities that constitute mathematics learning for 15-year-olds in participating countries. We begin with a brief overview of TIMSS and PISA before highlighting changes in the classroom activity that successive iterations of these surveys have revealed.

Country participation in TIMSS has steadily increased over the years reflecting increased interest at government and education system level in the performance of their students relative to those in other countries. In 2003, 46 countries participated from the continents of Africa, Asia, Australia, Europe, North America, Oceania, and South America (Gonzales et al., 2004). By 2019 participation had risen to 64 countries, representing a broad range of geographic, demographic, and economic diversity (Mullis et al., 2020). Although the focus of this book is on Western countries, TIMSS is relevant because it provides an international overview of mathematics education, allowing comparisons among countries and the identification of distinctive characteristics of mathematics teaching and curriculum objectives in particular countries of interest.

PISA, undertaken by the Organisation for Economic Co-operation and Development (OECD) assesses how well 15-year-old students can apply the knowledge and skills they have learned in the areas of reading, mathematics and science to real-life problems and situations. Seventy-nine countries participated in PISA in 2018 (Schleicher, 2019), which was the seventh cycle of the international assessment since the programme was launched in 2000. Each assessment focuses on one of the three subjects and provides a summary assessment of the other two. So, while Mathematics has been assessed by PISA once every 3 years since 2000, the mathematics

domain was the main area of focus only in 2003 and 2012. Mathematics will again be the major domain assessed in 2022.[1]

We focus on aspects of TIMSS and PISA that relate most directly to what teachers do, or are able to do, in the presence of students. In terms of constraints on teacher's activity with students, resources including teacher's expertise and time for mathematics teaching, are especially salient and hence considered here.

3.1   Resources for Teaching Mathematics

Hopper et al. (2017) explained how the TIMSS Context Questionnaire gathers data about two types of resources that affect the teaching of mathematics. These are Type I variables, beyond the direct control of the teacher but that, nevertheless, provide constraints and affordances for what teachers are able to do in their interactions with students (Type C). The first are general resources such as school infrastructure (e.g., buildings, and grounds, heating and lighting, classroom space), teaching supplies, and the availability of technology. The second resource type is specific to mathematics including such things as particular software, calculators, and instructional materials. Data are also gathered on the difficulty or otherwise of finding well-qualified mathematics teachers, and on the rates of attainment of tertiary discipline and pedagogical study deemed necessary for teaching mathematics that teachers have undertaken. While acknowledged as crude proxy for knowledge for teaching mathematics, the extent to which mathematics teachers have undertaken such studies contributes to Type E variables that inform and constrain Type D and hence Type C activities.

The amount of time that teachers are able to spend with their students constrains the kinds of activities in which they can engage. Lack of time is frequently cited by teachers as an obstacle to implementing innovative practices in their mathematics classrooms (e.g., Livy et al., 2021). There was a significant variation in the amount of mathematics instructional time across the 64 countries surveyed in TIMSS 2019. On average, the fourth-grade students received 154 h of mathematics instruction per year, which equated to approximately 17% of total instructional time. The average number of hours received by eighth grade students was 17 h less than in fourth grade (137 h or 13% of the total) (Mullis et al., 2020). The increase in the amount of mathematics instructional time since 2003 is noteworthy. Although the sample size in TIMSS 2003 was considerably smaller (19 countries at fourth grade and 35 countries at eighth grade) the data point to a smaller time allocation: on average, fourth-grade students in 2003 received 149 h of mathematics instruction per year, which equated to approximately 16% of total instructional time. The average number of hours received by eighth grade students in 2003 was 26 h less than in fourth grade (123 h or 12% of the total) (Mullis et al., 2004). The reduction in hours dedicated

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[1] The next PISA round was initially scheduled for 2021. It was postponed until 2022 due to the COVID-19 pandemic.

to mathematics instruction from Year 4 to Year 8 likely reflects the broader range of subject areas taught at eighth grade (Mullis et al., 2020).

3.2 Instructional Practices

Rožman and Klieme (2017) identified three major international trends in education based on contemporary educational policy and discourse. These were: an increased interest in regular assessment of student progress; greater advocacy of student-centred pedagogies; and promotion of reasoning and problem-solving rather than the development of computational and procedural skills as the goals of mathematics teaching. They investigated four cycles of TIMSS (1995–1999–2003–2007) at eighth grade across 18 countries. Only slight evidence of increased use of testing was found across TIMSS assessments from 1995 to 2007 (Rožman & Klieme, 2017). In relation to the second trend—greater advocacy of student-centred pedagogies – there was some evidence that associated pedagogical approaches, such as making connections between mathematics and student's daily lives and working in groups had increased in several countries, most particularly in East Asia. In relation to the third trend—the promotion of reasoning and problem-solving rather than the development of computational and procedural skills as the goals of mathematics teaching—contrary to expectations, there was an increased practice of computational skills, with a particular emphasis in Central and Eastern Europe. Despite an initial increase in the frequency of problem solving, there was a decrease from 2003–2007.

In the 2003 and 2007 TIMSS studies, Year 8 students were asked about instructional practices in their classrooms considered relevant to instructional quality (Eriksson et al., 2019). In their discussion Eriksson et al. (2019) focused on three items, namely: (1) we listen to the teacher give lecture-style presentations, (2) we relate what we are learning in mathematics to our daily lives and, (3) we memorise formulas and procedures. As Eriksson et al. (2019) pointed out there is no consensus as to the optimal frequency with which any of these practices should occur. The frequency of lecturing, for example, that might be considered beneficial depends upon what the teacher is aiming to achieve, that is their goals for teaching. As explained by Manizade, Moore and Beswick, a teacher adopting a behaviorist perspective is likely to be concerned with helping students to perform flawlessly the steps of a procedure to obtain correct answers to a class of mathematical problems. In this case telling students clearly the steps that need to be followed is likely to be effective. In contrast, from other perspectives such as social constructivism, where the goals of teaching relate to the quality of interactions among students and building subjective knowledge, much less frequent use of lecture style presentations would be deemed desirable.

TIMSS 2015 data indicated positive associations between instructional clarity and student achievement (Hooper et al., 2017) as did TIMSS 2019 (Mullis et al., 2020) which used updated scales to further explore this trend. Students at fourth grade in 2019 reported clearer instruction than did students in eighth grade: Most students in

fourth grade (95%) reported moderate to high clarity of instruction compared with only 46% of students in eighth grade.

TIMSS 2019, like TIMSS surveys since 1995, collected data on instructional practices and strategies. For mathematics these concerned how often students; worked on problems on their own, explained their answers in class, and decided on their own strategies for solving problems (Hooper et al., 2017). Just as the theoretical perspectives that teachers bring to their work influence the goals they have for their teaching (Manizade, Moore & Beswick) and hence the instructional practices that they are likely to adopt, the choice of items included in TIMSS studies reflect the theoretical perspectives, and their concomitant goals and practices, that are of interest to the test designers, influenced by theoretical developments and recent research on approaches to teaching mathematics. The three items listed from TIMSS 2019 suggest interest in the extent to which problem solving and reasoning, and collaborative or individual working, are fostered in mathematics classrooms. These are consistent with problem solving and social constructivist perspectives on mathematics teaching. Researchers have, across successive iterations of TIMSS, explored associations between particular instructional practices and mathematics achievement. As Eriksson et al. (2019) pointed out the results of these studies do not always support theoretical assumptions about what constitutes instructional quality. They suggest that instructional practices should only be considered characteristic of quality teaching if they are found empirically to support student achievement.

TIMSS video studies were conducted in 1995 and 1999. The 1995 study involved a total of 231 mathematics lessons in the United States (81 lessons), Germany (100 lessons), and Japan (50 lessons), while in the 1999 a total of 638 mathematics lessons were video recorded across the seven participating countries: Australia, Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland, and the US (Neubrand, 2006). Video studies offer an opportunity for teachers (and student) behaviors to be studied repeatedly from different theoretical standpoints, and to address different questions about what is happening in those classrooms. Researchers have been interested in such things as how teachers structure their lessons, the clarity of instruction, interruptions, and how homework is treated. For example, Neubrand (2006) re-analysed 22 lessons from each of the three participating countries in the 1995 study to explore the number and types of tasks that teachers offered their students in the three countries. The 1999 lessons have also been examined in terms of lesson structure, mathematical content, and instructional practices, and to discern differences in mathematics classroom activity in different countries. Hiebert et al., (2003) observed that while there were some similar features in the relatively higher achieving countries, there were also distinct differences. For example, eighth-grade lessons in all participating countries included both whole-class work and individual/small group work. However, lessons in Australia, the Netherlands and Switzerland allocated more time, on average, to students working individually or in small groups. Another finding of note was that across all of the participating countries, at least 80% of lesson time in eighth grade, on average, was dedicated to solving mathematics problems. But there was considerable variation in respect to drawing the relationships between mathematics problems and real-life situations

ranging from only 9% of problems per lesson in Japan to 42% of problems per lesson in the Netherlands. Regarding computers, relatively few eighth-grade lessons in the participating countries made use of them. However, 91% of eighth-grade lessons in the Netherlands used calculators; a percentage much higher than in the other countries which ranged from 31 to 56% of lessons (except in Japan where no reliable estimate could be reported due to their infrequent use). In summary, Hiebert et al., reported that 'no single method of teaching eighth-grade mathematics was observed in all the relatively higher achieving countries participating in this study' (2003, p. 15).

Eligible mathematics teachers and students in a representative sample of 150 PISA participating schools in eight countries (Australia, Finland, Latvia, Mexico, Portugal, Romania, Singapore, and Spain) responded to the OECD's Teaching and Learning International Surveys (TALIS) on classroom practice (OECD, 2017). Teachers and students were asked to rate teacher's use of eight classroom practices. These practices were clustered according to three broad teaching strategies: structuring practices, student-oriented practices, and enhanced learning activities. Structuring practices entailed the explicit specification of learning goals; student practice until all students have understood the content; and a summary presentation by the teacher of recently learned subject matter. Student-oriented practices were the differentiation of the work for students with learning difficulties or the ability to progress more quickly than their peers, and groupwork that allows students to devise a collective solution to a problem or task. Enhanced learning activities comprised students undertaking projects of at least one week's duration, an expectation that students explain their thinking, and encouragement to seek multiple ways to solve problems (OECD, 2017).

Both teachers and students reported that almost all mathematics teachers across participating countries used clear and structured teaching practices; specifically, explicitly stating learning goals; allowing students to practice until they understand the content; and providing summaries of recently learned content. The teacher's use of enhanced learning activities was also commonly reported by both teachers and students, suggesting strong encouragement of students to solve problems in more than one way, and a high expectation that students explain their thinking on complex problems. The use of project work lasting at least one week was less frequent. While used less often than the other two practices, most teachers and over half of students confirmed the use of student-oriented practices, i.e., giving different work to students according to their level of understanding, or the use of small groups for students to come up with joint solutions.

Structuring practices were the most frequently used teaching practices in mathematics classrooms, according to both teachers and students. According to the authors, "Since they (structured practices) aim to deliver an orderly and clear lesson, they could be seen as the necessary foundation for the development of any other practice. This would explain why they are so predominant in the teaching strategies implemented by teachers" (OECD, 2017, p. 7). Nevertheless, "classroom instruction time is a scarce resource, and an overemphasis on structuring practices could limit teachers in their use of other potentially more innovative strategies, such as enhanced learning activities and student-oriented strategies" (OECD, 2017, p. 7).

3.3 Teacher's Use of Technology

The growing presence of digital learning technologies has brought new opportunities and challenges for mathematics teachers. An array of mobile devices, application software and other online technologies have transformed the landscape of mathematics classrooms providing myriad pedagogical opportunities, notably in relation to problem-solving, experimentation and collaboration. Based on the most recent TALIS Vincent-Lancrin et al. (2019) noted that changes in the use of ICT in mathematics lessons has been a major driver of pedagogical innovation in mathematics classrooms, along with professional development of mathematics teachers through peer learning. However, the challenge for teachers to equip themselves with the requisite skills to effectively use new technologies and engage in higher-order pedagogical tasks is significant. An observation made by Handal and Campbell et al. in 2012 still has currency a decade on:

> In the case of online tools, there is a vast range of technologies available, but do teachers feel that they know how to find them and use them once located? A range of dynamic geometry software (e.g., Geometer's Sketchpad) and computer algebra software is available. These tools have a steep learning curve and teachers need to be able to model these technologies for students for use in the classroom. (2012, p. 394)

A corollary of a digitally-rich classroom is a shift in the role of the teacher and hence what they do in their direct interactions with students. This is particularly discernible in the context of the 'flipped classroom' where instructive videos typically replace 'traditional' homework tasks to allow more focused teaching in class time (Muir, 2020). In such circumstances where recorded teaching is made available to students to engage with in their own time, the teacher and each student are effectively interacting, albeit in a uni-directional way, asynchronously. Medley (1987) did not envisage interactions of this kind, but they have become increasingly common as technology has evolved and as circumstances have demanded the use of distance learning. Teacher behaviors as they engage in virtual asynchronous teaching are an aspect of Type C that warrants research. The content that is presented and whether or not it is presented in a way that elicits student-centered interactions depends on the theoretical perspective adopted by the teacher.

TIMSS 2019 investigated three areas relating to the use of technology: computer access for instruction; technology to support learning; and tests delivered on digital devices. Teachers were asked about availability of computers during mathematics lessons and the types of access i.e., whether each student has a computer, the classroom has shared computers, and/or the school sometimes gives access to computers. Teachers reported similar levels of access to computers at fourth and eighth grades (39% and 37% respectively), but there was variation in the level of access to computers across countries as well as in the types of access. The type of access most frequently reported for both fourth and eighth grades was that the school has computers that the class can sometimes use (29% and 28% respectively). Average student achievement was associated with access to computers at both grades, not

surprisingly given that access to computers would be related to socio-economic advantage (Mullis et al., 2020).

TIMMS 2019 also investigated the frequency with which teachers used computer activities to support learning in mathematics. Around two-thirds of students in both fourth and eighth grades were in classes in which their teacher reported that they "never or almost never" do computer activities to support learning (67% and 68% respectively). Average student achievement was lowest for students in these classes, with a 15 point-average difference at fourth level and an 18 point-average difference at eighth level (Mullis et al., 2020). The way in which teachers administer tests, and specifically whether they use computers or tablets for this purpose was also examined with eight grade students reporting the lowest occurrence of digitally delivered tests having the highest achievement.

4 Atypical Mathematics Teaching Practices

In this section, we consider studies that have addressed practices that have been less common in mathematics classrooms. We discuss the topics that have attracted researcher's attention when it comes to teacher's efforts to implement non-traditional practices and discuss aspects that have been most influential in shaping teacher's activity in mathematics classrooms in the last two decades.

Since 2000, smaller scale studies have emphasized the examination of pedagogical approaches based on constructivism, with many studies having involved examining the implementation and impact of particular practices. Teacher-student interactions have sometimes been observed directly, but artifacts such as teacher's lesson plans (Type D), have also been reviewed, and teacher actions inferred from them. Artifacts of this kind provide indirect insight into what teachers do in their classrooms but need to be interpreted carefully because of their indirectness. There are, for example, many reasons for which a lesson may not be implemented as planned. Small scale studies have focused on broad pedagogical approaches or perspectives (e.g., project-based learning, culturally responsive teaching), aspects of teacher's practices (e.g., questioning, types of listening), the organization of teaching and learning (e.g., flipped classrooms), and classroom environments. In the sections that follow we describe findings from these studies according to themes identified from the foci of the studies.

4.1 Pedagogical Approaches

Boaler (e.g., Boaler, 2001) has made extensive contributions to research on teacher's use of student-centered approaches complemented by practical work providing resources (underpinned perhaps by a social constructivist, cognitive learning theory, or structuralist perspective (Manizade, Moore, & Beswick)) and instructions for teachers to inform their classroom activity. In Boaler's work, the concept of rights of

the learners, that include such things as the right to be heard, make mistakes and be confused, requiring a degree of sensitivity from teacher's side (Kalinec-Craig, 2017) features as something that should guide teacher's interactions with students. What one considers to be the rights of a learner depends in part upon teacher's perspective on mathematics teaching and hence what the goal of teaching is. From a situated learning theory or social constructivist perspective it would be quite natural to allow students to voice their thinking whereas a teacher approaching their task from a behaviorist perspective might see this as detracting from the effectiveness of teaching aimed at the perfect performance of procedures. From this perspective, affording students rights necessarily constrains the actions available to and appropriate for teachers as they interact with students.

Fewer studies have considered how the student-centered approaches proposed are understood by teachers, or how they are translated in classrooms. Silver et al. (2009) analysed portfolio entries submitted by teachers. In the entries, teachers proposed lesson plans with pedagogical features to support the development of students' understanding. They found that teachers were not able to systematically embed innovative pedagogical approaches in their best practice submissions. While this study shed light on the degree of teacher's adaptation to some atypical practices, the study did not address the question of how each innovative, student-centred approach was understood by teachers; that is how the teachers defined and hence might enact the atypical practices they were proposing in their entries.

The research literature suggests that student-centered interactions and teacher's role in those interactions have been thoroughly researched and are well understood. Nevertheless, large-scale studies such as TIMSS and PISA suggest these approaches are not widely used. Reasons for the limited spread of student-centered approaches has been the subject of considerable speculation. For example, Buschman (2004) pointed to a "blame game", described as teachers commonly arguing that good activities don't exist and 'blaming' the supply of activities, as an explanation and canvassed many of the features of the debate about the uptake of atypical practices in which researchers in the field, have participated. These include: generic definitions of the approach in question (problem-solving as a loose term that refers to enhanced understanding, student centeredness and shifting the teaching from drilling to supporting genuine ideas); the realization that such practices have not been fully entertained by teachers, implying that the suggested practices would work as expected should the teachers only learn the way to acquire what is suggested to them; and providing informed, but not thoroughly evidence-based speculations about the situation.

Approaches that were innovative but not student-centered were hard to find in the body of research conducted in the last two decades, suggesting that perspectives that underpin teaching with features that could be characterised as student centred (e.g., social constructivism, structuralism, problem solving, culturally relevant pedagogy, and project and problem-based learning), are the lenses through which researchers have envisioned effective mathematics teaching. We struggled to find studies that examined innovative teacher-centered approaches and did not find studies taking a fresh perspective on behaviorist approaches.

There is, however, a body of research on cognitive load theory (Paas et al., 2004; Sweller, 2011), that has investigated teaching practices and techniques that reduce unnecessary load on students' working memory. Such an approach is, if not teacher-centred, at least teacher-led, and often considered as an opposing approach to student-led problem solving, inquiry-based learning or 'discovery learning' (Paas, 2004, p. 6), although the intention of the approach is not to avoid mental challenges, but to question the external interruptions that may appear in student-centred, inquiry-based or collaborative problem-solving settings. Best practices to reduce (unnecessary) cognitive load have been developed and delivered through laboratory studies, as well as within training programs for teachers (Van Merrienboer & Sweller, 2005). One can find comparative studies testing the effects of reduced cognitive load on student's learning (e.g., on geometry in Reis et al., 2012; the use of spreadsheets and sequencing in Clarke et al., 2005) but how teachers have applied those practices in their mathematics classrooms and the extent to which laboratory-based findings can be reproduced in classroom contexts seems less known.

The pedagogical approaches discussed above have their roots in ideas presented in earlier decades. For example, "a quasi-empirical" approach to mathematics teaching was proposed by Lerman (1990). In that approach, teachers were encouraged to take mathematical misconceptions as hypotheses (as a source of something productive) and investigate the conditions under which they might or might not work (and why). Similarly, Ball and colleagues (e.g., Ball & Bass, 2000) have contributed to the general understanding of student-centered, constructivist pedagogies. Schoenfeld (e.g., 1992) has been influential in elaborating and building understanding of problem-solving as a means of teaching mathematics. Influential elaborations such as these have likely contributed to student-centred, inquiry-based approaches becoming dominant in the small-scale intervention studies. Comparison of these studies with typical mathematics teaching practices discerned from large scale studies, along with studies that suggest many teachers may have deeply ingrained views aligned with a behaviorist perspective on teaching (Schoenfeld, 2018) offers an explanation for the limited traction that student centred teaching has achieved. Not only do theoretical perspectives constrain the behaviors of teachers in their interactions with students, but they also constrain the kinds of questions researchers ask, the way studies are designed, and the questions that remain unanswered.

In the next section we discuss approaches in mathematics classrooms, namely, the practices in, and organisation of, the environment of a mathematics classroom.

4.2 Aspects of Teacher's Practices

Burkhardt (2006) reviewed the benefits and the spread of teaching modelling in the mathematics classrooms, concluding that the approach is only moderately used despite the opportunities it affords for student learning. Boaler (2001) described research in which modeling was a practice that had made a difference in student's learning in an investigation contrasting mathematics teaching in two schools. She

concluded that teachers needed to change their practices to allow students to develop transferable problem-solving skills.

The concept of *robust understanding* was introduced by Schoenfeld et al. (2020) along with a framework for teaching in ways that support the development of student's robust understanding of mathematics (Schoenfeld, 2018). He described activities derived from three teacher's lessons and analysed them in terms of the framework. The three teachers differed in the aspects of the framework that they emphasized. Each was able to address some aspects but struggled in others. In general, teachers seemed to struggle to shift from pedagogies that develop procedural knowledge to facilitating more connected understanding, and to build on student's thinking, making sure everyone had access to opportunities to develop their agency (Schoenfeld, 2018). Similarly, Buschman (2004), noted that teachers often miss opportunities to build on student's ideas, and speculated that there is a need for more examples of the desired practice, more collaboration among teachers, and greater acceptance of making mistakes while adapting to new practices.

Others such as Conner et al. (2014) have examined ways in which teachers can support *argumentation,* while Handal and Bobis (2004) considered *thematically structured teaching.* Sullivan et al. (2003) investigated *context-based teaching* and Shahrill (2013), conducted a review of teacher's questioning, focusing on what makes questioning effective, rather than on what teachers are actually doing in relation to questioning.

A particular practice, "instructing between the desks" was investigated as part of the cross-cultural Lexicon project by Clarke and colleagues (e.g., Dong et al., 2015). In this project, aspects of teachers' practices were labelled in order to provide a vocabulary to make it easier for researchers and teachers to address the various aspects of teachers' conscious and unconscious actions in mathematics classrooms. Clarke and his team were able to identify significant cultural differences in the ways in which teachers facilitate students' learning. For example, instructing between the desks seems more casually and less systematically applied in many Western countries, but rigorously practiced as "Kikan-shido" in some cultures (O'Keefe et al., 2006). Linguistic aspects of mathematics teaching have also been addressed by Sfard (2021), who elaborated on the role of language in the mathematics learning process.

4.3   The Organization of Teaching and Learning

Flipped classrooms have attracted considerable attention from mathematics education researchers during the last two decades. The enactment of a flipped classroom relies on technology, as the learner needs to acquire some of the content through digital resources independently. The need for independence on the part of the learner has been suggested to require self-determination (Deci & Ryan, 2012) from the learner's side and being well informed of appropriate resources from the teacher's side (Muir, 2020; Muir & Geiger, 2016). Muir (2020) observed a teacher implementing a flipped classroom approach and concluded that with careful preparation, the teacher

was able to support all aspects of her students' self-determination (competence, autonomy, and relatedness), while also helping students to develop their conceptual and procedural knowledge.

In addition to the well-studied flipped classroom approach, we found several case studies of community engagement. Many of these studies were reported in conference proceedings, but there were also a few such cases documented in journal articles. For example, Leonard and Evans (2008) described an intervention in which teachers worked closely with local churches in urban settings to adapt practices from community building. The aim was to address social justice and improve cultural responsiveness. Leonard's and Evan's (ibid.) study serves as an alternative example of what teachers (with or without a research-connection) could engage with in order to widen their perceptions of what is possible to support mathematics learning, as well as to better meet the needs of their students as individuals with varying backgrounds.

4.4 Classroom Environments

Research studies are typically based on researchers' initiatives inspired by their beliefs about what constitutes good mathematics teaching. Teachers may adopt the new practice during an intervention, but reports of what happened before and after these interventions are rare.

Some researchers have made extensive efforts in creating resources to help teachers apply recommended ideas independently of participation in a project. Liljedahl (2019), for example, has suggested tangible changes in the classroom environment. His concept of "thinking classrooms" includes the use of vertical surfaces as a mean to support student argumentation. Working in small groups and documenting the mathematical work on vertical boards that everyone can see has attracted attention (as evidenced in teacher groups in social media) but is hard to find evidence of precisely how these practices have been adopted or the extent of their adoption.

Research literature is written and initiated by researchers, and when teachers share their ideas (for example, in professional journals or on social media), the accounts are mostly anecdotal. One of the authors of this chapter considers herself "an insider" in relation to what we can infer of teachers' attention to educational ideas in social media. Having her own media to spread research-based resources for teachers to use, she has learned that even if there is a "hype" from the teacher's side about a new practice, the real change may remain undone or only partially implemented. As Buschman (2004) explained, it is hard work for a teacher, who most likely has never been experienced alternative methods as a learner or observed them being used by colleagues, to adopt them, no matter how much value they might see in doing so.

In the digital era, online resources are also available for teachers to use for a range of purposes (e.g., as enriching the activities, outsourced feedback, creating excitement). Handal et al. (2013) reviewed more than one hundred mobile applications designed for mathematics learning. They categorized applications using three main clusters: explorative, productivity and instructive tools. It was noted that teachers

should understand an application's instructional value when deciding which to use as some are of little instructional value. They recommended a "watchful but enthusiastic eye" (p. 126) on new mobile learning developments in mathematics teaching.

Other examples of the ubiquity of digital resources vary from general organisation of teaching such as hybrid learning environments (Cribbs & Linder, 2013), to specific techniques, such as teaching with embodied learning technologies (Flood et al., 2020), or applications of known learning theories, such as cognitive load theory, in digital settings (Pass & Sweller, 2005). In an overview of the impact of the Internet on mathematics classrooms Engelbrecht et al. (2020) discussed the new meanings for old constructs such as 'tool', 'resources' or 'learning setting'. These new meanings, introduced in mathematics classrooms in the digital era include using Massive Open Online Courses and blended approaches (referred to as Principles of design), technologies in online contexts supporting social interaction and construction of knowledge, and online tools and resources (traditional resources in a digital form, as well as new conceptualisations of what is perceived as a mathematical activity).

In sum, the mathematics classroom as a physical environment has begun to be transformed along with the expansion of the digital world (Engelbrecht et al., 2020). Teachers teaching mathematics are no longer restricted to being the key source, let alone the sole source, of mathematical knowledge. What is more, Engelbrecht et al., (ibid.) discussed the Internet Era transforming the traditional teacher led push approach to mathematics teaching into a student led pull approach, increasing student engagement and agency. Again, the ways in which teachers have reacted to these recent opportunities is less documented (Clark-Wilson et al., 2020) but appears to vary from not using technology, supporting student's use of technology, through to deliberately eliciting student thinking with and through technology.

Finally, COVID-19 pandemic has accelerated the adoption of technologies in mathematics classrooms, and the impacts are yet to be fully identified. Some insights about impacts of digital technologies in mathematics education during the COVID-19 pandemic were discussed by Borba (2021). The sudden move to online classrooms around the world required teachers to react quickly and with minimal preparation. There is an urgent need to study how the mathematics learning process looks, and specifically what teachers do as they interact with students in new online settings on such a massive scale. The impacts of COVID-19 might have included a decrease in equity as a result of differing access to technologies according to student's socio-economic background (e.g., using a phone to attend the mathematics class instead of a computer) (Clark-Wilson, 2020). The pandemic necessitated all teachers of mathematics engaging with technologies to teach. Studies of teachers' activities with students will continue to need to include conceptions of mathematics classrooms that transcend physical boundaries.

5   Implications and Conclusions

The research considered in this chapter is far from exhaustive. Rather we surveyed a broad range of literature to identify the kinds of research being undertaken relating to teacher's interactive classroom behaviors, and the extent to which promoted practices are used beyond specific studies.

We distinguished between normative mathematics teaching practice and atypical mathematics teaching practices. Large scale studies such as TIMSS and PISA provide insight, albeit indirect, into what happens in the majority of mathematics classrooms. It seems that, in contrast with the student-centered approaches that have dominated mathematics education literature in recent decades, behaviorist approaches remain prevalent. Researcher's beliefs about, or theoretical perspective on, mathematics teaching inform and constrain their research (its design, conduct and reporting) just as teacher's theoretical perspectives in either the pre-active (Type D) or interactive phase of teaching (Type C) limit the actions that they perform in their classrooms. The mismatch between the teacher behaviors that researchers advocate and the pedagogies that students most commonly report experiencing raise the longstanding issue of how teacher's practice can be influenced in ways deemed desirable. Researchers' interests in particular perspectives on teaching mathematics seem also to have limited research on the practices that most commonly occur in mathematics classrooms. A better understanding of these practices, including the reasons for which teachers adopt and often stick with them, and the variations in context and the practices themselves that affect their efficacy would be valuable in its own right as well to inform efforts to influence teacher's interactive classroom activity.

In some classrooms technology has had a profound impact on pedagogical possibilities and has led to new ways of structuring teaching such as flipped classrooms. There has been recognition that in a digital world interactions between teacher and students can be both virtual and asynchronous. This development extends Medley's conception of Type C variables research on teacher behaviors during online synchronous or asynchronous teaching. It problematizes what it in fact means to be in the presence of students.

It is apparent that researchers bring their own theoretical perspectives and beliefs to their work, just as teachers do. The theoretical perspectives described by Manizade, Moore, and Beswick apply equally well to interactive teacher behaviors (Type C) and to pre- and post-active teacher behaviors (Type D). Our review has also highlighted the relative dearth, beyond large scale studies, of research on normative interactions in mathematics classrooms.

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Student Mathematics Learning Activities

Maria A. Timmerman

1   Introduction

In the late 1980s, to understand "good" teachers and improve teaching practices, Medley (1987) reviewed prior research on teaching and teacher education and identified 10 different variables that were studied to determine effective teaching (Introduction, this volume). Using a chain of effects of presage-process–product research, he reviewed studies that focused on measuring teaching and student behaviors that resulted in desired student learning outcomes. Further, he identified six of the 10 variables (Types A—F) as "online variables" (p. 105) that were in direct control of the teacher and these variables could be studied individually or in relationships between two or more variables. Using Manizade et al.'s (2019) adaptation of Medley's work for mathematics education (Introduction, this volume), this chapter describes an analysis and review of the literature relevant to the Type B variable, student engagement in mathematics learning activities, over the last three decades. According to Medley, *student learning activities* are defined in the following way:

> Pupil learning activities occur in the classroom. The principal means by which teaching can affect learning outcomes is through its influence on pupil behaviors in the classroom. The function of teaching is to provide pupils with experiences that will result in desired outcomes. It is axiomatic that all learning depends on the activity of the learner. (p. 105)

As mathematics education researchers, we are interested in examining relationships between how students engage in or approach student learning activities (Type B) that result in the successful achievement of desired student learning outcomes (Type A). It is the teacher practices in classrooms (Type C) that are needed to facilitate effective and equitable student interactions with learning activities in which

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students develop mathematical content knowledge and engage in the process of doing mathematics.

Yet, what are the characteristics of student engagement in learning activities that promote the development of content knowledge? What behaviors do students actively engage in while learning mathematics that reflect what it means to *know and do mathematics*? Can these learning activities be generalized across diverse K-12 classrooms, including settings that use a wide range of technological tools that support a "synergistic relationship" between technical and conceptual dimensions of mathematical activity (Zbiek et al., 2007)? How do teachers facilitate and enhance students' experiences while learning mathematics? One way to address these questions is to consider a review since Medley's work of how the global mathematics education community has described constructs that further explore students' development of mathematics content knowledge and engagement in learning activities while doing mathematics.

A historical review of reform-based mathematics curriculum initiatives provides insight into visions of various student learning activities, including the use of technology, which impact how students engage in knowing and doing mathematics (Sect. 2). To address the many names for these activities, I use Kobett and Karp's (2020, p. 40) terminology of *behaviors and dispositions* (i.e., proficiencies, processes, practices, competencies, and habits of mind) to identify the multiple and intersecting student experiences that are relevant to how students develop and show evidence of their mathematical thinking. Section 3 articulates multiple theoretical perspectives that capture how the process of student learning occurs in different learning environments. This is followed by studies relevant to student engagement in making sense of mathematics (problem-solving behaviors) and perseverance (productive dispositions) that are often linked to instructional practices to support desired learning outcomes (Sect. 4). For some studies, Medley's methodology concerns are addressed related to the quality and effectiveness of research. Lastly, a discussion of findings is presented and implications for future mathematics education research in the area of student mathematics learning activities and active student engagement in knowing and doing mathematics (Sect. 5).

2   Student Mathematics Learning Activities: An Overview

Over the past several decades, early reform initiatives in the United States [U.S.] (National Council of Teachers of Mathematics [NCTM], 1980, 1989, 1991, 1995, 2000; National Research Council [NRC], 2001) and other countries, such as Denmark, New Zealand, and Australia (Davidson et al., 2019; Hipkins, 2018; McDowell & Hipkins, 2018; Niss, 2003) have promoted new curricula frameworks to develop mathematics content knowledge and learning activities to improve student mathematics achievement. The organization of curriculum centered on content at different grade bands with some consideration of behaviors needed to engage students in learning mathematics. Student mathematics learning activities are a set of behaviors and dispositions students engage in to achieve learning goals that reflect an

in-depth understanding of mathematics. From the last three decades, this overview documents a shift toward a focus on student thinking needed to build a conceptual understanding of mathematics and identifying how students should experience solving mathematical tasks. A review of reform initiatives shows an evolution of specificity of learner activities envisioned to meet high-quality curriculum goals that support students' learning of mathematics with understanding.

Beginning in the mid-1970s and into the decade of the 1980s, school curriculum reform focused on accountability and measurable standards that demonstrated students' achievement in mathematics (Cuban, 1992; Pink, 1989). Teacher certification standards and higher student graduation requirements were raised in the hopes of improving the teaching and learning of mathematics. The 1983 publication, *A Nation at Risk* (National Commission, 1983) reported the failure of the U.S. school system with the decline of student test scores and achievement levels. Students lacked mathematical competence and they were unable to problem solve. At the same time, the business community became aware of a shrinking supply of skilled workers causing them to become involved in public school reform (Cuban, 1992; Martin, 1989; Sola, 1989). According to Martin (1989), businesses supported education initiatives because of the potential of providing skilled workers, including those able to work with the emergence of technology. Yet, the need for accountability prompted a return to teaching basic skills and the measurement of student behavioral objectives (i.e., achievement of *performance goals*) where students completed rote procedures and computations that could be easily measured.

During the decades of the 1970s and 1980s, what appeared to be missing was a focus on measuring student achievement of *learning goals* (Smith & Sherin, 2019). Moving beyond equating knowing mathematics as successfully completing procedures, researchers needed to show evidence of what students "understood" about specific mathematics content as a result of engaging in learning experiences in the classroom. In response to the needs of the discipline and society for the 1980s, NCTM published the *Agenda for Action* (1980)*,* which recommended future directions for improving the teaching and learning of mathematics. Based on reports of low mathematics performance, the student behavior of *problem solving* became central for engaging students in a mathematics learning activity and has remained a primary focus in curriculum initiatives over the last three decades.

In the 1990s, the NCTM trilogy of U.S. *Standards* reform initiatives (1989, 1991, 1995) provided a vision for the organization of school mathematics curriculum and evaluation, teaching, and assessment. The sets of standards described the nature of mathematics with an emphasis on students developing a conceptual understanding of mathematics rather than an acquisition of procedural knowledge, skills, and facts. Based on interpretations of Piaget's (1970) and Vygotsky's (1981) work, constructivist and social constructivist theories of learning supported a new vision of students constructing knowledge individually or collaboratively, rather than passively receiving knowledge. Mathematics represented a dynamic, changing discipline rather than a static body of knowledge. However, it is critical to state that the early sets of NCTM standards represented "statements of values" and that underlying assumptions about the teaching and learning of mathematics "were not well anchored in

either research or theory" (Kilpatrick, 2003, p. 1). Likewise, Lesh et al.'s (2020) recent review of learning theories in mathematics education found that the early "NCTM Standards themselves were not based on any research per se, but simply an envisioning of what mathematics education in classrooms (i.e., in practice) might look like and what the appropriate content might look like, keeping the learner in mind" (p. 862). One of the issues relevant to a lack of research may be attributed to transitioning from past theories and methods of measuring procedural, student performance goals to a vision of measuring conceptual, student learning goals often showed little, if any, research related to new ways of teaching and learning. This is because the sets of standards had not been implemented in many mathematics classrooms. Moreover, although the curriculum initiatives promoted mathematics content learning goals and engagement in student mathematical learning activities (i.e., behaviors and dispositions), teaching practices (Type C) that support student learning with understanding were missing.

In response to a lack of research and explicitly connected to an updated version of U.S. standards (NCTM, 2000), Kilpatrick (2003) asserted that a companion publication (NCTM, 2003) synthesized a review of the literature that informed the vision of school mathematics in the 1990s and 2000. In this publication, Sfard reviewed learning theory research and identified ten mathematical learner needs that were reflected in the curriculum changes of the NCTM standards. For example, she identified learners as having a "need for meaning and the need to understand ourselves and the world around us have come to be recognized as the basic driving force behind all our intellectual activities" (p. 356). Bringing the needs of learners to the forefront, researchers raised new questions about how to measure student behaviors and dispositions that provide detailed explanations of students' need for "meaning" while learning mathematics with understanding and what does this look like in mathematics classrooms.

Recognizing the ever-present dilemma of balancing the needs of mathematics (discipline theory) and the needs of the learner (psychological theory) in the organization of curriculum, Sfard asserted: "In our attempts to improve the learning of mathematics, we will always remain torn between two concerns: Our concern about the learner and our concern about the quality of the mathematics being learned" (p. 386). When one of these theories controls too much of the school mathematics curriculum, then disruption occurs within the entire curriculum. Over the last three decades and across different countries, the challenge of this dilemma has continued to be addressed with frameworks of curriculum initiatives that identify content knowledge students should know and processes students need to engage in while doing mathematics. Reviewing the relevant literature, a number of terms and documents pertaining to student behaviors and dispositions will appear in this section and be discussed further throughout the chapter. Brief, capsule definitions of these terms and documents are included in the Appendix. The goal of the following subsections is to identify and compare student mathematics learning activities (Type B) that have evolved with students becoming knowers and doers of mathematics.

2.1 Mathematical Processes

After much debate related to the dilemma Sfard (2003) articulated about balancing the needs of both the discipline and learners, the U.S. *Principles and Standards of School Mathematics* (NCTM, 2000) expanded the vision of mathematics education to include a more deliberate focus on school curriculum organized around the framework of *process standards* to promote learning activities students should engage in while doing mathematics. Rather than describe performance goals of doing procedures, the processes defined what mathematicians might do and say when problem solving. The process standards recommended providing all students opportunities to learn mathematics through engagement in five overlapping processes: problem solving, communication, representation, making connections, and reasoning and proof (NCTM, 2000). Problem solving is the primary action of mathematics activity and it has always been recommended as way to know and do mathematics (NCTM, 1980). The learning activity of reasoning develops through problem solving. Compared to an earlier set of process standards (NCTM, 1989), *representation* was added to the original four processes as a way to engage students in making their mathematical thinking explicit. To support students' development of mathematical reasoning and proof, Huinker (2015) extended Lesh et al. (1987) modes of representation: contextual, physical, visual, verbal, and symbolic, with an explicit focus on students building *representational competence* from which mathematical connections are made "between" and "within" representations. In Sect. 2.5, Zbeik et al. (2007) use an equivalent term of *representation fluency* as a construct to describe students' access and engagement with multiple representations in technological environments. The process standards inform ways students could participate while engaged in knowing and doing mathematics.

2.2 Mathematical Competencies

At the same time, in 2000, the Denmark Ministry of Education created a national committee to examine ways to improve mathematics teaching and learning. Their work resulted in the *Mathematical Competencies and Learning of Mathematics: The Danish KOM Project* (Niss, 2003). In this report, *mathematical competence* was defined as having "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role" (p. 7). The project identified eight mathematical competencies that demonstrated evidence of students' "mental or physical processes, activities, and behaviors" (p. 9). The competencies extended NCTM's process standards and included: thinking mathematically, posing and solving mathematical problems, modeling mathematically, reasoning mathematically, representing mathematical entities, handling mathematical symbols and formalisms, communicating in, with, and about mathematics, and making use of aids and tools (including

instructional technology) (Niss, 2003). This framework of mathematical competencies is relevant to Manizade et al.'s (2019) adaptation of Medley's Type B variable as they identify learning experiences students should engage in to develop a deep understanding of mathematics articulated in high-quality curriculum goals.

In a similar vein focused on identifying mathematical competencies, the Program for International Student Assessment [PISA] (PISA, 2021) measures to what extent 15-year-olds use their many years of building mathematical knowledge to solve real-world problems. In students' lives outside of school, they need to solve problems that often demand the use and integration of multiple mathematical topics, rather than only knowing how to use a single procedure learned in a mathematics lesson. PISA assesses different *mathematical competencies* that gauge students' mathematical literacy; that is, "an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts" (PISA, 2021). The PISA mathematical literacy framework lists multiple competencies under each of three clusters: reproduction, connections, and reflection. As a research fellow at the Australian Council for Educational Research (ACER) at the beginning of the last decade, Turner (2010) reviewed research analyzing PISA mathematics test items. To be successful in solving contextual problems, he found that students needed to activate prior mathematical knowledge. Further, he reported students' difficulty in problem solving when they needed to activate more rather than fewer mathematics competencies. Similar to Denmark's competencies, the PISA competencies included the following: communication, mathematising, representation, reasoning and argument, strategic thinking, and using symbolic, formal, and technical language and operations. Turner argued for teacher activities (Type C) in which they increased a focus on these competencies (Type B) to engage students in developing mathematical literacy.

Over the last two decades, a Ministry of Education-funded project, *Competencies in New Zealand Curriculum (NZC)* (McDowall & Hipkins, 2018; Hipkins, 2018), described an evolution and research base of *key competencies* for student learning in general and eight content learning areas for the twenty-first century. Connected to a PISA framework, a construct of competencies originated from an Organization for Economic Development (OECD) Definition and Selection Competencies (DeSeCo) Project which produced a framework to guide the development of PISA assessments (Hipkins, 2018). For the NZC, each learning area described "what they [students] will come to know and do" (Ministry of Education, 2015, p. 37) and identified five key competencies: thinking, relating to others, using language, symbols, and text, managing self, and participating/contributing. According to the Ministry of Education (2020), "Key competencies matter because they support dispositions that will enable young people to learn well now, and to go on learning throughout their lives… Dispositions mean learners are ready (i.e., being motivated to use particular knowledge, skills, and values to achieve the task at hand), willing (i.e., recognizing when it is relevant to draw on these), and able (i.e., knowing how to do so appropriately)." Similar to the framework of proficiency strands (NRC, 2001) and Kobett and Karp's explicit inclusion of "disposition" when describing students' knowing and doing mathematics, the NZC recognized the critical role of dispositions needed for current

and future student learning. In mathematics and statistics, "students explore relationships in quantities, space, and data and learn to express relationships in ways that help them to make sense of the world around them" (p. 17). When examining mathematical connections to four of the key competencies: thinking, relating to others, using language, symbols, and text, and participating and contributing, the NZC stated: "Students develop the ability to think creatively, critically, strategically and logically… They learn to create models and predict outcomes, to conjecture, to justify and verify, and to seek patterns and generalizations… [there is] a broad range of practical applications in everyday life, in other learning areas, and in workplaces" (p. 26). Within the NZC, three interrelated strands of eight levels of achievement objectives are identified: number and algebra, geometry and measurement, and statistics. Each level begins with this statement: "In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will solve problems and model situations that require them to:" (Ministry of Education, 2014). Similar to other frameworks of competencies described previously, there is a focus on students engaged in thinking, meaningful contexts, knowing, doing, and dispositions.

McDowall and Hipkins' (2018) review of large systematic studies that examined competencies in the NZC resulted in emergent themes that defined "four phases in the ways that key competencies have been understood and enacted in the overall school curriculum" (p. 2). Between 2006 and 2018, these phases provided a "trajectory of change" when considering the nature of student learning and how to weave the competencies into the curriculum. As an example, although there was overlap between the phases, in phase two (i.e., 2007–2011), "relationships between key competencies and ideas about learning to learn (an *NZC* principle) and lifelong learning (a part of the *NZC* vision)" (p. 7) came to the forefront. Research examined how the NZC was implemented across multiple schools and what barriers existed. A shift occurred in phase three (i.e., 2011–2014) with a recognition of a need for the "weaving of key competencies and learning area content" (p. 9); that is, relationships were examined between competencies and desired discipline-specific learning outcomes (Type A).

Moreover, "students' opportunities to develop their key competencies were closely tied to the pedagogy used by the teacher" (p. 9) (Type B and C variables). To engage students in learning activities, they needed tasks where they took "meaningful action in real-world contexts" (p. 10) and other pedagogical approaches included critical inquiry and experimental learning. To investigate phase four studies, which are ongoing, McDowall and Hipkins (2018) reported: (1) "Students should actively use and build knowledge, as opposed to just being consumers of knowledge produced by others;" (2) "There should be opportunities for students to collaborate in more demanding ways than simply group work;" and (3) "The diverse life experiences and ways of being that students bring to learning are seen as a resource for learning rather than a problem to be managed" (p. 12). Looking ahead to future research, Hipkins et al. (2018) examined the OECD 2030 Learning Framework (p. 2) and its alignment and implications for the NZC. As in the past, the 2030 framework identifies a focus on knowledge, skills, attitudes, and values leading to competencies for individual and

societal well-being. The OECD framework development is a collaborative, international project and a work-in-progress. It is intended to update the DeSeCo framework for PISA assessments and provide a pathway for future research connecting student learning activities, teacher activities, and student learning outcomes (Type A, B, and C variables, Introduction, this volume).

2.3 Mathematical Proficiency

In the same time period as the updated NCTM (2000) process standards, the National Research Council's [NRC] Mathematics Learning Study Committee published *Adding It Up: Helping Children Learn Mathematics* (2001) to identify how students attain *mathematical proficiency* through cognitive and affective engagement within these five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. By including the last strand, *productive disposition*, the NRC committee asserted the value of beliefs, attitudes, and emotions and their affective impact on students' engagement in learning mathematics. According to NRC, conceptual understanding is defined as the "comprehension of mathematical concepts, operations, and relationships" and productive disposition is the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 116). Making connections to the strands, Kobett and Karp (2020) mapped each proficiency to examples of what students' strength behaviors look like in a classroom setting. For conceptual understanding, they included a student question, "Why do we call some numbers square numbers? Why do we call some numbers cube numbers?" and explained: "When students make a comment that something doesn't make sense to them, that is an indication that they desire mathematics should be a sense-making activity" (p. 42). Not only was this student engaged in making sense of the meaning of different types of numbers, the student asked *why* questions to develop reasoning about the structure of numbers.

In Australia, the national curriculum standards identified *mathematical reasoning* as both a process that demonstrates mathematical thinking and a strategy for learning mathematics (Davidson et al., 2019). According to the Australian Curriculum and Assessment Reporting Authority (ACARA, 2017), reasoning is one of the four *proficiency strands* students engage in when "thinking and doing of mathematics." In other words, the process of reasoning provides insight into students' mathematical thinking and their engagement in student learning activities. The other three proficiency strands are understanding, fluency, and problem-solving. The four Australian proficiency strands "describe the actions in which students can engage when learning and using the content" (ACARA, 2017). Thus, the proficiency strands suggest a call for research that examines students' mathematical thinking when developing content knowledge (learning) and engagement in doing mathematics (using the content).

2.4 Standards for Mathematical Practice

In 2010, the U.S. created the national *Common Core State Standards for Mathematics (CCSSM)* (National Governors Association [NGA] Center for Best Practices and Council of Chief State School Officers [CCSSO], 2010), which included specific mathematical competencies for students called the *Standards for Mathematical Practice* (SMP). Many of the same international mathematical behaviors identified previously were stated: (1) make sense of problems and persevere in solving them, (2) reason abstractly and quantitatively, (3) construct viable arguments and critique the reasoning of others, (4) model with mathematics, (5) use appropriate tools strategically, (6) attend to precision, (7) look for and make use of structure, and (8) look for and express regularity in repeated reasoning. For the U.S., the CCSSM continued an evolution of reform visions stated in earlier initiatives and by other international researchers (Bostic & Sondergeld, 2015; Hipkins, 2018; Keazer & Jung, 2020; Kobett & Karp, 2020; Koestler et al., 2013; McDowall & Hipkins, 2018; NRC, 2001; Sanchez et al., 2015; Sfard, 2003; Turner, 2010). One purpose for creating the CCSSM was to provide consistency across the U.S. in K-12 grade-level curriculum standards rather than each state having different standards. The eight SMP described how students should engage in mathematics learning activities to become "doers of mathematics" (Kobett & Karp, 2020, p. 40).

In summary, when reviewing the aforementioned frameworks of curriculum initiatives, there is a shift toward making explicit how students should experience doing mathematics while making sense of their developing mathematical content knowledge. To demonstrate the evolution of student learning activities across different reform initiatives, a few mathematics educators have compared behaviors and dispositions found in the documents. Kobert and Karp described connections between the mathematical proficiency strands and SMP. If researchers use Manizade et al.'s (2019) framework (Introduction, this volume) for examining relationships between classroom Type C and B variables (i.e., teacher-student activities), studies could address Kobert and Karp's challenge: "We want teachers to think about how their students *respond to* and *interact* with mathematics learning via each of these components and that, in doing so, they listen for whispers of their students' previously undetected strengths" (p. 41). What research exists that documents how students engage in learning activities portrayed in frameworks of curriculum initiatives to develop a deep understanding of mathematics and how do teachers listen and respond to their students? Recently, Lesh et al. (2020) argued: "The mathematics education community still does not know how to operationally define measurable conceptions of almost any of the higher-level understandings or abilities that the CCSC Standards refers to as mathematical practices" (p. 863). In essence, when working with the complexity of studying the nature of students' mathematical learning with understanding and student engagement in a range of mathematical practices (i.e., behaviors and dispositions), do studies exist for the knowledge base that provide evidence of measures to define effective and equitable student experiences with learning activities in mathematics classrooms, including technology-based environments?

A potential line of research could take advantage of Koestler et al.'s and Kobett and Karp's alignment between the NCTM process standards and the Common Core standards of mathematical practice. These authors presented classroom vignettes for each SMP to illustrate how students engaged in doing these learning activities. Specifically, the problem-solving process standard was connected to all eight SMP. This suggests if researchers focused on students' engagement with the first SMP, *make sense of problems and persevere in solving them*, there is a strong possibility that students will be engaged in the other "higher-level" practices. Given that similar practices are articulated across international frameworks of curriculum initiatives, research is warranted to provide evidence of students' engagement in problem-solving behaviors (i.e., making sense of mathematics) and productive dispositions (i.e., perseverance).

2.5   Cognitive Technological Tools and Student Mathematics Learning Activities

In the *Second Handbook of Research on Mathematics Teaching and Learning*, Zbiek et al. (2007) articulated a perspective of multiple constructs researchers should use to examine students' mathematical understanding while engaged in technology-based learning activities. Reviewing earlier research, the authors used the term *cognitive technological (CT) tools* to represent a wide variety of technologies that reflect a technical dimension, conceptual dimension, and a "synergistic relationship" among these two dimensions. Focusing on the technical dimension, CT tools "must allow the user the means to take actions on mathematical objects or representations of these objects" (p. 1171). Examining the conceptual dimension, CT tools provide "reactive visual feedback" as "observable evidence of the consequences of the user's actions" (p. 1171). Zbiek et al. cautioned researchers against the study of mathematics teaching and learning in technological settings using only one dimension. This is attributed to the fact that student learning activities may include technical actions, such as solving equations and graphing, and simultaneously these actions are informed by students' conceptual understanding and reasoning, such as conjecturing, finding patterns, and generalizing. Similarly, in the recent *Compendium for Research in Mathematics Education*, Roschelle et al. (2017) described a change in technology media over the last two decades from static to dynamic representations whereby students learn mathematics with understanding over time. Roschelle et al. identified *dynamism* as a new construct that incorporates a "time dimension" for students making sense of mathematics through dynamic representations. Specifically, they asked: "How is a mathematical representation being connected to a student's experience of time to advance understanding of mathematical relationships?" (p. 863). To support students' learning of difficult mathematical topics, Roschelle et al. used the "design of dynamic representations to enable new means of access [for students] to the topic" (p. 865). In Sect. 3, two of the emerging theoretical perspectives are grounded in conceptual studies (Hackenburg, 2010; Simon et al., 2016, 2018) whereby students

use the dynamism of computer microworlds to support research focused on the interrelationship between technical and conceptual dimensions.

When students engage in doing technology-based mathematics learning activities, they may set goals and search to find appropriate CT tools that are needed to solve a mathematical task. Dependent upon the cognitive demand of a task, students can set different types of goals (i.e., performance or learning) which results in students exhibiting different types of behaviors. When using these CT tools, Zbiek et al. (2007) identified two types of activities students engage in when solving tasks: *exploratory* and *expressive* (p. 1180). Building on mathematical modeling research (Bliss & Ogborn, 1989), students engaged in doing exploratory activities will follow teacher instructions to use specific CT tools and procedures. On the other hand, expressive activities allow students to select their own CT tools and make their own decisions on how they will solve a technology-based task. Mathematics curricula often include "explorations" for students to engage with different learning activities and dependent upon how much teacher direction (Type C) is given, elements of both exploratory and expressive activity can be observed. Examining how students engage in doing mathematics through these two forms of activity will often result in different student learning outcomes (Type A). As an example of expressive activity, Zbiek et al. described the role of "play" in learning where students were allowed the freedom of unstructured play and time to try a range of different actions with CT tools to determine what was possible or not possible as they viewed the results of their actions. Students engaged individually or with partners and eagerly called out what they observed in a technological setting. However, the conundrum of the "play paradox" (Hoyles & Noss, 1992) comes to the forefront, where many CT tools offer students such a wide range of processes for solving problems, that they may never encounter the mathematical content a teacher intended or what the designers of a technology-based activity planned. Zbiek et al. offered mixed research results on the productive use of unstructured, expressive play versus structured, exploratory play to engage students in learning and doing mathematics.

Moreover, in a technological setting, researchers have examined both types of activity (i.e., exploratory or expressive) that students engaged in and made observations of students' corresponding behaviors which "lead to insights about the appropriateness of their use of those tools and about their understanding of mathematics" (Zbiek et al., 2007, p. 1184). Specifically, inferences about students' mathematical thinking were supported by students' actions with CT tools, which in turn, reflected students' mental actions. To categorize student behaviors, Zbiek et al. introduced the construct of *work method* which draws upon the research of Guin and Trouche (1999) and Trouche (2005). In a 1999 study of 17- to 18-year-old students' engagement with mathematical tasks that included an option to use symbolic calculators, Guin and Trouche reported five different student work methods: random, mechanical, resourceful, rational, and theoretical. As an example, students using a *random work method* would search using trial and error to find a CT tool action that would give any answer (i.e., correct or incorrect) for a mathematical task. Yet, students' engagement in a random process of finding any result often provided evidence of students missing the mathematical analysis of a problem. In other words, students

accepted the results without any reflection related to the underlying mathematics which hindered their ability to achieve mathematical learning goals.

Revisiting the development of frameworks for mathematics curricula designed for student engagement with learning activities, some researchers (Sandoval et al., 2000 and Hong & Thomas, 2002 as cited in Zbiek et al., 2007) have identified the construct of *representational fluency* as a lens to study students' learning by noticing how and why students interact and make sense of multiple representations of the same mathematics entity. How might students think differently about possible models and strategies for problem solving in a technological environment that provides quick access to multiple representations? Also, how could the selection of mathematics content go beyond traditional school mathematics due to the potential capabilities of CT tools? Consistent with other researchers, Zbiek et al. described representational fluency as "the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (p. 1192). Access to technology can provide learners with opportunities to use different actions to 'try out' multiple representations and make sense of expected or unexpected results. As students reflect on their actions and begin to understand the meaning of each representation, they have an opportunity to develop representational fluency which could lead to a deep understanding of mathematical concepts.

Taken together, addressing research studies examining student engagement in learning activities (Type B) portrayed in frameworks of curriculum initiatives, including technological environments, provides insight relevant to both cognitive and affective aspects of student learners as they become knowers and doers of mathematics. To address Lesh et al.'s (2020) concerns, researchers can ask: How have we transitioned from measuring student learning for lower-level procedural outcomes toward analyzing student learning associated with desired higher-level thinking student outcomes (Type A)? One way researchers may respond is to consider a review since Medley's work of important constructs that interpret existing research and target new areas of research with a focus on the complexity of the learning and teaching process; that is, the interrelationships between teachers, students, mathematical activities, curriculum content, and the added effect of technology. In the next section, three theoretical perspectives provide explanations relevant to how and why student behaviors and dispositions develop in the way they do within different learning environments.

3   Theoretical Perspectives

Within a framework for research relevant to study student behaviors and dispositions, questions can be raised that warrant further investigation about how and why students engage in learning activities. What kinds of interactions provide students with learning opportunities to develop mathematical knowledge with understanding and do mathematics? Are there patterns in how students become "knowers and doers

of mathematics" or is it idiosyncratic for individual students? Middleton et al.'s (2017) recent review of *engagement* research articulates the complexity of studying the phenomenon of student engagement while learning and doing mathematics. They reported four individual and overlapping components of engagement: behavioral, cognitive, affective, and social. There are research challenges in providing explanations that attend to the four different components of engagement in learning activities to move our understanding of students' mathematical thinking forward. Jansen (2020) elaborated and defined engagement with mathematics as "an interactive relationship students have with the subject matter, as manifested in the moment through expressions of behavior and experiences of emotion and cognitive activity, and is constructed through opportunities to do mathematics" (p. 273). To advance research relevant to student learning activities, researchers could consider Jansen's recent focus on cognitive and social aspects of behaviors "in the moment" to provide evidence of what engagement might look like for students building mathematical content knowledge. In Siedal and Shavelon's (2007) meta-analysis of studies of teaching effectiveness related to student learning during the period 1995 to 2004, they articulated the role of student learning activities needed to build understanding:

> We assumed that learning is a set of constructive processes in which the individual student (alone or socially) builds, activates, elaborates, and organizes knowledge structures. These processes are internal to the student and can be facilitated and fostered by components of teaching. Moreover, we assumed that higher order learning and a deep understanding of learning content is based on the quality of knowledge building and, thus, on the execution of learning activities. Learning activities should evoke both basic information processing and domain-specific processing. Consequently, we assumed the area of executing learning activities to be most proximal to knowledge building. (p. 462)

Relevant to Manizade et al.'s (2019) framework of examining relationships between variables to determine "good" teaching (Introduction, this volume), Siedal and Shavelson's meta-analysis reported constructivist and social constructivist paradigms of knowing in studies that made connections between students' execution of student mathematics learning activities (Type B), desirable student learning outcomes (Type A), and interactive teaching behaviors (Type C). Different theories of learning hypothesize frameworks centered on student engagement in mathematical learning activities and consequential desired student learning outcomes. As researchers interpret particular aspects of the learning process, it is framed by their own construction of theories to explain what they notice in students' behaviors and dispositions.

In this section, I describe three theoretical perspectives that provide explanations of student engagement in learning activities which are needed to develop mathematical content knowledge with understanding and engage in processes envisioned in frameworks of curriculum initiatives over the last three decades. Departing from describing student learning activities in mathematics classrooms, two researchers' conceptualizations of learning are examined through individual dyads and one-on-one teaching experiments using technology-based problems (Hackenberg, 2010; Simon et al., 2016, 2018; Tzur, 1999; Tzur & Simon, 2004). According to Tzur (2004), teaching experiments allow a teacher-researcher to present tasks, use ongoing

analysis of students' current cognitive constructs, and design more tasks that promote students' engagement in constructing higher-level mathematical thinking. On the other hand, consistent with Medley's call for research in classroom settings, Liljedahl (2016) studied connections between teaching practices and student engagement in learning activities in mathematics classrooms (Type CB research).

3.1 Developing Schemes: Progressive Coordination of Actions

Hackenberg's (2010) model of students' reversible multiplicative schemes is an important contribution to the evolution of research on students' engagement in mathematics learning activities. Synthesizing prior studies of students' development of fraction knowledge (Steffe, 1994; Tzur, 1995, 1999, 2004), Hackenberg identified three areas of research that informed key theoretical constructs for her study: (a) building on students' prior knowledge and everyday experiences with fractions; (b) student learning activities for fraction knowledge—partitioning and unitizing; and (c) three of Kieren's (1980) five subconstructs of fractions—quotients, operators, and measures of length. Further, she studied the process of *reversibility* in developing multiplicative relationships. Solving a problem with a sequence of actions in one direction is not easily decomposed to reorganize a scheme in the other direction. Before reporting on the results of Hackenberg's study, her interpretation of scheme theory is described to explain one theory about how learners develop mathematical knowledge. Similar to Medley's review, she drew upon theories of Piaget and Vygotsky to explain how students learn mathematics.

Hackenberg defined mathematical learning "as a process in which people make accommodations in schemes in ongoing interaction with their experiential world" (p. 385). According to von Glasersfeld's (1989) interpretation of Piaget's theories, a *scheme* consists of three parts: (a) an individual recognizes a situation or experience from a previous situation, (b) engagement in an activity associated with this situation, and (c) expecting the same result or outcome experienced when previously engaged in the activity. When examining fraction knowledge that is needed to develop multiplicative schemes, learners engage in activities, such as partitioning, dis-embedding, iterating, and splitting (see Steffe & Olive, 2010; Tzur, 1995, 1999, 2004 for details of these operations). A perturbation occurs when a learner's current schemes no longer appear useful because they do not fit past learning experiences. To eliminate perturbations, schemes either remain stable, or become modified contingent upon a learner's actions and reflections. For Hackenberg, a perturbation explains any reorganization of a learner's existing schemes. Through repeated experiences, a process of *reflective abstraction* internalizes knowledge based upon the entire cycle of perturbation, action, and reflection. If a learner coordinates a scheme successfully using accommodation and does not need to physically act on parts of a task while describing his or her reasoning, an *anticipatory scheme* is constructed.

Hackenberg's research design allowed her to engage four sixth-grade students in problems to facilitate each learner's construction of anticipatory fraction schemes for reversible multiplicative relationships. Data collection consisted of videotaped episodes with cameras focused on interactions between a pair of students and researcher, and a recording of students' computer or written work. Students used the JavaBars computer program (Biddlecomb & Olive, 2000) to facilitate a meaningful interpretation of the fraction construct of measure as length. Olive (1994) stated that microworlds are "tools for the teacher/researchers to construct situations in which they can use their emerging models of the children's mathematics" (p. 71).

Using retrospective analysis of the video files, Hackenberg examined each student's cognitive structures and how schemes changed over time. She reported that students constructed schemes to solve tasks when a fraction relationship existed between known and unknown quantities. One pair of students demonstrated use of fraction anticipatory schemes. Only one of the four students also engaged in reversible schemes when constructing reciprocal relationships. Hackenberg found that students' construction of anticipatory schemes for multiplicative relationships required a coordination of three levels of units prior to engaging in an activity. Teaching experiments using technology-based problems offer an environment where researchers can examine students' engagement in exploratory or expressive activities (Zbiek et al., 2007). Further, researchers could study how these two activities in technological settings are related to scheme theory to provide an explanation of student actions and reflections when they are building mathematical content knowledge and doing mathematics.

3.2 Learning Through Activity: Progressive Coordination of Mathematical Concepts

In a similar vein, building upon Piaget's (1980) theoretical construct of reflective abstraction, Learning Through Activity [LTA] (Simon et al., 2016, 2018) is a research model that examines how learners engage in learning activities to develop mathematical concepts. In an evolution of research on student learning activities, prior LTA research from the past 10 years provided insight for an emerging integrated theory relevant to students' conceptual learning and instructional design. Using Manizade et al.'s (2019) framework, the LTA research model potentially informs future research making connections between Type D, C, B, and A variables (Introduction, this volume). Specifically, the LTA model seeks to answer this question: "How do humans learn mathematical concepts, and how can instruction be designed to enlist these learning processes in service of learning particular mathematical concepts?" (Simon et al., 2018, p. 96). Further, what is the process that engages a learner to move forward from constructing one concept to a higher-level concept in a learner's network of knowledge for different mathematical concepts? And how can this learning process be promoted?

To study these questions, Simon et al. (2018) proposed an elaboration of the construct of reflective abstraction with two refinements: Focusing on new concepts developed from prior concepts rather than using schemes, and a shift away from earlier work of abstractions attributed to a reflection of activity-effect relationships (Simon et al., 2004). The authors asserted that perturbations do not provide evidence of how learning occurs and scheme theory does not explain what a learner "attends to" in order to achieve a learning goal. Moreover, they no longer viewed reflective abstraction as a chronological sequence of actions for developing a new concept, but a construction of higher-level concepts based on lower-level actions. Balancing the needs of mathematics and a learner, Simon et al. (2018) described developing concepts as a "bi-directional" process; "that is, how one explains conceptual learning is dependent on the nature of a concept, and the nature of a concept is, in part, determined by the process through which it is constructed" (p. 98). A concept consists of a goal (e.g., solve a task) and an action a learner takes to achieve the goal. When engaged in mathematical activity, learning may not occur if there are no prior actions (i.e., mental activities) a learner can access. In LTA's model, actions are considered components of concepts, which transforms the construct of reflective abstraction from a *coordination of actions* to a *coordination of existing concepts* (Simon et al., 2016). Student learning activities provide opportunities for learners to construct mathematical concepts if they are aware of a sequence of available mental actions they have already constructed.

As an example of progressive coordination of concepts, Simon et al. (2018) analyzed the 5-year Measurement Approach to Rational Number (MARN) Project data. Similar to Hackenberg's (2010) study, the same program, JavaBars, was used to facilitate students' construction of fraction and multiplicative concepts. A teacher-researcher interacted one-on-one with a student to avoid the influence of others' thinking that is often encouraged in classroom settings. The task sequence research design included: "(1) Assess the relevant understanding of the learner; (2) Specify the learning goal (intended abstraction); (3) Identify an activity or activity sequence that the learner already has available that could be the basis for the new abstraction; and (4) Design a sequence of tasks that is likely to bring forth the learners' use of this activity and lead to the intended abstraction" (Simon et al., 2016, p. 67). When students engaged in carefully designed tasks intended to promote conceptual learning, individual learning processes illustrated "in the moment" thinking and student focus while solving the task.

Building on Tzur and Simon's (2004) hypothesis that two stages, participatory and anticipation, are necessary to develop mathematical concepts, LTA researchers (Simon et al., 2016, 2018) proposed that an initial reflective abstraction is only the first of two stages for building a mathematical concept. For the first, *participatory* stage, a learner engages in an activity and uses existing concepts to begin to develop new mathematical knowledge. The analyses of MARN data provided evidence that learners may not be able to use their initial abstraction (concept) created one day for a similar task the following day. Only when a learner could call upon an earlier abstraction (concept) in different contexts, LTA researchers identified this second

stage as *anticipatory*. Simon et al. (2018) reported that a fourth-grade student coordinated pairs of actions when determining a composite fraction amount of a whole number quantity. Higher-level conceptual knowledge was built upon prior existing knowledge. The two-stage distinction represents a new aspect of research when analyzing qualitative data of student engagement in learning activities. Still, LTA researchers point out that future research is needed to provide a more detailed explanation of how teachers can promote a transition from students' participatory stage to an anticipatory stage for developing conceptual knowledge. What is the role of teacher activities (Type C) to facilitate this transition of students engaged in knowing and doing mathematics (Type B)?

To inform data analysis and instructional design, LTA researchers (Simon et al., 2016, 2018) also continued to study the development of a *reversible concept* Hackenberg (2010) and other researchers (Steffe, 1994; Tzur, 2004) have examined as a necessary part of conceptual learning. A student may construct a reversible concept when he or she does not need to engage in lower-level actions where the original concept was developed. Using the context of Cognitively Guided Instruction (CGI) research-based addition and subtraction tasks (Carpenter et al., 2015), LTA researchers built a *typology of reversibility* for six potential tasks (see Simon et al., 2016, 2018 for details of reversible concepts). Consistent with Hackenberg's (2010) findings for reversibility, Simon et al. (2018) reported that a learner may have an original concept and not easily construct reversible concepts. The typology of reversibility has informed these researchers' decisions related to the design of instructional tasks used during the LTA teaching episodes.

Overall, LTA's theoretical perspective focuses on explaining the process of building conceptual knowledge through students' engagement in learning activities as a progressive coordination of mathematical concepts. Using ongoing data analyses, individual tasks and sequences of tasks are modified dependent upon a learner's progress. If no new concept is developed, more of the same or different experiences are needed to facilitate student reflection and a new abstraction. A challenge for researchers is to reflect upon ways to apply LTA's theory beyond individual students engaged in teaching experiment settings and implemented in whole-classroom settings. To this end, in the next section, I provide an example of student construction of mathematical knowledge and engagement in learning activities in the context of classrooms.

3.3   The AHA! Experience: Proxies of Student Engagement

Medley (1987) recommended five different types of future research needed to inform effective teaching practices, with two types focused on student learning activities in classroom settings: Type BA, "research relates learning outcomes to pupil learning experiences" and Type CB, "research relates interactive teacher behavior to pupil learning activities" (p. 110). For Type CB relationships, Medley posed the following two questions for researchers to examine: "The teacher whose pupils have the best

learning experiences in school (Type B)? The teacher whose classroom behavior conforms most closely to some conception of 'best' practice (Type C)?" (p. 106). Using Manizade et al.'s (2019) framework (Introduction, this volume), studies are needed that focus on student–teacher interactions between student learning activities and interactive teacher behaviors that engage students in becoming knowers and doers of mathematics.

As an example of Type CB research which evolved from 10 years of earlier research in Canada, Liljedahl (2016) proposed nine elements of critical teaching practices that are needed for teachers to orchestrate and sustain student thinking in mathematics classrooms. Moreover, he identified student *proxies of engagement* to describe and measure the effectiveness of the nine elements of teaching practices to facilitate student learning. In many of his classroom observations, he reported how teachers implicitly assumed "that the students either could not or would not think" (p. 362). This may be related to established classroom norms that supported learning in traditional ways which hindered students' ability to engage in thinking and problem-solving behaviors recommended by reform curriculum initiatives.

Liledahl argued for a transition moving away from a non-thinking toward a thinking classroom; that is, "a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (p. 362). Consistent with other researchers' (Cobb, 1994; Cobb et al., 1992) calls for the coordination of Piaget's (1970) constructivist and Vygotsky's (1981) sociocultural perspectives, Liljedahl assumed that knowledge is constructed both individually and collectively, during social interactions with others while engaged in doing mathematical activities. For Cobb (1994), these two complementary perspectives address how theories of learning emerge; that is, "the sociocultural perspective gives rise to theories of the conditions for the possibility of learning, whereas theories developed from the constructivist perspective focus on both what students learn and the processes by which they do so" (p. 18). As described earlier, Hackenberg's and Simon et al.'s research approach of teaching experiments provided explanations for the process of student learning outside mathematics classrooms.

To inform Liljedah's (2016) study of teaching and learning practices in secondary mathematics classrooms, it is useful to review his perspective on the process of mathematical learning "in the moment" during group work and individual problem solving. In 2005, experiences in his mathematics course for prospective elementary school teachers (PTs) affected their thinking about teaching and learning mathematics. An AHA! experience occurred when "a problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight, in a moment of illumination" (Liljedahl, 2005, p. 219). If a student was "stuck" working on a problem, but experienced an AHA! moment, she or he became "unstuck" and continued to make progress. Liljedahl studied the learning process of how this sudden insight or AHA! experience happened and how it affected the PTs' ability to make sense of problems and persevere. Some PTs often identify themselves as failures in mathematics based on a lack of successful learning experiences and they exhibit high math anxiety in mathematics courses. Given the vision of mathematics

curriculum initiatives for learners to develop a deep understanding of mathematics, a potential increase in the intensity of affective responses may result in promoting more negative attitudes when compared to learning routine procedures. Liljedahl's conceptual framework included attention to the affective domain for learning mathematics; that is, examining the constructs of beliefs, attitudes, and emotions (McLeod, 1992). Beliefs reflect low levels of affective involvement, are relatively stable, and develop over a long period of time. According to McLeod, attitude "refers to affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (p. 581). By contrast, the emotional aspects of learning are unstable and connect more to "in the moment" feelings that are "fleeting" (McLeod, 1992).

To study the process of how learning occurs when students experience insight during an AHA! experience, Liljedahl (2005) examined how "moments of illumination" were related to positive emotions and how they changed PTs beliefs and attitudes about doing mathematics. For an assignment, PTs wrote about an AHA! experience while problem solving. Analyzing responses, Liljedahl reported four affective themes: anxiety, pleasure, change in beliefs, and change in attitudes. He found that repeated positive emotional AHA! experiences produced positive beliefs and attitudes about mathematics and students' abilities to do mathematics. As an example, one PT wrote: "AHA moments are those great moments of deeper understanding and clarification of problems where incorrect or incomplete understanding is overcome. These moments inspire us and encourage us to keep going despite the frustration and anxiety that often tends to overwhelm us in times of difficulty when attempting to solve a problem" (p. 231). Engaged in making sense of mathematics, this PT became aware of her need to persevere, as moments of insight can lead to an understanding of mathematics. Liljedahl hypothesized two explanations for a high degree of change in the affective domain: "Positive emotion that is achieved during an AHA! experience is much more powerful than the emotions that are achieved through non-illuminated problem solving" and "Having solved something challenging, or understood something difficult, besides being a great accomplishment is also a measure of what is possible" (p. 231). AHA! experiences promoted changes in PTs' behaviors and dispositions; that is, engagement in student learning activities of problem solving and perseverance.

Liljedahl (2016) extended his work and investigated engagement of secondary mathematics students who worked together in small groups of two to four to solve problem-solving tasks. He studied the interaction between Type B and C variables by examining the effect of different teaching practices and how students engaged in problem solving. To inform his observations, he used Mason's (2002) framework of noticing; that is, "*Noticing* refers to the act of focusing attention and making sense of situational features in a visually complex world" (Jacobs & Spangler, 2017, p. 771). From data analysis, he proposed nine elements of effective mathematics teaching practices for building and sustaining a thinking classroom (see Liljedahl, 2016; for list/analysis of practices). Using an iterative design-based research approach, each element provided opportunities for teaching practices to be refined or dropped, depending on how students engaged in mathematical thinking while problem solving. Still, Liljedahl reported that it was challenging for teachers and students to shift from

traditional, familiar classroom norms. To resolve this issue, he used a "contrarian" approach in which an ineffective practice was changed to the exact opposite and then implemented in mathematics classrooms.

Liljedahl measured the effectiveness of teaching practices by studying "proxies of engagement—observable and measurable (either qualitatively or quantitatively) student behaviors" (p. 366). He referred to these behaviors as "proxies" because he did not have direct access to student thinking and he could not tell if the mathematical thinking was an individual construction, or, collective thinking due to interactions with others. He reported eight student behaviors and dispositions: (1) time to task, (2) time to first mathematical notation, (3) eagerness to start, (4) discussion, (5) participation, (6) persistence, (7) non-linearity of work, and (8) knowledge mobility. As described in Sect. 2, linkages can be made between Liljedahl's student engagement in learning activities (Type B) and those listed in various frameworks of curriculum initiatives. In response to Lesh et al.'s (2020) concerns of the need for "measures" of higher-level student understanding, Liljedahl provided a framework of student behaviors and dispositions that could be used in future studies to provide evidence of the effects of students' engagement in learning activities while building content knowledge and doing mathematics.

Moreover, Middleton et al. (2017) reported researchers studying student engagement experiences often approach their studies using a lens of an observational study. Also, interview data can provide more detailed insights on the observed behaviors. For his 2016 study, Liljedahl conducted follow-up interviews to confirm teachers' interpretation of student behaviors. Similar to other research perspectives focused on how student mathematical learning occurs and described in this section, Liljedahl asserted that we need "to honor the activities of a thinking classroom through a focus on the processes of learning more so than the products and it needs to include both group work and individual work" (p. 382). That is, as Medley (1987) recommended for the future evolution of research for teaching, there is a need to focus on the interplay between elements of teaching practices and student engagement in learning activities (Type B and C variables) rather than examining only student learning outcomes (Type A).

In summary, the last three decades of frameworks of mathematics curriculum initiatives impacted researchers' approaches to studying the needs of the learner and needs of the discipline for effective mathematics teaching and learning. The complexity of studying student engagement in higher-level thinking with understanding has called for an examination of student learning activities through a lens of various theoretical perspectives that provide explanations relevant to how and why student behaviors and dispositions develop in the way they do. Given the different perspectives relevant to students' development of mathematical thinking with understanding and doing mathematics, theories have emerged in particular settings using teaching experiments in technological settings and mathematics classroom environments. As students become knowers and doers of mathematics, Chan and Clark (2017) address the difficulty in conducting valid and reliable research studies of student learning in classroom settings, as there is a "tension between the need for control in an experimental environment and the freedom needed for the participants

to interact and behave as they would in a naturalistic classroom setting remains a challenge in the design of research studies investigating student learning in social settings" (p. 954).

Nevertheless, different theoretical perspectives allow researchers to gain insight into potential refinements in the conceptualization or design of studies that examine student learning activities and active student engagement within diverse individual and whole-class settings, including CT tool environments. This could result in unique insights emerging from studies making explicit connections between Type A, B, and C variable relationships. The next section characterizes a selection of studies of student mathematics learning activities identified earlier in Sect. 2 that encompass most behaviors and dispositions into two main activities: (1) making sense of mathematics (i.e., problem-solving) and (2) perseverance in doing mathematics: productive disposition, productive struggle, and productive failure. Taken together, the studies extend the mathematics education knowledge base of the effects of student learning activities when students engage in developing mathematics knowledge with understanding and doing mathematics. Each study includes a brief description of methodology to address Medley's (1987) quality concerns related to conceptualization, instrumentation, design, and statistical analysis.

4   Making Sense and Perseverance Involved in Learning Mathematics Knowledge

4.1   Problem Solving

Mathematicians, mathematics educators, and teachers have described the problem-solving process in multiple ways (Schoenfeld, 1992) which has led to the development of research agendas focused on examining student behaviors supporting the development of mathematical knowledge (Lesh & Zawojewski, 2007; Schoenfeld, 1992; Schoenfeld & the Teaching for Robust Understanding [TRU] project, 2016). According to Santos-Trigo's (2020) recent review of mathematics education research literature, *problem solving* is defined as "the systematic study of what the process of formulating and solving problems entails and the ways to structure problem-solving approaches to learn mathematics" (p. 687). Over the last three decades, studying the behaviors and dispositions of student engagement in problem solving has continued to be a research priority with an emphasis on detailed accounts of teacher expectations for problem solving and student interactions in mathematics classrooms. This is attributed to the shift of focus on teachers understanding students' mathematical thinking "in the moment" and making connections between Type B and C variables (Manizade et al., 2019). Lesh and Zawojewski (2007) described students' engagement in problem solving as using "several iterative cycles of expressing, testing and revising mathematical interpretations—and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics

within and beyond mathematics" (p. 782). As described earlier, teaching experiment methodology (Hackenberg, 2010; Simon et al., 2016, 2018) has provided an opportunity for researchers to examine students' thinking "in the moment" and explain how students develop mathematical conceptual understanding.

An emerging field of research is investigating student learning activities (see Sect. 2) identified in the Common Core standards of mathematical practice [SMP] (Bostic & Sondergeld, 2015; Gilbert, 2014; Sanchez et al., 2015) and similar mathematical competencies (Hipkins, 2018; McDowell & Hipkins, 2018; Niss, 2003; NRC, 2001; Turner, 2010) that focus on students' sense-making and extends Polya's (2004) problem-solving research. A new term of mathematical *sense-making* defines the needs of a learner when engaged in problem solving as a critical component of what it means for students to know and do mathematics. A limited number of qualitative studies (Bostic & Sondergeld, 2015; Kapur, 2014; Warshauer, 2015) have examined research questions focused on students' problem-solving experiences in mathematics classrooms. Although the term *problem solving* is not explicitly stated in the SMP, the meaning is implicit and places a priority on problem solving as students "make sense" of mathematical content.

The literature revealed various teacher interpretations (Type C) of student problem-solving behaviors (Type B) as envisioned in frameworks of mathematics curriculum initiatives. In an exploratory study, Keazer and Jung (2020) designed a survey for 71 PTs in which they responded to questions about student mathematics learning activities. For example, PTs read a paragraph description of the first SMP and were asked to think about their future teaching when responding: "Which aspect of SMP1 do you think will be most difficult for you to develop in your students? Why?" (p. 82). Separate statements of the SMP1 description were matched alongside PT responses that described anticipated difficulties when engaging students in these behaviors and dispositions. The PTs selected: *They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt*, with the highest frequency as the most difficult learning activity to develop; the second highest activity was: *Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, 'Does this make sense?'* Encouraging their future students to plan, use more than one strategy, and reflect on the problem-solving process as "making sense" did not appear to be a strength. Close to one-third of the PTs shared that they themselves struggled with some of the expected learning goals of SMP1. Consequently, it was a major challenge for many PTs to anticipate how they would engage students in learning activities (Type B) in their future mathematics classrooms.

Keazer and Jung's findings led to their design of a conceptual framework matching student behaviors and dispositions articulated in the SMP1 sentences to Polya's (2004) four problem-solving phases. Citing the research of Schoenfeld and the TRU project (2016) with a focus on the cognitive demand of tasks dimension, they proposed using the SMP1-Polya framework to facilitate prospective and practicing teachers' understanding of different levels of sense making (i.e., problem solving). According to Keazer and Jung, "SMP1 aligns with level 3 sense making, in which the

teacher supports students in mathematical exploration and productive struggle that results in understanding and engagement in mathematical practices" (p. 88). Making connections explicit between sections of SMP1 sentences and Polya's problem-solving phases could provide an entry point for supporting teachers' understanding of student engagement in problem-solving experiences. With the high frequency of two SMP1 statements in the PTs responses, the student behaviors of Polya's second phase, *devise a plan*, and fourth phase, *look back*, continued to show the need to engage students in problem solving or making sense of mathematics to develop a progression of understanding mathematical concepts. For researchers interested in understanding different levels of students' sense making that supports participatory and anticipatory conceptual development, problem-solving activities may provide an opportunity to examine LTA's theory of progression of concepts (Simon et al., 2016, 2018) beyond individual students to small- and whole-group work methods in mathematics classrooms.

4.2   Productive Disposition

Building upon Liljedahl's (2016) theoretical perspective that includes affective factors of learner engagement, recent studies are focusing on student "persever-ance" in solving problems. As described earlier, the NRC (2001) defined an affective strand of *productive disposition* as viewing "mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 116). Gilbert (2014) broadened the meaning of productive disposition to include learning activities in which students are "making sense of problems and persevering in solving them" and linkages to motivational theory. Observing students actively engaged in doing mathematics, researchers could ask: What do strengths-based learners look like when they exhibit the characteristics of a productive disposition in mathematics classrooms? According to Kobett and Karp (2020), "They are just curious and fasci-nated. They work diligently, even when faced with obstacles. They try again when stymied. They understand that learning mathematics can be hard work and they will, therefore, often continue to work well after their peers have given up" (p. 43). For further investigation, how might researchers study and measure these characteristics of students displaying productive disposition?

As an example, in October 2005, Gilbert (2014) surveyed a sample of 140 prealgebra students who volunteered to participate from two California middle schools. She hypothesized a relationship between productive disposition (Type B) and an achievement-related (Type A) variables. Specifically, she studied a relation-ship between students' abilities to attend to precision when they critiqued another student's work. To examine student learning activities, Gilbert stated, "The behav-iors required to demonstrate these SMP thus relate to psychological constructs that go beyond ability beliefs (e.g., efficacy) and utility value (i.e., usefulness of mathe-matics)" (p. 340). First, students responded to survey questions that measured moti-vational constructs associated with productive disposition, such as, "My main goal

in math is to learn as much as I can" (mastery-approach goal) (p. 342). Second, students completed an assessment item which measured their ability to add fractions with unlike denominators. Third, using an open-ended question, students were given a student's incorrect work, ½ + ¾ = 4/6, and asked to write an explanation to the student indicating why the answer was right or wrong.

Gilbert reported using reliability and factor analyses with an examination of correlations that documented the subscales measured distinct constructs of productive disposition. She found that 44% of the students responded with a *more precise* critique of a student's incorrect strategy by engaging longer and suggested at least two steps to correct the student's work. Also, a multivariate analysis of variance supported the hypothesis that students who responded with a more precise critique of a peer's work reported a higher productive disposition than students who responded with a basic critique. Two motivation constructs: (1) productive disposition and (2) mastery approach goals and negative emotions, showed statistically significant differences between the two groups. Based on survey responses, more precise critique students reported higher mastery-approach goals and less frequent negative emotions compared to basic critique students. The results of this study suggest more research is needed to focus on NCTM's (2014) effective teaching practices (Type C), including "building procedural fluency from conceptual understanding" (Type B, mastery-approach goals), where the procedure of adding fractions is built upon a foundation of conceptual understanding. Using multiple representations of fractions, students could be provided with opportunities to make connections between concepts and procedures situated in a classroom where meaningful mathematics discourse occurs.

4.3 Productive Struggle

Beginning elementary school teachers often say that students should not "struggle" or be confused in learning mathematics and if they do struggle, a teacher may restate the same strategy for students to follow. Keazer and Jung (2020) reported a few PTs stated they needed to "show and tell" (Type C) all possible strategies to students rather than engage them in productive struggle. However, researchers have reported the positive effects of productive struggle whereby the act of struggling is crucial for students learning mathematics with understanding (Hiebert & Grouws, 2007; Keazer & Jung, 2020; NCTM, 2014; Schoenfeld & TRU, 2016; Warshauer, 2015). Hiebert and Grouws (2007) defined *productive struggle* as a student learning behavior that promotes students making sense of mathematics and is necessary to develop conceptual understanding. In a similar manner, Dingham et al. (2019) identified productive struggle as "intellectual effort students expend to make sense of mathematical concepts that are challenging but fall within the students' reasoning capabilities" (p. 91). Schoenfeld and the TRU project (2016) identified five dimensions of mathematics learning activities that were necessary to ensure that classroom environments supported students as "powerful thinkers." In response to the needs of the discipline, one dimension focused on the cognitive

demand of tasks in "which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged… The level of challenge should be conducive to what has been called productive struggle" (p. 1). Similar to behaviors and dispositions described in Sect. 2, productive struggle engages students in perseverance when solving challenging problems. Schoenfeld and the TRU project (2016) reported teachers categorized at the highest level supported "students in productive struggle in building understandings and engaging in mathematical practices" (p. 24). Likewise, NCTM (2014) explicitly addressed the need for teachers to engage students in productive struggle: "Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and support to engage in productive struggle as they grapple with mathematical ideas and relationships" (p. 48).

Warshauer (2015) studied what different types of student struggle looked like in six U.S. middle school mathematics classrooms and how teachers responded to their students' struggles (Type CB research). His conceptual framework centered on the "process of struggling to make sense" (p. 378) for a deep understanding of mathematics, the relationship between the students' struggles and the types of mathematical tasks explored, and the dynamic, social nature of interaction when teachers responded as helping or hindering student learning. Given the complexity of studying student–teacher and student–student interactions, he conducted embedded case study methodology (Yin, 2009) using instructional episodes. Multiple sources of data allowed for triangulation of the data to establish dependability, confirmability, and transferability when he reported findings of the study.

Warshauer developed a productive struggle framework for reporting the frequency of four different types of student behavior of struggle: get started, carry out a process, uncertainty in explaining and sense-making, and express misconception and errors (Type B). As an example, "confusion about what the task was asking" or a "gesture of uncertainty or resignation" (p. 385) described students struggling at the beginning of the problem-solving process. It should be noted that there is a parallel alignment in some of his framework categories to Polya's (2004) four phases of problem solving; that is, "get started" with Polya's first phase and "carry out the process" with the third phase. Similar to Keazer and Jung's (2020) study, connecting student struggles to some of Polya's problem-solving phases could provide researchers with a new lens for analyzing students' sense-making through existing problem-solving literature.

For student–student interactions, Warshauer reported students' "uncertainty in explaining and sense-making" when their explanations lacked clarity and did not make sense to other students, or they struggled with appropriate responses. He found evidence of proportional reasoning misconceptions such as using additive rather than multiplicative thinking for the meaning of ratios. For teacher-student interactions, Warshauer reported the frequency of four different types of teacher responses to student struggles: telling, directed guidance, probing guidance, and affordance (Type C). The first two types of responses did not engage students in productively understanding the concept of proportional reasoning. As might be anticipated, a telling response often enabled a student to move beyond being stuck, but used a teacher's thinking rather than student thinking. Often, a procedure was stated for a student to

follow which resulted in lowering a problem's level of cognitive demand. Both the last two types of teacher responses supported students' thinking without lowering the level of cognitive demand.

Warshauer identified three outcomes of student struggles: productive, productive at a lower level, and unproductive. Productive interactions included: "(1) maintained the intended goals and cognitive demand of the task; (2) supported students' thinking by acknowledging effort and mathematical understanding and (3) enabled students to move forward in the task execution through student actions" (p. 390). He reported 42% of student struggles met all three criteria, 40% of the interactions only used the second criteria, and 18% of struggles were unproductive. For unproductive struggles, students were not "making progress toward the goals of the task; reached a solution but a task that had been transformed to a procedural one that significantly reduced the task's intended cognitive demand; or if the students simply stopped trying" (p. 391). In essence, teachers balanced how much they pressed students to persevere based on students' levels of tolerance for frustration at different levels of cognitive demand. Productive struggle depended on keeping tasks at higher cognitive-demand levels, supporting students' perseverance, and teachers who provided guidance and affordance. These results promote the future use of a productive struggle framework as a tool for researchers examining students' productive struggles (Type B) and teacher-student interactions (Type BC research).

4.4 Productive Failure

Research on examining students' productive struggle when attempting to make sense of mathematics content and persevere in solving problems, is related to engagement in another Type B variable: *productive failure* (Kapur, 2010, 2014; Simpson & Maltese, 2017). Failure can be defined in many ways, such as, giving up or stopping engagement in an activity, not reaching the intended goal, or incorrect problem solutions. Further, failure can bring to the forefront negative connotations such as "negative emotional states (e.g., fear, anxiety, depression), low perceptions of self, diminished sense of belonging, less academic risk taking, and avoidant behaviors" (Simpson & Maltese, 2017, p. 223). These negative behaviors and dispositions suggest that failure may decrease students' desire or ability to continue to problem solve. Still, what might happen if we view failure as a "necessary and sufficient condition" for students' engagement in learning activities? In what ways might students' metacognitive analysis of their problem-solving process while stuck on a problem make errors explicit, or, how may critiquing their peers' use of models and strategies support learning? According to the Partnership for 21st Century Learning (2019), creativity and innovation are enhanced through failure; that is, persistent attempts are part of innovative practices marked by "a long-term, cyclical process of small successes and frequent mistakes" (p. 4). How might this process of success and failure be part of learning activities?

Simpson and Maltese (2017) studied the role of failure in the development of science, technology, engineering, or mathematics (STEM) professionals. They interviewed 99 STEM professionals about their experiences in entering and pursuing a STEM-related career. Using life history interviews, they focused on how participants' failure shaped: outlooks connected to failure, career trajectories within STEM fields, and provision of additional skills. They reported about one-fifth of the professionals described failure as a positive experience. However, when using a follow-up survey and asked if "the term failure was an accurate representation or label of their experiences, 67% disagreed and claimed words and phrases such as inadaptability, setback, unsuccessful, not living up to expected outcomes, defeat, and learning opportunity as more suitable" (p. 228). Rather than considering failure as an end to becoming a STEM-related professional, they reported two-thirds of respondents saw failure as a minor setback that motivated them to move past difficulties in coursework or professional projects. Also, they described the trait of "persistence" as the most "important quality to possess when experiencing instances of failure" (p. 233). As described earlier, perseverance is a productive student mathematics learning activity envisioned by curriculum initiatives over the last three decades.

In a study of ninth-grade students who lived in the national capital region of India, Kapur (2014) proposed that engaging students in problem solving which initially resulted in productive failure would ensure "correct conceptual knowledge and mathematical procedures over faulty ones" (p. 1009). For Kapur, the term *productive failure* meant that students' initial individual problem-solving attempts were unsuccessful in finding correct solutions, and became productive when supported with appropriate mathematics classroom instruction. Similar to productive behaviors researched over the last decade, Kapur hypothesized relationships between individual student failure (Type B), sequence of teaching phases (Type C), and student outcomes (Type A). For Kapur's (2014) study, in one classroom, students first participated in a problem-solving (PS) phase for standard deviation (SD) problems that was followed by a direct instruction (DI) phase. In the comparison classroom, the same teacher first taught students using DI followed by a PS phase. During the PS phase, students solved a SD practical problem individually and the teacher encouraged them to use multiple strategies and find as many solutions as possible. For the more traditional DI phase, the teacher showed four examples of SD problems, gave time for individual student practice, and provided student feedback related to common SD misconceptions.

Similar to Gilbert's (2014) study, Kapur examined aspects of both cognitive and affective behaviors and dispositions using surveys and mathematics content knowledge measures. He designed four instruments to measure students' learning of SD concepts and procedures. These included a pre- and post-test of SD knowledge and survey questions relevant to engagement and mental effort. Kapur reported that the class of students who began instruction with a PS phase provided an average of six different solutions to a SD practical problem. The number of solutions served as a "proxy" measure of students' prior knowledge activation. By comparison, the other class of students beginning with DI, only demonstrated an average of three different solutions.

Examining affective behaviors and dispositions, data collected from survey questions provided evidence of significantly greater mental effort of productive failure students (PS phase first) compared to DI students (PS phase second) during both phases of instruction. Yet, Kapur found no significant difference between the two sequences of instruction on math ability or prior SD knowledge. Analyzing posttest data and the two different sequences of instruction, Kapur reported "significant multivariate main effects only of math ability and condition" (p. 1013). Although there was no significant difference between students' procedural knowledge in either classroom, students engaged in the PS phase first, significantly outperformed students receiving the DI phase first on posttest conceptual understanding and transfer items. No significant correlations appeared in the data for students beginning with DI.

Kapur's research supports a learner's perspective that is relevant to NCTM's (2014) teaching practices whereby teachers provide students time to think, make conjectures, and use their own strategies while problem solving: "Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies" (p. 17). Kapur's study provides specificity for this teaching practice (Type C) by supporting engagement in student learner activities (Type B) that may include productive failure first at the beginning of a lesson. After experiencing a PS phase followed by more instruction, students engaged in more mental effort and demonstrated more conceptual understanding than students who experienced DI (teaching as telling) at the beginning of a lesson. Thus, it appeared that productive failure provided students with an opportunity to learn from their own failed solutions and they were ready to engage in classroom-based instruction with a focus on important mathematical ideas relevant to SD. For teachers who believe it takes too much time to allow students to think and engage individually in the PS process, Kapur found that "time on task, the number of problems solved, and materials for each of the phases were identical in both [classes]" (p. 1010).

5   Discussion of Findings and Future Implications

What can be learned from this selected analysis and review of student mathematics learning activities that actively engage students in knowing and doing mathematics? How has research evolved over the last three decades to support students' development of mathematical content knowledge and engagement in processes (i.e., behaviors and dispositions) that have been identified in multiple frameworks of international mathematics curriculum initiatives? How has the increased availability of CT tools for students enhanced researchers' observations and inferences of students' thinking, including technologies to advance research methodologies? What theoretical perspectives have researchers refined for examining the nature of students' construction of mathematical content knowledge with understanding to provide explanations relevant to how and why student behaviors and dispositions develop in the way they do within different learning environments? Lastly, how has this

chapter informed future research needed to advance our understanding of student learning activities?

To address these questions, this chapter's review and analysis of three decades of research highlight the contributions of selected studies related to understanding the nature of student mathematics learning activities and the resulting impact on students' knowing and doing mathematics. The findings offer insights for researchers, curriculum designers, administrators, teachers, parents, students, and other stakeholders involved in mathematics teaching and learning, situated in both non-technological and technological environments. First, a major theme in this chapter of studies of student learning activities was researchers' increased focus on reviewing multiple characterizations of mathematical behaviors and dispositions to refine competency frameworks to study how students actively engage in the processes of learning mathematics. An evolution of similar and interrelated learning activities from different countries provided details about what processes to study and how to analyze the effect of students' learning experiences, including two main learning activities of making sense of mathematical knowledge and perseverance in doing mathematics. At the beginning of the twenty-first century, Sfard (2003) asserted that learning activities should "engage students in what may count as an authentic activity of mathematizing rather than in learning ready-made mathematical facts" (p. 354). There has been growth in researchers' understanding of what constructs to study related to student mathematics learning activities (Type B) and various theoretical perspectives that provide explanations of students' engagement in knowing and doing mathematics.

Although Kobert and Karp (2020) created an alignment of student behaviors and dispositions between the five strands of mathematical proficiency (NRC, 2001) and the eight standards of mathematical practice (NGA Center for Best Practices and CCSSO, 2010), few studies have focused on this alignment and what can be learned to inform our understanding of student engagement in learning activities. Researchers could further examine the relationships among these multiple frameworks in curriculum initiatives and the impact of using different (albeit similar) frameworks (see Sect. 2) to examine students' active engagement in learning mathematics. What is the same and what is different in using these identified mathematical behaviors and dispositions to investigate students' knowing and doing mathematics? As another example, how might researchers take advantage of Koestler et al.'s (2013) and Kobett and Karp's alignment between the process standards (NCTM, 2000) and standards of mathematical practice (NGA Center for Best Practices and CCSSO, 2010)? What insights might emerge when researchers "synergize" these two frameworks together to inform research studies about students' engagement in mathematics learning activities? To respond to engaging students in an "authentic activity of mathematizing" that mathematicians display when they know and do mathematics (see Sect. 2), researchers could build upon a rich tradition of studying students' problem-solving behaviors (e.g., Polya's problem-solving phases) with a further examination of critical connections between frameworks and conceptualizations of higher-level mathematical processes. Further, research could focus on at least three of Liljedahl's (2016) proxies of student engagement (i.e., discussion,

participation, and persistence) to examine and explain students' perseverance while problem solving, both individually and in groups, to make inferences about students' mathematical thinking.

Second, research over the past three decades has extended our understanding of how and why the process of student engagement in learning and doing mathematics occurs in different learning environments. Using teaching experiment methodology, a small number of studies have articulated emerging theoretical perspectives that focused on analyzing students' development of mathematical concepts in technological settings outside the classroom (Hackenberg, 2010; Simon et al., 2016, 2018). From these studies, observations and analyses documented how students engaged in a sequence of learning activities using CT tools that were intended to promote students' reflective abstraction and reversible thinking for rational number concepts. How might a similar cycle of students' engagement in learning activities including CT tools and coupled with researchers' noticing and analyses provide a research pathway to further our understanding and infer students' mathematical thinking for reversible thinking in different conceptual areas? Simon et al. (2018) proposed that researchers could positively contribute to addressing unsuccessful mathematics instruction for specific conceptual areas (e.g., fractions, ratios, proportions, and other) through implementing the LTA research model.

One result of the last decade of research, Simon et al. (2018) refined an earlier theoretical framework of scheme theory and moved research forward with a better understanding of the constructs of student learning to create the LTA theory which resulted in analyzing students' progressive development of concepts. In what ways could researchers use the LTA research model of task sequence design and analyses to investigate students' engagement in mathematical processes that focus on a progressive concept development and lead to intended abstractions in non-technological environments? Further studies of students' learning with understanding in different areas of mathematics could provide more useful explanations as to how and why students' knowledge changes or does not change "over time" or "in the moment." Moreover, if researchers look beyond using teaching experiment methodology, what can be gleaned from the LTA approach to investigate small- and whole-group student engagement in knowing and doing mathematics in classrooms? In particular, findings from Liljedahl's (2005, 2016) studies should be explored using his new conceptual framework to measure and expand our understanding of a relationship between mathematics teaching practices and student learning activities (Type CB research) that seems necessary to build and sustain thinking classrooms. Furthermore, the past two decades of research studies about student mathematics learning activities has shown an increased focus of examining the interrelationships between Type B and C variables (Keazer & Jung, 2020; Schoenfeld & TRU, 2016; Warshauer, 2015) to inform our understanding of the effects of students' engagement in learning activities. Yet, more studies are needed to explore research questions about relationships among Type B and A variables (Gilbert, 2014) and Type A, B, and C variables (Kapur, 2014) to improve student learning outcomes (Type A).

A third finding of this chapter is the identification of some of the important constructs (e.g., expressive activity, exploratory activity, representational fluency,

and others) that are needed to inform research related to technology-based mathematics teaching and learning (Zbiek et al, 2007). These constructs should be further explored to refine our current understanding of links between student engagement in the processes (i.e., behaviors and dispositions) of learning mathematics and students' use of CT tools. As an example of future research for studying "promising variables" with student-tool relationships, Zbiek et al. proposed: "Students' dragging behavior [with CT tools] could be viewed as an intervening variable between the mathematical activity and student achievement" (p. 1201). In other words, using Manizade et al.'s (2019) framework for examining relationships between Type B and A variables (Introduction, this volume), researchers should investigate the potential of a new "intervening variable" between two adjacent variables in the adaptation of Medley's (1987) work that could provide evidence of how students engage in learning and doing mathematics in technological settings.

Given the documentation of some unproductive student work methods, Zbiek et al. call for "research that identifies constructs that are associated with the development of *judicious use* [italics added] of technology" (p. 1186); that is, examining teacher activities (Type C) which facilitate students being aware of their need to focus on the mathematics content of a task and use productive work methods (Type B). Researcher observations of successful and unsuccessful student behaviors when using CT tools may provide insight into how the successful use of technology can be sustained and ways to change unsuccessful student behaviors.

Further questions that warrant researchers' investigation of students' mathematical learning and engagement with technology-based activities include: If students encounter an unexpected result with one representation (using CT tools), do they stay with that representation, or switch to another representation that provides more insight as a way to solve a given task? What is the role of teacher activities (Type C) in engaging students in their development of representational fluency? As described in Sect. 2, access to technology introduces the "play paradox" where unstructured, expressive activity can enable some students to avoid the intended mathematical content of an activity. How might studies of understanding students' development of representational fluency provide evidence of the effect of exploratory activity and expressive activity in technological settings? Zbiek et al. advocate for studies of "how the representational fluency of a group relates to the representational fluency of individuals in the group" (p. 1194). Also, is there a relationship between the construct of representational fluency and student work methods (Zbiek et al.)? Within technological environments, researchers should consider many of these questions and examine relationships between Type A, B, and C variables to inform the knowledge base of student mathematics learning activities.

Fourth, reviewing different conceptualizations of student engagement in mathematics learning activities focused on not only identifying behaviors and dispositions that actively engage students in knowing and doing mathematics in existing studies but also to suggest new directions in building the knowledge base related to student mathematics learning activities. A clear trend of this chapter's selected review of studies about student learning activities focused on how and why students make

sense of mathematics "in the moment" and perseverance to know and do mathematics "over time." Whether using teaching experiments or classroom settings, researchers investigated and explained students' engagement in learning mathematics with understanding and doing mathematics. As an example, Liljedahl's (2005) study of PTs' problem-solving activities and their "AHA! moments of illumination" promoted positive changes in their mathematical understanding (i.e., cognitive construct) and productive dispositions (i.e., affective construct). Complementary relationships between cognitive and affective constructs of students' mathematical learning experiences could inform future research design for individual studies or sets of related studies. Moreover, studies with an increased focus of examining interrelationships between Type B and C variables provided evidence of how teachers responded to individual students' use of or lack of problem-solving strategies and informed their decision-making on next steps in a lesson or sequence of lessons (see Sects. 3 and 4).

To advance our current understanding of mathematics teaching and learning, there is a continued need to review and extend the knowledge base related to the development of student behaviors and dispositions that actively engage all students in knowing and doing mathematics. One way to move the knowledge base forward is a consideration of the results of the past decade with an increasing availability of wide-ranging technological methodologies that can provide data about students' engagement in mathematics learning activities to both teachers and researchers (Type CB research). To address the gap between research and practice for understanding and improving students' mathematical learning experiences, Cai et al. (2018) proposed the collection, analysis, and use of "continuous data on the learning experiences of each student" (p. 363) to facilitate researchers' understanding of explicit connections between teaching practices (Type C) and student learning activities (Type B). Yet, questions need to be considered if technological and methodological tools exist without overwhelming both researchers and teachers with too many data? According to Cai et al. (2018), the "capacity to capture, process, and store comprehensive cognitive and noncognitive data longitudinally for every student either already exists or is on the near horizon" (p. 364). Two years later, Cai et al. (2020) described current digital tools for collecting and managing student data but acknowledged that technological tools that could be used "during [classroom] lessons to monitor small-group discussion, analyze student work, and even gauge students' affect" (p. 392) are still under development. Still, examining the future potential of technology to access student mathematical thinking for each student in the next decade, Cai et al. (2018) have proposed a framework for collecting, analyzing, and using data on students' mathematical experiences that uses a three-part time frame: (1) in the moment, (2) short term, and (3) long term (p. 366). In addition, both cognitive and noncognitive learning experiences, such as "students unexpected responses" and "students' engagement with tasks" are identified and have been reported in prior studies of student mathematics learning activities (see Cai et al., 2018, for further framework details). Future research is needed to ground this framework in the data across multiple diverse settings.

Notwithstanding and looking to the next decade, Bartell et al. (2017) asserted that the "CCSSM, with its implicit political and economic goals and its lack of explicit attention to race, gender, class, and so forth, is not framed to support equity" (p. 9). Consequently, Bartell et al. designed a framework to connect research-based equitable mathematical teaching practices (Type C) with all the SMP (Type B) to explicitly address issues of equity. Making connections, they identified nine core teaching practices described in the chapter on culture, race, and power in the *Second Handbook of Research on Mathematics Teaching* (Diversity in Mathematics Education , 2007) and more recent research (see Bartell et al., 2017, for details of the practices). As an emerging field of research, their framework offers existing and new research connections between student mathematics learning activities and equitable mathematical teaching practices (Type BC research). Each part of the framework provides multiple entry points for research supporting what it means for students to actively engage in effective and equitable mathematical learning activities. Students' engagement in mathematical behaviors and dispositions needs to be studied in particular contexts and situations to inform and extend the knowledge base of what works and does not work for all students to become knowers and doers of mathematics.

Brief, Capsule Definitions of Terms and Documents for Chapter 6

- *Assessment, curriculum and evaluation, and professional standards for school mathematics:* A trilogy of documents that provided a vision for the organization of curriculum reform in the U.S. in the 1990s (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 1995).
- *Behaviors and dispositions:* Identification of student experiences, such as, proficiencies, processes, practices, competencies, and habits of mind (Kobett & Karp, 2020, p. 40) that demonstrate how students develop and show evidence of their mathematical thinking.
- *Cognitive technological (CT) tools:* Consists of tools that support a "synergistic relationship" between technical and conceptual dimensions of mathematical activity in technological environments (Zbiek, Heid, Blume, & Dick, 2007).
- *Competencies:* Frameworks for knowing and doing mathematics, such as, (1) Denmark's (2003) mathematical competencies that provided evidence of students' "mental or physical processes, activities, and behaviors" (p. 9); (2) Program for International Student Assessment [PISA] (PISA, 2021) assessed mathematical competencies as "an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts" (PISA, 2021); and (3) Identified in the New Zealand Curriculum (NZC), competencies "that describe what they [students] will come to know and do" (Ministry of Education, 2015, p. 37).

- *Conceptual understanding:* Student learning is defined as the "comprehension of mathematical concepts, operations, and relationships" (National Research Council [NRC], 2001, p. 116).
- *Direct instruction (DI):* Traditional, instructional methods where students watch, listen, and take notes about problems that teachers provide procedures and solutions for students to follow and use (Kapur, 2014).
- *Learning goals:* Focus on student "understanding" where students build knowledge; "Explicitly state what students will understand about mathematics as a result of engaging in a particular lesson" (Smith & Sherin, 2019, p. 14).
- *Learning through activity*[LTA]: A research model that examines how learners actively engage in learning activities through a progressive coordination of mathematical concepts (Simon, Kara, Placa, & Avitzur, 2018; Simon, Placa, & Avitzur, 2016).
- *Mathematical sense-making:* Student engagement in processes, such as problem solving, to learn mathematics with understanding; one aspect of what it means to know and do mathematics.
- *National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO]:* Authors of the U.S. Common Core State Standards for Mathematics (CCSSM), 2010.
- *Organization for Economic Development (OECD) Definition and Selection Competencies (DeSeCo) Project:* Created a framework to guide the development of PISA assessments.
- *Performance goals:* Focus on the end result or product of students' engagement in learning mathematics: "What students will be able to do as a result in engaging in a lesson" (Smith & Sherin, 2019, p. 14).
- *Principles and standards for school mathematics:* Updated U.S. document that provides a vision for curriculum reform at the beginning of the twenty-first century (NCTM, 2000).
- *Problem-solving:* Defined as "the systematic study of what the process of formulating and solving problems entails and the ways to structure problem-solving approaches to learn mathematics" (Santos-Trigo, 2020, p. 687).
- *Process standards:* Five processes that define what mathematicians might do and say when engaged in doing mathematics: Problem solving, communication, representation, making connections, and reasoning and proof (NCTM, 2000).
- *Productive disposition:* An affective construct defined as learners having an "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p. 116).
- *Productive failure:* Students' initial problem-solving attempts are unsuccessful and became productive when supported with appropriate mathematics classroom instruction (Kapur, 2014).
- *Productive struggle:* A student learning behavior that promotes learners making sense of mathematics and is necessary to develop conceptual understanding (Hiebert & Grouws, 2007); "Intellectual effort students expend to make sense of mathematical concepts that are challenging but fall within the students' reasoning capabilities" (Dingman, Kent, McComas, & Orona, 2019, p. 91)

- *Proficiencies:* Frameworks for students' engagement while learning mathematics, such as, (a) Cognitive and affective proficiencies for five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001); and (b) Reasoning as one of the four proficiency strands students engage inwhen "thinking and doing of mathematics" (Australia Curriculum and Assessment Reporting Authority [ACARA], 2017).
- *Prospective elementary school teachers (PTs) and AHA! Experience:* Students engage in problem solving and experience how "a problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight, in a moment of illumination" (Liljedahl, 2005, p. 219).
- *Representational fluency:* Within or outside technological environments, "The ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (Zbiek et al., 2007, p. 1192).
- *Research for principles and standards for school mathematics:* Research literature that informed the U.S. vision of school mathematics in the 1990s and 2000 (NCTM, 2003).
- *Scheme:* A cycle of perturbation, action, and reflection in which an individual anticipates, acts and mentally prepares, and assesses the outcome of his or her actions (Hackenberg, 2010; Steffe, 1994; von Glasersfeld, 1995)
- *Standards for Mathematical practice (SMP):* Eight mathematical competencies identified as a national Common Core State Standards for Mathematics (CCSSM) in the U.S., 2010.
- *Student learning activities:* "In the classroom… All learning depends on the activity of the learner" (Medley, 1987, p. 105).
- *Student engagement:* Defined as "an interactive relationship students have with the subject matter, as manifested in the moment through expressions of behavior and experiences of emotion and cognitive activity, and is constructed through opportunities to do mathematics" (Jansen, 2020, p. 273).
- *Teaching for robust understanding [TRU] project:* Framework of five dimensions of classroom activity that supports professional development (PD) to engage teachers in creating a classroom student learning environment that facilitates the development of powerful thinkers (Schoenfeld & the TRU project, 2016).

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Student Mathematics Learning Outcomes

Jelena Radišić

1 Introduction

Within the context of presage-process–product research and its pursuit of 'good' teaching (Type C), learning outcomes (Type A) represent the end goal and final criterion on which any assessment of teaching must be based (Medley, 1977, 1987a; also see Figs. 2 and 3 in Manizade et al., 2022). In this respect, 'good' teaching can be considered teaching that produces the maximum learning outcomes and progress to meet the prescribed education goals. However, the very idea of learning outcomes and educational goals has changed throughout the years, mainly because the understanding of what education, and here mathematics education, should entail has changed (Kilpatrick, 2020a; Manizade et al., 2022).

Since the 1980s, worldwide, the increasing demand for knowledge in many areas of life and work has placed the burden of productivity on education systems (Klieme et al., 2008). Consequently, this has led to a stronger focus on 'outputs' and 'outcomes' at all levels of the educational system and their transferability to the job market. In such a society, mathematical knowledge, ability, skills and(or) competence are seen as an essential prerequisite in encountering the challenges of the world today (Boesen et al., 2018; Ehmke et al., 2020; Freeman et al., 2015; Gravemeijer et al., 2017; OECD, 2016). Such a need has also led to a broader understanding of what being 'mathematically' equipped means. It includes both posing and answering questions in and by means of mathematics (i.e., reasoning, modelling, problem-solving), as well as handling the language, constructs and tools of the field (i.e., formalism and language, handling different representations, handling material aids and tools for mathematical activity, digital tools included; Niss et al., 2017;

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Niss & Højgaard, 2019). At the same time, mathematics itself consists of different subfields, each of which may employ somewhat different mathematical tools. Some have argued that the development of the underlying frameworks that bring together these constituents has, in return affected teaching to a certain extent (Type C) (e.g., Boesen et al., 2014). Such discussions are coupled with considerable differences of opinion regarding which teaching methods are effective, which may help sustain different learning goals and desired outcomes (e.g., Blazar, 2015; Hiebert & Grouws, 2007; Hill et al., 2005).

Indeed, efforts to improve the quality of teaching largely depend on the effectiveness and availability of accurate, detailed and objective evaluations of teaching (Medley, 1987a, 1987b). Students' learning outcomes represent one of the most favoured criteria, especially amid policy, under the assumption that it is reasonable to judge teaching by its results, just as we do for most other activities in life (Darling-Hammond & Rustique-Forrester, 2005). Additionally, both affective and self-belief constructs may be specified as learning outcomes (Ramseier, 2001), here with the rationale that competent participants within a field also hold certain beliefs about the field itself (Aditomo & Klieme, 2020; Radišić & Jensen, 2021). Furthermore, facilitating students' development of positive self-beliefs and interest in mathematics increases the probability that even students with lower skills can gain the opportunity to move forward in developing own skills and can gradually become individuals who apply reasoning or problem-solving in daily situations (Callan et al., 2021; Freeman et al., 2015; Radišić & Jensen, 2021; Verschafel et al., 2020).

Against this background, in this chapter, we focus on how learning outcomes have been defined and some of the conceptualisations that have been used for the purpose. Afterwards, the process of assessing learning outcomes within the classroom context, here regarding international large-scale assessments (ILSAs), will be discussed in more detail. From the perspective of Medley (1987b), successful assessment of student outcomes involves three essential steps (p. 170). These comprise standard tasks or a set of tasks that must be alike or equivalent for all students. Thus, the differences in the quality of performance will not arise because of dissimilarities in the tasks. Next, a detailed, objective and accurate documentary record is required. Finally, a scoring key with clear procedures for developing the criterion score from the record is compulsory, ensuring that the same quality can be obtained no matter who does the scoring. Robust conceptualisation, instrumentation, design and statistical analyses are crucial prerequisites for these three steps to be fulfilled. ILSAs may be seen as the perfect examples of student outcomes assessments in trying to realise all these conditions.

Subsequently, this chapter will also address how technology shapes our understanding of students' learning and outcomes. Finally, we revisit the idea of an 'outcome', examining it in the context of individual students' characteristics (Type G), namely self-beliefs and interest in mathematics.

2 Being Competent in Math: A Desirable Learning Outcome

Historically, becoming proficient in mathematics has taken on different meanings and expectations (Abrantes, 2001; Kilpatrick et al., 2001; Niss et al., 2017). Ever since the 1930s, when it came to defining mathematics learning outcomes, the focus was primarily on knowledge and understanding of the mathematical content, that is, definitions and theorems and a clear set of associated procedural skills. However, these ideas of being proficient in mathematics were soon confronted (Niss et al., 2017). Examples can be found as early as in the work of Polya (1945), who writes that if a teacher only focuses on 'drilling' their students in routine operations, that same teacher destroys students' interests, hampering their intellectual development. Since the 1980s, the National Council of Teachers of Mathematics (NCTM) in the US has strongly advocated that problem-solving should be the focus of school mathematics; at the same time, basic skills should be defined to include more than merely computational ability (1980, p. 1). Similar conceptions were nurtured elsewhere, leading to a firm footing in the understanding that the overall enactment of mathematics and general mathematical thinking is and should be embedded in the different aspects of daily activities, school curriculum included (Kilpatrick et al., 2001; Niss & Højgaard, 2019; Niss et al., 2017). At the same time, it was acknowledged that none of these different aspects could stand-alone or contradict one another (Kilpatrick et al., 2001; Niss et al., 2017; RAND Mathematics Study Panel, 2003). Notably, despite the development in what it means to master mathematics, desired mastery, that is, the outcome, was inevitably used as a criterion to assess student learning progress in math, the quality of teaching, or even the system (Klieme et al., 2008). Moreover, although the criterion-oriented outcome evaluation still has a strong foothold in both practice and research, the concept of 'mathematical knowledge' has lost its supremacy, and the idea of competence has gained momentum both in mathematics education research and neighbouring fields like educational psychology (Guskey, 2013; Niss et al., 2017; Kilpatrick, 2020b; Sternberg & Grigorenko, 2003; Sternberg, 2017; Weinert, 2001). There, the idea of competence was mainly discussed within and in connection to the notion of ability and intelligence (Sternberg & Grigorenko, 2003; Sternberg, 2017).

The concept of competence is one of the most fleeting in the educational literature (Kilpatrick, 2020b), and arriving at a collective meaning of competence across different fields is even more difficult. The concept possesses a myriad of similar terms like mastery, proficiency or skill (Niss et al., 2017). Furthermore, one distinguishes between a 'competence' with a broader meaning compared with the term 'competencies', which refers to the various facets of competence (Blömeke et al., 2015).

With the notion that 'many theoretical approaches and no single conceptual framework' (Weinert, 2001, p. 46) can be found, Weinert recognises seven different ways of defining competence. Weinert's framework includes general cognitive competencies, specialised cognitive competencies, the competence–performance model, modifications of the competence-performance model, cognitive competencies and

motivational action tendencies, objective and subjective competence concepts and action competence. The idea could be observed as a critical shift within the broader context of the presage-process–product research because it moves away from utilising a strict cognitive lens. Interestingly, starting in the late 1980s, NCTM Curriculum and Evaluation Standards for School Mathematics (1989), besides a focus on cognitive competencies, already included motivational action tendencies, as recognised by Weinert, in the form that students should learn to value mathematics and become confident in their ability to do mathematics. Revisions in 2000 eliminated these attitudinal and dispositional aspects (NCTM, 2000).

Conversely, specialised cognitive competency frameworks dominate the mathematics field. Also, being context-dependent implies that their advancement can only be perceived as a result of an individual's interaction with relevant situations and experiences, such as one encounter during the mathematics class.

In contrast, Bloom's taxonomy (1956), with its attempt to outline the cognitive goals of any school subject, can be seen as a predecessor of context-dependent frameworks today. Its categories of knowledge, comprehension, application, analysis, synthesis and evaluation, which were later revised by Anderson and Krathwohl (2001), do not withstand the criticism that the taxonomy itself does not genuinely fit the mathematics field's needs (Kilpatrick, 2020b). Nevertheless, a competency framework for mathematics may still include a division of the processes alone, leaving out the mathematical content or combining the processes and the subject's content. Examples of the former are seen in the frameworks proposed by the National Research Council in the United States and the well-known KOM[1] project (Niss, 2003; Niss & Højgaard, 2019) linked to the reform of the Danish education system. Reforms in other countries, under similar influences, followed (e.g., Abrantes, 2001; Boesen et al., 2014; Nortvedt, 2018).

The KOM framework

The KOM project defines mathematical competence as a 'means to have knowledge about, to understand, to exercise, to apply and relate to and judge mathematics and mathematical activity in a multitude of contexts, which do involve, or potentially might involve, mathematics (Niss, 2003, p. 43). It distinguishes between the eight competencies needed for mastering mathematics and is divided into two groups (Niss, 2003, 2015). The first gathers aspects of involvement with and in math—thinking mathematically, posing and solving mathematical problems and modelling and reasoning mathematically. The second group of competencies addresses dealing with and managing mathematical language and tools—representing mathematical entities, handling mathematical symbols and formalisms, making use of aids and tools and communicating in, with and about mathematics (Fig. 1).

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[1] KOM = Kompetencer og matematiklæring; in Danish: Competencies and the Learning of Mathematics.
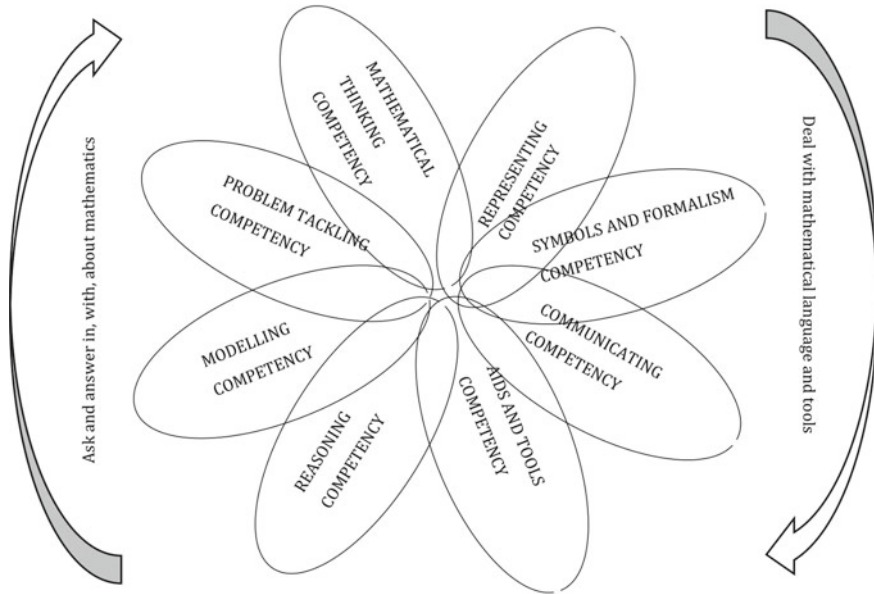
Fig. 1 Competencies within the KOM framework (adapted from Niss, 2003)

At the same time, it is proposed that the entire set of dimensions has both analytical and productive sides (Niss & Højgaard, 2019). In addition, each competency can be developed and employed only by dealing with specific topics in math, but their choice is not predetermined and may transcend the subject.

Parallel to the use and development of each of the competencies, the framework also proposes three types of 'overview and judgements' that students should develop through their study of mathematics. These include its application, its historical development and its unique nature, which combined with the eight competencies, may be used (a) descriptively—to describe mathematics teaching and learning; (b) normatively—by proposing outcomes for school mathematics; and (c) metacognitively—aiding teachers and students in monitoring what they are teaching or have learned so far. Overall, the critical impact of the KOM framework was in its introduction and description of the concepts of mathematical competence and mathematical competencies (Kilpatrick, 2020b), as well as the possible roles these may play in the process of teaching and learning mathematics (Niss & Højgaard, 2019). Ultimately, it should be noted that although KOM authors recognise the importance of affective and dispositional factors as part of mathematical mastery, these factors are not considered within the KOM framework. Mathematical competence and competencies are, in essence, cognitive constructs.

Strands of mathematical proficiency

In contrast to the competency approach, which focuses on what it takes to *do* mathematics, Kilpatrick, Swafford and Findell (2001) focus on mathematics learning.

Mathematical proficiency is the key concept used for this purpose. The basic premise is that mathematical proficiency should not be seen as a unidimensional trait. Instead, it should combine five *strands* that are mutually intertwined and codependent (Kilpatrick, 2001). Central to this understanding is a concept from the findings in neighbouring fields (i.e., 'cognitive' sciences, as referred to by Kilpatrick) that having a deep understanding involves learners being able to connect pieces of existing knowledge. In turn, such a 'connection' is an essential factor in facilitating whether learners can use what they know effectively while solving (mathematical) problems.

The following strands are pertinent to the framework (a) *conceptual understanding*—comprehension of mathematical concepts, operations, and relations; (b) *procedural fluency*—a skill needed to carry out procedures accurately, fittingly and flexibly; (c) *strategic competence*—an ability to articulate, formulate, represent, and solve mathematical problems; (d) *adaptive reasoning*—a capacity for logical thought, justification, explanation and reflection; and (e) *productive disposition*—a habit of inclining to see mathematics as functional, useful, and meaningful, coupled with a belief in one's own efficacy (Kilpatrick et al., 2001). To reach mathematical proficiency cannot be achieved by attending to merely one or two of these strands, but rather, this calls for instructional programmes that address them all (Kilpatrick, 2001), an argument resonating well with the earlier ideas of Polya (1945) (Fig. 2).

Fig. 2 Five proficiency strands. (adapted from Kilpatrick et al., 2001)
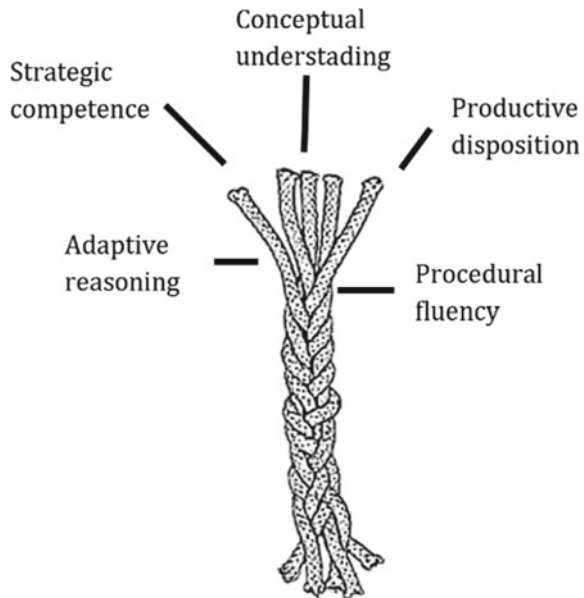
Kilpatrick et al. (2001) argue that, although there is no perfect fit between the proposed strands and different kinds of knowledge and processes identified by researchers within mathematics education or the adjacent fields on the factors contributing to learning, the strands do resonate with a substantial body of literature on the topic. Examples can be found in the investigation of motivation, which is considered a component of productive disposition and metacognition, contributing to strategic competence. Finally, proficiency strands include motivational action tendencies (Weinert, 2001), compared to the KOM framework, which remains solely in the cognitive sphere.

How is mathematical competence conceptualised in the ILSA?

Among the frameworks that combine the processes and mathematics content, we choose to discuss two coming from the ILSA domain here, given their prevalent influence on understanding student competence in mathematics worldwide. The first is linked to Trends in International Mathematics and Science Study (TIMSS), which was initiated in 1995 and as a follow-up to the IEA's[2] previous studies during the 1960s through the 1980s. TIMSS uses the curriculum as the principal organising concept to see how educational opportunities are provided to students and the factors that affect how such opportunities are used by the students (Mullis, 2017). The TIMSS curriculum model has three aspects. It comprises the intended curriculum, the implemented curriculum and the attained curriculum. These three aspects combined represent the mathematics students are expected to master. Since 1995, the TIMSS framework has been provided for grades four and eight, with recurrent improvements given each four-year cycle (Mullis, 2017). For example, the TIMSS assessment frameworks for 2019 were updated from those employed in 2015. In this way, the participating countries are provided with a chance for an update regarding their national curricula, standards and mathematics instruction for every cycle, keeping the frameworks relevant and coherent with the previous assessment. At the same time, in each cycle, a particular emphasis is given to a specific aspect of the assessment. In the TIMSS 2019 cycle, the focus was on the transition to eTIMSS. This transition implied conducting the assessments in the eTIMSS digital format, hence providing an enriched measurement of the TIMSS mathematics (and science) frameworks. Here, the mathematics frameworks were updated to utilise both digital and paper assessment formats. About half the countries participating in TIMSS 2019 transited to eTIMSS, and the process has continued into the 2023 cycle.

In 2019, the frameworks for both grades four and eight were organised around two dimensions: (1) content dimension (i.e., subject matter to be assessed) and (2) cognitive dimension (i.e., thinking processes to be assessed; Lindquist et al., 2017). See Table 1 for details.[3]

The content domains differ between grades four and eight, thus reflecting the topics taught at each level. For example, the 'number' is emphasised more in grade

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[2] International Association for the Evaluation of Educational Achievement.

[3] For examples on TIMSS tasks from the 2019 cycle see https://timss2019.org/reports/achievement/ and accompanying exhibits for grades four and eight.

Table 1 Domains of mathematical competence within the TIMSS framework (adapted from Lindquist et al., 2017)

	Grade 4		Grade 8	
	Content domain			
Number	Whole numbers, expressions, simple equations, relationships, fractions and decimals	50%	Integers, fractions and decimals, ratio, proportion, and per cent	30%
Algebra			Expressions, operations, equations, relationships and functions	30%
(Measurement and) Geometry data (and Probability)	Measurement, geometry	30%	Geometric shapes and measurements	20%
	Reading, interpreting, and representing data, using data to solve problems	20%	Data, probability	20%
	Cognitive domain			
Knowing	Recall, recognise, classify/order, compute, retrieve, measure	40%	35%	
Applying	Determine, represent/model, implement	40%	40%	
Reasoning	Analyse, integrate/synthesise, evaluate, draw conclusions, generalise, justify	20%	25%	

four than in grade eight, at which point algebra is also introduced. Although in grade four, the section on 'data' focuses on collecting, reading and representing data, the interpretation of data, basic statistics and the fundamentals of probability are the focus in the eighth grade. Also, about two-thirds of the items demand that students use applying and reasoning skills. The cognitive domains are alike for both grades, with less of an emphasis on the 'knowing' domain for grade eight. Altogether, they largely resemble earlier Bloom's taxonomy categories (1956).

Similar to the notion of Bandura on the distinction between knowing and being able to use one's own skills well when under diverse settings (1990), the question on which skills young adults at the end of (compulsory) education would need to be able to play a constructive role as citizens in society was the guiding principle of OECD policymakers in setting up an international programme to assess the outcome of schooling (OECD, 1999, 2003, 2013a, 2018; Trier & Peschar, 1995). Unlike TIMSS, the Programme for International Student Assessment (PISA) crosses the boundaries of school curricula by taking a functional view (Klieme et al., 2008) with the idea of being prepared to cope with the demands and challenges in the future. This

cross-curricular competence or life skill becomes central within the PISA framework (OECD, 1997, 2013a, 2018).

In the context of mathematics, the PISA framework initially defined *mathematical literacy*[4] as 'an individual's ability, in dealing with the world, to identify, to understand, to engage in and to make well-founded judgements about the role that mathematics plays, as needed for that individual's current and future life as a constructive, concerned and reflective citizen' (OECD, 1999, p. 41). Also, the mathematics framework drew a clear parallel to the eight competencies of the KOM framework, but here with the label of skills (e.g., modelling skill, Niss, 2015). Succeeding frameworks alternate *ability* with the 'capacity to reason mathematically and to formulate, employ and interpret mathematics to solve problems in a variety of real-world contexts' (OECD, 2018).

PISA 2003 was the first to focus on students' mathematical literacy. Its framework entailed situations/contexts (i.e., personal, educational/occupational, public, and scientific) in which the problems were situated. The mathematical content categories (i.e., quantity, space and shape, change and relationships, uncertainty) and the processes (i.e., thinking and reasoning; argumentation; communication; modelling; problem posing and solving; representation; using symbolic, formal and technical language and operations; and the use of aids and tools) were employed to solve them (OECD, 2003). The latter serves the purpose of supporting matematisation, representing constitutive parts of a comprehensive mathematical competence (Niss, 2015). In the 2003 framework, relying on the work of Niss and colleagues is even more prominent.

With the 2012 PISA round, the mathematics framework grew significantly, eventually including a more progressive organisation of the contexts (i.e., societal was introduced), content (i.e., data were combined with uncertainty) and processes that have undergone a more significant change. This change has led to the following division of the processes students engage in as they solve problems: formulating situations mathematically; employing mathematical concepts, facts, procedures and reasoning; and interpreting, applying and evaluating mathematical outcomes. The associated underlying fundamental mathematical capabilities, which replaced earlier mathematical competency (Niss, 2015), include communication, representation, devising strategies, mathematisation, reasoning and argument, symbolic, formal and technical language and operations and mathematical tools (OECD, 2013a).[5] Redefining fundamental capabilities served the purpose, among other things, to clearly set the stage for a scheme to analyse the requirements of PISA items.

Finally, PISA 2022 has aimed to consider mathematics in a 'rapidly changing world driven by new technologies and trends in which citizens are creative and

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[4] For a more thorough discussion on the notion of mathematical literacy see Jablonka (2003). Overal, the argument drawn is that literacy focusses on the individual's ability to use the mathematics they are supposed to learn at school.

[5] Example items may be found at https://www.oecd.org/pisa/test/PISA%202012%20items%20for% 20release_ENGLISH.pdf.

engaged, making nonroutine judgements for themselves and the society in which they live' (OECD, 2018, p. 7) (see Fig. 3).

These shifts focus on the capacity to reason mathematically. At the same time, the effect technology has created, fosters the need for students to understand computational thinking concepts that are part of mathematical literacy. The theoretical foundations of the PISA mathematics assessment are still based on its fundamental concept of mathematical literacy, here relating mathematical reasoning and the three processes of the problem solving (mathematical modelling) cycle. The framework defines how mathematical content knowledge is organised into four content categories that are coupled with four context categories that situate the mathematical challenges students face. Novel to the framework is a more detailed description of
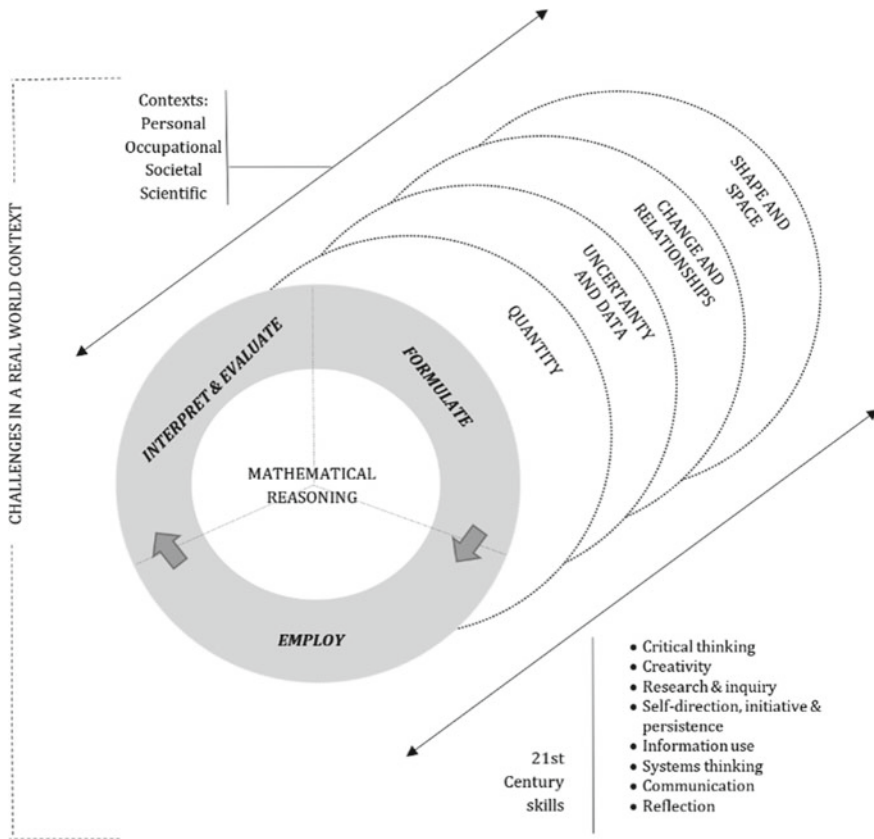

Fig. 3 PISA 2022 mathematics framework—the relationship between mathematical reasoning, the problem-solving (modelling) cycle, mathematical contents, context and selected twenty-first-century skills (adapted from OECD, https://pisa2022-maths.oecd.org/); *Note:* This is an adaptation of an original work by the OECD. The opinions expressed and arguments employed in this adaptation are the sole responsibility of the author of the adaptation and should not be reported as representing the official views of the OECD or of its member countries

mathematical reasoning that includes six basic understandings that deliver structure and support. The basic understandings include (1) understanding quantity, number systems and their algebraic properties; (2) valuing the power of abstraction and symbolic representation; (3) seeing mathematical structures and their regularities; (4) distinguishing functional relationships between quantities; (5) using mathematical modelling as a lens onto the real world; and (6) understanding variation as the fundamentals of statistics.

To sum up, irrespective of whether a competency framework is hierarchical (e.g., Bloom), whether it addresses topic areas in mathematics (e.g., TIMSS) or not (e.g., KOM framework), or what its primary use is (normative vs, descriptive), the frameworks serve the purpose of demonstrating that the learning of mathematics and outcomes at the end is more than acquiring a myriad of facts. Instead, mastering mathematics as an outcome of learning (Type A) involves grappling with its content and is more than carrying out well-rehearsed procedures. Although school mathematics is often seen as a simple match between knowledge and skill, competency frameworks challenge this view, affecting curricular contents more and more (Abrantes, 2001; Boesen et al., 2014; Nortvedt, 2018). Even if it may appear that the frameworks do not communicate well, fundamentally, some form of mathematical modelling is described in each (i.e., in PISA 'formulate', 'employ' and 'interpret' or in TIMSS by 'applying' and 'reasoning'). Still, how explicitly this is stated (e.g., in KOM modelling competency), of course, varies. What separates KOM from the PISA and TIMSS frameworks is that the latter consider the reality of ILSA; that is, clear operationalisations are needed for measurement to take place (Medley, 1987b)—'elements have to be separable to be measurable' (Niss et al., 2017, p. 241). A clear example of this principle can be found in the introduction of the fundamental capabilities within the 2012 PISA mathematics framework. In the following section, we focus on how learning of mathematics and its outcomes are captured.

3 Assessment of Student Outcomes

It has been argued that as long as there were students learning mathematics in one form or another, they experienced some form of assessment, either to observe the impact of the teaching they have been exposed to or how much of the content they have mastered themselves (Niss, 1993; Suurtamm et al., 2016). Thus although it may seem that discussing student learning outcomes has picked up in intensity recently, especially when the results of ILSAs are concerned, measuring student outcomes has a long tradition. Furthermore, Kilpatrick (1993) maintains that the notion of assessing the mathematics students have learned was unavoidably entwined with the questions of who should receive additional mathematics instruction and how that instruction should be brought about. Thus, assessment and measuring outcomes are an integral part of teaching and learning from the beginning (Suurtamm et al., 2016).

According to Niss et al. (1998), in mathematics, the term assessment refers to the identification and appraisal of students' knowledge, insights, understanding, skills,

achievement performance and capability in math. Pegg (2002) later contests this, stating that the dominant view of assessment in mathematics has been focused on content, specific skills and the production of these in a given situation. In addition, when assessments are being made, they are never free of context, serve different stakeholders and are bound by the available resources. So within the classroom context, although assessing students' problem-solving skills may be inviting, many teacher-made tests will still focus on computational skills because these are often less time-consuming (Palm et al., 2011).

Conversely, one of the principal reasons assessment of student outcomes has attracted increased attention from the international mathematics education community is that during the past couple of decades, the field of mathematics education has developed considerably (Suurtamm et al., 2016). Nevertheless, assessment practice seems to be somewhat lagging (e.g., paper and pencil is still the dominant format), and the ideas of mathematics as a hierarchically organised school subject and a vehicle for regulating education still seem to be alive (Kilpatrick, 1993, 2020a). Thus, the challenge of assessing students' learning gains in mathematics still focuses on producing measures that allow for an understanding of how students come to use mathematics (Type A) in different social settings and how one can create mathematics instruction that helps them use mathematics even better (Type C) (Blazar, 2015; Blömeke et al., 2016; Hiebert & Grouws, 2007; Kilpatrick, 2020a; Manizade et al., 2022; Medley, 1977, 1987a).

However, one needs to acknowledge that the variety of assessment practices has been increased. Still, Nortvedt and Buchholtz (2018) recognise that discussions within the field of mathematics education are often influenced by discussions in neighbouring disciplines (e.g., educational psychology), ultimately affecting the purpose, conceptualisation and chosen outlets of assessment. To date, the purpose of assessing student outcomes in mathematics has varied. Although debates are still very much alive on what the purpose of mathematics education is (Niss, 2007; Niss et al., 2017) or the optimal teaching practices (e.g., Blazar, 2015; Hiebert & Grouws, 2007; Hill et al., 2005), the same goes for what should be the primary purpose of assessments in and of mathematics. Although some strongly argue that assessments should be used mainly to improve learning (e.g., Black & Wiliam, 2012; Niss, 1993), the formative-summative debate—coupled with the existence of ILSAs and national tests—ignites the ongoing discussions, despite attempts to merge some of the contrasting perspectives (e.g., Buchholtz et al., 2018; Nortvedt & Buchholtz, 2018). Each type of assessment may target different audiences and needs. While some have the purpose of informing policy-making, others are intended to inform the teacher teaching a particular group of students (Nortvedt, 2018). These goals can also be combined within a particular assessment session.

'One size of assessment does not fit all'

Student learning outcomes are assessed in different contexts and for different purposes (Klieme et al., 2008; Niss, 1993; Kilpatrick, 1993, 2020a; Suurtamm et al., 2016). These equally include ILSAs (e.g., TIMSS and PISA), evaluations of implemented programmes or classroom assessments. Thus, an assessment is of central

importance in education (Taras, 2005). Furthermore, given that the realisation of many educational decisions, choices and interventions depend on assessments, their accuracy in monitoring learning outcomes is pivotal (Klieme et al., 2008). At the individual level, assessments provide teachers with an opportunity to promote individual learning. However, they may also be detrimental in granting students an opportunity to continue education in the desired field (e.g., entry test to the STEM field in higher education). Conversely, assessments that report results at an aggregated level (i.e., country score in PISA) assess institutions or systems and advise and inform decision-makers and policy. Thus, 'one size of assessment does not fit all' purposes (Pellegrino et al., 2001, p. 222).

However, in improving learning outcomes, the discussion is often set on the formative–summative divide. If we assume that the central aim of educational research is to improve teaching and learning processes, formative assessment can be seen as one such tool (Black & Wiliam, 2012; Niss, 1993; Pinger et al., 2018; Taras, 2005; Thompson et al., 2018). Formative assessment is founded on the notion of evaluating students' understanding and progress regularly throughout the process of teaching and making use of this information to improve both teaching and learning. Consequently, teachers can use this information to adapt their (mathematics) instruction, aligning it with students' needs and providing them with feedback to improve learning. For assessment to be regarded as formative, it is fundamental that the assessment information is used successively to alter students' learning processes (Black & Wiliam, 2009, 2012). Providing students with feedback is a powerful tool for changing learning processes and, as a result, is regarded as a key strategy in realising formative assessments (Hattie & Timperley, 2007; Klieme et al., 2008).

At the same time, the assessment of individual achievements may also entail the summative evaluation of competencies, either at the individual or aggregated level (Klieme et al., 2008; Taras, 2005). Such evaluations help determine whether a student has reached a certain level of competence, for example, upon completion of upper secondary education. As such, these evaluations have significant consequences—representing a high-stake test for the student (de Lange, 2007; Klieme et al., 2008). However, if a student takes part in an ILSA survey such as TIMSS and PISA, the practical consequences of taking a test are nonexistent (e.g., getting a low mark) at the student level (low stakes). In contrast, the consequences of the same assessment at the system level may be detrimental and lead to practical decisions at the system level (e.g., PISA shock in Germany, de Lange, 2007).

International large-scale assessment and the field of mathematics

Robust conceptualisation, instrumentation and design have long been recognised to be essential to successful assessment (Medley, 1987b). Large-scale studies utilise complex and often representative samples, offer multi-layered, rich data and results, allow for the latter's generalisability and are created to describe and inform about a particular system rather than an individual student (Middleton et al., 2015). Of course, such studies are also rather costly. Nevertheless, in past decades, many countries have opted for some version of large-scale assessment studies. Examples may be found in national mathematics tests in Norway (Nortvedt, 2018) and Sweden (Boesen et al.,

2018) or in the National Educational Panel Study (NEPS) in Germany (Ehmke et al., 2020) and National Assessment of Educational Progress (NAEP) in the United States (NCES, 2021). Common across these is that they attempt to capture what it means to be mathematically competent in one way or another. However, despite this 'common' goal, it can be questioned to what extent they all measure the same thing or the '*what*' of the assessment (Nortvedt & Buchholtz, 2018). A joint criterion or framework is missing, similar to measuring and comparing temperatures in different capitals across the world without referencing either Celsius or Fahrenheit scales while doing so (Cartwright et al., 2003). ILSAs produce such a frame of reference, thus attracting much attention in educational research and outside the field, that is, policy and media, when discussing the quality of education in different countries and how that quality can be improved (de Lange, 2007; Nortvedt, 2018).

The origins of ILSAs date back to the 1960s, with the International Association for the Evaluation of Educational Achievement (IEA) being established in 1959. Its purpose was to conduct international comparative research studies focused on educational achievement and its factors. In this initial stage, the aim was to understand the vast complexity of the aspects influencing student achievement in different subject fields, with mathematics being one of them. The famous metaphor used by the founding researchers was that they 'wanted to use the world as an educational laboratory to investigate effects of school, home, student and societal factors' (Gustafsson, 2008, p. 1). The argument was that an international comparative methodology was essential for investigating the effects of many of these factors. The first study investigating mathematics achievement in 12 countries started in 1964. The Six Subject Survey, conducted in 1970–71, followed the first study, gathering information on the subjects of reading comprehension, literature, civic education, English and French as foreign languages, and science. Throughout the 1980s, mathematics and science studies were repeated.

During the 1990s, the IEA was transformed. The TIMSS was born, creating a slight shift in the focus[6]—describe the educational systems that partake in the study. The published international reports primarily describe the outcomes alongside background and process factors. There is no attempt to explain the variations in outcomes between school systems—inferences about causes and effects are also omitted. The latter, causes and effects, are left to participating countries, with caution being urged when claiming causality because of the cross-sectional design of ILSAs (Rutkowski et al., 2010).

In many cases, the results were used to evaluate educational quality as a basis for national discussions about educational policy (Cai et al., 2015; Gustafsson, 2008; Middleton et al., 2015). This goal was even more prominent with the establishment of PISA in 2000 (OECD, 2001). The volume and frequency of ILSAs have increased (i.e., every three and four years), along with the number of participating countries (e.g., 58 in grade four and 39 in grade eight in the 2019 TIMSS cycle). Both these aspects have contributed to comparing and contrasting the systems and particular

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[6] 'Third' in the title Third International Mathematics and Science Study shifted later into 'Trends' and the same acronyms continue to be used.

groups within a system, for example, boys and girls in grade four in Norway or across the Scandinavian countries (Cai et al., 2015), helping in identifying the affordances and strengths within and across each (Mullis et al., 2016; Mullis, 2017; OECD, 2013b, 2016).

Here, in terms of methodological challenges, ILSAs have been questioned on whether a single assessment format or particular test can grasp the full image of being skilful in mathematics, the 'how(s)' of assessment (Nortvedt & Buchholtz, 2018) and provide comparable measures of curriculum effects across countries (de Lange, 2007). Jablonka (2003) addresses this situated nature of mathematics competence in the context of PISA, stating that the contexts used in the assessment will be familiar to some students more than others (e.g., students across Europe, compared to students in many of the African countries). Cultural differences are an essential aspect in understanding students' mastery of mathematics as a field (Manizade et al., 2022). This variation is visible in the ILSA results. For example, PISA 2012 reports a significant variance across countries (OECD, 2013b), with as many as, on average, 43% of the students reporting perceiving themselves as not being competent in mathematics. Just within Europe, in the same cycle, the number of students who scored low in mathematics (below level 2) was between 10.5% (Estonia) and 60.7% (Albania). However, according to de Lange (2007), follow-up discussions about the outcomes of ILSAs are often about politics rather than performance, and the consequences of having similar or dissimilar results as a neighbouring country may not be taken upon in the fashion it was established or envisioned earlier. Taking the example of PISA, Baird et al. (2016) claim that the connection between PISA results and policy is not consistent. PISA's 'supranational spell' (p. 133) in policy connects to how its results are used as a magical stick in political discourse, as though results invoke particular policy choices. Instead, they divert from the ideological basis for reforms, indicating that the same PISA results could motivate different policy solutions.

Most often, when a new set of results in PISA or TIMSS for mathematics come out, policymakers, media and a part of academia focus on the country rankings using the number and position in the league tables as an indication of system quality. Auld and Morris (2016) dispute such a view, claiming it reduces the complexity of the information ILSAs may provide while decreasing opportunities to identify insights that could be used to learn valuable lessons about school effectiveness and inform national educational policies.

In observing the benefits of partaking in ILSA programmes and their relevance to mathematics as a field, Sälzer and Prenzel (2014) argue that ILSAs provide a standard or benchmark against which countries can measure themselves. In addition, the abundance of data collected about schools, processes and outcomes allows for profound insights into policy and decision making and observing particular patterns relevant to the teaching and learning process. Cai et al. (2016) are even more explicit regarding the affordance of mathematics education gains from ILSAs; these include understanding students' mathematical thinking, classroom instruction, students' experiences with teaching and students' disposition for mathematics. All of these are highly valid to mathematics education researchers, as well as school leaders and teachers.

Moreover, with the recent uptake in the use of technology in mathematics class-rooms, ILSAs can also be a vehicle for understanding what it means to be competent in mathematics in a digital environment (Stacey & Wiliam, 2013).

How does technology affect our understanding of student outcomes in mathematics?

The use of technology, especially within the past decade, has influenced how mathematics is viewed (Manizade et al., 2022). Technology has enabled a 'transformation of [the field] from static to dynamic symbolic systems through which teachers and learners can access knowledge and think' (Hegedus & Moreno-Armella, 2018, p. 1). It has also set a new understanding of students' competence—including handling digital tools (Niss et al., 2017)—affected curricular goals (Gravemeijer et al., 2017) and initiated the need for different kinds of assessments to probe students' skills in a new way (Li & Ma, 2010; Stacey & Wiliam, 2013). Several studies have analysed technology-enhanced learning environments in mathematics classrooms (e.g., Higgins et al., 2019; Hillmayr et al., 2020; Pape et al., 2013). In their meta-study, Li and Ma (2010) show that the effect of technology may vary. For example, technology can promote elementary over secondary students' achievement or special needs education students over the general population. In addition, the positive effect of technology has been found to be more significant when combined with constructivist instructional approaches compared with the traditional ones. Drijvers (2015) provides caution, stating that the integration of technology in mathematics education is a subtle question, whose success and failure occur at the levels of learning, teaching and research (p. 147).

Over the past decade, digital assessments have emerged primarily in the context of large-scale assessments of students' outcomes, both within the national (e.g., Norway, Japan, USA) and international contexts. TIMSS and PISA are clear examples of the latter. It has been argued that computer-based assessments (CBAs) possess several advantages. They allow complex stimuli, response formats, and interactive testing procedures and may incorporate computerised adaptive testing (Klieme et al., 2008). Regarding the latter, the task (stimuli) presented is designed to fit the individual ability level of the test taker (student) in real-time. Feedback procedures may also be incorporated (Chung et al., 2008), thus allowing the assessment of learning progress (i.e., 'dynamic testing'). Moreover, CBAs allow for the production of complex and interactive stimuli that would be very expensive or difficult to realise on paper. Consequently, the practice may afford the assessment of new competencies previously not accessible through more traditional procedures.

Software suitable for use in the mathematics classroom is also increasingly available. Its use within the classroom context is advocated for by the argument of improving the quality of mathematics teaching and assessment to be more realistic and attuned to the needs of the new generation learners (Gravemeijer et al., 2017; Hoogland & Tout, 2018). The possibility of simulating real-life situations in the assessment situation makes CBA an example of what Weinert (2001) describes as context-specific cognitive dispositions.

Although digital tools may enable new and enhanced possibilities for the learning, teaching and assessment of mathematics (Drasgow, 2002; Drijvers, 2015; Higgins

et al., 2019; Hillmayr et al., 2020; Pape et al., 2013), only their appropriate use and equal availability to all participants will produce a positive impact (Gravemeijer et al., 2017; Higgins et al., 2019; Li & Ma, 2010). With this in mind, a request for any assessment is to afford all students the optimal opportunities to demonstrate what they have learned and can do (Niss, 2007). Such demand holds the same for technology-assisted assessments. Although the recent shift towards assessments focusing on problem-solving and modelling may benefit from the technology, others argue that assessments of student outcomes combining affordances of technology and nonstatic item formats allows students to demonstrate mastery of a broader range of mathematical skills (Hoogland & Tout, 2018; Stacey & Wiliam, 2013). Conversely, although a shift from traditional paper-based to computer-assisted assessments may be favourable to a task requiring students to model a solution, Jerrim (2016) warns of the possible adverse effects on student outcomes. These risks are primarily in danger of appearing if mathematics teaching does not include the use of such tools. An interesting finding here may be found in PISA 2012. Together with the regular paper-and-pencil test, a computer-based assessment in mathematics was offered as an option. Among the European countries that took advantage of this, only a handful of countries remained at the same competency level when paper- and computer-based assessment results were compared (OECD, 2013b).

Furthermore, studies show that the impact of computer-assisted assessment also relies on students' prior experience. In some cases, this may include general computer skills (Falck et al., 2018; Stacey & Wiliam, 2013), whereas in others, this tackles the understanding and use of specific tools (Hoogland & Tout, 2018) or item formats (e.g., real-life problems). Hillmayr et al. (2020) show that overall, digital tools positively affect student learning outcomes. However, the provision of teacher training on digital tool use significantly moderates the effect. The effect size is more prominent when digital tools are used in addition to other instruction methods and not as a substitute, with intelligent tutoring systems or dynamic mathematical tools being more beneficial than hypermedia systems.

New opportunities with CBAs have opened doors for their use in a different context that is still relevant to mathematics teaching and learning. Nevertheless, many of these applications are driven by the rapid development of computer technology rather than well-founded models and theories. Thus, much empirical and theoretical work is needed to link complex measurement potentials to particular learning outcomes and (or) instructional practices hence maintaining rigour in the conceptualisation, instrumentation and design (Medley, 1987b) of future assessments.

4   Optimal Self-beliefs and Motivation for Math: A New-Old Learning Outcome

So far, we have discussed students' learning outcomes as achievement outcomes. However, both an array of motivational and ability–belief constructs may be specified as learning outcomes (Ramseier, 2001). Although Weinert (2001) recognises 'cognitive competencies and motivational action tendencies' as one of the seven ways to define competence, in Medley's reflections (1987a), individual student characteristics (Type G) are seen as mediating the relationship between A (outcomes) and B (students' learning activities). Medley (1987a) states, 'Even if two pupils have identical learning experiences, they do not show identical outcomes because of differences in these characteristics' (p. 105). To date, such a lens has dominated the field. ILSAs are a clear example, with domain achievements measured separately from attitudinal constructs and the latter often being reported concerning achievement. An exception may be found in the framework of Kilpatrick et al. (2001), which include motivational action tendencies.

The tradition of observing 'attitude' towards mathematics in mathematics education research was apparent as early as the 1950s (Zan & Di Martino, 2020). However, one fundamental characteristic of the research in that period was the absence of a proper definition or theoretical background. Schukajlow et al. (2017) recognise an increased interest in attitudinal constructs (i.e., motivation, affect, ability beliefs) in mathematics education over the last decade, and similarly, Zan and Di Martino (2020) attribute the beginnings of modern research on these topics to McLeod (1992), who include attitude among the three factors that define the affective domain.

At the same time, neighbouring fields—namely educational psychology—have flourished in different conceptualisations and frameworks that explain what drives human action (e.g., Ryan & Deci, 2016; Eccles & Wigfield, 2020; Hidi & Renninger, 2006). Several of these frameworks are used in mathematics education research, each presenting its terms (Schukajlow et al., 2017). Among them, attitudes, self-beliefs, intrinsic motivation and interest are probably the most commonly used. Amid the diverse frameworks aiming to explain students' motivation, expectancy-value (EV) theory covers a variety of aspects that affect the decisions students make by relating students' expectancies for success and subjective task values to their achievement and achievement-related choices (Eccles & Wigfield, 2020; Wigfield & Eccles, 2000).

Within the EV framework, expectancies for success originate from a person's domain-specific beliefs that are based on experience or beliefs about their ability to succeed in future tasks like solving a mathematical problem. Though the tags these beliefs may have had are somewhat different, confidence, self-efficacy and self-concept are all found under the category of ability beliefs (Lee & Stankov, 2018). Furthermore, the EV model recognises four subjective task values. These include intrinsic value, attainment value, utility value and cost (Eccles & Wigfield, 2020; Wigfield & Eccles, 2000). Intrinsic value relates to the anticipated enjoyment that one expects to gain from doing a task. The dimension itself is similar in certain respects to the concepts of interest (Hidi & Renninger, 2006) and intrinsic motivation

(Ryan & Deci, 2016). Attainment value relates to identity and how important the task is for the individual. Utility value indicates how useful the task is for other goals. Again, in certain respects, the utility value is related to the idea of extrinsic motivation (Ryan & Deci, 2016). Finally, cost indicates the time, effort, stress and other valued tasks put away to fulfil the current task in which an individual participates.

Today, although mastering math is seen as a requirement in meeting the demands of modern life (Boesen et al., 2018; Ehmke et al., 2020; Freeman et al., 2015; Gravemeijer et al., 2017; OECD, 2016), research demonstrates that students' task values and ability beliefs are fundamental to their optimal outcomes in mathematics (Dowker et al., 2016; Marsh et al., 2012; Schöber et al., 2018; Skaalvik et al., 2015; Stankov & Lee, 2017; Wang, 2012, Watt et al., 2012). For example, in PISA 2012, a rise in the degree of one standard deviation in self-efficacy was linked to an increase of 49 score points in achievement—the equivalent of more than one school year (OECD, 2013b). Similarly, students experiencing low-ability beliefs are potentially at risk of underperforming (OECD, 2013b).

Nevertheless, the relationship between achievement in mathematics and different motivational and belief constructs is not always straightforward; neither one portrays a unique image (Zan & Di Martino, 2020). For example, Wang (2012) argues that task values are stronger predictors of engagement and choices to stay in the field of mathematics, while expectations of success predict more immediate student achievement outcomes. Watt et al. (2012) similarly conclude on intrinsic value, linking ability beliefs to staying within the field (i.e., career choice in math). Prast et al. (2018) argue for a unique contribution of perceived competence in predicting subsequent achievement in mathematics.

Gender differences in students' ability beliefs in mathematics are relatively common (Nagy et al., 2010), mostly favouring boys (e.g., Geary et al., 2019). Likewise, although it has been shown that among ability beliefs, positive self-concept is conducive to student learning and achievement in mathematics (Marsh et al., 2017), the relationship itself can be direct, indirect (Habók et al., 2020) or reciprocal (Schöber et al., 2018).

Despite this somewhat diverse image on how students' ability beliefs and task values, namely intrinsic value, contribute to achievement in math, that is, what the genuine relationship between what Medley (1987a) labels as the A and G type variables, there is growing support regarding the development of positive self-beliefs and interest in math. The latter has gained a foothold given that both self-beliefs and interest are regarded as facilitators in students becoming individuals that engage in mathematical reasoning, apply problem-solving in daily situations and even choose careers in mathematics (Freeman et al., 2015). From the perspective of lifelong learning, this is crucial given its end goal of building highly competent, engaged individuals. The reasoning backs the notion that competent participants within a field are also those who possess certain beliefs about the field itself (Aditomo & Klieme, 2020), like the use of mathematical reasoning in everyday lives. Also, supporting students' development of positive self-beliefs and interest in math increases the likelihood that even students with lower competence will acquire the opportunity to move forward in developing their skills and gradually become individuals who engage in,

for example, applying problem-solving or some form of mathematical modelling in daily situations (Callan et al., 2021; Radišić & Jensen, 2021).

Echoing what Maehr (1976) noted many years ago, that motivation is one of the more essential and seldom studied educational outcomes, Anderman and Grey (2017) conclude that motivation matters. However, coupled with ability beliefs, motivation is still not considered a prized outcome in (mathematics) education. Undeniably, 'achievement' repeatedly triumphs motivation. Although, across many countries, decision-makers proudly acclaim the extent high achievement students have reached in a particular domain, like mathematics, little focus is given to whether those students subsequently wish to continue pursuing a career in mathematics (Anderman & Grey, 2017).

5   Concluding Remarks

Starting from the presage-process–product paradigm and reasoning formed primarily in the period after Medley's reflections (1977, 1987a, 1987b) on relevant research variables in understanding mathematics teaching and student outcomes as its ultimate goal (Manizade et al., 2022), in the present chapter, an attempt had been made to provide an overview of the main lines of rationale in mathematics education research on student learning outcomes and their assessment. A point of departure in this process has been capturing the basic ideas on what it means to be proficient in mathematics and how students' outcomes could be understood in light of such ideas. The focus was on different conceptual frameworks instead of particular theoretical background. In doing so, different frameworks were presented, with no ambition to capture all of them but instead to sketch the flow of ideas pre- and post-Medley times. This was achieved by showing dominant orientations (e.g., the dominance of cognitive and context-specific frameworks), their possible similarities (e.g., KOM and PISA framework) and dissimilarities in the understandings each of them provides (e.g., TIMSS and PISA).

A discussion on some core aspects of assessing student outcomes followed, capturing its historical perspective within mathematics education, including the foundations of the formative versus summative assessment and through the lenses of ILSAs, which have strongly affected the assessment process. Although there was no particular aim to investigate all methodological challenges related to assessments as such, major ideas were discussed by keeping in mind the principle conditions Medley (1987b) mentions (e.g., robust conceptualisation, instrumentation, design) when discussing the successful assessment of student outcomes. The section ended with deliberation on the technology intake and need to link complex measurement potentials to particular learning outcomes and (or) instructional practices (Manizade et al., 2022).

Finally, an argument was raised on how student outcomes could be envisioned today and the extent this widens or blurs the relationship between the A–G variables proposed by Medley (1987a). To date, it remains crucial to grasp at what it means and

requires to master mathematics. Understanding the role of dispositional factors (e.g., ability beliefs and task values) in the conceptualisations of mathematical competence is still required (Niss et al., 2017), especially given their fleeting position across existing frameworks (e.g., included as one of the proficiency strands but absent from the vast majority of other frameworks). Possible recognition could lead to a broader and fuller understanding of what it means to be proficient in mathematics without solely relying on the cognitive aspect of being competent. Ultimately, this may lead to different methodological choices on measuring mastery and shifting the balance from cognitive to noncognitive learning outcomes, which, in return, affect choices such as applying problem-solving or some form of mathematical modelling in daily situations or even pursuing a career in mathematics. Only then could the enactment of mathematics, coupled with teaching and assessment, be genuinely in agreement.

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Units of Analysis of Research on Teaching Mathematics that are not Under Teachers' Control: Offline Variables

Individual Student Characteristics, Abilities and Personal Qualities and the Teacher's Role in Improving Mathematics Learning Outcomes

Rhonda M. Faragher [ID]

1 Introduction

Teachers of students in general mathematics classrooms accept and welcome the learners they are assigned to teach. Students' characteristics influence the planning teachers undertake, the learning activities they provide, and the learning outcomes achieved by their students as a result. This chapter explores the impact of student characteristics that are beyond the control of teachers, and yet are within their powers through their actions to make a considerable difference to the mathematics learning outcomes of their students (Manizade et al., 2019).

In recent times, two significant developments—the recognition of streaming (the practice of grouping students "within-grade-level on the basis of *perceived* ability" (Forgasz, 2010, p. 57)) as harmful and the recognition of inclusive education as beneficial—have changed the nature of general mathematics classrooms. The research on assigning students to mathematics classes based on their achievement has long shown that this is a harmful practice (Hunter et al., 2020; Wilkinson & Penney, 2014; Zevenbergen, 2005). Streaming is detrimental to not just low and average groups but also has limited benefits along with possible risks for high achieving students (Linchevski & Kutscher, 1998; Parsons & Hallam, 2014).

In a separate development, the inclusive education movement, where all learners are welcomed and supported in general learning environments, has also become prominent. General Comment Number 4 of the United Nations' Committee on the Rights of Persons with Disabilities (2016) distinguishes inclusive education from integration, segregation and exclusion. Inclusive education requires the learning support needs of all learners to be met within the general classroom where all learners are working towards the learning outcomes of their class program. Again the research

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evidence is unequivocal (Hehir et al., 2016)—all learners do better in inclusive education environments. Through international charters and conventions, the provision of inclusive education has become a requirement of States Parties to these agreements. In the case of the United Nations Convention on the Rights of Persons with Disabilities (United Nations, 2006), all but seven countries in the world have signed the convention, thereby signalling their intention to provide inclusive education for all learners with disabilities.

For mathematics education, these two evidence-based movements—heterogeneous rather than streamed groupings and inclusive education—combine to provide strong impetus for the need to teach all learners in general classrooms. In practice, this means that teachers can expect to teach students across the breadth of human variation. This might seem overwhelming and indeed it would be if the approach were to attempt to teach individual lessons to each student in the class. Research in this field has led to alternative approaches that indicate approaches that teachers might use to plan effective mathematics lessons for all learners, including those with intellectual disabilities in the one classroom. These will be canvassed in this chapter. In this way, it is argued that teachers use to advantage the characteristics learners bring to the classroom, leading to improved mathematics learning outcomes for all.

2   Literature Review

Individual student characteristics are beyond the teacher's control. If we assume inclusive classrooms, the students they teach are also beyond the teacher's control. Teachers do not exclude students who have particular attributes, such as intellectual disability. The example of intellectual disability is taken here because it is possibly the area of student diversity that presents as most instructive for understanding the impact of learners' characteristics on mathematics learning outcomes. The impact of learner attributes on the presage, process, and products (PPP) of mathematics teaching must be understood for the best learning outcomes to be achieved.

One challenge encompassing both theoretical and methodological aspects of this area relates to the historical approach to the education of low attaining students in mathematics. The emergence of two education systems—special education and mainstream—has meant the mathematics education of students with intellectual disabilities and learning difficulties mostly had been undertaken by special education teachers. In recent times, inclusive education has been adopted around the world with the aim of educating all students in mainstream classrooms (Florian, 2012) and therefore the mathematics education of learners with intellectual disabilities has become the task of general mathematics teachers.

Mathematics education research typically takes a different approach to research from that of special education research in the field of mathematics (Xin & Tzur, 2016). It is not a happy marriage and reconciliation is not easy. Behaviourist approaches to mathematics learning, with a focus on remediating deficits and explicit teaching of procedures (often with mnemonics to aid recall of the procedures, see, for example,

Flores, 2010), have been favoured by many special education researchers in mathematics (for a review, see Tan et al., 2019), though not all (see, for example, Browder et al., 2012). Special education research focuses on interventions and understanding learner deficiencies. Mathematics education research focuses more on pedagogy appropriate for learning mathematics. The systematic review by Tan and colleagues, sampled literature published between 2006 and 2017 related to mathematics education of students with intellectual disabilities. From their review, they distilled three categories: Deficit-oriented, Discursively-aligned and Socio-political. 48% of reviewed articles fell within the sub-category "Behaviorism" of the "Deficit-oriented" category, where they note, "In these studies, mathematics education is characterized as reproducing or memorization of prescribed facts or procedures. … Students in these studies are shown a particular sequence of solving mathematics problems, followed by assessing their ability to exactly follow each step in the pre-defined procedure." (Tan et al., p. 6). The goal of the mathematics taught was found to be functional, "research focused on the importance of mathematics fact recall as they connected their work to functional life skills and the importance of mathematics for everyday tasks such as shopping" (p. 6).

By contrast, general mathematics education research emphasizes the value of mathematics as a discipline and favours less utilitarian views of the purpose of learning mathematics, for all students including those with disabilities (Scherer et al., 2016). In the Tan et al. (2019) review, a category of research was identified that aligned with this view of mathematics education research.

> In contrast to the dominant forms of research in mathematics education involving students with intellectual disabilities that focuses on direct forms of instruction and basic mathematics skills development …, undermining conceptual construction and understanding of mathematics, the studies in this category presume, to a greater extent, that students with an intellectual disability are mathematics thinkers and doers, capable of a range of mathematics engagement. (p. 8)

For the purposes of this chapter, providing an overview of current understandings in this field, three areas of diverse student characteristics are reviewed: mathematics learning disabilities, which are inherent to the student; learned difficulties, which are not; and other learner characteristics, not related to intellectual development, that have an impact on mathematics learning leading to learning difficulties.

2.1 Current Understandings of Mathematics Learning Difficulties, Disabilities and Dyscalculia

Some students struggle to learn mathematics. This is hardly a revelation. No matter whether the assessment is norm-referenced (with a proportion of the population automatically performing worse than expected for their age) or criterion-referenced (with a possibly similar subset of the population not meeting learning targets for their grade level), teachers, parents, and the students themselves have known that

learning mathematics does not come easily for all. The source of these difficulties is of particular interest because that affects approaches teachers might take to improve learning outcomes.

It would seem that the source of some mathematics learning disabilities is neurobiological in origin leading to differences in cognition. Developmental Dyscalculia (DD), possibly with sub-types (Skagerlund & Träff, 2016), is a condition caused by atypical development of the parts of the brain that support the understanding of quantity such as numerical magnitude processing, the development of a mental number line, and the ability to calculate using known facts (Kaufmann & von Aster, 2012; Peters & De Smedt, 2018; Skagerlund & Träff, 2016). Diagnosis is not undertaken based on brain imaging studies, however. Instead, diagnosis is usually made based on the determination of "a serious impairment of the learning of basic numerical-arithmetical skills in a child whose intellectual capacity and schooling are otherwise adequate" (Kaufmann & von Aster, 2012, p. 769). This comparison with progress in other areas of schooling or in comparison with other intelligence measures leads to the challenge of diagnosing DD in learners with intellectual disabilities who have relatively more difficulty in acquiring a sense of number than other areas of learning (Cuskelly & Faragher, 2019). The significant challenges of defining mathematics learning disabilities by discrepancy from "normal" achievement are well articulated by Scherer and colleagues (Scherer et al., 2016). Lewis (2014) draws attention to the challenges of diagnosis based on an arbitrary cut-off score and instead advocates for analysis of 'persistent understandings' that are divergent from correct mathematical understanding as an indication of mathematics learning disability.

Some aspects of difficulties learning mathematics overlap with language difficulties. Dyslexia, a learning disorder characterized by difficulties with language and reading, has a documented overlap with difficulties retrieving arithmetic facts (Peters & De Smedt, 2018) and is also neurobiological in origin. The area of the brain affected is not the same as that proposed for DD. Aspects of mathematics affected by dyscalculia also differ from that of dyslexia. Some aspects of arithmetic, such as retrieval of number facts, appear to be more language based (Dehaene, 2011). The impact of limitations of remembering arithmetical facts could arguably have been overstated, being a carry over from a time when the inability to calculate by written methods severely hampered further work in parts of mathematics reliant on these processes. With the ready availability of alternative methods of calculation, it might be the case that number and quantity need no longer be seen as core aspects of mathematics upon which the rest of the discipline relies. More research is needed in this area (Verschaffel et al., 2016).

Beyond those learners with mathematics learning disabilities such as DD, there are those with learning difficulties. The proportion of individuals who meet diagnostic criteria for DD is thought to be in the range of 5–7% (Butterworth et al., 2011). However, "a much larger number of children and adults experience less severe or less specific difficulties with mathematics which are nevertheless sufficient to cause significant educational and occupational difficulties" (Dowker & Kaufmann, 2009, p. 339). These learning difficulties with mathematics are the result of factors separate from the neurobiological functioning of the learner and reflect low achievement due

to other factors. These factors include poor teaching, environmental factors, affective factors (Lewis, 2014), minority status (Hunter et al., 2020), previous academic attainment, gender, age, health, family socio-economic characteristics (such as parental education, income and health), and school characteristics (Parsons & Hallan, 2014).

Understanding the impact of factors that lead to mathematics learning difficulties has been the focus of mathematics research endeavors (Faragher et al., 2016; Scherer et al., 2016; Vale et al., 2016) and the concern of teachers. Lindenskov and Lindhardt noted in their study (2020, p. 65) the concern of teachers with the paucity of mathematics learning experiences they felt compelled to offer students who had difficulties learning mathematics.

The distinction between the two groups—disabilities and difficulties—is critical for intervention. Underpinning both, though, is the need for, and value of, good teaching with the right support.

2.2   Current Understandings of Learned Difficulties

Low attainment in mathematics, as the argument is building, can be the result of learning disability leading to different development, or learning difficulty due to factors in the ecosystem of the learner. Low attainment can be the result of a mismatch of mathematics education approaches with the needs of the learner (Lewis, 2014; Lindenskov & Lindhardt, 2020).

The third group of low attaining learners is comprised of a group who have acquired their difficulties with mathematics while they have been learners at school. These are those students who did not commence school with a disability or difficulty but through engagement with the school mathematics environment, they acquired difficulties with mathematics. This third group can be designated as those with *learned* difficulties. A group of learners in this category that has received research attention in recent times are learners with mathematics anxiety (Dowker et al., 2016). This group of students can become so fearful of mathematics that they will actively avoid even those mathematics tasks that are easily within their capability (Wilson, 2018).

Learned difficulties with mathematics can be pervasive, leading to limitations on the mathematics individuals are willing to undertake in the contexts of their lives. This important research finding, documented in many government reports including the influential Cockroft Report from the UK (1982), has led to impacts on conceptualization of numeracy—the use of mathematics in life contexts. These models acknowledge that it is insufficient to know mathematics and make sense of contexts; affective attributes such as willingness to undertake mathematics are also essential to consider (Goos et al., 2015).

While the three groups proposed (learning disabilities, learning difficulties and learned difficulties) have been presented as disjoint, intersection between groups may well exist. Students with learning disabilities could quite conceivably gain more pronounced difficulties via the same mechanisms as other students e.g. development of mathematics anxiety.

2.3   Interventions

Having outlined three main groups of low attaining students in mathematics—those with learning disabilities, learning difficulties, and learned difficulties—attention is now turned towards what research indicates might be done to improve the attainment of those students.

Overcoming a neurological developmental difference requires specific attention to the cause, with brain-based interventions implicated. Some studies have tested the impact of computer interventions based on repeated task training aimed at developing new neural pathways (Räsänena et al., 2009) with some localized benefits. Unfortunately, transfer to areas of mathematics learning beyond the immediate training tasks was not found. It would seem that treatment approaches that correct neurological impairments to mathematics learning are yet to be uncovered. Instead, or in the meantime, teachers must take other approaches. It is not acceptable to do nothing—ways around the barriers to learning must be found if students are to be successful at learning mathematics in inclusive classrooms. Lewis' work (2014) indicates the need for teachers to deeply comprehend the way a student is understanding or making sense of a mathematical concept and basing teaching to build from that conceptualization towards a correct mathematical understanding.

For students with learning difficulties, Lindenskov and Lindhardt (2020) indicated teachers in their study held a commonly accepted view that low attaining students required training and task repetition. These teachers also noted, however, "the low motivation of students who are vulnerable partly due to the monotonous and tedious tasks they were offered" (p. 65). After engagement with the research project, these teachers subsequently noted: "that the early use of calculators may prevent students' low calculating skills to slow down processes towards conceptual understanding" (p. 66). This is an important example because it moves beyond a focus on number facts and arithmetic, characteristics used by much of the research literature to identify mathematics learning disabilities and difficulties. The calculator is used to move beyond those early aspects of arithmetic to a much more productive mathematical focus on understanding.

With respect to learned difficulties, overcoming detrimental thinking patterns acquired in the school years can be exceedingly difficult to change (Dowker et al., 2016). Some approaches have been trialed based on techniques used in psychology to treat other forms of anxiety. From a mathematics education perspective, this is an area of low attainment where the intervention should surely be to prevent anxiety from developing in the first place.

In this section of the chapter, a brief review of research understandings of types of student difficulties with learning mathematics has been given. It is now time to turn attention to the research investigating work that teachers might undertake to account for these student attributes in lesson planning and teaching.

3   Impact of Students' Attributes on Their Learning Outcomes

Where do teachers need to take account of these diverse attributes? In the Medley (1987)/Manizade et al. (2023) framework, we are looking at the impact of Type G (student attributes) on the relationship between Types B (learning activities) and A (learning outcomes). It is not possible to teach separate lessons to each student in the class, nor desirable, and yet the current focus on learning trajectories and more traditional approaches of teaching from where the student is at might suggest that is required. The implicit assumption is that mathematics is inherently hierarchical and rarely questioned (Forgasz & Cheeseman, 2015) and therefore, learners follow the same path, though perhaps at different rates. There is an alternative view, that there are many paths to mathematical achievement.

If we were to make the assumption, or perhaps working hypothesis, that alternative pathways to mathematics attainment are possible, we could then consider some alternative approaches to curriculum design and the planning of mathematics learning activities by teachers. In this section, three will be considered: Universal Design for Learning, the use of digital solutions, and year level adjusted curriculum.

3.1   Universal Design for Learning

Universal Design (UD) was first used in architecture where it was suggested that new buildings and spaces could be made accessible by their design from the beginning. The underpinning idea required consideration of access for individuals with disabilities, and all others, as a key principle in the design phase of architectural planning. In brief, UD involves the design of products and environments to be useable by all people to the greatest extent possible without the need for adaptation or specialized design. The impact of UD in public building design is obvious, once noticed. The frustration of lack of access or the expense of adding facilities to provide access for an individual after a building is completed is avoided. In countries where UD is written into building design codes, we come to expect that there will be ramps or lifts, easily accessible light switches and power points, good lighting and accessible toilet facilities. If we need them, they are there, if we do not, they do not impede our use.

The idea of UD was expanded into other areas and by the late 1990s, it was applied to education. Meyers and Rose, researchers at the Center for Applied Special Technology (CAST) developed an application of UD to learning situations and coined the phrase Universal Design for Learning (UDL) (CAST, 2020). In an analogy to universal building design, UDL emphasizes meeting as many learning support needs as possible in the one lesson plan. The key feature of the designed curricula is the promotion of access, participation, and progress in general education for all learners. UDL becomes a way of thinking: planning (presage) always to provide

multiple ways of presenting information, engaging with content, and demonstrating accomplishment. Diversity is expected, planned for, and valued for adding richness and alternative ways of thinking about the topic.

The use of rich problem solving tasks, an established practice in mathematics pedagogy (Chan & Clarke, 2017), is an example of how teachers might engage and challenge all learners in the one classroom with the one task (Lindenskov & Lindhardt, 2020; Sullivan, 2017). There are many sources of tasks of this nature and in the collection of Downton et al., (2006) they also provide work samples from students demonstrating different ranges of performance and accomplishment on each task. For example, Mason et al., (2010, p. 184) offers the following task: "What numbers have an odd number of divisors?" In order to plan to use this in an inclusive classroom, a teacher would consider what students would need to know to be able to make a start. What "enabling prompts" (Sullivan et al., 2006) would be needed? Understanding the question is likely to be needed, including definitions of keywords, such as "divisor". Ways to find divisors (factors) of numbers would be needed. Demonstrating an example with blocks would be one way. For example, a teacher might show how to find the factors of 12 by taking 12 blocks and arranging them in rectangles. The side lengths are the factors. A teacher would also need to prepare ways to extend the problem for learners who have solved the original task. Mason et al. suggest "Is there a number with exactly 13 divisors?" as one option.

Just as universal building design will not meet the needs of some users with very specific needs, some learners require very specific adjustments. These adjustments can be added into the planning stage when those requirements are known. For example, one learner with Down syndrome in a senior secondary mathematics class, who was involved in the research project discussed in the later section (see Faragher, et al., 2019, for details), required step-by-step instructions for using his graphics calculator. His teacher prepared these adjustments as part of her lesson planning for specific topics. Any additional adjustment she made, she also provided for other students. In the case of the graphics calculator instructions, she made and laminated two copies—one for the student with the individual plan, and the other was placed on the table at the front of the class for use by any other student (and many did).

3.2 Digital Solutions

It is undeniable that technological advances have made astonishing possibilities for adjustments in mathematics classrooms. They have also fundamentally changed the nature of numeracy. Numeracy is the use of mathematics in life contexts and therefore, how we engage with these contexts and the technology available for our use, changes our numeracy needs. Implications for learners with intellectual disabilities have been discussed previously (Faragher, 2019). When we consider the impact of Type B activities (Student Mathematics Learning Activities) on learning outcomes, it can be argued that there is a fundamental change needed here: in what students are required to do, and what they need to be taught for numeracy development.

Digital solutions also affect how we know what students are able to do and understand as a result of learning. It is possible to go far beyond the time-honored techniques of tests and examinations to gather evidence of students' mathematics learning. For students who find writing difficult, we can record them demonstrating techniques or presenting their work to peers. For students who have limited expressive language, we can observe them making choices or undertaking adjusted tasks. Video records can capture the 'aha' moments. In a recent research study discussed in Sect. 5.2 below (see Faragher et al., 2019, for background), such a moment was captured on video. The researcher and teacher were working in a secondary mathematics classroom with a student with Down syndrome. The student had limited expressive language. He was working on trigonometry with other students in the class, including his friend without a disability who was helping him learn to distinguish right-angled triangles from other triangles. The moment when the student reached for a right-angled triangle from a collection of possibilities, rather than testing at random, was observed by the researcher and the teacher, and captured on the video recording. In this way, the video recording is an example of a digital solution where a record of learning is made that can be analyzed and used to confirm learning. Of course, a camera has to be on at the time and except in the context of research studies, it is unlikely to be the case in routine classroom activities. However, the use of video can be used strategically. For example, in the concluding phase of a lesson, where a teacher wishes to gather evidence of learning, particularly from a student with communication limitations, a video clip could be taken of the student demonstrating the performance of a task.

Individual learner characteristics (Type G) affect the types of activities they are given (Type B) that allow them to demonstrate their learning (Type A). Digital options allow many more valid approaches to the assessment of learning with the likelihood of uncovering Type A outcomes that may never have been imagined. A study of the use of technology for formative assessment of mathematics was undertaken by Dalby and Swan (2019). They considered "the potential for iPad technology to facilitate and enhance formative assessment processes by contributing to the construction of richer and more efficient processes, that bring benefits to student learning" (p. 835). In the research, six lessons were co-designed with teachers and researchers, and these were then trialled in two secondary schools in the UK. The research used "a cyclical process of design, testing, feedback, reflection and redesign" (p. 836). Data analysis of the process of formative assessment used coding and categorization from which five categories emerged. The analysis indicated the potential for the use of technology for assessing mathematics using existing pedagogies while cautioning that "the greatest challenge for teachers in using technology in the classroom is not the technology but an understanding of the process by which it can enhance student learning." (p. 843). More research into the use of digital solutions for assessing learning, particularly of students with intellectual disabilities, is needed to understand its full potential.

Sometimes, the promise of digital solutions is promoted as the panacea for improving the learning of low attaining students, and particularly those with disabilities. It is clear that while these tools do hold great promise, they are not sufficient in

themselves and the other variables around teacher classroom practice need greater exploration. In this chapter, the possibilities the digital context affords to teachers are recognized in conjunction with other aspects of their work.

3.3   Adding Adjustments to Year Level Curriculum

The learning theorist Bruner argued that it is possible to teach any topic to a school aged child in an intellectually honest manner (Bruner, 1960, 1977). Over the years since then, examples from mathematics have emerged where students have indeed been taught seemingly more sophisticated mathematics than their years or curriculum attainment would suggest would be possible. Intriguing examples have emerged from Italy, a country that has not had a special education system for more than 50 years and so has had opportunity to explore possibilities of curriculum innovations. In a paper by Monari Martinez and Bennetti (2011), we see examples of students with significant intellectual disabilities being taught and achieving learning outcomes in areas of mathematics such as algebra and coordinate geometry. Perhaps the most astonishing is a student who learned to use the distance formula to find the distance between two points and then graphed these on centimetre graph paper. Subsequently, the student came to understand measuring with a ruler as she learned the ruler could be used to obtain the same answer as she had already calculated.

Learning how to measure with a ruler through co-ordinate geometry is beyond intriguing to be completely counter-intuitive. Replication studies are needed to determine if the specifics of that particular study can be repeated. Further research with similar results from other areas of mathematics is already emerging, however. Studies from the United States also indicate what is possible (Browder et al., 2012; Creech-Galloway & Collins, 2013). In the United States, the "No Child Left Behind" legislation encouraged the development of teaching approaches to support the requirement that all students would be assessed on the curriculum aligned with their grade level (Browder & Spooner, 2014).

Known variously as "age-appropriate", "grade-aligned" or "year-level" curriculum, teachers have devised innovative approaches to learning design (Browder & Spooner, 2014). I choose to use the terminology of year-level adjusted curriculum (YLAC) because some students are older or younger than their class peers due to a number of possible factors including transfer across school districts, ill-health, and delayed entry. The purpose of adjusting the year-level curriculum is to begin with the curriculum being planned for the class and then meet specific learning needs by planning adjustments.

In the YLAC approach, teachers begin with the curriculum for the year level they are teaching; that is, they start with the lesson as they intend to plan for their assigned class. This approach is discussed in more depth elsewhere (Faragher, 2017; Faragher et al., 2019), and outlined here. Using principles of UDL or other planning methods, teachers plan multiple approaches for each aspect of their lesson, ensuring the learning support needs of as many students as possible are provided for in the

standard plan. This means that enabling prompts to assist learners to enter a learning task are provided as well as extending prompts to ensure all learners, and particularly gifted and talented students, are challenged in the lesson (Sullivan et al., 2006). Similarly, provisions for students with language, social-emotional, physical, and sensory needs are planned.

Once the lesson has been planned, teachers then consider specific additional adjustments that may be required by some learners. In research studies exploring the practices of teachers who were including students with Down syndrome in regular primary and secondary mathematics classes (Faragher & Clarke, 2020; Faragher et al., 2019), teachers considered each stage of the lesson and thought about where the student might face barriers. At this point, teachers would look for ways to work around the barriers. These situations arose where students' impairments were having an impact on their work, such as difficulties hand-writing or using the layout of a calculator. On occasion, the barriers were to do with intellectual disability, though these were mostly attended to in the general plan.

In these three approaches to lesson planning (UDL, digital solutions, and YLAC), the impact of the characteristics of learners (Type G), is clear. Diverse classrooms bring a diversity of Type G variables that directly affect the work of teachers. In the decades since inclusive education has become policy around the world, the nature of teachers' work has fundamentally changed as well. This work, and the impact of Type G variables on learner outcomes, needs much greater exploration. In the following section, a more detailed analysis of examples from two recent studies into YLAC provide an illustration of the interaction between Type G factors and Type B (student learning activities) and Type A (learning outcomes). Through this analysis, an overt investigation of the interaction of these variables can be considered.

4   Learning Year Level Curriculum

At the turn of this century, a significant research project in Australia was undertaken to track the early numeracy development of children (Clarke et al., 2002). The project developed task-based interviews for teachers to use with their students to track their mathematical development. In a subsequent study, the interview was adapted to explore the mathematical development of young children with Down syndrome. While the interview had been used with children enrolled in special schools (Clarke & Faragher, 2004), to our knowledge, it had not been used with children with Down syndrome. Down syndrome is a genetic condition leading to varying degrees of intellectual disability. Difficulties with number have been documented in research literature for decades (and often incorrectly generalised to difficulties with mathematics in general). Other areas of mathematics attainment have rarely been studied until recent times (Faragher & Clarke, 2014).

In our research project (Clarke & Faragher, 2014; Faragher & Clarke, 2014), we became aware that some children seemed to be making greater progress than others and that there appeared to be a teaching effect. We wished to know more about the classroom environments where these children were developing their early mathematics knowledge.

4.1 Learning Year Level Mathematics Curriculum in Primary Schools

So began a study of learning year-level mathematics curriculum in primary classrooms by children with Down syndrome. We were focused on the work of teachers, rather than the students themselves as we observed their teaching practice over one school year. In our project, professional learning workshops were interspersed with classroom lesson observations and interviews with teachers about their work. As reported in a recent paper (Faragher & Clarke, 2020), there were indeed practices that teachers adopted, in response to the learning characteristics of their students (Type G), that had an impact on the learning outcomes of students with Down syndrome (Type A) within inclusive classrooms. In that study, it was clear that effective teachers expected all students to think mathematically and were focused on ways to engage their students with Down syndrome in cognitively challenging mathematics. They made judgments in lessons about when to withhold from telling a student the answer, and instead, encouraged them to persist. Teachers also had a clear focus on the mathematics of the year-level and they communicated that focus to teaching assistants (adult helpers without teaching qualifications) assigned for the lesson. The mathematical focus of the lesson for the student with Down syndrome was the same as for the rest of the class and teachers made adjustments to enable that to occur. Consideration of the learning needs of the student occurred at the planning and lesson implementation stages. A key finding was that "the provision of reasonable adjustments in mathematics is highly skilled work exemplifying high-quality mathematics teaching. This involves knowledge of the learner, the mathematics and how to teach it" (Faragher & Clarke, 2020, p. 141). In the context of education research variables, here is the interplay between Type G variables (individual student characteristics, abilities, and personal qualities) and resulting learning outcomes (Type A). The provision of reasonable adjustments to allow students with disabilities to access the curriculum is a requirement in law in many countries, and specified in the UNCRPD. Determining and planning reasonable adjustments requires a teacher to carefully consider the learning characteristics of students at all stages of the presage, process, product model of classroom learning.

In our work with primary school teachers, it was evident that good teaching and the right support had an important impact on the mathematics learning outcomes of their students with Down syndrome. In the Medley/Manizade et al. model (2023), we can see the interaction of research types at play here and these will be discussed in the implications section below.

4.2 Learning Year Level Secondary Mathematics

The primary mathematics research led naturally to consider possibilities in secondary mathematics. In particular, a parent had learned of early findings of the primary project and was keen to have her school consider implications for her son (called "Brian" in the project) in year nine. At the time, he was struggling with single digit addition and was being given worksheets with simple examples, such as $5 + 3$, written vertically. Brian was acutely aware that this work was childish and what is worse, he found it difficult. His teacher asked for advice and following a short conversation, planned an adapted worksheet on the topic being taught to his class—linear functions. A clever aspect of the planned adjustment was that Brian ended up doing lots of single digit addition, but now in the context of algebra where he was substituting for variables. He now enjoyed doing this work. As a result of this initial success, the school became instigators of a broader research project that involved the teaching teams of five students with Down syndrome in three Australian states over two years.

Background to the research study, including the methodology and methods involved have been discussed previously (Faragher et al., 2019). In this chapter, the mathematics learning of three of the students is studied through the analytical lens of the Medley/Manizade et al. (2023) framework. In analyzing the case studies, the focus will be on how individual student characteristics (Type G) mediate the connection between student mathematics learning activities (Type B) and the resulting learning outcomes (Type A).

In the sections below, three of the students will be introduced—Brian, Jay, and Mary. I have used pseudonyms for each. First, I present a brief description of their learning context before moving to give a short overview of key aspects of the different variable types.

4.2.1 Brian

Brian has Down syndrome and attended a mainstream Catholic boys' school. At the time of the study, Brian was in his final two years of secondary school, studying Prevocational Mathematics. This is a subject designed for students who require mathematics beyond school in areas such as employment and trades. It is not designed for students intending to study at university.

Brian's teacher had a genuine expectation that Brian could be successful at learning the mathematics content of his course. She focused her planning on considering barriers Brian might face and then working out what adjustments might be needed to assist him. In an interview, when asked about additional planning load, she agreed there was extra work involved but she discounted this as a problem. Her point was that while she was spending time making resources and undertaking task analysis for various procedures, this was an investment in future work because "there will always be students of mine who need this support". She made a particular point

of making two copies of any resource—one for Brian, and one for the rest of the class. Students in one observation lesson were seen to access the instruction guide for calculating means, medians, and modes.

Learner characteristics affected the teacher's planning of assessment activities that would be supported by the teacher aide. She felt that a page of exercises would be daunting and so she took the required exercises and reprinted them in a more engaging format. The advantage of electronic texts was clear here, as the reformatting was relatively straightforward. She often used color coding so that Brian would be able to have a visual cue to the type of task involved. She had developed this strategy through her experience of working with Brian over time.

Most of the assessment for this Prevocational course was undertaken by assignments involving rich tasks that students completed during class time. One example was a task where students were to design a car park. Another was to investigate various data representation approaches in the quantity control of matches. Each required mathematics techniques from different branches of mathematics, in these cases, geometry and statistics. Planning the work for Brian involved breaking the assessment activities into smaller sections, considering the mathematics required and then undertaking a task analysis to break the learning into small steps. A further consideration was the work to be undertaken by the teaching assistant. At the start of each lesson, the teacher would spend a short time with the teacher aide (not necessarily the same person each lesson) explaining the mathematics and indicating what she required Brian to do. It was not her expectation that the teacher aide would be solely responsible for teaching Brian.

During class, Brian worked on the assigned tasks, supported by his teacher aide who assisted with understanding the task instructions. The teacher would move around the room assisting students as they needed. She would also routinely move to check on Brian's progress. In the observation lessons, the interaction between the teacher, the teacher aide and Brian was noted. On one occasion, Brian needed assistance with a particular part of the carpark assignment. He was incorrectly using his calculator to find the perimeter. When his teacher approached, she sat down in the chair beside Brian to assist. The teacher aide listened briefly then moved away to assist other students. This seamless interaction was a way the two adults supported each other and the learning in the classroom.

Type G Individual Student Characteristics, Abilities, and Personal Qualities

Brian was eager to learn. This was apparent throughout the observations. More specifically, he was eager to learn mathematics that was being taught to the other students in the class. His enthusiasm was described in his exclamation "I just love it" which exemplified the finding on affect reported in a previous paper (Faragher et al., 2019). At the final observation visit, the class was in the last few weeks of secondary school and their assessments were largely complete. The other students were keen to leave the secondary mathematics lessons behind and talk about post-school parties. Not Brian! His teacher explained that a visit would be productive because she was still preparing mathematics lessons for Brian (and any other student who wanted—though there were no other takers!).

Mathematically, Brian continued to have difficulty with arithmetic and used a calculator to do any calculations required. Most lessons he was supported by a teacher assistant.

Type B Student Mathematics Learning Activities

The mathematics learning activities that Brian was engaged in during observed lessons were significant to inclusive practice and the resulting learning outcomes. Brian in his early years in secondary school indicated the desire to study mathematics that was like his peers' program. Success with adjusted activities in the junior school led to the expectation of participation in and the possibility of success with learning year-level adjusted mathematics in the senior school. In the senior school, exit assessment is required. The assessment tasks developed by the mathematics department were designed to meet the requirements of the state syllabus. In preparing adjustments for these tasks, his teacher had to present work with the right level of challenge. This was not always easy to judge and the teacher talked in her interviews about the need to make further adjustments sometimes during a lesson. She relied on the teacher assistant and her interactions with Brian to discern when the level of challenge of the tasks was too little or too much.

Type A Student Mathematics Learning Outcomes

Brian's learning outcomes fell into two types—mathematical and non-mathematical. As Brian was studying a senior subject, his work was assessed to state standards. He received a passing grade for Prevocational Mathematics with the culmination being the award of the Queensland Certificate of Education. Brian became adept at the use of mathematical equipment, including graphics calculators and spreadsheets, and indeed, he required these to remove the calculation load enabling him to engage in the mathematical thinking and processes of the subject.

Beyond the achievement of mathematics learning outcomes, other learning outcomes were evident. Brian enjoyed his engagement in the senior mathematics classes and spoke with pride of his work. His teacher reported that his learning behaviour had improved in other subject areas in the school that were not part of the research study. This transfer is reminiscent of findings in the area of quality of life, a framework for understanding disability. Researchers in that field have found that interventions aimed at improving quality of life in one domain have led to unintended benefits to other domains of life (Brown et al., 1989).

Impact of Type G on B and A

Brian's learning characteristics could not be ignored by his teacher and had a significant impact on her work. The success Brian experienced in lessons (as evidenced through his obvious enjoyment in lessons, engagement with tasks, and through wanting to keep learning at the end of the school year) and in his exit assessment on completion of the course, are success measures for the activities planned by Brian's teacher. These activities were carefully constructed based on his teacher's deep knowledge of him as a learner. She explicitly considered his learning characteristics to develop activities that would build his mathematical understanding.

An intriguing, and rarely considered side benefit in the PPP analysis of mathematics education research, is the impact on the learning activities offered to the other students in the class. Because the teacher had to take account of Brian's significant learning support needs, she provided activities that were supportive of the learning needs of other struggling students in the class.

4.2.2   Jay

Like all students in the study, Jay has Down syndrome and an intellectual disability. In the first year of the study, Jay was in his second year of secondary school, attending a mainstream, co-educational Catholic school. The school has a tradition of educating students from diverse backgrounds, with a variety of learning needs and accomplishments. Jay's class included students from a range of nationalities, many recently arrived in the country.

His teacher had many years' experience and Jay's class was the bottom stream of mathematics for the year level (students were assigned to classes based on achievement, with four classes—one for high, two for moderate and one for low achievement). The bottom stream class still had the expectations of meeting the year level curriculum learning outcomes. In preparing for Jay's class, his teacher, made few adjustments. This was a bottom stream class and Jay was able to undertake activities along with other students. The teacher used whole-class teaching of mathematics techniques with worksheets to allow students to practise.

Assessment of the junior secondary mathematics involved tests and assignments. Jay undertook the same tests as the other students in his class, which were designed for the bottom stream class. In the second year of the project, Jay was observed working on an assignment where he had the same task as others in the year level and was required to undertake exploratory data analysis of various data sets, including constructing back to back Stem and Leaf plots. He was observed using his laptop to complete the questions that were not modified.

Each observed lesson there was at least one teacher aide assisting the teacher and on occasions an additional cultural liaison assistant. Jay required little support from the teacher aides. For example, in one observed lesson, his teacher aide, who used a wheel chair, positioned his chair beside but a little behind Jay. From time to time he drew Jay's attention away from his worksheet, to the teacher giving instructions at the whiteboard at the front of the class. The teacher aide assisted students nearby when they required support.

Following the practice on the worksheet tasks, the teacher returned the focus to the whiteboard, calling for responses from students. During this time, he asked direct questions of Jay, based on what he had noticed Jay was able to do on his sheet.

One observed lesson was taught by a casual teacher, replacing the teacher on sick leave. This relief teacher assumed Jay would need easier work than the other students and had prepared a simpler level worksheet on the topic ratio, the same topic as the rest of the class. When the worksheet phase of the lesson was commenced, the teacher immediately gave Jay the easier worksheet. I asked the teacher if Jay

might also have the worksheet being given to the rest of the class. She immediately agreed and gave the second sheet to Jay who was initially concerned. He was unsure what sheet he should do and his desire to complete work required reassurance from his teacher that he only needed to complete one of the sheets. Jay completed the unmodified worksheet without error.

Type G Individual Student Characteristics, Abilities, and Personal Qualities

Although in year 8, Jay had been assessed by the school as being at a year 3 level in mathematics. Jay is a serious student, dedicated to his work. He is strongly motivated to complete assigned tasks. In observed lessons, he was never seen to be off-task. Amid exuberant adolescents, he sat quietly working away on his mathematics. For each of the observed lessons, Jay was prepared for his class, fastidious in his attention to having the right equipment, including his laptop, textbook, and notebooks. It was perhaps remarkable to note that Jay was arguably the most dedicated student in his class. Some of the adolescent behaviours expected in bottom stream classes, including disengagement, off-task distractions, and lack of motivation were exhibited by other students, but not Jay. In every observed lesson, Jay diligently completed the tasks assigned to the class, mostly without adjustments. An observable aspect of Jay's work was his attention to the steps in a process. His dedication and emphasis on task completion were attributes that supported his work in this class, where his enjoyment was evident in following the steps in mathematical procedures and exercises that were required to be successful in that class.

These Type G variables are quite different from those usually reported for students with intellectual disabilities. Those reports, such as in research and professional literature more commonly focus on learning deficits. School documentation that indicated year 3 level in mathematics further emphasize deficit and when considering Type B variables, have a direct impact on tasks offered to students.

Type B Student Mathematics Learning Activities

By being included in the general mathematics class, Jay experienced the same learning activities as his classmates in the whole class instruction phase. In the phases that focused on consolidation and practice, usually through worksheets or computer based exercises, differentiation of learning activities occurred. Being the bottom stream class of the year level, the level of mathematical challenge had already been reduced for the class and Jay did not indicate he needed further adjustment during the observed lessons. However, as was noted earlier, on at least one occasion, the learning activity he was initially provided was made simpler than it needed to be.

Type A Student Mathematics Learning Outcomes

For Jay, the learning outcomes were measured through the use of worksheets and assignment tasks. As part of the data collection for the project, photographs of these completed sheets were taken. One lesson culminated in the completion of exercises on index notation where he completed each exercise without error. The lesson observation and the learning artefacts can only provide evidence of the completion of

procedures—following the steps of the process. We can know little of Jay's under-standing of the structures of mathematics underpinning these steps. However, it must also be noted that such evidence was not obtained from other members of the class, either. All were asked to demonstrate that they could complete the assigned exercises and the requirement was met if the steps were completed without error.

Impact of Type G on B and A

The Type G variables—the learner characteristics—that Jay brought to his mathematics lessons might reasonably be presumed to offer expectations of mathematical success. Those Type A outcomes were initially at risk, though, through presumptions of more pervasive mathematics learning difficulties than actually existed. The relief teacher did not know the student and made the assumption, perhaps based on school documentation, that he would need a simpler sheet. She automatically gave the easier worksheet to Jay, without checking, and without offering the sheet to any other learner. This indicates the danger of offering learning adjustments before a student demonstrates the adjustments are required.

High expectations and presumed competence are needed to counteract the pervasive deficit discourse about learners with intellectual disabilities.

4.2.3   Mary

Mary lives in a small town in a farming district. She attends her local school and at the time of the project was in her first year of secondary school. Mary seemed to enjoy her time at school and liked interacting playfully with her teacher and teacher assistants. In the first observation, she was being taught alongside two other students with significant, but very different learning support needs arising from other disabilities than Down syndrome. The class had 30 students and was taught by the teacher with two assistants working with the students with disabilities to the side of the class. In the first year of the project, Mary's teacher prepared different activities for Mary from the main lesson she planned for the class. This approach changed as the research progressed. At the first observation, Mary was being taught in a segregated approach and over the two years, she gradually became more included in the general class activities. This mirrored Mary's developing social inclusion (Koller et al., 2018). For example, her teacher told of her walking to school each day, with a mobile phone should she need assistance when originally she had been driven to school. Similarly, on the first observation day, Mary had been having lunch on her own in the school library. On later visits, she spent lunchtime in the school playground.

In this section, the teacher's changing inclusive practice will be presented and then, using the Medley/Manizade et al. (2023) model, these data will be analyzed to suggest how teacher reflection on the impact of Type G variables (individual student characteristics, abilities, and personal qualities) can lead to profound changes in their inclusive mathematics teaching practice.

The first lesson observed was from a year 7 statistics unit where students were learning about Stem and Leaf plots. The lesson was adjusted for Mary. She collected

data about screen time usage, as the other students did, however, instead of using the data to construct a Stem and Leaf plot, she was given another task of putting numbers in order. In the post-lesson conversation, the researchers and the teacher discussed the possibility of teaching Mary to construct a Stem and Leaf plot. After the next lesson, the teacher sent the team a photo of the student's work sample where she had completed a Stem and Leaf plot. This led the teacher to reflect on the structure of her teaching and in the following lessons, she adopted inclusive teaching practice, planning the one lesson with adjustments for Mary based on her learning characteristics.

In the next year, the now year 8 class was learning about adding and subtracting positive and negative integers. The teacher found a lesson from a website that focused on rich tasks for gifted and talented learners. In a pre-lesson conversation, she discussed how she had thought about how she would engage Mary in the context of the lesson. The task was a game that was to be played in pairs where each player had a hot air balloon with hot air and sandbags affecting the height. She found a video clip of hot air balloons to introduce the lesson to ensure that Mary would understand the context of the game. Because the lesson involved a game, she chose the student who she would assign to play with Mary. The activity game cards were prepared for each group in the class and were made by the teacher before the lesson.

At the start of the lesson, the video was played and Mary moved her chair so she could be directly in front of the screen. When the class broke into groups to play the game, the teacher assistant sat with Mary and her classmate and helped them to learn the rules of the game. Once the students were settled to the task, the teacher aide gradually withdrew. The teacher then asked the assistant to complete some administrative work on the computer at the teacher's desk. The students continued to play the game without further teacher assistance.

Type G Individual Student Characteristics, Abilities, and Personal Qualities

Mary's learner characteristics had a significant impact on the planning that her teacher undertook. In addition, she also considered the learner characteristics of the other students in the class, specifically when she selected a student to partner with Mary in the mathematics game. She chose a student who liked working with Mary and who would not over support her.

Type B Student Mathematics Learning Activities

Mary had the opportunity to learn year-level mathematics through engaging in learning activities with her class peers. Her teacher had embraced the idea of an inclusive year-level curriculum and developed her planning whereby she no longer prepared separate lessons for learners with intellectual disabilities. Instead, she made adjustments to ensure Mary could engage with lessons. Mary, in turn, responded by expressing her enjoyment of working in the mathematics lessons. She also enjoyed working with her classmates.

Type A Student Mathematics Learning Outcomes

Achievement of learning outcomes was indicated by completed tasks. Lesson arte-facts were collected, such as photographs of completed activities. Other indications of Mary's learning were also obtained by observing her responses in the lessons. In the lesson on operations with integers, Mary quickly learned the rules of the game. The winner was the balloon that reached the top of the vertical axis first. Each turn of the cards involved turning over a positive or negative sign and then a positive or negative number. Mary loved to win and when she turned her cards and saw that she had to move the balloon down, she refused to move the balloon. It was obvious that not only did she understand how to operate with integers, but also, she could do so in her head. Furthermore, she insisted on playing until she won, gaining much practice in the process.

Impact of Student Characteristics on Learning Activities and Outcomes (Type G on B and A)

As for Jay, Type G variables initially led to restrictions in the mathematics offered to Mary. There was an assumption that her intellectual disability would require different activities for the lesson (Type B), leading to detrimental impacts on the learning outcomes (Type A). What also was evident was how quickly the teacher was able to make profound changes to her practice through reflection on the lesson and engage-ment in a conversation with researchers about the lesson. Here is an indication of the role of researchers in a teacher's professional ecosystem. Teachers do not work in isolation and having a colleague to support reflection and stimulate professional growth has clear benefits for students' learning outcomes.

   The teacher already had the required pedagogical skills and practices in her reper-toire. Her initial approach appeared to be governed by what she believed was a necessary response to the Type G learning support needs of her student. As her knowledge of the student grew, and encouragement to try adjusting the year level curriculum had an impact on the learning outcomes (she could see what the student could achieve) she fundamentally changed her teaching approach to a genuinely inclusive mathematics classroom.

   A striking development was the use the teacher made of rich tasks from a collection of resources to challenge gifted and talented learners. She saw value in the use of these tasks in meeting the diverse learning needs of her students.

4.3 Learner Characteristics, the Mathematics They Engage in and the Learning Outcomes They Achieve

In times past (and not that long ago), students with learning disabilities were rarely given the opportunity to learn mathematics at their year-level. Exploration of exam-ples where teachers have adjusted the year-level curriculum for their students at

primary and secondary level are instructive for considering how learning characteristics attenuate relationships between mathematics learning activities (Type B) and the resultant learning outcomes (Type A). A focus on students who struggle with learning mathematics allows us to examine what might be possible for all learners.

What is clear is that these learner attributes in no way predetermine the learning outcomes. Learners need the opportunity to engage with year level mathematics, adjusted to be at the level of productive challenge (Gilmore & Cuskelly, 2014). Before and during lessons, teachers make adjustments in dynamic ways where they bring their understanding of mathematics, the learning outcomes intended, and their knowledge of the learner to the decisions they make. Dynamic judgments about the level of challenge and required support allowed teachers to respond where initial over-support was being made to students' learning activities. Over-support has the potential harm of reducing the mathematics learning outcomes, and it was important that teachers responded and corrected this at the point when it occurred.

In all the lessons observed in both the primary and secondary projects reported earlier in this section of the chapter, the students were engaged in learning mathematics for their year level. In contradiction to what is commonly portrayed or anticipated by guides to teaching students with mathematics learning difficulties, the students in our project were rarely off task, when the work was the same topic as their peers. Indeed, as we saw with both Jay and Brian, these boys were at times more focussed than their peers without disabilities. The impact on learners, particularly those in the secondary years, was pronounced with benefits not only to their mathematics learning outcomes but also to their self-concept as learners of mathematics and learners in general.

4.4   Implications from These Studies

In the examples from YLAC mathematics research projects, we see students with significant individual characteristics and abilities likely to have an impact on their mathematics learning. These attributes are outside the control of the teacher and yet as has been seen, teacher actions can ameliorate student attributes and lead to productive learning outcomes. Rather than these attributes determining the learning outcomes, a case can be made that while student attributes might affect the work of teachers and the learners themselves, mathematics learning outcomes at the year level are indeed possible.

Teachers in our studies underpinned their work with the expectation that their students could be successful at year-level mathematics. If there were barriers, they worked to find ways around through adjustments to learning materials or approaches to the topics. A common feature was that small suggestions regarding possibilities for changing approaches to be more inclusive were taken up by the teachers in creative and reflective ways, leading to new approaches in their classrooms or different responses from students.

These studies of the practices of effective inclusive mathematics teachers indicate the dynamic nature of teaching and responding to the learning of students due to the characteristics they bring with them to the learning process. Teachers' planning is affected before, and during, the lesson as teachers think and problem solve and make decisions in the moment. It is clear that individual student characteristics affect learning outcomes but in surprising ways. Creative, reflexive teachers respond to these characteristics by changing, adjusting, and developing the mathematics learning activities offered not just to the students with learning disabilities and difficulties, but to all learners in the class. Learning outcomes are likely to be improved for all. As a way out of low attainment, this is a critical aspect of mathematics education and an imperative of mathematics education research to further explore its possibilities.

This research also calls into further question the persistent, detrimental practice of streaming based on previous attainment. By planning for all learners, anticipating diversity and providing learning adjustments as required to year level curriculum, teachers have an alternative to separating students into coarse class groupings with known detrimental impacts.

5   Implications

Learning characteristics of students with intellectual disabilities cannot be ignored—they have too great an impact on teachers' work. When embracing the teaching of diverse learners, teachers commence planning with learners' characteristics central to their thinking.

Medley (1987) defined Type G variables as "individual student characteristics, abilities and personal qualities which determine outcomes of any specific learning experience". Recent research, as outlined in this chapter, raises questions about the determinism of outcomes. Similar questions also emerge from a greater understanding of the impact of disability on learning, a field that continues to evolve and with greater opportunities for learning afforded by inclusive practice around the world, continues to surprise teachers and researchers alike. Perhaps a more accurate definition for Type G variables might be those learner variables that "affect outcomes", rather than determine outcomes. These learner variables do not determine outcomes, at least in a predictable way. Indeed, by continuing to explore ways to make learning accessible for students who are variable in their characteristics, abilities, and qualities (Type G), there is the capacity for teachers to be surprised (Russo et al, 2020). This surprise is likely to affect preactive teacher activities (Type D) with flow on implications right through to student learning outcomes (Type A).

In addition to modification to the definition of Type G variables, there are implications for the placement of these variables in the framework. Currently, Type G variables are shown to act between Type B and Type A. New ways of thinking about learning for students with learning difficulties and disabilities, challenges us to think about new ways of representing where Type G variables affect the PPP process.

Teachers do not plan a lesson and teach it to the class only to find that learner characteristics are affecting how that information is received. When working with diverse learners, teachers factor in learner characteristics much earlier on, and throughout the PPP process.

In this chapter exploring the impact of learner variables on student outcomes, a focus has been taken on learners who struggle with mathematics. These students present a considerable challenge to teachers as they work to improve learning outcomes. In most countries around the world, it has only been in recent times that students with significant mathematics learning difficulties and disabilities have been included in mainstream classrooms with the expectation that they can achieve learning outcomes. A deep understanding of the practice of teachers in these contexts is still emerging (Tan et al., 2019). It would seem that there is much more to learn about Type G variables and their effect on student learning outcomes and especially so in inclusive mathematics classrooms. Furthermore, there is much to learn about how teachers grow professionally through reflecting on the interplay between learner variables, student activities, and the resulting learning outcomes.

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Individual Student Internal Contexts and Considerations for Mathematics Teaching and Learning

S. Megan Che and T. Evan Baker

1 Introduction

School classrooms are enormously intricate, complex, dynamic, and, to an extent, unpredictable spaces where diverse humans meet together to deepen and expand their intellectual prowess. As the introductory chapter to this volume more fully explicates, the field of mathematics education research has devoted much intensive time and effort to improving our understandings of the myriad factors pertaining to teaching and learning in mathematics classrooms. Specifically, the chapters in this volume are connected in our shared endeavors to elaborate on one of the contexts adapted from Medley's (1987) framework, which include internal and external student and teacher contexts as well as characteristics and qualities of students, teachers, and learning environments.

Within this larger project, this chapter presents, discusses, and problematizes the progression of our field's understandings of individual student internal contexts through considerations of (1) meanings of internal/external (subject/object) dichotomies, (2) individual student cognitive processes, (3) individual student affective processes, (4) how these individual student cognitive and affective experiences connect with (are informed by and inform) each other as well as broader communities such as mathematics classroom learning environments and home environments, and (5) implications for teachers and teacher educators.

Specifically, we situate the notion of an individual student as an entity in continuous dialectic with environmental influences, to the point that—at one scale, it

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becomes meaningless to distinguish between individual and environment. As we begin this chapter, we explicate the scale we choose to use for our chapter: the scale at which individuals are distinguishable from each other and their environments but also inseparable from each other and their environments. In so doing, we elaborate a connected perspective of student identity. We then similarly situate the experiences of cognition and affect prior to deeply considering the progression of our understandings of students' cognitive and affective processes. We proceed to articulate and examine ways these individual experiences connect with multi-individual communities like home environments and mathematics classrooms. Our chapter ends with a consideration, from the lens of social justice, of implications of these understandings for teachers and teacher educators.

2 Context

The title of this chapter presents a few boundaries and restrictions on our field of view in an attempt to focus the reader's attention on what we (the authors) would like for you to be attending to. Aware of the risk of reductively simplifying and nonchalantly utilizing heavy-handed attention-directing tactics, we here attempt to openly articulate our meanings for these boundaries vis-à-vis the purposes of this chapter. The first of these boundaries is the concept of an "individual". As von Glasersfeld (2013) points out, humans dialectically and continually construct and reconstruct themselves based on one's analysis (however (un)aware one may be of this analysis) of how one's peers view one. That is, I continually (re)construct my notion of myself as an individual based, to a practically meaningful extent, on who I think people around me think I am. Thus, my individuality is inseparable from the constructions of people around me, on which I (at least partially, and however unconsciously) base my own notion of who I am; identity construction is an intertwined, reflexive process of "understanding who I am and whom you see" (Walshaw, 2010, p. 490). For teachers, this means that the self-fulfilling prophecy is at best incomplete (Wineburg, 1987); students do not simply live up to the expectations of teachers because teachers' expectations are not transmitted directly and unfiltered to students. Students construct their ideas about teachers' expectations and perspectives of them for themselves, based on their experiences with teachers. To the extent that teachers' constructions of students influence those students' identity constructions of themselves as students, these students respond to what they (the students) *think* their teacher's expectations are of them. In constructing ourselves, we do so by playing at shadows, mirror images, or doubles (Žižek, 1989; Turner & Oronato, 1999). A potential implication for teachers of reflexive, complex, and non-linear processes of identity construction is that our beliefs are not imparted in a direct one-to-one fashion onto students' psyches (Walshaw, 2010). As a teacher, for instance, if I fervently believe that all students are capable of doing important mathematics, that belief risks residing only within myself since—unless my students realize/internalize that

I am constructing them as capable, talented, and smart—they may not (re)construct themselves as such in our class.

The notion of individuality is further quickly complicated when one considers that not all aspects of an individual, or not all of one's identities, may be equally prone to this process of reflexive reconstruction. Additionally, not every "other" may hold equal sway over one's reconstruction of themselves. Certain, perhaps more peripheral characteristics or personality traits may vary relatively widely over the course of a year, a month, a week, or even within a day (Markus & Kunda, 1986; Turner & Onorato, 1999). Other, perhaps more central, identity aspects tend to be more stable, though still dynamic (Markus & Kunda, 1986; Turner & Onorato, 1999). With time, experience, and attention teachers can develop an awareness of which aspects of their students' individualities they might have the potential to influence and which aspects are more deeply ingrained. Further, as teachers understand from their own experience, the quality and quantity of leverage we hold over the individual constructions of others are variable from person to person and also across time; we may be able to more firmly convince a student one day of our belief in their potentialities than we are on another day, and we may be less successful in our convincing than a different teacher of that same student.

Because of the dynamic nature of at least some aspects of identity, it is important for us to understand that one individual student may exhibit, enact, and construct different student identities in different academic disciplines (Aydeniz & Hodge, 2011). Even within one discipline (mathematics, for instance), an individual student's identity may be changing and changeable. To encapsulate these dynamic processes and complexities, we see (mathematics) identity as a fluid construct that dialectically shapes and is shaped by social context; for our chapter, this context is a complex range of individual, cultural, and social influences in a learning environment that are often in states of tension between conflicting roles and relationships that are activated at any moment in a mathematics classroom (McAdams, 2001; McCaslin, 2009; Nasir, 2002).

In the following sections, we focus closely on students' construction of mathematics and students' psychosocial construction of themselves as students of and doers of mathematics. In this opening section, however, we seek to foreground the interdependency of our individualities as we simultaneously acknowledge that, at some scale (for instance, the scale of visible physical humans in a classroom) we exist as sets of seemingly separate individuals brought together in community. We posit that, from a different perspective (for instance, the not-directly-visible cosmos composed of strands of relationship, influence, power, and control) and—for many of the purposes of mathematics teaching and learning—we function more as interconnected beings, inextricable from the perceptions of our surrounding environment, which we ourselves also influence. As such, we (co)construct as we are (co)constructed by the webs that enmesh us; the ontological status of our identities is unclear and perhaps unknowable (Turner & Onorato, 1999; von Glasersfeld, 2001; Walshaw, 2010; Walkerdine, 2003; Žižek, 1989, 1998). For us, the implications of the uncertain ontology of identity are that we expect (and, at times, seek) the unexpected. In the indeterminacy of identity, for us, resides the potential for curiosity and

wonder not just about our natural worlds but about our 'own' selves and those selves of our students. This affords us opportunities for expansiveness and responsiveness.

This complex indeterminacy of identity also poses research challenges, not the least of which is a variety of theories of identity. In the preceding discussion, we have implicitly placed ourselves in a more poststructural view of identity than psychological or socio-cultural (Grootenboer et al., 2006) because of our emphasis on dynamic process (becoming) and on the relative nature of identity, but even that placement is murky since our perspective shares many salient features with a socio-cultural perspective, including the embodied and connected nature of identity. The experience of not being quite able to provide a static, definitive definition for a critically important aspect of student mathematical reality can be frustrating as well as confusing; in our work, we are becoming more comfortable with uncertainty and more cognizant of the value in process rather than product. That is, we see the *process* of thinking through and with these various perspectives to be increasingly important to our research methods and methodologies. Our distance from a definitive "answer" to the nature of identity is becoming less of a challenge for us to navigate because we see that distance as providing space for (re)thinking and (re)envisioning what we think we know.

Another boundary we establish in our title is that of focusing on internal rather than external contexts. What might we mean by that? Encircled and embedded as we are within dialectical, dynamic, non-linear and non-deterministic connection, all reverberating within material and historical situations, experiences, and narratives, on what level does it make sense to distinguish between internal and external? The demarcation of internal contexts serves, for the purposes of this chapter, to establish an operationalization of what the *object* (see Deleuze & Guiattari, 1994; Gallagher, 2000; Russell, 2001; Wittgenstein, 1969 for a sampling of philosophical considerations of self, subject, and object) of this chapter is—mathematics students' psychosocial processes of identity and content construction in school mathematics. For much of this chapter, we will examine students' constructions of themselves and their mathematical contexts with an aim of better understanding how students come to understand themselves as mathematics students as well as how students understand mathematics. This understanding of students' understandings can, hopefully, further our reflexive (re)construction of ourselves as mathematics teachers and as mathematics teacher educators. Therefore, our meaning for the word "internal" in our title signifies that the primary basis of understanding for this chapter derives from insights students construct within themselves about themselves as mathematics students as well as the perceptions they have (co)constructed for/within themselves about the discipline of school mathematics.

Understanding that these student insights about themselves and about mathematics are situated within (that is, formed by and also forming) threads and strands of material and historical circumstance, community (defined at various scales), and family (expansively conceived), we want to reiterate that we are not attempting to pretend away those strands but rather, for the moments of the reading and writing of much of this chapter, to foreground the students in the strands rather than the strands. In the penultimate section of this chapter, we (re)focus on the ways in which

students interact and relate with (that is, how they are formed by and how they form) the threads and strands connecting them with events and circumstances at broader scales than an individual. Additionally, situating these insights in historical circumstance, community (e.g., family, school, classroom, peer, biosystem) commits us to (co)construct the student as an active, self-cognizing agent (Davis & Sumara, 1997; Newell, 2008) rather than (co)constructing the student reductively as a product of context and environment.

One last boundary we have built as we define what this chapter might mean for us and for you is the demarcation of which aspects of individual identity we intend to include; for our purposes of articulating our field's progression of understanding of student internal context in school mathematics, we will consider student cognitive (psychological) and (with) affective (social) contexts. Because of the importance of both knowing and feeling in mathematics classrooms, we feel that giving emphasis to these aspects of identity, for this moment, might afford us a clarity of insight relevant to mathematics teaching and learning that could be potentially obscured if we attempted a more distributed examination of student identity. Medley (1987) framed these internal contexts as variables that affect student response to mathematics teacher behavior. While we concur that student internal context often influences how a student responds to activities in their mathematics classrooms (activities which Medley framed as Type C, interactive mathematics teacher activities, and Type B, student mathematics learning activities), our dialectic and indeterministic perspective of student internal context is informed also by critical, postmodern, and constructivist insights that have largely emerged in the decades since Medley established his framework.

As we traverse the unfolding of our field's understandings of student internal cognitive contexts, we will be seeking insight about how students come to know themselves—with the obvious implication that this knowing is ever incomplete and is continually ongoing, that this knowing is much more a process than a product (Davis, 2004). Particularly, we will discuss many of our field's contributions to questions of how students know themselves as mathematics students, what knowing means for mathematics students, and what students know they know (or not) about mathematics. As important and interesting as it is for mathematics teachers and mathematics teacher educators to attain deep insight, from students, about how mathematics students come to know about themselves and about mathematics in school, it is also interesting and important for us to understand students' affective and sociological realities and identities in school mathematical environments. In the next section, we will articulate trends and patterns in our field's progression of understanding about how students feel in school mathematics classrooms—how students feel about themselves as mathematics learners, and how students feel about mathematics (or school) mathematics.

3   Students' Internal Contexts

In this section, we detail our field's insights of student processes of generating mathematical understandings, focusing on cognitive, sociological, and psychosocial perspectives of student mathematical experiences. Additionally, in the second subsection, we discuss mathematics students' perceptions of themselves as mathematics learners, including our understandings of mathematics students' identities, self-concepts, self-perception, and self-beliefs. In the third subsection, we discuss students' internal contexts vis-a-vis the subject of mathematics itself, so we investigate students' perceptions and processes of forming attitudes about mathematics content. Throughout, we emphasize those many places where the insights in these subsections connect and overlap with those that we have highlighted in other subsections; these are not mutually exclusive categories we are setting up but rather focal points in connection with each other.

Students' cognitive processes as mathematics learners

In the decades prior to the accessibility of translations of Piaget (especially Piaget, 1972) and Vygotsky (see Vygotsky, 1978a, 1978b) in the U.S. in the 1970s, the educational research community's understandings of students' epistemological processes were influenced by a behaviorist management perspective, which emphasized connections between repetition, routinization, and skill performance (Doll & Broussard, 2002); though there existed progressive counter-narratives to the reductionist and utilitarian outcroppings of behaviorism, particularly from Dewey, one of the first philosophers to emphasize students' active roles in learning (Dewey, 1933/1998; Bruner, 1990). From a behaviorist epistemology, students learn mathematics by repeatedly performing small chunks of mathematical operations through steps provided by a teacher, and fluent performance as well as immediate recall of procedure are prioritized (Doll & Broussard, 2002). Constructivist epistemological paradigms have, for the past several decades, expanded and complicated educational scholars' understandings of student processes of making sense of mathematics by illuminating the active role of student cognition in learning processes (Doolittle, 2014). In mathematics education, our understandings of constructivist cognition have been bolstered in no small measure by our long-standing research connections with cognitive psychologists:

> Cognitive psychologists have provided the concept of 'well-organized' schemata to explain how people impose order on experiential information. Assimilation, accommodation, and mode of functioning in response to new information are important in the enterprise of schooling […] Schema use must be a dynamic, constructive process, for people do not have a schema stored to fit every conceivable situation. In this view, acquisition of knowledge implies changes in schemata, not just the aggregation of information. (Romberg, 1992, p. 62)

Cognitive constructivism espoused by Piaget and the radical constructivism of Von Glasersfeld (2013) are focused on how individuals internally, actively *construct* knowledge as they seek to make sense of lived experiences (Bruner, 1990); the

emphasis is understanding people's internal cognitive structures and processes rather than on an imposition or interaction of external 'knowledge' (Schunk, 2020). Social constructionism (Vygotsky, 1978a, 1978b) tends to highlight the social nature of knowledge construction and orients us to the importance of interaction in processes of knowledge construction. Connecting these two orientations is the premise that, rather than environmental stimuli producing knowledge (or adaptations), it is an individual's active processing of stimuli in relationship to that individual's cognitive structures that brings about knowledge (Huitt, 2003). Doolittle and Hicks (2003) distill constructivist epistemology into four tenets:

1. Knowledge is not passively accumulated, but rather, is the result of active cognizing by the individual.
2. Cognition is an adaptive process that functions to make an individual's cognition and behavior more viable given a particular environment or goal.
3. Cognition organizes and makes sense of one's experience, and is not a process to render an accurate representation of an external reality.
4. Knowing has its roots in both biological/neurological construction and in social, cultural, and language-based interactions (pp. 77–78).

Given these understandings of individual student cognitive processes of learning, several potential insights and implications for teachers emerge, including the indirect nature of teaching (Ackermann, 2001). That is, the content that teachers may try to impart into students is not transmitted in a direct, unfiltered manner. Instead, students actively respond to content from teachers (stimuli) by connecting it to (and connecting to it) their pre-existing cognitive structures. Doolittle and Hicks (2003) discuss several additional learning principles which can inform constructivist pedagogy, including:

- The construction of knowledge and the making of meaning are individually and social active processes
- The construction of knowledge involves social mediation within cultural contexts
- The construction of knowledge takes place within the framework of the learner's prior knowledge and experience
- The construction of knowledge is integrated more deeply by engaging in multiple perspective and representations of content, skills, and social realms
- The construction of knowledge is fostered by students becoming self-regulated, self-mediated, and self-aware (Doolittle, 2014, pp. 498–490).

For mathematics teaching and learning environments, specifically, constructivism is strongly connected with ontological questions about the nature of mathematics, because one corollary of certain constructivist views (particularly from radical constructivism) is that the process of coming to know is an adaptive process grounded in experiential realities and that knowing is *not* a process of discovering external, independent, pre-existing realities (Lerman, 1989). Contrasting with an enduring popular view of mathematics (the "Romance of Mathematics" as Lakoff and Nuñez (2000) call it (p. 339)) as existing outside of the mind of a knower, as Lakoff and Nuñez point out, "Ideas do not float abstractly in the world. Ideas can

be created only by, and instantiated only in, brains" (p. 33). As Lakoff and Nuñez detail precise ways in which humans have used language to develop mathematical metaphors that extend our very limited innate mathematical capabilities, they connect the ways in which human minds make sense of experiences that are external to us:

1. There are regularities in the universe independent of us.
2. We human beings have invented consistent, stable forms of mathematics (usually with unique right answers).
3. Sometimes human physicists are successful in fitting human mathematics as they conceptualize it to their human conceptualization of the regularities they observe in the physical world. But the human mathematical concepts are not out there in the physical world (pp. 345–346).

Many other constructivists in mathematics education assert, rather than claiming that mathematics does or does not map an external reality, that—because we construct our understandings from the basis of our own experiences and previous knowledge— we cannot know whether a mathematical concept exists in an objective reality (Von Glasersfeld, 1995; Steffe & Gale, 1995; Simon, 1995). Instead, our test of emergent knowledge is not an independent, objective existence or truth but the extent to which that mathematical knowledge works in our lived realities; that is, the extent to which mathematical insights are "viable" (Von Glasersfeld, 1995). As students construct their mathematical knowledge, they do so by coordinating mathematical material or mental actions into organized, goal-directed action patterns (Steffe, 1991). The goal towards which students are oriented is that of resolving the perturbation or disequilibrium that arises when students have a novel experience and "restoring coherence" to their experiential worlds (Cobb, 1994). Further, as students interact with peers and teachers in a mathematics classroom environment, they have opportunities to test and refine the viability of their mathematical conjectures, contributing to an emergence of a socioculturally-embodied mathematical knowledge (Cobb, 1994; Cobb & Yackel, 1996).

In the next subsection, we leverage these current constructivist understandings of student cognition to investigate how these cognitive processes might connect with mathematics students' perceptions of themselves as mathematics learners. In the last section of this chapter, we explicate several insights specific to mathematics teachers relating to a constructivist epistemology of student learning.

Students' identity constructions as mathematics learners

During the past several decades, mathematics educators have (re)formulated a variety of constructs to facilitate our understandings of how students view themselves as learners of mathematics, including mathematics self-concept, self-efficacy, mathematics identity, and mathematical disposition. Di Martino and Zan (2011) emphasize that, far from being disconnected with cognitive processes, the interactions between emotional and cognitive dynamics constitutes the concept of affect. This affective sphere in mathematics education, which Di Martino and Zan (2011) frame as interactions between Emotional Disposition, Perceived Competence, and Vision of Mathematics, constitutes a fundamentally important "internal representation system" (p. 1).

Medley's (1987) framework articulates these internal contexts as characteristics (for us, specifically, identity, self-concept, self-efficacy, disposition) of students that affect their response to behaviors of mathematics teachers. In this subsection, we examine our understandings of how mathematics students construct themselves as learners.

As we discussed in the opening section, for us, (mathematics) identity dynamically dialectically shapes and is shaped by a complex range of influences existing in a social context like a mathematics classroom. There is no shortage of reasons to devote energy to better understanding student mathematics identity, given its central location to mathematics learning. As Andersson et al. (2015) point out, "When considering how students' affective responses impact on their willingness to engage in learning mathematics, the notion of identity becomes particularly important because it provides ways to understand the complexity of students' decision making". Grootenboar and Zevenberguen (2008) also affirm the importance of identity: "The teachers' role is temporal, and at the end of the teaching period it is the students' mathematical identities that will endure." Boaler and Greeno (2000) and Boaler (2002) raise the point that mathematics learning environments provide stimuli for students as they construct their mathematics identities and that the range of identities students construct may be linked to the mathematical learning environments they experience. Specifically, there are indications that students learning in traditional lecture-based environments as compared to discussion-based environments may construct different mathematical identities (Boaler, 2002).

Just as we in this chapter at times highlight certain frames of reference (the scale of an individual rather than that of a classroom, for instance), mathematics education researchers frequently foreground particular aspects of mathematics identity with a view to deepening our insights relative to that specific aspect. Research in mathematics education and psychology indicate that two prominent aspects of individual student mathematics context that strongly connect to student mathematics performance are prior academic (and mathematics) performance (sometimes referred to as intelligence) (Deary et al., 2007; Frey & Detterman, 2004; Gustafsson & Undheim, 1996; Kuncel et al., 2004) and motivation (Gose et al., 1980; Schicke & Fagan, 1994; Spinath et al., 2006; Steinmayr & Spinath, 2009). These two characteristics are clearly connected in feedback loops wherein strong motivation can fuel higher performance, which can fuel further increases in performance as well as strengthened motivation; the reverse can also hold wherein lower academic performance can dampen motivation, which can contribute to further declines in performance (Guay et al., 2003; Marsh & Yeung, 1997). In the remainder of this section, we foreground the notion of motivation before focusing on the influences of prior performance; then we revisit the connections between these aspects of student mathematics identity in a culminating discussion of the construct of mathematical disposition.

Mathematics education and psychological researchers rely on a variety of interacting constructs to formulate different theories of motivation such as Bandura's (1986) social cognitive theory (Schunk & DiBenedetto, 2020), expectancy-value theories (Eccles & Wigfield, 2002), and self-determination theory (Ryan & Deci, 2002); see Schukajlow et al. (2017) for a more comprehensive review of motivational theories in mathematics education. Psychological constructs that contribute

to student motivation include expectancy-value (Eccles et al., 1983), task interest (Cleary & Chen, 2009; Cleary & Kitsantas, 2017), and math anxiety (Pajares & Graham, 1999) among others. However, research indicates the possibility that self-efficacy may correlate with mathematics performance more strongly than other motivational constructs (Cleary & Kitsantas, 2017; Pajares & Graham, 1999). Because of this potentially stronger correlation of student self-efficacy with student performance, the notion of self-efficacy merits further elaboration.

The set of a student's mathematical self-perceptions, particularly self-efficacy and/or self-concept (Steinmayr & Spinath, 2009) are central to student internal context and to their mathematical identities. Though these two constructs sound and seem very similar (and, indeed share several commonalities (Bong & Skaalvik, 2003)), many educational psychologists have constructed both theoretical and empirical distinctions between self-concept and self-efficacy (Zimmerman, 2000; Bong & Skaalvik, 2003; Parker et al., 2013; Chmieleqski et al., 2013). Self-concept and self-efficacy are similar in that they are taken to (at least partially) explain an individual's thought, emotion and action in a given context where the individual's perceived skills and abilities come into play. Further, both self-concept and self-efficacy are domain specific (Bong & Skaalvik, 2003), so an individual can have a strong self-concept or self-efficacy in one area but not in another. Bong and Skaalvik (2003) turn to Bandura (1986) to draw distinctions between the two constructs:

> While self-concept represents one's general perceptions of the self in given domains of functioning, self-efficacy represents individuals' expectations and convictions of what they can accomplish in given situations. For example, the expectation that one can high-jump 6 ft is an efficacy judgment (Bandura, 1986). It is not a judgment of whether one is competent in high jumping in general but a judgment of how strongly a person believes that [they] can successfully jump that particular height under the given circumstances. Self-efficacy researchers thus tend to emphasize the role played by specific contexts in efficacy appraisals. (p. 5)

In the context of mathematics classrooms, then, mathematics self-concept discloses individuals' perceptions about themselves in the area of mathematics (or perhaps in more specific domains like algebra or geometry) while mathematics self-efficacy, following Bong and Skaalvik's (2003) articulation, is a more bounded, context-dependent estimation of the extent to which students believe they can succeed at given specific mathematical tasks to certain levels. Both mathematics self-efficacy and mathematics self-concept can influence students' mathematical identities, mathematical dispositions, mathematical classroom experiences and the mathematics they construct in practically meaningful ways, not the least of which is by fueling oneself with the motivation to persevere.

Bandura (2010) noted that an individual's self-efficacy relates to their belief in their ability to exert an influence over events pertaining to their lives, a belief which connects right to the foundation of human motivation and emotional well-being as well as (academic) performance. Bandura (2010) points out that, unless individuals believe that they can produce "desired effects by their actions, they have little incentive to undertake activities or to persevere in the face of difficulties." As one may expect, mathematics education researchers have devoted much energy to

intensive studies of student mathematics self-efficacy, from investigating potential sources of student mathematics self-efficacy (Lent et al., 1991; Lopez & Lent, 1992; Usher & Pajares, 2009) to exploring connections between self-efficacy and mathematics performance (Hackett & Betz, 1989; Pajares & Graham, 1999; Pajares & Miller, 1995) to the potential roles of mathematics self-efficacy in achievement, problem solving, and even career choice (Betz & Hackett, 1983; Lopez & Lent, 1992; Pajares & Miller, 1994; Randhawa et al., 1993). Researchers also point to the importance of mathematics self-concept to persistence (Parker et al., 2013) and to mathematics performance. Seaton et al. (2014) suggest that addressing students' mathematics self-concept may be as influential to their mathematics performance as building their mathematical fluency.

These decades of research indicate that the way students view themselves as learners of mathematics is centrally important to the mathematics those students might construct. In the next section, we discuss how students' views of themselves interfaces with their views of mathematics as they engage in processes of learning mathematics.

Students' views on mathematics

Understanding the ways students construct mathematical knowledge by actively seeking to make sense of their realities through adaptation to new experiences connects to the previous discussion of students' mathematical identities, because, as we saw, the particular characteristics students construct about themselves as mathematics learners can influence their construction of mathematical insights. Another aspect of mathematical learning environments also connects, however, to ways students might be primed to undertake mathematical knowledge construction, and that is students' views of the field of mathematics (Kilpatrick et al., 2001). As McLeod (1992) notes, students in school mathematics settings regularly experience a range of emotions; the frequency, ferocity, and severity of these emotions relates to a student's affective attitude towards the discipline of mathematics itself. McLeod (1992) asserts that, "the improvement of mathematics education will require changes in affective responses of both children and adults" (p. 575).

In school mathematics settings, students' views about the nature of mathematics (what mathematics is, what mathematics is like, what mathematics is not—from psychosocial perspectives) is informed by their experiences as mathematics students. These mathematical experiences are categorized by Medley (1987) as Type C, Interactive Mathematics Teacher Activities and Type B, Student Mathematics Learning Activities. Unsurprisingly, students who have experienced success in school mathematics performance tend to continue to experience success in school mathematics performance (Archambault et al., 2012; Reynolds, 1991). Concerningly, however, students who have less successful school mathematics prior achievement tend to struggle with mathematics performance (Archambault et al., 2012; Reynolds, 1991). These trends make sense both from cognitive and affective perspectives, because, if a child has somehow succeeded in learning mathematics content, it is reasonable to surmise that they may be successful in learning more, and perhaps related, mathematics content. Further, it is reasonable that a student experiencing success

in school mathematics likely is boosted in their mathematics identity (self-concept, self-efficacy), which can increase the chances that that student persists and perseveres as well as enjoys mathematics while connecting school mathematics to other aspects of their lives. We have realized for decades, from the work of Eccles and others, that students' impressions about and experiences with mathematics in school inform and relate to both their mathematics identities and their construction of mathematical knowledge (Eccles, 1983; Meece et al., 1990; Simpkins et al., 2006).

However, decades after the National Council of Teachers of Mathematics (NCTM, 1989, 2000) embarked on its monumental reform movement, students are still frequently taught mathematics from a traditional perspective where mathematics is seen as a static discipline, a "sets of preexisting facts and procedures that is passed along from teacher to student in an authoritarian manner" (Wilkins & Ma, 2003). In such classrooms, students' activities are dominated by silent, individual seatwork and rote note taking, so many students suspect that mathematics is about memorization of content (Wilkins, 2000). In such classroom environments, students often reasonably surmise that mathematics is dry, boring, and potentially a waste of time. Responding to this concerning student view of mathematics, mathematics educators have, in the past few decades, begun to focus increasingly on supporting productive student mathematical dispositions, which Kilpatrick et al. (2001) characterized in their report *Adding it Up* as an intertwining of students' views of mathematics with their views of themselves as mathematics learners:

> Productive disposition […] includes the student's habitual inclination to see mathematics as a sensible, useful, and worthwhile subject to be learned, coupled with a belief in the value of diligent work and in one's own efficacy as a doer of mathematics. (Kilpatrick, 2001, p. 107)

Gresalfi (2009) emphasizes that, just as the content students learn is inseparable from the ways in which they learn it, students' dispositions—their social, affective, and motivational factors such as persistence, collaboration, and engaging with novel problems—are both central to and inseparable from learning processes. Dispositions involve students' ideas about, perspectives towards, and their interactions with content; dispositions "capture not only…what one knows but how [they] know it…not only the skills one has acquired, but how those skills are leveraged" (Gresalfi, 2009, p. 329). Students' mathematical dispositions are clearly pertinent to mathematics teachers, who may seek to better understand their students' mathematical dispositions in relation to classroom mathematics practice. Clark et al. (2014) emphasize four aspects of student identity that can provide information for mathematics teachers relative to their students' mathematical dispositions, which includes students'

- perceptions of their mathematics ability and the ways these perceptions influence their mathematics performance
- perceptions of the importance of mathematics inside and beyond their current experiences in the mathematics classroom
- perceptions of the engagement in and exposure to particular forms of mathematical activity and the ways these engagements influence students seeing themselves as mathematics learners, and

- motivations to perform at a high level and attributions of their success or failure in mathematical contexts (p. 251).

4   Connections Among Individual Student Internal Contexts and Broader Social Scales

Contexts of social systems (students, teachers, schools, families, communities, or disciplines of study) are filled with relationship and connection. Although, in all of the previous sections of this chapter, we have repeatedly attempted to plant this theme, we use this section to specifically foreground these influences and connections seemingly external to the scale of an individual student. These influences include relationships between and among students and their classroom peers, their teachers, their school community, their home contexts, their local social community, their individual and familial histories and material contexts. Because of the limitations both of language itself and of our (lack of) prowess with language, we are constrained to list (rather incompletely) these relationships as if they are connected only to the student and are separate from their own interconnections, but this is inaccurate. All of these relationships are in dynamic and nondeterministic (although self-organizing) conversation with all of the other relationships (and more), pinging and twinging in transformative interdependence such that one apparently isolated experience that may, for instance, directly connect only the student and their teacher activates this entire biosystem of relationships (though perhaps not all to the same extent), buzzing them alive with energy that can (un)make and/or (trans)form them. Thus, when we have the privilege to interact with a student, we must also be interacting with their entire relationship biosystem.

For us, this illuminates the impossibility of cleaving a mathematics student from their ambient realities and contexts. As Aydeniz and Hodge (2011) explain,

> students' identities in relation to science or mathematics cannot be fully understood without considering the multiple communities in which students participate including home community, school community, and the online social communities that now define most students' daily social lives in western societies. (p. 513)

For us, this means that, in a classroom relationship biosystem, it is not possible, for instance, to affirm a student's mathematical contributions while delegitimizing a seemingly distinct aspect of that student, such as their non-English primary language, because that student's construction of mathematics is inextricably, meaningfully connected to that core, identity-influencing experience of being, for instance, bilingual. Additionally, we maintain that it is also not possible to generatively value, say, a student's mathematical persistence while delegitimizing, however indirectly, a seemingly different but core aspect of that student's reality, such as their gender identities. When we (as people in general or especially as teachers) are dismissive, even offhandedly, of the efforts of persons (like our students, for instance) to have their humanity embraced and legitimized, we risk hindering for ourselves, and perhaps for some time, the potential to affirm students in their mathematical

processes. Because students' identities are complex, dynamic, and interconnected, teachers cannot assume that they can separate a student's mathematical identity from the student's holistic identity.

Further, a stunting dismissal of our fellow humans' identities and lived realities tends to atrophy classroom relationship biosystems, as one might imagine, at (various) scale(s) and can profoundly influence students' mathematical (just to name one) identities, even for students apparently adjacent to the target of the dismissal. That is, a delegitimization (very often) resulting in a degeneration of relationship can tinge not just the holistic biosystems of the persons directly involved (the doer of the delegitimization and the direct recipient(s)), but also those of all of the persons cognizant of the delegitimization; vicarious experiences can powerfully impact our efficacies (Zimmerman, 2000). Teachers, in practice, interact not just with a student but with their entire relationship biosystem, and these interpersonal interactions contribute to classroom-level interactions, which also interact with the holistic relationship biosystems of all aware persons in those interactions as well as impinging upon the classroom-level network of relationship.

To dismiss or belittle aspects of students' selves has potential restrictive ramifications for how a teacher can meaningfully interact with students in their mathematics classroom. Conversely, there is a potential for a more expansive student–teacher connection when teachers' interpersonal behaviors consistently communicate how highly each student and their intellectual contributions are valued. Student perception of teachers' behaviors indicating tolerance, care for student wellbeing, and relative lack of authoritarianism is an important feature of student–teacher connections (Van Petegem et al., 2008), so implications for teachers are profound on the scale of students and, we emphasize, on the level of the classroom community relationship biosystem as well. Teachers, because of their position in the classroom, have considerable opportunity and responsibility to mindfully facilitate equitable status relationships with all students; otherwise, students' academic progress and classroom participation (among other potential damages) are at risk (Alexander et al., 1987; Cohen & Lotan, 1995; Fuller & Clarke, 1994).

The interconnected nature of student mathematics cognition, identity, and disposition to wider contexts like mathematics classrooms, schools, and communities uncovers several issues pertaining to student access to opportunities to develop and nurture productive identities and dispositions within dynamic and rigorous classroom environments. Access to important mathematics content is far from equitably attainable (Reddy, 2005), as academic curriculum tracks (Oakes, 2005), along with intergenerational and geographically-dependent disparities in school funding structures (Kozol, 2012) pose non-trivial barriers and opportunity gaps (Horn, 2012) to the fulfillment and facilitation of every students' potential to be their most full mathematical selves.

Much as student internal context cannot be bifurcated into disconnected pieces like cognitive vs. emotional processes, individual students cannot be separated from their multi-leveled social and historical contexts. The discussion in this section illustrates the futility, as understood within mathematics education scholars, of attempting to isolate mathematical cognitive processes in students and interact (or operate

with) solely those processes. Perhaps even more critically, this section emphasizes the responsibilities and obligations of teachers to advocate for student growth, development, and health in holistic and connected ways.

5   Implications for Teachers

Though the reasons for inequitable access to high quality, rigorous, engaging mathematics learning opportunities are many, systemic, and extending beyond bounded educational structures, we focus in this section on several aspects of mathematics learning environments that teachers and students can more directly influence, and which are closely connected to our previous discussions of student mathematics learning processes, student mathematical identity constructions, and student attitudes and views towards mathematics. These aspects include the existence and roles of status in mathematics classrooms and socially just and affirming pedagogies in mathematics.

Socially just and affirming mathematics pedagogies not only provide opportunities for mathematics teachers to provide students with enactive experiences, which strongly inform students' efficacy beliefs (Zimmerman, 2000) and which can foster productive mathematics dispositions, but they can also facilitate students' cognitive growth. For instance, culturally responsive teaching is one such pedagogically affirming approach, which Hammond (2014) characterizes as

> An educator's ability to recognize students' cultural displays of learning and meaning making and respond positively and constructively with teaching moves that use cultural knowledge as a scaffold to connect what the student knows to new concepts and content in order to promote effective **information processing**. All the while, the educator understands the importance of being in a relationship and having a social-emotional connection to the student in order to create a safe space for learning. (p. 15, emphasis in original)

Culturally responsive teaching in mathematics asks teachers to be open and expansive to a variety of 'real worlds' that their students navigate and negotiate daily (Gay, 2002). Further, socially just and affirming pedagogy expects teachers to validate students lived realities in part by creating space for those realities in mathematics learning processes. Simultaneously, culturally responsive mathematics teaching expects that high quality, rigorous, important mathematics teaching and learning occurs in classrooms. Our understandings of such justice-oriented pedagogies indicates that powerful and affirming mathematics teaching draws on and leverages broader contexts that students inhabit beyond their individual selves and expands students' and teachers' awareness of the ways mathematics holds power to critically analyze and interpret our worlds, potentially opening up spaces for justice-oriented agency and action (Aguirre & Zavala, 2013).

Just as our field has pointed to connectivity and generativity over the past several decades, it has also emphasized the capability of (all) students to deeply think and reason, so an affirming classroom relationship system is not devoid of student questioning and debating. When responding to a student contribution, for instance, an

affirming expectation can be that the student will articulate, explain, and justify their contribution and, if necessary, attempt to persuade peers in the classroom to agree with their mathematical justification. An affirming mathematics classroom relationship system does not let just any suggestion prevail; it acts—with often analytical purpose—to ascertain which suggestions are viable for the mathematics community and to justify why that viability exists. At the same time, however, we are increasingly (if belatedly) aware of important commitments to social justice that mathematics teachers entail as political agents in an unjust social system. We can only weakly attempt to maintain the neutrality of mathematics and the teaching of mathematics; this neutrality is exposed as a fiction, which, for us, implies that mathematics teachers' socio-political self (and community) awareness is more and more paramount. Beyond implementing mathematical instructional practices that deeply and actively engage students in contextual and meaningful mathematics, Aguirre and Zavala (2013) argue that culturally responsive mathematics teachers must

- develop a socio-cultural political consciousness
- understand and embrace social constructivist and socio-cultural theories of learning
- get to know and leverage the mathematical resources of students, their families, and their communities.

Socially justice-oriented mathematics classrooms can provide a potential-filled connective space for nurturing and facilitating the aspects of student internal and individual contexts we have discussed in this chapter.

6   Concluding Comments

The adapted framework for this volume articulates a number of factors, characteristics, and contexts relevant for teaching and learning in mathematics classrooms. Each of these is important on their own, and perhaps even more so in concert with other contexts; for the authors of this chapter, we agree with Dewey (1906) and many other educational scholars that students and their contexts are crucially important to mathematics learning. In this chapter, we have provided an overview of several influential aspects of individual student internal context (Type H), including student mathematical identities, self-efficacy, and disposition. We have also emphasized the significance of the connections between all of these aspects and other educational and social considerations and contexts; students exist, as we all do, within a cosmos of relationship rather than in an isolated vacuum.

Given this overview, several avenues for future research emerge in our field, as we hope they do for the reader. Though each person may see different research potentials arising from the work scholars in our field have already done, the potentials we see are for an increased presence of postmodern research perspectives on student internal context and for greater consideration of critical theoretical approaches in our consideration of student and educational contexts. Postmodern perspectives disrupt

static boundaries and binaries as well as linear, predictable pathways while emphasizing the importance of context and acknowledging the existence of indeterminacy (Stinson & Bullock, 2012). These perspectives offer promise to our work in student mathematical internal context because of the dynamic complexity of identity and its non-linear enmeshment in broader conditions; we have the potential to (re)envision our notions of student identity and student mathematics performance in ways that are open to irregularity and spontaneity while maintaining rigor in research. Simultaneously, we see ourselves as undertaking an important responsibility to refrain from treating students as isolated subjects; in socially aware and critical educational research, it is incumbent upon us to deepen our understandings of the systemic nature of concealed, asymmetric relationships of power (Stinson & Bullock, 2012) and the ways those of social contexts of inequity reveal themselves in children's educational lived experiences and identities.

One social context that has emerged in the past two decades as a potentially powerful research focus is that of online communities and the potential to inhabit yet another identity as a virtual being in virtual worlds. Though mathematics researchers have been studying the connections between technology and student mathematics motivation, achievement, and attitude (Higgins et al., 2019) since the mid-1980's, the possibility to understand how online spaces and realities impact and are impacted by students' online mathematical identities is more recently being realized (Rosa & Lerman, 2011), especially in the context of gamification (Lo & Hew, 2020). Technology has progressed from desktop computers placed in classrooms to hand-held devices providing not only unprecedented access to information but also potential for identity transformation and (re)construction. The responsive, adaptive, and dynamic aspects of critical postmodern research perspectives seem well poised to contribute to our understandings of students' mathematical identities and internal social contexts in a variety of technological mathematical learning environments, including gaming environments, online mathematics classrooms, and social media environments while also pushing us to better understand patterns and asymmetries in student access to important online mathematics learning communities.

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External Context-Related Research: Digital Resources as Transformers of the Mathematics Teachers' Context

Ghislaine Gueudet and **Birgit Pepin**

1 Introduction

The contribution of this chapter addresses current issues associated with the evolution of research in mathematics education related to the external context for mathematics teachers' professional activity. The external context combines many elements, such as for example materials, facilities, community support (Manizade et al., 2019). We contend that, in this digital era, digital resources play an essential role in this external context. We primarily focus on research concerning the external context and related with digital resources: this includes research about the digital resources themselves, as well as research about for example community support for the integration of digital resources by teachers, or educational policy linked with digital resources.

In other words, we have concentrated on Medley's (1987) Type I (external context) variable in the context of digital resources. By doing that, we focus on the "materials" variable (Manizade et al., 2019), with a specific focus on digital materials, and on other materials when they are combined with digital materials. For the sake of the size of the chapter, we do not review research concerning school administration (e.g., Hunter, 2019), supervision (e.g., Yang et al., 2021), community support (e.g., Nicol, 2018), and parental support systems (e.g., Wadham et al., 2020), when they are not linked with digital resources.

Returning to our focus, digital technologies have led to tremendous changes in these external context variables: not only changes in the access to available digital

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materials, but also changes in what can be called "community support" through digitalization (e.g., on platforms), or more generally for supporting the integration of digital resources. Research in mathematics education about these external context variables has undergone very significant changes. The factors 'causing' these changes in the focus of research studies were related to the technologies themselves, but also to other factors, such as events in the society impacting the educational system (e.g., the COVID-19 pandemic), have played a role. Moreover, these studies not only concentrate on the changes in the external context (Type I) variables; we evidence in this chapter that they also address the influence of digital resources as elements of the external on several online variables and their interactions, in particular D (teachers' pre-post-out-of-class activities) and E (teachers' competence, knowledge and skills), and also the process variables B (students' mathematics learning activities) and C (teacher-student interactions in class).

Given the large number of research publications concerning the teaching of mathematics in the digital era and related to the selected external context variables, we restricted our search to the identification of important trends. In this chapter, we address the following research question:

> Which are the evolutions of research in mathematics education about digital resources as context for mathematics teachers' professional activity?

We considered recent research literature, research published between 2016 and 2020. This was done, because during that period a large body of research emerged that addressed the changes due to digital resources. We also included selected seminal pieces cited in the literature, covering the last 20 years. This included conference proceedings of the following conferences: Congress of the European Society for Research in Mathematics Education (CERME10,[1] 2017), CERME11, 2019); Mathematics Education in the Digital Age (MEDA,[2] 2018; MEDA,[3]); International Conference on Technology in Mathematics Teaching (ICTMT13,[4] 2017), ICTMT14,[5] 2019); International Conference on Mathematics Textbooks Research and Development (ICMT3,[6] 2019). Further, we included journal articles: we searched the 2016–2020 issues of Educational Studies in Mathematics (ESM), Journal for Research in Mathematics Education (JRME), ZDM Mathematics Education, Digital Experiences in Mathematics Education (DEME). Moreover, we searched the following books: Hoyles and Lagrange (2010, ICMI Study 17 about technology), Clark-Wilson et al., (2014, 2021), Drijvers et al. (2016), Monaghan et al. (2016), Trouche et al. (2019). We systematically searched for papers or chapters about digital technologies and digital resources as contexts in mathematics teaching: we used the keywords "technology", "digital technology", "digital resources", "digital platforms", "digital

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[1] All the CERME proceedings are available at http://erme.site/cerme-conferences/.

[2] https://www.math.ku.dk/english/research/conferences/2018/meda/proceedings/.

[3] https://www.jku.at/linz-school-of-education/steam/meda-conference-2020/.

[4] https://hal.archives-ouvertes.fr/hal-01632970.

[5] https://duepublico2.uni-due.de/receive/duepublico_mods_00048820.

[6] https://tagung.math.uni-paderborn.de/event/1/.

tools", crossed with "teaching", "teacher", "teacher professional development". We excluded papers whose central focus concerned topics addressed in other chapters of this book (e.g., mathematics teacher affect studies; studies on mathematics teacher professional development; mathematics teacher knowledge). Nevertheless, we did not restrict ourselves to the research about digital resources because studying the influence of a given digital resource on the teaching of mathematics often includes studying its use by teachers, or indeed the knowledge development through the use of such resources. At the end of this process, we retained 160 papers and chapters. We noted for each of these papers the questions addressed, and the main results obtained.

We have chosen the following organization for this chapter: After this Introductory Sect. 1, we present theoretical elements guiding our review of the literature (Sect. 2). In Sect. 3, we discuss evolution of research about educational policies and about teachers' professional activity, including assessment. Section 4 focusses on research about the quality of digital curriculum resources, while Sect. 5 concerns selected current evolutions. In Sect. 6 we present our conclusions.

2 Theoretical Frames guiding our Review

In this section we introduce the concepts that guided our review of the literature. We present in particular what we mean by (1) educational technology as compared to digital curriculum resources; and (2) mathematics teachers' professional activity for the purpose of this chapter.

2.1 Digital (Curriculum) Resources and Educational Technology

The literature reviewed in this chapter concerns what we call *digital resources*. Digital resources can be defined as materials that have been conceived and created digitally or by converting analogue materials to a digital format. Examples of digital resources are simulations, models, graphics, e-books, and e-notes intended to make learning more engaging, accessible and contextualized. Over the past decade there have been numerous research studies investigating the use of digital resources for mathematics teaching (e.g., Clark-Wilson et al., 2014; Drijvers et al., 2016; Hoyles & Lagrange, 2010).

Within the general category of digital resources, we distinguish between *digital curriculum resources* (DCRs) and *educational technologies* (ETs), following Pepin et al. (2017a) who defined DCRs as follows:

> It is the attention to sequencing—of grade-, or age-level learning topics, or of content associated with a particular course of study (e.g., algebra)—so as to cover (all or part of) a curriculum specification, which differentiates DCRs from other types of digital instructional tools or educational software programmes. (p. 647).

ETs can be defined as the digital tools that are used in and for education by students or teachers (e.g., platforms). Once these tools are used for teaching (and learning) a particular curriculum content, and built into for example a lesson plan, they would have become DCRs.

Pepin et al. (2017a) observed that research about DCRs pays particular attention to:

1. The aims and content of teaching and learning mathematics;
2. The teacher's role in the instructional design process (i.e., how teachers select, revise, and appropriate curriculum materials);
3. Students' interactions with DCRs in terms of how they navigate learning experiences within a digital environment;
4. The impact of DCRs in terms of how the scope and sequence of mathematical topics are navigated by teachers and students;
5. The educative potential of DCRs in terms of how teachers develop capacity to design pedagogic activities.

For our review, it makes sense that we bring these two together (DCRs and ETs), as teachers are working in environments that are influenced by both. Nevertheless, as we will see in what follows, the distinction between them can contribute to refining our understanding of the research literature: for example, conceptualizing *quality* (see Sect. 4) is a need that emerged from the studies about DCRs.

2.2   Teachers' Professional Activity

Reviewing the literature about digital resources as external context for teachers' professional activity depends on the perspective chosen on this professional activity. Indeed, the external context and this professional activity are intertwined.

Borba and Villarreal (2005) started with the premise that technologies have changed humankind, and emphasized that:

> [...] humans-with-media, human-media or humans-with-technologies are metaphors that can lead to insights regarding how the production of knowledge itself takes place [...] this metaphor synthesizes a view of cognition and of the history of technology that makes it possible to analyze the participation of new information technology 'actors' in these thinking collectives in a way that we do not judge whether there is 'improvement' or not, but rather identify transformations in practice. (p. 23)

The "Humans-with-media" perspective challenges the borders between what is external and what is internal for the teachers interacting with a context comprising digital resources. In terms of the framework of research on teaching mathematics (Manizade et al., Chap. 1), it invites to consider that some of the digital media do not only belong to the offline Type I variables, but can also be considered as belonging to the online Type E variables, e.g., because the teacher-with-media can be seen as a hybrid entity.

Using a historical lens, when digital resources became available for teachers and mathematics classrooms, and teachers were increasingly encouraged to use those 'tools', the research literature also reflected this turn. Questions such as the following were asked: What is the teacher-tool relationship (e.g., Brown, 2009)? In which ways does the 'tool' influence teachers' practices, or indeed their knowledge development concerning the use of the 'tool'? Particular theoretical lenses were developed to face the challenges associated with answering such questions. The instrumental approach to didactics (e.g., Guin et al., 2005) and the documentational approach to didactics (e.g., Trouche et al., 2020a) provide us with theoretical tools which have been useful to face this challenge in our review. The instrumental approach (Guin et al., 2005) was developed to study, and theorize, the integration of computer tools into mathematics education. It distinguishes between an *artifact*, a product of the human activity, designed for a goal-directed activity, and an *instrument* developed by the user along their activity for a given goal. The subject (e.g., the student) develops an instrument, incorporating the artifact (external) and knowledge (internal). Two different subjects even with the same goal do not develop the same instrument. The development of an instrument is called instrumental genesis. This genesis comprises two inseparable processes: instrumentation that describes how the features of the artifact influence the subject's activity; and instrumentalization which describes how the subject modifies the artifact, according to their pre-existing knowledge.

The instrumental approach has been used in mathematics education research to analyze how students learned with educational technologies (the calculator, in particular). The concept of orchestration was introduced by Trouche (2004) to address the question: "How do teachers use technology in class, and why do they use it this way?". With the perspective of the instrumental approach, this question was formulated as: "How do teachers orchestrate the students' instrumental geneses with a given educational technology?". The instrumental orchestration was defined as the systematic organization, arrangement and didactical use of artifacts in the classroom, and therefore, concerns both Type C (interactive mathematics teacher activities) and Type D (pre- and post-active mathematics teacher activities) of Medley's variables.

Introduced by Trouche (2004) and refined by Drijvers (2012), the concept of instrumental orchestration has been a first step in the development of studies referring to the instrumental approach and investigating the teacher's role. This direction of research has rapidly developed, with authors considering teachers' instrumental geneses. In particular, Haspekian (2014) introduced the concept of teachers' double instrumental geneses: the teachers had to learn the technical functionalities of the artifact, and at the same time had to learn how to use the artifact for their teaching goals. While these studies from the instrumental approach still focused on educational technologies, the available DCRs were rapidly growing, and opening new avenues for a mathematics teacher's (curriculum/tool/task) design activities (e.g., Pepin et al., 2017b), individually and collaboratively.

The proliferation of available DCRs and the need to understand its consequences for teachers' professional activity led to the introduction of the *documentational approach to didactics* (DAD, Gueudet & Trouche, 2009; Gueudet et al., 2012; Trouche et al., 2019, 2020a). This approach considered the interactions between

teachers and (digital) *resources* mobilized for their teaching. Referring to Adler (2000), the term resource was used with a very general meaning, namely anything that can re-source the teacher's practice is a resource. All the elements of teachers' professional external context: (digital) curriculum resources, students' productions, discussions with colleagues can constitute resources. Even elements of their personal context can become resources for their teaching: discussions with a member of a family, a journal where the teacher notices interesting statistics etc. Teachers' documentational work (searching for resources, selecting them, modifying them and using them in class) is central to their professional activity.

The documentational approach drew on the instrumental approach and introduced a distinction between a given set of resources, and a *document*, developed by the teacher about their use of these resources for the goals of their activity. The document connects the recombined resources and the teacher's professional knowledge. The development of a document was called *documentational* genesis. Like the instrumental genesis, it encompasses two associated processes of instrumentation and instrumentalization.

The DAD viewed a teachers' professional activity as continuous design work and considered teachers as (co-)designers. The availability of a wealth of resources, digital resources in particular, opened new possibilities for teachers but also created new complexity, requiring the development of teachers' (co-)design capacity (Pepin et al., 2017b). According to the documentational approach, Medley's variables Type C (interactive mathematics teacher activities) and Type D (preactive mathematics teacher activities) are strongly linked, and the DAD can be considered as a conceptualization of the links between variables of Type I, C, D and E.

At the end of this review of theory, we retain certain points that seem particularly important to us, and we make certain choices for the rest of the chapter:

- In what follows we use "digital resources" as the most general term. We acknowledge that it is complex to distinguish between Digital Curriculum Resources (DCRs) and Educational Technologies (ET), and that both are often combined in teachers' practice. Nevertheless, this distinction can be useful for some aspects of the literature.
- There are still many terms relevant for our study that are not always precisely defined (e.g., "digital platform" can be used for very different digital resources, depending on the cultural context in particular).
- The instrumental approach introduced the distinction between an artifact (external) and an instrument (both external and internal) developed by teacher interacting with this artifact. The documentational approach introduced a similar distinction between resource (external) and document (both external and internal). While we do not consider here studies focusing on teacher knowledge, we included in our review studies focusing on the interactions between teachers and digital resources.).

3 Evolution of Research about External Context Variables linked with Digital Resources and about their Influence on Teacher Work and Teacher Knowledge

In this section we analyze the evolution of research considering external context variables (Type I) and their influence on the online variables Type C, D and E, keeping our focus on teachers working with DCRs. While Medley (1987) considered that external context variables influenced the Type E-D relation (between teachers' competencies, knowledge and skills and teachers' pre-post-out-of-class activities), we align here with the new model proposed by Manizade et al. (Chap. 1 in this book) by also considering their influence on the Type D-C relation (between teachers' pre-post-out-of-class activities and teachers' interactions with students in class).

We claim that one way that research about external context variables has evolved concerns investigations about educational policies (including official curricula and reforms) addressing the provision and use of DCRs. We present this research and its evolution in Sect. 3.1. The research about teacher integration (or non-integration) of DCRs has also evolved during the last 20 years. Since this integration is strongly influenced by educational policies, we consider that research about teacher integration addresses the influence of Type I variables on Types C-D-E, and will discuss this in Sect. 3.2. One of the levers used by educational policies to influence teacher integration of DCRs is assessment; we focus on this issue in Sect. 3.3.

3.1 Educational Policies as Context for Teachers' Work with DCRs

Educational policies, including curriculum reforms, were not listed by Medley (1987) amongst the examples of Type I variables. Nevertheless, educational policies of their respective countries and institutions are an important element of the teachers' external context. In the "Challenges in basic mathematics education" brochure, Artigue (2011) stresses that "Quality education for all today cannot be achieved without taking technological factors into account" (ibid p. 35). Within mathematics education research, work on educational policies, and how they contribute to shaping the teachers' use of DCRs (how this Type I variable influences Types C and D) or how educational authorities use DCRs in their attempts to shape teachers' practices has developed during the last 20 years.

Educational Policies and Access to Technology

Between 2000 and 2010, many studies investigated how educational policies and projects at a national scale tried to promote through different means the use of technologies in the mathematics classroom (UNESCO, 2005). These studies, often comparing different national situations, examined in particular the issue of access to technology. Specifically, how the policies try to develop this access, and does the

actual provision of computers permit the design by the teacher of classroom orchestrations where students exploit the potential of relevant software in their mathematical activity?

Julie et al. (2010) described the situations in four countries (Russia, Hong Kong, Vietnam, South Africa) and one region (Latin-America). They noted similarities in the educational policies of these countries and particularly the acceptance at the political and bureaucratic level of the use of digital technologies for mathematics teaching and learning. The translation of policy into practice took very different forms (in terms of equipment in computers, Internet access, provision of digital resources, and teacher education), according to the different economic situations of these countries. Nevertheless, in all countries they observed that unequal access to technologies remained, and that the actual use of digital technologies in schools was rare.

Sinclair et al. (2010) compared five projects concerning the use of technologies in the teaching and learning of mathematics that had been undertaken at a national scale in different parts of the world. These projects were: Mexico's Enciclomedia, Italy's M@t.abel; the US's Sketchpad for Young Learners, Lithuania's Mathematics 9 and 10 with The Geometer's Sketchpad, and Iran's E-content initiative. The authors introduced three axes, for their comparison of the projects: (1) The curriculum axis (Technology activities support existing curriculum vs. Technology activities encourage new content); (2) The teacher practices axis (Technology activities reify existing teacher practices vs. Technology activities endorse new practices); and (3) Activity design ("Open" activity design for students vs. "Closed" student activity design). Their analysis led them to observe shifts in the projects, such as increasing participation of the teachers as co-designers and epistemic value (supporting the learning of mathematics) of the technologies being progressively foregrounded relative to its pragmatic value (e.g., obtaining a numerical result). Nevertheless, at least in some of the countries, difficulties of access to computers were an obstacle for the implementation of these projects.

Analyzing Evolution of the Policies and their Implications

The work by Trouche et al. (2013) can be considered as a transition between the 'early' (2000–2012) works about educational policies and technologies, where the issue of access was central, to more recent works (2013–2021) where DCRs are used by educational authorities to support teacher design, and at the same time to try to influence teacher classroom practices.

Trouche et al. (2013) analyzed the issues connected to policy implications on two continua/ dimensions, as shown in Fig. 1.

1. bottom-up to top-down policy approaches (e.g., "A top- down policy could be a national directive of imposing access to graphing calculators during national examinations; whereas support for teachers who start to design their own online resources can be seen as a bottom-up policy" (p. 2).); and
2. access—support approaches (e.g., "In the United States, the National Council of Teachers of Mathematics (NCTM), in its 2008 Position Statement, claims
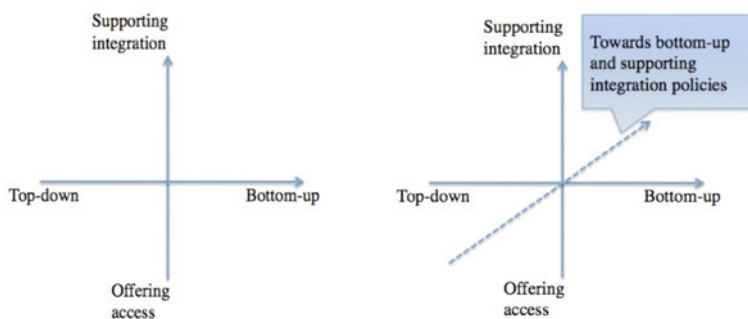
Fig. 1 The two policy dimensions (left) with potential orientation towards bottom-up and supporting policies (right), extracted from Trouche et al. (2013)

that "all schools must ensure that all their students have access to technology" but also that "Programs in teacher education and professional development must continually update practitioners' knowledge of technology and its classroom applications" (NCTM, 2008, p. 13).)

This evolution of the policies as envisaged by Trouche et al. (2013) is also linked with evolution in research foci. Researchers in mathematics education have increasingly investigated how the educational policies support teacher integration of technologies, and teacher design. The researchers themselves sometimes participate in this effort, by developing curricula in particular. This evolution of the policies (and of associated research) is linked with another kind of evolution: the educational authorities increasingly use DCRs to provide resources for teacher design (supporting this design, acting on Type D variables), with further aim of influencing the classroom practices (acting on Type C variables).

Use of DCRs for Supporting Teacher Design and Shaping Teacher Practices

The Cornerstone Maths project in England is an illustrative example (e.g., Clark-Wilson & Hoyles, 2019) of systematically scaling-up of innovations involving DCRs. It began in 2011 by designing curriculum units that embedded digital technology for learning mathematics (called dynamic mathematical technology, DMT). Such technologies were said to offer the potential for teachers and pupils to (re-)express their mathematical understandings. The national curriculum for mathematics (introduced in England in 2012) specifies the content of the school mathematics curriculum (5–16 years) but offers little pedagogical guidance with regard to the use of technology, implying that teachers should use their judgement about when ICT tools should be used, and how. Consequently, there were no government-funded initiatives to support either secondary school mathematics teachers to develop ways of integrating DMTs into their classroom practices or for mathematics departments to embed such approaches within their school-designed schemes of work. Some use of technology across the secondary curriculum was expected and lightly monitored within the school inspection regime. There was the need to support within-school

upscaling. The team judged it important to design and test an in-school professional development toolkit as that could help instructional leaders in schools to support other colleagues and to develop as leaders. The present step (reported in the article) concerned the issue of supporting a large-scale and sustained use of this curriculum units, and this was done via a web-based toolkit. However, there were also difficulties in the context: due to a shortage of mathematics teachers, which acted as a barrier for schools to sustain innovations and innovative practices. The study is also interesting in terms of the importance of the schools as sites for supporting professional development of teachers with respect to their work with digital curriculum resources.

While researchers sometimes participate in the design and dissemination of DCRs to contribute to teacher professional development, the national educational authorities more generally have offered resources to teachers. Their aim is to influence teachers' practices in and out-of-class, and to contribute to their professional development (in particular in the context of reforms). In many countries, digital platforms propose DCRs to teachers.

Concerning platforms and their use by teachers, there are issues involving expectation management. For example, in relation to 'design'- developments, whilst in some countries (and schools) mathematics teachers are to some extent expected to (co-) design their curriculum, in others teachers are expected merely to follow the approved textbook (Trouche et al., 2019). However, the free availability of an enormous number of DCRs, leaves the teacher at a loss in regard to assessing the quality of the available DCRs (see Sect. 4 below), and how to design or amend DCRs? The availability of free resources is also of economic importance, as it raises the issue of competition with commercial resources (e.g., textbooks). In some countries, government institutions provide access to or design DCRs themselves. Others offer opportunities for teachers to engage in the creation of resources. The DCRs' design issues cannot be seen on two dimensions; they are more complex, involving a variety of 'systems' and agents with commercial and economic considerations.

In their investigation of digital platforms for mathematics teacher design (Gueudet et al., 2021), the international team members analyzed the affordances and constraints of commonly used digital education platforms available for mathematics teachers (often provided by governments). They used the documentational approach and the concept of 'connectivity' (Pepin, 2021), introduced by Gueudet et al. (2016) in their study of e-textbooks. These authors distinguish between: (1) macro-level connectivity (e.g., connections made between the book and other websites, or between the resource systems of users); (2) micro-level connectivity (e.g., internal mathematical connections made by the authors between different representations; between the mathematical content and real-life contexts). Transferring this concept to digital platforms, Gueudet et al. (2021) compared three contrasting cases of platforms in three European countries (France, Netherlands, and Denmark), in terms of potential instrumentation and instrumentalization processes for users, and of micro- and macro-level connectivity. They found important differences between the platforms that were strongly linked with national educational policies and national perspectives on teachers' work. For example, in Denmark the use of the platforms was compulsory, and their features were chosen to compel teachers to design objective-driven lessons,

according to new national standards. In France, the digital platform was designed to support the implementation of the new curriculum. In the Netherlands, the platform was linked with a policy supporting the use of open educational resources.

As claimed by these authors, digital platforms or other digital resources offered by the institution can be seen as interfaces between educational policies and teacher's practices (in class and out-of-class). Hence studies about educational policies are strongly linked with the studies about teachers' integration of digital resources that we discuss in the next section.

3.2   Teachers' Integration of Digital Resources

In this section we analyze the evolution of research about teachers' integration of digital resources. We argue that the studies about integration firstly tried to identify the factors supporting or hindering the use of technologies by teachers in class. The influence of Type I variables (e.g., educational policies) on the uses in class (Type C variable) was identified by these studies; then they noted the importance of teacher knowledge (Type E) as a factor of integration. As the environment for teachers in terms of available resources has become more complex, specific theoretical frameworks have emerged that allow for the consideration of interactions between variables of Types C, D, E and I.

Early research mostly focused on questions concerning the factors explaining the integration or non-integration of educational technologies. The factors identified were firstly external variables, and integration was considered in terms of in class use. We consider these works to address the influence of Type I on Type C variables. Some of these variables were linked with the national or regional educational policies, in terms of equipment, and technical support offered to the teachers in their schools (Thomas, 2006). These policies also led to the presence, or not, of the technologies in the mathematics curricula, and in the official examinations (Trouche, 2016), and this had a strong influence on the extent of integration. Other external factors concerned the level of the school environment, the school culture and the interactions among colleagues in the school (Forgasz, 2006). If the use of technologies was promoted by the school, with support staff, or by way of a collective project drawing on some kind of technology, the integration was favored. On the other hand, if a group of teachers in a school felt that the use of technology was an additional constraint imposed by the superiors, this constituted a strong obstacle to integration in that school.

Progressively, the studies considered Type I variables as factors explaining integration and Type E variables like teachers' experience and teachers' knowledge (e.g., Attard et al., 2020; Geiger et al., 2016; Goos, 2014). This draws a more complex landscape, involving the interaction of four types of variables. The model (framework of research on teaching mathematics) introduced by Manizade et al. (Chap. 1, this book) considers the influence of Type I variables on the E-D and on the D-C relations. But in this model the E-D-C relation is presented as linear: teacher knowledge influences teacher preparation which in turn influences teacher activity in-class. In the studies

we consider here, E-D-C can be viewed as a triangle, with two-way interactions along each side of the triangle. Indeed, the researchers investigated how Type E variables influenced the use of technology in-class (Type C) and out-of-class (Type D), still taking-into-account the external context (Type I). The theoretical perspective of the instrumental approach (Guin et al., 2005) strongly associates the technology (Type I), the teacher's knowledge (Type E), and her practice both in class (Type C) and out-of-class (Type D) and has been used by some authors to study the interactions between teacher knowledge and their use of technology.

Assude (2007) introduced the concept of 'instrumental integration stages'. She proposed four stages of increasing technology use in the classroom. In the *instrumental initiation* stage, the teacher wants the students to learn how to use the software; in the *instrumental exploration* stage, the students explore the software through mathematical tasks; the *instrumental reinforcement* means that the software in used to reinforce mathematical knowledge, and finally in *instrumental symbiosis* stage the software and the mathematics are combined in the students' mathematical activity. Assude (2007) explained that these stages do not correspond to stages of professional development. Rather, even in the same lesson, the teacher could propose software use corresponding to instrumental reinforcement at some point, and to instrumental exploration at another moment. Assude (2005) also foregrounded the importance of time as factor hindering or favoring the integration of technology by teachers. The consideration of time economy as an essential variable was realized in particular by Ruthven (2009), who proposed a theoretical framework combining variables of different natures and including this notion of time economy. This framework called "the Structuring Features of the Classroom Practices" (SFCP) associates five features (considered here as variables) that explain how a teacher integrates a new digital technology:

- the *working environment*: classroom equipment, support in the school (Type I);
- the *activity format*: the teacher and their students have a usual activity format (Type C);
- the *curriculum script*: professional knowledge (Type E);
- the *time economy* (Type I);
- the *resource system*: mathematical tools and curriculum materials in use in the classroom (Type I, with interactions with Type C).

The SFCP framework foregrounded the importance of considering different features for understanding the integration or non-integration of a given educational technology. It played an important role in the evolution of research from studies focused a single educational technology to studies considering sets of resources, including digital resources of different kinds. Considering these five features led to studies evidencing the interactions between Type I, Type E and Type C variables.

The SFCP framework (Ruthven, 2009) was together with the instrumental approach one of the sources of the documentational approach. One of the main developments brought about by the studies referring to the documentational approach has been the strong association of the teacher's work in the classroom and outside the

Fig. 2 The Chinese abacus, material (on the left), virtual (on the right) (Gueudet & Poisard, 2018)

classroom, because teacher design as an essential and continuous process takes place in class and out-of-class leading to associations among variables of Types C and D.

Studies referring to the documentational approach (Trouche et al., 2019) considered complex sets of resources (e.g., Gueudet & Poisard, 2018; Wang et al., 2018). The integration by a teacher of an available resource meant that the teacher, using this resource, developed one or several documents. For example, Gueudet and Poisard (2018) studied the integration by a primary school teacher of a set of resources[7] designed by a research team for the teaching of number, using the Chinese abacus, both material and digital, as seen in Fig. 2.

The mathematics teacher planned for her students to use manipulatives. She integrated both the material and the digital abacus in her lesson. This was observed by the researchers through the analysis of the documents developed by the teacher for different aims of her activity. In this case the virtual abacus was never considered as isolated, but as associated with the material abacus, lesson plans, examples of students' productions, and other resources designed by the researchers.

To summarize, the integration of digital resources by teachers is viewed with the perspective of the documentational approach as their integration in teachers' resource systems (Trouche et al., 2020a). Studying this integration process requires to consider the work of the teacher in class (Type C) and out-of-class (Type D), and teacher knowledge (Type E) previously developed that will influence the use of digital resources or developed along the use of these resources. The digital resources offered (e.g., by the educational authorities, but also simply available on the web) are Type I variables; but along their use in class and out-of-class the teachers develop documents, which are mixed entities: research about documents associates Types I-E-D and C variables.

One of the factors shaping links between education policies, digital resources and teachers' work concerns assessment. We consider research about assessment in the next subsection.

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[7] http://seminaire-education.espe-bretagne.fr/?page_id=611.

3.3 Assessment, Digital Technologies and Digital Curriculum Resources

The amount of research about assessment has significantly increased since 2000, and this also concerns research about assessment involving digital technologies or digital curriculum resources. Assessment is both a Type I variable, as an aspect of the curriculum, and an outcome of interactions between Type C and D variables, since the teacher designs assessment for their students and implement them in class. Hence, the research about assessment and digital resources concerns Types I, C and D variables. Since most of this research developed during the last ten years, we do not analyze in detail its historical evolution. The most important historical evolution that we want to stress in this subsection is the emergence of research about assessment, and especially about digital resources and assessment.

Stacey and William (2013) introduced a useful distinction, further developed by Drijvers et al. (2016), to categorize this research. Assessment *with* technology concerns the use of technology during an assessment, such as when the students are allowed to use CAS in a written exam. Assessment *through* technology concerns digital assessment, for example with online exercises. According to the distinctions we use here, assessment with technology is linked with DCRs.

Digital technology has been introduced internationally in mathematics curricula. This introduction has been followed by the integration of technology in both formative and summative assessment (Stacey & William, 2013). Teachers designing assessments with technology had new possibilities in their choice of tasks: they could propose rich problems that the students would not be able to solve without technologies (e.g., Leung & Bolite-Frank, 2015). These new possibilities were associated with a new complexity. In particular when designing summative assessments, teachers need to find a delicate balance between proposing tasks that are too complex and creating the possibility of a black-box use of technology. How teachers design assessments with technology also strongly depends on national educational policies, including whether and how technology can be used in mathematics exams (Drijvers et al., 2015). Moreover, technology can be used in very different ways in mathematics exams. Jankvist et al. (2021), presented contrasting examples of the use of CAS in different countries. In Denmark, for example, CAS has been used in the final exams in upper secondary school since 2005, and in lower secondary school since 2013. Jankvist et al. (2021) offered the example of a task presented at the final exam to Grade 9 students, showed that the students could solve this task without any mathematical reasoning, using the CAS as a black-box. They contrasted this example with a task given in Germany for the upper-secondary final exam in Bavaria in 2014. That involved complex mathematical modelling and would have been very difficult to solve without CAS. These differences in exams have strong impacts on the teachers' practices with educational technologies in class.

In the literature assessment *through* technology is called Computer-Aided-Assessment (CAA) or sometimes Computer Assessment System is also used, but we use here CAA in order to avoid a possible confusion with Computer Algebra

Systems. Sangwin et al. (2010) identified three possible outcomes generated by a CAA: a numerical mark; written feedback or statistics concerning a cohort's achievement. A numerical mark and automated feedback on technical errors offer to teachers the possibility of concentrating their own feedback on understanding (Olsher et al., 2016). Statistical overviews of cohort achievement are also a new element in teacher's external context, which can lead to adaptations of the content of the course. These can include adaptions, for example, when using clickers at university level (e.g., Lockard & Metclaf, 2015).

Other outcomes of CAA have been identified. For example, the FASMED project (Formative Assessment in Science and Mathematics Education), Aldon et al. (2017b) also foregrounded the possibilities opened by CAA in terms of automatically generated feedback or statistical overviews. They added possibilities for tracking students' learning paths, through access to statistics, and also to rich data about the students' mathematical activity.

Finally, technological advances have opened the way for a new kind of association between assessment *with* technology and assessment *through* technology resulting from automated scoring. Drijvers (2018) studied the automated scoring of students through digital means, using Intelligent Tutoring Systems. While online assessment sometimes means multiple-choice quizzes focused on technical skills, these new automated scoring tools have the potential to assess complex reasoning, and students' productions with educational technologies, particularly Dynamic Geometry Systems. These new tools offer possibilities for the design of digital assessment, associated with a subtle automatic scoring. As designers, teachers can propose dynamic and interactive tasks; as graders, they save a lot of time and have access to analyses of their students' work (Drijvers, 2018).

To summarize Sect. 3, it can be said that the research in mathematics education concerning the influence of digital resources and other associated Type I variables (e.g., educational policies) has evolved during the past 20 years towards more complexity and more refined analyses of the interactions between different kinds of variables. This refinement has been associated with an increasing complexity of the teachers' working environment in terms of digital resources, which has contributed to the development of specific theories. These theories have evidenced the need for considering different dimensions and the ways they are linked. These dimensions corresponded to Type I (e.g., the digital resources themselves, but also, for example, the time economy), Type C (teachers' practices in class), Type D (teachers' practices out-of-class) and Type E (teacher knowledge).

4   On the Quality of Digital Curriculum Resources

In this section, we review the literature concerning the quality of digital resources in teachers' contextual environment, as we assume that a conceptualization of quality influences teachers' choice of resources. Considering that over the past decade a considerable amount of research in mathematics education has investigated teachers'

lesson planning and enactment of designed lessons involving the use of digital resources (e.g., Aldon & Trgalová, 2017; Clark-Wilson et al., 2014, 2021; Trouche et al., 2019), it is surprising that a similar amount has not attended to the resources' quality within teachers' contexts of teaching. However, there are a number of studies attending to this issue in teacher design of their curriculum. In conceptualizing the quality of DCRs, we have to distinguish between (1) different aspects of quality criteria, and (2) quality of which kinds of digital resources.

4.1 Quality of Dynamic Mathematics Materials

Whilst there is no consensus of what 'platform', actually is, many dynamic mathematics materials are 'deposited' on platforms of some kind. The literature presents research on numerous online platforms providing a large number of Open Educational Resources (OER) for teaching mathematics: e.g., GeoGebra Materials, 2016; LearningApps, 2016; I2Geo, 2016. Teachers find it difficult to choose amongst the enormous quantity of resources and note inconsistency in their quality (Trgalová et al., 2011). Quality variability is particularly likely if the platform is not supported by 'gatekeepers', that is a dedicated 'editorial team' that checks on the quality of 'self-made' resources that are often freely available or shared by different types of users (Camilleri et al., 2014).

There are several platforms that provide mechanisms for assessing the quality of their resources, in order to be able to rank the materials according to their quality. In the context of the Intergeo project, for example, Trgalová et al., (2011, p. 1163) identified nine "relevant indicators" of the quality of dynamic geometry resources on their platform I2Geo: "metadata, technical aspect, mathematical dimension of the content, instrumental dimension of the content, potential of the DG, didactical implementation, pedagogical implementation, integration of the resource into a teaching sequence, [and] usage reports." In that project a questionnaire was developed based on these nine quality indicators. The assessment of the quality of a particular resource on the I2Geo platform required users to respond to nine broad statements, which can be extended optionally to 59 questions (ibid).

In her search for quality aspects of 'dynamic materials', Kimeswenger (2017) interviewed experts in electronic resource development, who described their views on educationally valuable use of dynamic materials. The analysis of the expert interviews revealed eight core "quality dimensions" as crucial factors: (1) author, (2) mathematical content, (3) resource type, (4) supporting the learning of mathematics, (5) integration into teaching, (6) advantages of dynamic material, (7) design and presentation, and (8) technical aspects. She also provided examples of these quality criteria. For example, for the criterion "Supporting the learning of mathematics", the following question is asked: "Does the dynamic material support the learning of mathematics?" and in of the following ways:

- Allows students to explore with the dynamic construction;
- Allows students to discover mathematics;
- Encourages students to make their own assumptions;
- Encourages students to formulate insights.

She also emphasized that the majority of experts stated that there is/was/should be a strong correlation between the 'quality' of the author and the created material, and the authors' views on learning.

In another study, Ladel et al. (2018) developed the ACAT framework for the evaluation of apps, in order to provide information on quality of apps and also on the various possibilities for teachers to evaluate apps in an efficient and reliable way. Artifact-Centric Activity Theory (ACAT) is a model developed to capture complex situations that arise when digital technology is introduced in classroom situations. They proposed five steps and questions for the evaluation: (S1) What is the mathematical *object* of the app? (S2) How do students *interact* with the mathematical object, mediated by the app? (S3) How does the *interaction* develop? (S4) Is the app suitable for teaching and learning the mathematical object? (S5) How can the app be used in classroom instruction?

Leaning on selected theories in mathematics education (e.g., cognitive load), Donevska-Todorova and Weigand (2018) developed three design principles for 'resources and tasks for technology-enhanced teaching and learning mathematics'. These were; (P1) Reduction of the total cognitive load by decreasing extraneous cognitive load; (P2) Reduction of the total cognitive load by decrease of the intrinsic cognitive load; (P3) Connection of active engagement and focus on mathematical content. Donevska-Todorova (2019) also developed a framework for evaluating the quality of tablet apps in primary mathematics education and their integration in student-centered learning environments. Focusing on the didactical potentials of tablet-apps, she identified six overarching categories: (1) mathematical content and relation to curriculum, (2) communication, collaboration and cooperation, (3) differentiation, (4) feedback and assessment, (5) connections and networking and (6) logistics. She claimed that the proposed model may become "meaningful for teachers' decision making when selecting and implementing touchpad-apps in their instructional practices but also for developmental surveying of existing apps, their re-designs and further novel designs involving identified potentials" (p. 121). Based on this framework, Donevska-Todorova and Eilerts (2019) also developed review criteria related to a particular content area: *space and shape.*

4.2 Quality of E-Textbooks

The e-textbook can be seen a system of digital curriculum resources. In their efforts to identify aspects of the quality of e-textbooks, Pepin et al. (2015) distinguished between three models of currently available e-textbooks – dynamic, evolving or "living", and interactive. In the "dynamic" model, a static textbook (traditional or digital)

is linked to other learning objects. In the "living" model, textbooks are dynamically and cumulatively authored by a community, often a community of teachers (e.g., Gueudet et al., 2013). The third model of e-textbooks – interactive – is based on a toolkit model, and is anchored in a set of learning objects, where tasks and interactive materials can be linked and combined in different ways. These distinctions also relate to the quality aspect of 'coherence'. Drawing on Gueudet et al. (2013) and Yerushalmy and Chazan (2008), distinguished between two types of coherence in textbooks. First, coherence of the design of a textbook encompasses aspects such as mathematical correctness, epistemological stance toward mathematical topics, sequencing that avoids gaps in the mathematical progression, consistent handling of mathematical objects, and consistency with national curricula. These aspects of coherence are constituted in the textbook's expositions, its tasks, and ways in which technology is made available to students. The second type of coherence-in-use is the coherence of what teachers actually propose to their students, drawing on the textbook, or on other curricular material. The e-textbook is changing the boundary between coherence of design and coherence in use. Issues pertaining to sequencing and availability of technology, which have been considered aspects of design of a linear textbook, are becoming aspect of coherence in use, as teachers re-design the textbook (e.g., Gueudet, et al., 2018). In order to help teachers to use digital resources in ways that provide a coherent learning trajectory for students, Confrey and her team (e.g., Confrey et al., 2017) have designed tools and materials to help teachers develop learning trajectories through a "bag of resources" in alignment with particular standards (in this case US Common Core State Standards).

4.3   Quality of Dynamic Mathematical Tasks

Concerning the quality in dynamic mathematics tasks, one of these quality aspects related to authentic tasks, which require realistic objects and questions (e.g., Jablonski et al., 2018). An example is MathCityMap which takes up the idea of outdoor mathematics through the creation of math trails by using an app and a web portal in which every registered user is allowed to create and publish their own tasks. Through a constantly growing community and the provision of a particular quality of the published material, the system is based on a multistep review process and several criteria for published tasks. Criteria for tasks in a MathCityMap math trail include the following: (1) Uniqueness (every task should provide a picture that helps identify the object of the task and what the task is about); (2) attendance (authenticity- the task can only be solved at the object location); (3) activity (embodied mathematics, i.e., mathematics can only be fully comprehended through an active experience); (4) multiple solution (solvable in different way); (5) reality (meaningful relevance); (6) hints (every task should provide at least one hint in terms of solving the task); (7) school math and tags (the task should feature a connection to school math); (8) solution formats (e.g., each task should be based on a meaningful answer format, such as intervals for measurement tasks); (9) tools (the task should be solved without

special and extraordinary tools). A math trail idea is a combination of different tasks that should harmonize as a trail. Therefore, the whole trail comes into the review process after every task of a trail has been through it.

4.4 Quality of Curriculum Programs

Choppin et al. (2014) created a typology for analyzing the quality of digital curricula in mathematics education. They documented two distinct curriculum types, individualized learning programs and digitized versions of traditional textbooks. In order to help educators better understand the characteristics of these materials, they developed and applied a framework to analyze a representative sample of digital curriculum programs. The framework has three distinct themes:

Theme 1 relates to students' interactions with the programs, and was subdivided into three categories that describe students' interactions with the programs:

1. Student learning experiences (what students see and do in the program);
2. differentiation/individualization (features that enable teachers to select content according to their perceptions of students' abilities); and
3. social/collective features (features of the programs aimed at virtually connecting groups of students or other stakeholders).

Theme 2 concerns curriculum use and adaptations, that address the flexibility of each program in terms of providing tools and resources to sequence and design lessons for teachers. Choppin et al. (2014) analyzed programs according to four categories that provide teachers the ability to:

1. Map and sequence lessons;
2. Design content of lessons;
3. Locate and use multi-media presentation materials; and
4. Make and store notes for future planning.

Theme 3 encompasses the analysis of assessment systems. As assessment systems offer the potential for online assessments and the ability to automatically analyze and report assessments, they proposed criteria for the analysis of the assessment systems, built into the programs, and focused on the following four categories of functionality:

1. Create assessments;
2. Record and store results of assessments;
3. Generate dashboard or other summaries of data; and
4. Generate and transmit reports/results to multiple audiences, including teachers, parents, and administrators.

Choppin et al. (2014) claimed that while the programs offered some of the features identified as transformative, particularly with respect to assessment systems that rapidly and visually report student performance, there were many features that did not take full advantage of the digital medium.

To summarize this section, it appears that there is a huge variety of DCRs. As different DCRs (and types of DCRs) have different affordances and constraints (also as compared to analogue materials), perceptions of what is 'quality' also vary: from interactive, over add-on, to dynamic materials, to name but a few of the quality notions. Moreover, notions of didactic quality seem to change their 'appearance' in teachers' work with digital resources (e.g., what does consistency or coherence means in e-textbooks).

5 Recent Developments and Future Directions for Research

We have seen in the previous sections that the progress of research on teaching taking-into-account digital resources is manifested through development of theories, which propose different dimensions to understand the interactions of teachers with available digital resources, and the consequences of these interactions. We foresee that this progress will continue, since teachers' working environment (and hence their professional activity in class and out-of-class) continue to evolve with new elements, predictable or not at this stage.

In this section, we consider three themes that correspond to recent evolutions in the external contexts of mathematics teachers' professional activity and that are giving rise to a growing body of research. We have chosen themes illustrating different elements of the external context for teachers: (1) official curricula and the integration of programming in these curricula; (2) the collective work of teachers in different kinds of teams or networks related to digital resources; and (3) the COVID-19 pandemic, which foregrounded the importance of digital tools, in particular for distance teaching (and learning).

5.1 Introduction of Programming in the Mathematics Curricula Internationally and Consequences for Teachers

Research in mathematics education about programming is not new (e.g., Papert, 1993), albeit the interest for programming in mathematics education declined at the end of the 90s and beginning of 2000s.

Then a major change happened in the official curricula internationally: between 2010 and 2020 that saw programming introduced in primary and secondary school curricula of many countries (see e.g., Haspekian, 2017; Misfeldt et al., 2020; Modeste, 2015). In some countries programming was introduced as a specific discipline and has been taught by computer science teachers. In others it has been inserted in mathematics curricula and has been taught by mathematics teachers. These changes in curricula have led to a renewal of research about programming and computational thinking in mathematics education. While the early works in the 70s and 80s

were mostly focused on students' learning and their development of computational thinking, researchers now acknowledge the importance of the teacher (Benton et al., 2017; Pérez, 2018), and the need to investigate how teachers integrate programming in their mathematics courses.

In the Scratchmath project in UK, Benton et al. (2017) designed a curriculum for primary school teachers, and the team studied teachers' implementation of this curriculum. The authors observed that some primary school teachers were not familiar with programming, and that the concept of an algorithm was difficult for them. Nevertheless, the framework designed by the researchers supported teachers in their implementation of strategies with their students. The choice of strategies depended in particular on their confidence with Scratch. The teachers also made different choices in terms of emphasizing programming, or mathematics. This research was similar to other studies evoked in Sect. 3.2 concerning teacher integration of digital resources, and how they were integrated.

The issue of the links between programming and mathematics that these developments draw attention to has been investigated in several studies. Pérez (2018) proposed a framework evidencing different dimensions of computational thinking; this framework has been actually developed has a tool for secondary school mathematics teachers engaging for the first time with the teaching of programming and facing the need to combine mathematical thinking and computational thinking. Misfeldt et al. (2020) examined the official curricula in Denmark, Sweden and England, and examined the enacted curriculum through selected cases. They identified four possible types of relations between mathematics and programming: "(1) specific relations to mathematical concepts or processes [.]; (2) explicit relations to mathematics [.]; (3) implicit relations to mathematics, [.]; and (4) no or weak relations to mathematics." (ibid. p. 259). How teachers can and do combine mathematics and programming in this new context is a promising and important direction for research.

The role of the teacher in courses combining programming and mathematics has already been the subject of research at university level, where programming has been present in some courses since the early 2000s. One example is the MICA courses (Mathematics Integrated with Computers and Applications) at Brock University in Canada. Buteau and Muller (2014) evidenced that teachers in these courses also intervened as policy makers, and that this role was essential for implementing and sustaining the intervention at the departmental level. In their recent research, Buteau et al. (2020) used the instrumental approach and the theory of orchestration, to study how a teacher in MICA courses supported students' instrumental geneses with a programming language, for mathematical investigations. They showed that the lab setting was a key element in the teacher's orchestration, where the work of the students on their projects were supported. Lockwood & Mørken (2021) called for more research exploring the relationships between computing and mathematics at university level, and this is also certainly a promising direction for research.

Studies about teaching programming and mathematics have much in common with the recent studies evoked in Sect. 3.2, about how teacher integrate digital resources. They have also investigated how a Type I variable (the introduction of programming in the official curriculum) affects Type C, D and E variables. We note, nevertheless,

that the nature of Type E variables in this case is specific, since it questions the links
between mathematics and programming.

5.2 Teacher Collaborative Activities

Research on mathematics teachers' collective work with DCRs has developed signif-
icantly over the last 20 years, and particularly in the most recent of these years.
This includes research on teacher's work in established communities as well as in
spontaneously set-up communities with a common purpose, and it also includes the
collective work online, in schools, at home or in institutions that offer collective work
as professional development.

Regarding collective work in organized teacher collectives, Gueudet et al. (2016)
provided a window into the collective design of an e-textbook, which was made
possible by new "digital" opportunities: e.g., platforms, discussion lists. The context
of the collective work was provided by the French Sesamath teacher association and
their design of a Grade 10 e-textbook in terms of the "functions" chapter. This study
concerned the influence of Type I variables on the collective teacher design (Type D).
Here the Type I variables include the digital platform, but also from the point of view
of an individual teacher, member of the group, the other members of the group (in
this case mathematics teacher and computer science specialists). At the individual
level, these variables also influence a member of the group in terms of professional
knowledge (Type E), and the Type D-E variable interaction, as described in Medley's
framework.

In terms of teachers working 'spontaneously' with colleagues, Trouche et al.
(2020b) reported on the collective work of an experienced mathematics teacher at
secondary level, who has also worked as a teacher educator in a university depart-
ment. They investigated her work and professional development with colleagues
(e.g., lesson planning), with a particular interest in the digital resources, including
both digital curriculum resources (e.g., e-textbooks, online resources) and digital
technologies (e.g., for communicating, sharing). Results show that her transition to
DCRs was a critical process in her professional learning trajectory. Of importance
were the notions of *resource system* for studying the teacher's activity as a whole,
and of *documentational trajectory* for studying the teacher's activity over the time. In
other words, they point to a teacher's resource system (an organized system of digital
and analogue resources) and his/her collective work (over time) as major ingredi-
ents for professional learning and development. The authors claimed to contribute
to a better understanding of the impact of digital resources on mathematics teachers'
work and professional learning over time, and of the ways the context of collaboration
shapes their professional work and learning. Hence, they consider the interactions
between variables of Type I (digital resources, colleagues) and variables of Type C,
D and E (since the practice and the knowledge of the teacher evolved).

Other recent research concerns teacher collective work with DCRs in a context
of preservice or in-service teacher education, hence interactions between Type I

and Type J offline variables (with consequences for Types C, D and E variables). Although the collective work in the context of teacher education has been researched for more than 20 years, the use of various digital means opened new possibilities, in particular in terms of blended or distant learning, that have been recently investigated. For example, in a study by Borba et al. (2018) online pre-service teacher distance education is the context. The purpose of this study was to analyze the role of digital technologies in two specific contexts: how teachers, tutors, and students play a role in producing interactive DCRs, and how digital technologies themselves can play a role in teaching distance learning courses. However, for these roles to emerge, the authors pointed to the need for participants in online courses to interact collaboratively. Their results showed that the roles are related and that digital technologies transform both teacher and student roles and participation in the virtual classroom, with the result that an 'agency of media' (meaning here the possibility to combine different media, to change media when relevant) emerges in online mathematics education.

Lesson Study (LS, Takahashi, 2014) provides context in which teachers collaborate to design lessons (through cycles of plan—teach—reflect). LS has been investigated for more than 20 years by mathematics education researchers; but they are renewed by digital resources, allowing in particular the organization of blended training. Joubert et al. (2020) reported on a Lesson Study in a blended approach to support isolated mathematics teachers (who could not meet face-to-face), to use and integrate mobile technology in their teaching. They identified eleven aspects playing an important role in the processes: technology; collective/group; learning management system; online facilitation; technological pedagogical content knowledge; (mobile) learning strategies; a lesson planning form; backward design; time; photos, videos and reports; and reflection questions. The eleven aspects that emerged led to the development of a framework consisting of three dimensions of LS, namely Collaboration, Instructional Development, and the Iterative Improvement Process, supported by the identified aspects.

Massive Open Online Courses (MOOCs) are another kind of digital curriculum resource that now contribute in mathematics teachers' in-service education. Hollebrands and Lee (2020) reported on the design of three MOOCs for mathematics teachers' professional learning. The designs were based on principles of effective online professional development that included: self-directed learning, learning from multiple voices, job-connected learning, and peer-supported learning. The team examined how these design principles were enacted in the development of the MOOC-Eds and how they influenced the engagement of 5767 participants. Evidence showed that the three MOOC-Eds were successful in "allowing two experienced mathematics teacher educators to design engaging experiences for teachers that have shown to have positive impacts on their beliefs, perspectives and practices in teaching mathematics and statistics" (p. 872). The authors claimed that scaling-up professional development for teachers requires much more than simply transforming typical in-person experiences into online videos and readings. As they grounded their design in an interconnected model of professional growth (Clarke & Hollingsworth, 2002) and used best practices from mathematics teacher education and design principles

for online teacher engagement, they claimed that they could establish a large-scale professional development program that engaged and impacted teachers from around the world.

An assessment of design principles used to guide the development of MOOCs for teachers was conducted by Aldon et al. (2017a). They examined how instructors' practices influenced collaboration and participation in MOOCs implemented in France (eFAN Maths MOOC) and Italy (UniTo: Geometria MOOC and Numeri MOOC). The MOOCs from these countries supplemented discussion forums with the use of other collaborative tools (e.g., Padlet, social networks, collaborative project spaces). There were differences noted in how the instructors facilitated collaboration. With those in the French MOOCs focused on fostering local collaboration while the Italian MOOCs encouraged collaboration among all participants within the MOOC. The study pointed to the importance of examining not just the design of a MOOC for teachers, but also how such MOOCs are enacted and experienced by participants.

Many possibilities for combining digital resources and mathematics teachers' collective work exist, and can have different consequences for teacher knowledge and teacher practice (within a teacher training program or more informally). Cai et al. (2020) suggest that digital technologies can contribute to the design of shared knowledge base for mathematics teachers and for researchers in mathematics education. The effective realization of these new possibilities constitutes a challenge for the mathematics education research communities and a promising direction for future research.

5.3   Digital Resources in Mathematics Education, Equity, and COVID-19

The socio-economic environment in which students live is also a critical component of the professional context for teachers. Research in mathematics education is increasingly taking this context into account, and there is interest in how teaching can contribute to equity (Forgasz & Rivera, 2012). Questions have been raised in particular about the use of technologies because students have different accesses and relationships to technology, depending on their socio-cultural background, how can teaching be equitable when teachers use technology in their mathematical courses? Can they use technology to create opportunities for students from different socio-cultural backgrounds? Forgasz et al. (2010) present a synthesis or research investigating such issues. They showed that obstacles to the use of technologies linked to issues of access seem to have decreased in rich countries, whereas they remain prevalent in developing countries. They also presented teaching interventions (in rich countries) where technology was used to create mathematical learning opportunities for all students.

While resources in the form of computers, software, and Internet access have tremendously increased since these early studies, important disparities in terms of

access to digital technologies remain at an international level (e.g., Bethell, 2016). The external context for mathematics teachers is thus very different according to the country in which teachers work, and we acknowledge that research synthesized in this chapter mostly addresses the context of teachers in rich countries. Nevertheless, even in these rich countries, socio-economic differences exist between different schools. In the U.S. Kitchen and Berk (2016) argue that the use by teachers of computer assisted instruction in schools that predominantly serve low-income students may favour work on technical tasks, instead of problems fostering a rich mathematical activity. This reduces the opportunities of learning for these students.

Research in mathematics education has increasingly considered equity issues, and how digital resources can contribute to equitable teaching. Referring to the framework guiding this book, we consider that this research investigates how a Type I variable (digital resources) can be used to counterbalance negative effects of another Type I variable (the socio-economic background) on the relations between processes (Types C and B) and product (Type A, learning outcomes). For example, in a study conducted in a primary school in 'unfavorable' (in socio-economic terms) contexts in Mexico, Sandoval and Trigueros (2021) observed that when primary school teachers create a classroom culture grounded on mutual respect, listening to each other, and combined this with the use of software supporting students' problem-solving activity, all the students can grasp the important mathematical ideas.

Finally, major changes in the mathematics teachers' external context in recent years have been due to the COVID-19 pandemic. From Kindergarten to University, teachers all over the world were forced to teach online of at least some of the time over several months. This dramatic context is also a new theme (or a large set of new themes) for research.

From the first lockdown, researchers in mathematics education launched questionnaires to investigate the consequences of this situation for teachers' practices, including naturally their use of technologies.

Drijvers (2020) and his colleagues, for example, conducted a study entitled "Math@Distance study" in Flanders, Germany and the Netherlands. They asked 1719 secondary school mathematics teachers about their teaching practices during lockdown. The use of digital resources was an important aspect in their study. They observed that the use of video conferencing software drastically increased. More surprisingly, the use of online exercises and online learning environments decreased. During the synchronous video lessons, the teacher presents, the students answer questions; but the collective work of students was scarce. Hodgen et al. (2020) reached similar conclusions, analyzing questionnaires and interviews with 49 heads of mathematics departments in secondary schools in England. Moreover, disadvantaged pupils were less engaged in the teaching due to problems of access, low parental support, and new personal and familial difficulties. Solomon (2021) stressed that equity is one of the most difficult challenges in the COVID-19 context; at the same time this context presented new opportunities for teachers to access student thinking using some of the technologies utilized during distant teaching.

Technological equipment and online teaching practices have changed since these "early" in the pandemic chronology studies. We assume that "Which digital resources

can support teachers, and students, in secondary school mathematics for distant or hybrid teaching in a context of pandemic?" will remain an important research question for some years to come. This is because the pandemic unfolds over several years and the research will need several years of setbacks to understand these phenomena.

6 Conclusions

The question leading this chapter was:

> How has the evolution of research in mathematics education about digital resources impacted the context of mathematics teachers' professional activity?

Reviewing the relevant literature, we have observed a very large number of changes in the research studies. We have selected and presented particular directions in these that seemed to be the most pertinent. Our focus was not only on digital resources themselves (e.g., e-textbooks, mathematical software, digital platforms, online assessment systems, tools for distant collaboration, videos, and other kinds of digital media), but on various aspects of teachers' external context linked with digital resources: community support, or time economy for example. Moreover, we have shown that the research studies on these topics strongly associate Type I, Type E, Type D and Type C variables.

The changes we observed and insights we gained can be summarized as follows:

- *Evolution in the research about educational policies*: early studies considered the policies in terms of material equipment of the schools, and then the place of the educational technologies in national curricula. They evidenced some discrepancies between the intended curriculum and the enacted curriculum, linked with a lack of equipment, and teachers' professional development concerning the use of technologies. The role of technology in national assessments (often very limited) was an important factor explaining the discrepancies. Recent research has been more focused on DCRs (e.g., digital platforms), proposed by the educational authorities to support teachers' design, in particular in a context of reforms.
- *Evolution in research about teachers' integration of digital resources*: the research questions evolved from the integration of a single educational technology by a teacher to questions about complex sets of resources available in a digital environment. This evolution in the questions being asked was linked with the development of theoretical frameworks and new conceptualizations of digital resource integration by teachers. New questions arose about the role of the teacher. In a digitalized context, students develop as self-directed learners together with support from their peers, and teachers become the scaffolders of knowledge development. The research also highlighted new requirements for the teachers, including a need to change their perspective on the mathematics (e.g., seeing programming as an integral part of mathematics). Finally, an increasing number of studies considered the potential and actual collective dimensions of teachers' work and how these have been impacted by digital resources.

- *Evolution in the research about the digital resources*: Concurrent with development and use of new digital resources, new issues have emerged and have been developed. These included the quality of digital resources. Research has produced different kinds of tools for assessing this quality, and revealed the need to re-conceptualize quality, to consider new possibilities for connectivity, and new perspectives on the teachers as designers of their own curriculum (Type D and Type C variables, since the design takes place out-of-class and in-class). It has become evident that new technologies and digital resources necessitate and drive new pedagogical approaches. In other words, questions are not only concerned with how the teacher may be able to suitably integrate resources, but with the digital resources themselves (e.g., digital learning environments) require and force teachers to take a different stance and build their 'teaching' (or coaching) around the new digital environment.

Different causes were combined to produce these changes. Each time that a new digital resource is introduced in school, it is a new element in the external context for teachers and opens the way for research on the potential of this digital resource, on its actual use, on its impact on teaching and on teacher knowledge (Types C, D and E). The general evolution of research on mathematics education has also influenced research about DCRs as part of teachers' context (this can involve any of the variables). New research issues (e.g., assessment; teachers' and students' collective work) encompass studies about digital resources and mathematics teaching. The socio-political turn, and the value of research addressing equity issues is also an important trend in recent research present in the literature we reviewed.

We foresee further evolution in research in all the directions mentioned above that stress the need for more research on:

- Educational policies pertaining to the offering of digital curriculum (e.g., digital platforms) and the tensions between supporting teacher creativity (with these resources) and efforts of the national agencies offering the resources to help teachers align with education reforms;
- Provision and quality of particular DCRs (e.g. for particular mathematical topic areas, including programming);
- Digital assessment procedures, developing from simple tests to complex digital environments where students can work collaboratively on tasks;
- Distant and hybrid teaching at all school levels, and its links with equity issues.

Moreover, in an external context requiring teachers to become the designers of their own curriculum, more research is needed on educative digital resources for teacher professional development (Type J variable). We contend that digital resources as elements of the external context for mathematics teachers' professional activity are often underestimated, and their affordances, constraints, and potential to drive under-researched. For us, this review was an eye-opener, and we believe that there are many avenues for mathematics education research in this field.

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Competency Framework for the Qualification of Facilitators of Mathematics

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1 Introduction

As pointed out by Medley (1987), research related to teaching is always focused on student learning outcomes (Type A). The factors relating to teachers which directly influence these outcomes include four online variables. Teachers' competencies, knowledge, and skills play a critical role in this regard (Type E) and are connected to the variables describing proactive (planning, evaluation in Type D) and interactive (process of teaching in Type C) observable teacher behaviors relating to their performance. The overall basis for teacher competencies are teachers' pre-existing characteristics which they already have prior to admission to teacher education (Type F).

For optimal performance, it is crucial to strengthen teachers' competencies (Type E); this is a core goal when conducting mathematics teacher training and reflection on experiences (see Type J). This *offline variable* is intended to foster teachers' personal

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and individual professionalization as a core presage variable in terms of presage-process–product Research (PPPR, see Medley, 1987, p. 108; Manizade et al., 2019). Mathematics teacher training includes pre-service activities at university as well as professional development (PD) programs for teachers already in service.

The core issue in teacher training is linking the cognitive side of knowledge with experience (Type J) gained in either pre-service or in-service training. Schön's concept of "The Reflective Practitioner" (1983) highlights this link as the initial connection between teachers' practice and knowledge, and thus also sheds more light on research into professionalization. The aim of any professionalization process is to lead teachers to recognize new ideas and innovative approaches for their teaching and enable them to implement these in the classroom. This process must be initiated and supported during PD programs. It is important that facilitators provide teachers with opportunities to reflect on and enhance all facets of knowledge because—in addition to the central role of knowledge in thinking, acting, and learning—learning is an active, constructive process, with knowledge and learning rooted in contexts and cultures (Brown & Borko, 1992; Putnam & Borko, 2000). It is worth emphasizing here that it is knowledge and beliefs themselves which are the critical targets of change for classroom implementation, since these largely determine what teachers do in the classroom. Accordingly, during PD programs, these components must be considered targets of change (Putnam & Borko, 2000). Borko and colleagues (2014) describe delivering appropriate PD programs as challenging and see facilitators as responsible for appropriately implementing such learning opportunities.

Although facilitators play an important role, a complete description of the competencies required to fulfill the complex tasks they must perform to conduct effective PD is lacking. Even though facilitators are usually experienced teachers, teaching experience in itself does not guarantee that a teacher has the necessary competencies to help other teachers develop their own mathematics teaching (Even, 2005). Our paper follows in the footsteps of Manizade and colleagues' (2019) framework along with Medley's (1987) concept of "mathematics teacher training and experience" (Type J). As mentioned in the book's introduction, over the years there have been many approaches to assessing the quality of teacher education and training and to evaluating (via empirical studies) the influence of corresponding variables on the development of teacher competencies. The importance of PD has also increased in recent decades, and thus corresponding variables such as teacher engagement and participation in PD have also been the focus of research. However, rather than looking at variables such as the design and quality of professional development, the focus here will be on the competencies of facilitators themselves with the aim to describe the range of competencies required by facilitators in mathematics. This chapter's key aim is to advance teaching research and build on Medley's framework by focusing on learning outcomes to include the effectiveness of those involved in educating teachers themselves.

Scope of Our Work

To describe the competencies needed by facilitators in mathematics education and understand their responsibilities regarding chain effects on students' learning

outcomes, it is important to build on existing research findings and insights from the field of teacher professionalization (Sect. 2). Epistemologically, learning is an active process of constructing new knowledge (von Glasersfeld, 1998) and it is therefore important to design any learning process as an interplay of content and learners' perspectives. Facilitators in PD programs must regard teachers as learners. However, unlike students and pupils, teachers are adults and experienced professionals, and facilitators must motivate them and initiate PD in an appropriate way. Therefore, this paper examines the field of general adult education, with a focus on the specific needs of mathematics teachers as adult learners. The first topic under discussion is thus mathematics education. Following this, we start our review with findings and insights regarding adult learning in general (Sect. 2.1) and then specifically from the perspective of mathematics education (Sect. 2.2). An overview of both fields is necessary to examine the various challenges that facilitators must address when leading mathematics teachers to reflect on and develop their practices and teaching routines.

Section 3 provides insights into how the role, tasks, and competencies of facilitators are currently conceptualized. To better understand and categorize our understanding of facilitators, the role is first explicitly defined (Sect. 3.1). Once again, the (overlapping but distinct) fields of PD in adult education and mathematics education are both considered in this part of the paper. In the field of adult education (Sect. 3.2), the review is based on existing competency frameworks already discussed in the literature (Wahlgren, 2016). However, these frameworks require adaptation and further development to be effectively applied within the specific context of mathematics education. Some of the existing competency frameworks within general adult education will also be discussed to provide a historical review of the development of these considerations (Sect. 3.2). This is followed by a focus on the teaching profession in mathematics education (Sect. 3.3). The challenge—as will be shown in the following sections—is to develop a competency framework that fits the professional requirement profile as well as the cultural and structural framework. To emphasize the complexity of this challenge, we will conclude this section by presenting such an evolutionary process based on the literature review performed within the context of the DZLM expert network (Sect. 3.4).

In the outlook, we will summarize the findings and lines of development from the last twenty years and take a look at further challenges in the field of mathematics teacher training and reflection on experiences (Sect. 4). In doing so, technological developments and their influence on teaching and training will be presented as an example.

Methodology of Our Literature Review

Depending on the topic and the existing literature, different types of literature reviews may be appropriate (Higgins & Green, 2008). To provide an overview of the research evidence on the competencies of facilitators in mathematics, we chose to conduct an integrative review to cover the breadth of both fields of interest, general adult education and mathematics education (Whittemore & Knafl, 2005). An integrative review represents a holistic approach offering the possibility of bringing together

studies using a wide variety of methods, meaning that qualitative and quantitative methods are considered in addition to theoretical and empirical ones. This enables us to not only depict the current state of research but also to create direct links to possible areas of application. Due to the involvement of several people (the four authors) in the research, a certain level of objectivity can be assumed in the selection and evaluation process.

Since reviews usually include all studies that are found to consider the research field, the number of studies included can vary dramatically. In the first search run for the field of general adult education, a very broad constellation was found extending in very different directions. Narrowing the search to the competencies of mathematics teachers and existing research on mathematics facilitators produced relatively focused and clearer results. Due to the wide range of work in both content areas, we decided to focus mainly on articles and studies looking closely at teacher competencies and their development and which attempt to formulate quality standards for facilitators. Furthermore, we also set restrictions in terms of publication date and study design. For example, for general adult education, we looked predominantly at literature from the last twenty years or which (in some cases) introduced changes or innovations to previous frameworks. For the requirements and competencies of mathematics facilitators, such restrictions were unnecessary due to the small field of research. By means of communicative validation, the focus was then condensed to the articles that have been integrated here. The results of the research have been discussed and checked how the authors assess the validity of the results (according to Mayring, 1990). When selecting articles, in addition to general keywords, particular attention was paid to their compatibility with teacher education and, based on the journals consulted in the database search, to high-quality international peer-reviewed journals. More narrowly, the discipline of mathematics was also a key focus, since it soon became obvious that subject-specific content was important.

We searched the literature using various keywords such as professional development, competency framework, competencies of facilitators, and requirements for facilitators in both content areas. We used several databases to include global studies. The databases included ERIC, ELSVIER, and even MathEduc, which was available until December 2019. These journals included the International Journal of Science and Mathematics Education (IJSME), International Journal of STEM Education, Journal of Mathematics Teacher Education (JMTE), Journal of Mathematical Behavior (JMB), Journal for Research in Mathematics Education (JRME), Psychology of Mathematics Education (PME), Mathematics Education Research Journal (MERJ), and ZDM—Mathematics Education. The International Handbook of Mathematics Teacher Education was also consulted, as Volume 4 is specifically addressed to facilitators in mathematics education. With reference to facilitators of adult education more generally, the following journals were reviewed: the Journal of Teacher Educator (JTE), Journal for Research on Adult Education (ZfW), Adult Education Journal, Teaching and Teacher Education, Educational Psychology and Educational Researcher (ER). Finally, in reviewing the articles that were crucial for us, we also consulted others that were listed in these articles if further insights could be generated. It can be noted that all mathematics-specific articles specifically related

to PD competencies, rather than PD design and quality, have been listed in this paper. Because there is such variety in the field of general adult education, we primarily selected those that provided an overall view of development cycles or used a Delphi study, as there is much variation to be found at the national level. All 78 referenced articles that were relevant to our work are mentioned and cited in this chapter.

2 The Evolution of Framing Teachers' Professionalization

2.1 PD in General Education

Medley's (1987) description of PPPR focused on processes to improve student learning outcomes. The general steps he identifies are nonetheless valuable and provide crucial foundation for further concretization. The modeling he used to describe each teaching and learning process—at the teacher-student level in his case—can be elevated and applied to the facilitator-teacher level too, thus providing a perspective on facilitators' competencies as well.

Innovations and change processes due to new curricula or new administrative conditions pose many challenges for schools and teachers, who need intensive support and assistance to tackle them. This requires PD programs that reach as many teachers as possible by scaling-up PD programs of a high standard. In Hattie's meta-study (2009, p. 119 ff.), he calculates the influence of content-related in-service PD programs with an effect size of $d = 0.62$ and classifies PD programs as an important intervention with significant effects on improving teaching quality. Facilitators of teacher PD programs, in turn, need comprehensive, evidence-based qualifications to be able to successfully design and implement teacher PD programs. The meta-study by Timperley and colleagues (2007) also reported the effects of teacher training on student outcomes. This study discovered that there was an average effect of $d = 0.66$, but variations were found by school subject and student level. For example, the effect in mathematics was $d = 0.50$. From the various meta-analyses, it can be concluded that teachers who regularly participate in PD programs sustain and even enhance their professionalism throughout their working lifespan, thus also effecting pupil learning outcomes. For this to occur, Lipowsky and Rzejak (2015) note that in-service teachers need regular PD. This in-service PD must involve a sufficient quantity of high quality learning opportunities planned and implemented by facilitators.

The insights provided by Medley (1987) in his description of state of the art teaching research can also be applied to learning outcomes among teachers following PD programs. In other words, the first question is which competencies teachers need to carry out effective teaching in schools. Based on the answer, the competencies that facilitators should have to deliver effective PD programs should be identified. Simultaneously, research that examines teacher competencies and pre-existing characteristics must also be considered, as this strand of research can shed further light

on the knowledge needed by facilitators to adequately enhance the competencies that teachers require.

A detailed illustration which highlights that the design and use of CPD programs are not the only initial and preparatory component of the transfer process in teacher professionalization can be found in Lipowsky's (2014) "offer-and-use model on PD level" (Fig. 1). It provides a highly differentiated framework to concretize in detail the challenges of the offline variable "mathematics teacher training and experiences" (Manizade et al., 2019) to achieve a successful transfer process and successful CPD at every level (see also Guskey, 2002), including for student learning outcomes.

Most research on PD focuses on describing the design of PD programs and assessing their quality. For example, the meta-study by Darling-Hammond and colleagues (2017), which reviewed 35 effective PD, identifies seven characteristics of effective teacher PD, that mainly relate to PD program design and implementation (e.g., content focused, sustained duration, or opportunities for feedback and reflection).

In addition to this focus on content, how the PD is carried out and how spontaneous situations are managed and properly moderated are also important. These areas relate to learning processes in subject lessons and active learning using the theory of adult
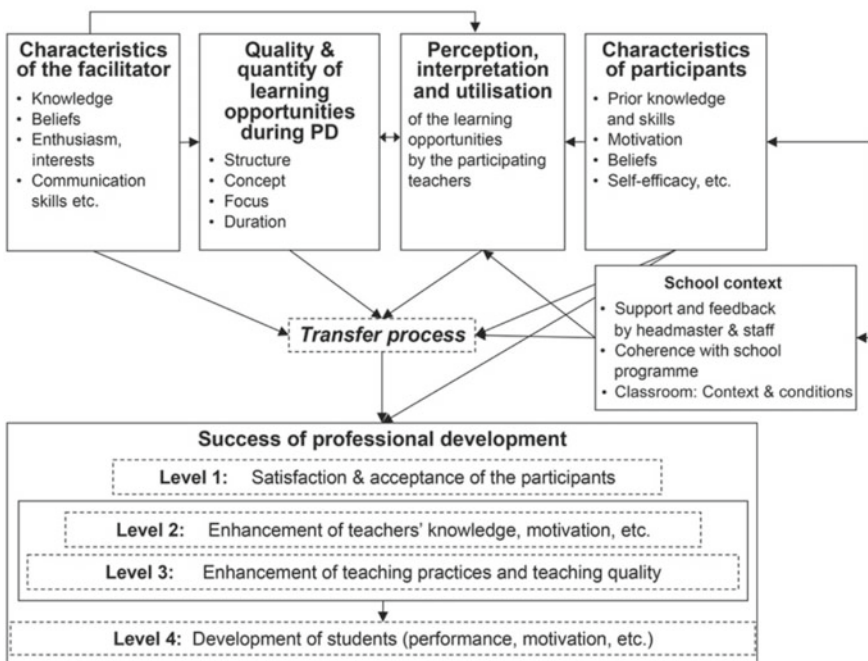

Fig. 1 Offer-and-use model for research on teachers' professional development. Lipowsky (2014, p. 515), English version: Lipowsky and Rzejak (2015, p. 30)

learning should be considered here. Collaboration should also be supported in in-service contexts, and various models of effective practices should be applied. The format of permanence should be another focus. Finally, regarding feedback and reflection opportunities, coaching and expert support are mentioned. However, like in the illustration of Lipowsky's model (2014), the competencies of the facilitators themselves are a crucial initial component.

2.2 PD in Mathematics Education

Focusing on the context of mathematics, Sztajn (2011, p. 221) pointed out that the attention paid to research in the field of mathematics teacher education increased significantly in the 1990s. When this is connected to Medly's framework (1987) and its adaptation in the context of research on mathematics teaching and mathematics teacher education by Manizade and colleagues (2019), it becomes apparent how this field has evolved to reflect current research.

In their meta-study, Timperley and colleagues (2007) looked at the effects of teacher training on student outcomes in a differentiated way, depending on school subject and student level. In mathematics, they speak of an effect size of $d = 0.50$ for student learning outcomes, although this differentiated view relates only to 11 core studies. For mathematics specifically, stronger effects were found in studies that focused on building teachers' content knowledge and pedagogical knowledge than studies that looked only at content knowledge. As reflected in several research papers on PD programs and most notably in the review by Sztajn and colleagues (2017), there is a growing body of empirical research that reveals the structure, content, and impact of effective CPD in mathematics education. Predominantly, these studies provide insights into the characteristics of PD programs that provide appropriate learning opportunities for teachers.

Other studies indicate that PD opportunities for mathematics teachers are recognized as a critical factor in increasing student achievement. Figure 2, which depicts the chain of effects from the competency level of facilitators (teacher leaders (TL)) to student learning outcomes, illustrates that the design and use of PD programs is not the only initial and preparatory component in the transfer process within teacher professionalization. Borko and colleagues (2014, p. 149 ff.) see the quality of mathematics instruction as the central factor influencing student learning. In this regard, the emergence of teacher competencies for quality instruction is seen as starting from high-quality PD programs. Accordingly, facilitators must be able to consider and implement all aspects of PD programs so that their influence on the quality of mathematics instruction is sufficient. This also requires establishing qualification standards for this group of adult learners (PD for TLs).

As can be seen in Fig. 2, frameworks usually describe facilitators from the classroom level (see Carroll & Mumme, 2007; Perks & Prestage, 2008; Jaworski, 2008; Hauk et al., 2017; Prediger et al., 2019). Thereby, knowledge on the lower level is always a component of the level above. Here, Carroll and Mumme (2007) see
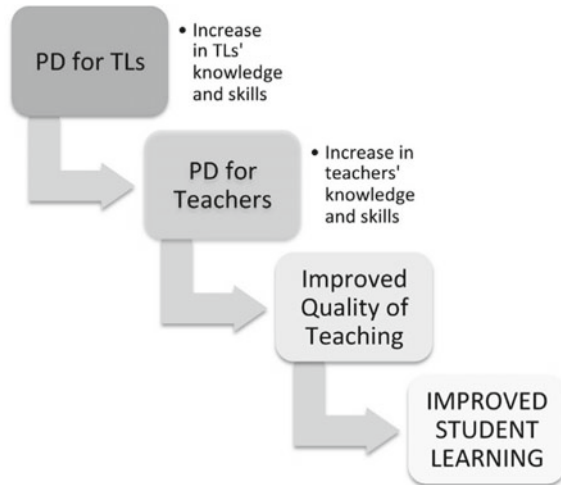
mathematical knowledge at the classroom level: a teacher's mathematical knowledge for the mathematics teacher educator within a larger context. That includes, for example, knowledge of teachers' professional learning. Perks and Prestage (2008) add an additional aspect they call "professional traditions" describing a structure similar to Carroll and Mumme's in which each level is nested within the next. This new area includes knowledge of school curricula or practices as well as research at classroom level. At teacher PD level, this is expanded to include knowledge of systems, institutions, and teachers' own research efforts.

However, the framework focuses more on the types of knowledge that must be brought into the classroom and less on the interactions between people and content. Thus, despite very similar nesting in both frameworks, the focus lies on different areas. In the work of Hauk and colleagues (2017), the relationships between knowledge and thinking types associated with the development of mathematical knowledge for teaching are presented. To illustrate this, they have used a concrete case of a specific type of elementary-middle school. Like Carrol and Mumme, the "three-tetrahedron model (3TM) of professionalization research" (Prediger et al., 2019) revisit the interaction between individuals—actors—, but rather than focusing on a specific type of school, it is intended as an overarching framework (Fig. 3, Prediger et al., 2019, p. 410). Following this, the 3TM instead describes rather the mesh of the different levels and is not to be understood as a process model as in Medley (1987).

In Germany, these research topics are the focus of the work of the German Center for Teacher Education in Mathematics (DZLM), a nationwide institution for the development and research of in-service PD programs for mathematics teachers (Prediger et al., 2019), concentrating on the qualification of facilitators. In the 3TM, individual levels are described and related to one another (Prediger et al., 2019). The elements of the three-tetrahedron model are the most important reference points for facilitator activities (see in more detail Prediger et al., 2019):
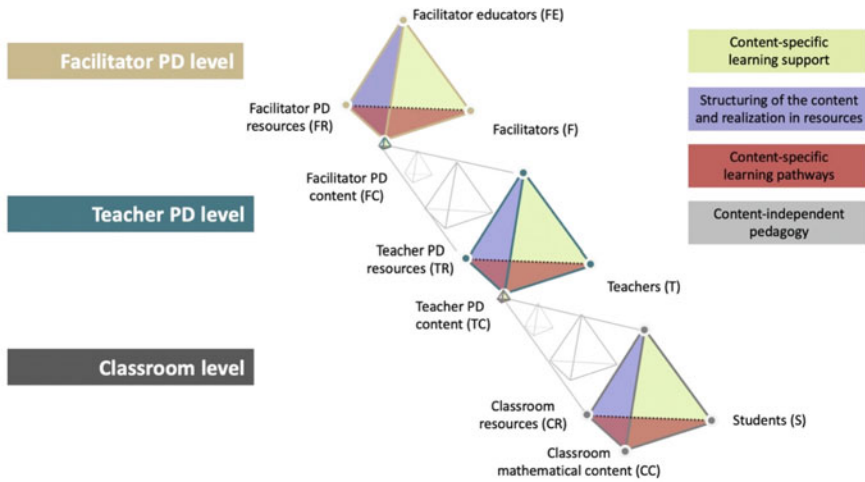
Fig. 3 Three-tetrahedron model (3TM) for content-related PD research (Prediger et al., 2019, p. 410)

- The lowest level is the classroom tetrahedron, in which the pedagogical triangle has been extended by the corner "classroom resources." This classroom tetrahedron, as a whole, is PD content.
- The teacher PD level regards teachers learning this content—so here too, there is a tetrahedron with the relevant actors (facilitators as teachers; teachers as learners) and a corner for resources, especially resources seeking to further education.
- Facilitators themselves are "learners" at the top level. Here, facilitators are involved in continuous qualification programs, which vary greatly in quality and quantity in different systems depending on local framework conditions.

Regarding the development of a competency framework for facilitators, it is important to realize that facilitators are related to all elements of the 3TM, since they are both learners on the qualification level and teachers on the PD level, while the classroom level, as the PD content, is always on the facilitators' minds. In addition, depending on the framing of the educational system in question, facilitators may act as teachers as well. In most systems, facilitators are also teachers and are active in the development of their own schools. They can act as colleagues among peers, accompany a quorum as a regular guest, or provide impetus as external experts.

For the qualification of facilitators, all relevant aspects of the individual must be considered (e.g., Bromme, 1992; König & Blömeke, 2009). This involves a cognitive perspective on the facilitators' knowledge and orientations and a situated perspective on their work as a facilitator (Prediger, 2019). These perspectives are complementary and may be located within a continuum from disposition to performance (Blömeke et al., 2015; Depaepe et al., 2013). In this paper, the focus is on the cognitive perspective. We will discuss competencies, which represent the latent characteristics required to perform effectively as facilitators (Weinert, 2001, p. 27). These cover

facilitators' knowledge, beliefs, and attitudes, which can be learned and improved through institutional learning opportunities (Klieme et al., 2008; Weinert, 2001).

Interest in PD in mathematics has increased, with consideration given to location and structure as well as interrelationships within the impact chain, but as Sztajn and colleagues (2017) pointed out, in terms of what is known about PD, the knowledge gap still includes what facilitators should be required to know and be able to do and what is associated with their preparation of PD. This field of interest has become more significant in more recent working groups. Since 2011, there have been several working groups at PME addressing this field, and attention has also been paid to mathematics facilitators at CERME and ICME. There have also been three major international publications, namely JMTE (2018, Vol. 21(5)), The International Handbook of Mathematics Teacher Education (2020, Vol. 4), and the book "The Learning and Development of Mathematics Teacher Educators" by Goos and Beswick (2021) which specifically address the knowledge, skills and development of facilitators in mathematics education.

Although the topic has been identified as important, there is still little research on this phase of teacher education. This is at odds with the attention given to the design of high-quality learning processes for teachers and facilitators, as previously mentioned. It is interesting to see which research strands have stood out in this area of mathematics teacher training over the past 20 years. While comprehensive and accurate knowledge of facilitator competencies was neglected in the past, it is now recognized as being an area of great interest which is worthy of specific consideration. We therefore aim to address the competencies of facilitators as a new presage variable.

3 Facilitators' Competencies—A New Presage Variable

As initiating PD with adult learners presents specific challenges for facilitators to overcome, we first focus on our understanding of facilitators as well as their role and then take a closer look at the field of general adult education. Following this, we look in detail at findings in the field of facilitators in mathematics education and highlight developments in this area of research.

3.1 The Role of Facilitators

In most school systems, there are people entrusted with the task of planning, organizing, and carrying out CPD programs for in-service teachers. In many (but not all) school systems, these people have worked or still work as teachers and are often, in a sense, "self-made" (Zaslavsky, 2008, p. 93). They usually devote themselves to these activities in addition to their work as teachers and often have few systematic qualifications for this activity. The many different designations in use—such as mathematics trainers, moderators, multipliers, teacher educators, didacticians, specialists,

coaches, or facilitators (see also Bernhardsson & Lattke, 2011, p. 19)—show the heterogeneity of the work they carry out. Henceforth, we will call them facilitators, as we feel this term best expresses their role in guiding teachers to undertake change processes more easily.

A "*facilitator*" is a person who opens new possibilities and accompanies others on development processes; in contrast, "teachers' leader" or "teachers' educator" designates a hierarchical relationship (e.g., in teacher PD programs with structurally conditioned relationships of dependence). Lunenberg and colleagues (2014) identify six different roles: teacher of teachers, researcher, coach, curriculum developer, gatekeeper, and broker. A look at these various roles once again highlights the manifold requirements for facilitators in terms of both knowledge and competencies. The diverse roles a facilitator might perform require expertise, skills, and special abilities such as accompanying, demonstrating, counseling, mentoring, evaluating, empowering, cooperating, and so on (Shagrir, 2013). Smith (2005) and Zaslavsky (2008) concretized these requirements by listing facilitators' typical characteristics (e.g., Shagrir, 2013; Smith, 2005; Zaslasvsky, 2008). Smith (2005) identifies specific qualities and behavior for facilitators, who he states should:

- be self-aware to reflect on their actions and discuss them,
- have in-depth professional knowledge based on theory (on testing in practice),
- be involved in research (to be involved in creating new knowledge) and in the writing processes of the curriculum,
- be good teachers and have experience in different age groups (school levels),
- have a comprehensive understanding of the education system and
- have reached a high level of professional maturity and autonomy.

These requirements, as Shagrir (2013) points out, obligate facilitators not only to establish clear work procedures at each stage, but also to sustain the relationship between the field of practice and teacher education institutions. Interestingly, these characteristics are all interdisciplinary and their application to mathematics and mathematics teachers is only implicit. Therefore, it is unsurprising that similar issues are also discussed in adult education research, with the added discourse of the respective professional field.

3.2 Facilitators in General Adult Education

Why do we need such competency frameworks in adult education at all? The necessity arose over time. Just as with other professional training strands, as adult education became an increasingly important field of action, professionalization had to occur, since the demands on teachers were constantly growing. Adult education represents a significant challenge and cannot be undertaken lightly, as pointed out by MacKaye back in 1931 when he described it as an "act of war" for which one must prepare tactically (as cited in Rossman & Bunning, 1978, p. 140). Delivering it is a challenging task which therefore requires in-depth qualifications. In the first

half of the twentieth century, numerous programs were developed in which studies were conducted to identify the central requirements and the core activities of adult trainers. As Rossman and Bunning (1978) noted, one of the first texts on training adult educators was published in 1948 by Hallenbeck. Since then, more and more attention has been paid to this topic, and over time, awareness has arisen that this activity must be taken up as a profession.

The literature shows that over several decades, various frameworks have been developed for adult education to teach skills, knowledge, and competencies (see Wahlgren, 2016). During this period, the question of adult educators' competencies has been studied from different perspectives and in different contexts. Three main positions can be distinguished according to Wahlgren (2016): Delphi studies, national curricula for adult educators, and studies on competencies for vocational educators. Wahlgren (2016) gives an overview of these developments and draws attention to the fact that these findings were mostly gained through Delphi studies by experts in this field. It is noteworthy that the differentiation between skills and knowledge is no longer made in more recent studies, and that these two concepts are no longer even the focus of research. Instead, attention is focused on the concept of competency: "In the more recent study, a distinction between knowledge and skills is no longer made, but the concept of competencies is still used." (Wahlgren, 2016, p. 346.) In addition, Wahlgren emphasizes that communication skills and the related ability to identify students' needs and experiences have consistently been found to be essential, even though different studies identified different emphases and rankings (Wahlgren, 2016, p. 346). One of the most recent large-scale Delphi studies on adult learning relating to facilitators' core activities was published by Bernhardsson and Lattke (2011), and makes it possible to compare different competency frameworks both over time and across countries, as Wahlgren (2016) has done.

There are numerous efforts underway to formulate uniform frameworks that can be used not only across occupational groups but also across countries. However, this is particularly difficult for teachers, as not only does each country have specific school qualification frameworks, but teachers as a professional group differ greatly from other professional groups such as lawyers or police officers. In this context, the prominent project QF2TEACH is worth mentioning. In QF2TEACH, the core competencies of teachers for continuing education are developed in relation to the European context (see Bernhardsson & Lattke, 2012). Such projects usually focus on comparing different activity profiles, but also on the degree of concretization, which varies across frameworks. Our aim is thus to identify the overarching structures that are considered relevant.

There are certainly substantial differences between the various occupational fields that may explain the far greater differences in the related formulation of competency frameworks. Facilitators must be experts in their specific domain (in this case mathematics education) and must know about the individual characteristics that teachers need to successfully carry out their profession. In the frame of DZLM, we aimed for a competency framework for facilitators of mathematics education. As a starting point we choose the GRETA competency framework of the German Institute for Adult Education (DIE, Leibniz Centre for Lifelong Learning) in the field of general

adult education, for two reasons. First, it fits well with the existing requirements (the abbreviation GRETA in German stands for "basics for a standardized process for recognizing teachers' and trainers' competencies in adult and continuing education"; www.die-bonn.de/greta; see Fig. 4), which are as follows:

- The cultural context (cf. Wahlgren, 2016) emerges from it.
- The specific requirements of this professional field are considered.
- The level of detail corresponds to that which appears to be suitable for the later use of the framework.
- Docking with the required subject area of mathematics is possible.

The GRETA framework offers an interdisciplinary structural competency framework covering all the basic competencies required to be able to teach well in adult and continuing education and it highlights the importance of including content-specific
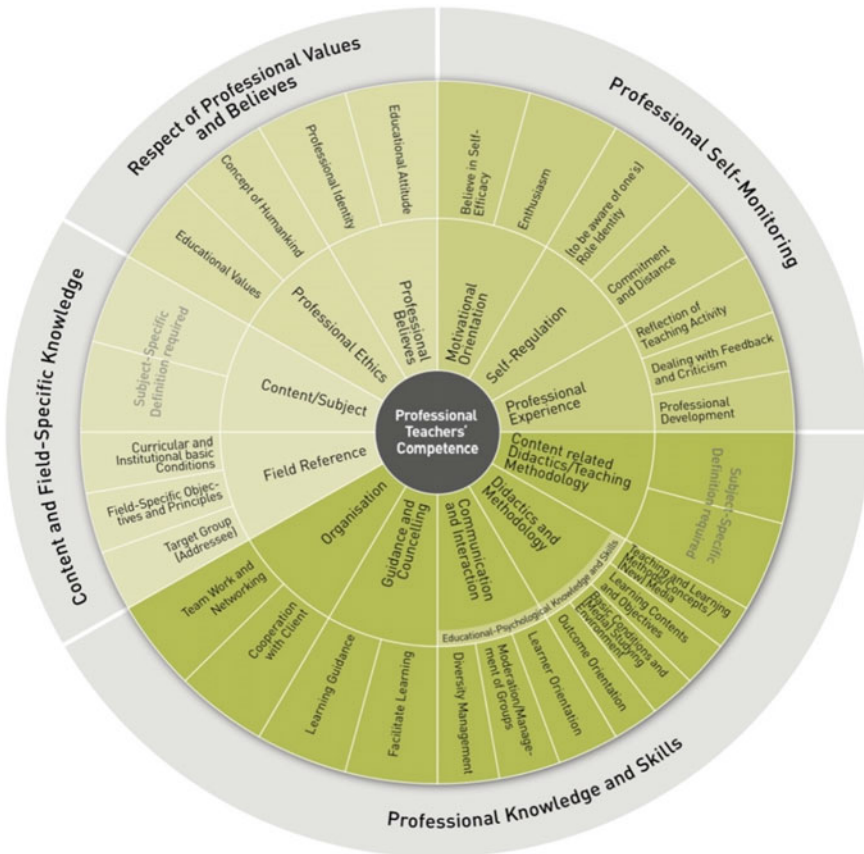

Fig. 4 GRETA competency framework (Strauch & Lencer, 2017). https://ec.europa.eu/epale/en/blog/greta-competence-model-teachers-continuing-training

competencies. This framework uses Baumert and Kunter's framework (2013) as orientation and was developed through a Delphi-process with educational practitioners and stakeholders. The comprehensive framework of Baumert and Kunter (2013) refers specifically to mathematics teachers. It relies on Shulman's (1986) structural knowledge dimensions: content knowledge (CK), pedagogical content knowledge (PCK), and general pedagogical knowledge (PK). Baumert and Kunter (2013) also included existing organizational knowledge, coaching knowledge to communicate professionally with parents (see Bromme & Rambow, 2001), and beliefs and values as separate categories, with fluid transitions.

The GRETA framework was developed in a Delphi-process with educational practitioners, researchers, and stakeholders, meaning manifold perspectives were involved. Facilitators for every subject area must consider aspects of general adult education because program participants are adult professionals. In addition, they must be experts in the specific domain (in this case, mathematics education) and must know about the individual characteristics that teachers need for the successful accomplishment of their profession. In this regard, the DIE has developed an interdisciplinary structural competency framework covering all the basic competencies required to be able to teach well in adult and continuing education.

The GRETA framework is designed to identify all relevant competency aspects (outer ring), domains (inner ring), and facets (middle ring) via an assessment procedure (Lencer & Strauch, 2016). The framework comprises an even more holistic understanding by providing four aspects of competency (see Fig. 4): professional knowledge and skills, content and field-specific knowledge, professional self-monitoring, and professional values and beliefs. The framework is applicable to all fields of adult education so the areas pertaining to subject-specific competencies have been left blank and must be filled in for the discipline under consideration (mathematics education in our case).

3.3   Facilitators in Mathematics Education

As noted in Sect. 2.2, several working groups have been at PME since 2011, and facilitators in mathematics education have also been a focus for CERME and ICME since 2021. In addition, since 2018 there have been three major international publications specifically dedicated to this field.

In contrast to the previous section which dealt with adult education in general, this section now focuses on the target group of facilitators and on facilitators in mathematics education specifically. A general adult education framework is insufficient as a competency framework for the qualification of facilitators in mathematics education, as content-specific concretizations must be made. For example, in the meta-study by Timperley and colleagues, they note that, when it comes to the school sector,

> Experts need more than knowledge of the content of changes in teaching practice that might make a difference to students; they also need to know how to make the content meaningful to teachers and manageable within the context of teaching practice. We are calling these skills provider pedagogical content knowledge. (Timperley et al., 2007, p. xxix)

This makes it clear that, above all, domain-specific knowledge is also highly relevant to the content in which one is acting.

Here, we take a more concrete approach and focus on teachers and facilitators in mathematics. By adopting this specific focus, more concrete facets of interest can be identified, which can then be elaborated as individual categories—always in comparison to an underlying general framework of adult education. Consequently, a concrete examination of the subject-specific challenges involved in the discipline of mathematics must take place. Teacher beliefs sometimes play an important role in mathematics education because they guide actions determining how the subject is taught (Kunter et al., 2013). For instance, mathematics as a discipline may be conceptualized in a more receptive way, with a focus on algorithms and automation. Alternatively, mathematical thinking and problem-solving may be the focus (e.g., Rott, 2020). The content of qualification programs for facilitators would differ according to these perspectives.

At the same time, it must be kept in mind that such a framework does not apply equally to every facilitator. The many designations used for people who carry out this role not only testify to the heterogeneity of their tasks but also make it clear that very different roles and activities are linked to the diverse requirements and competencies. Facilitators also play decisive roles in PD in terms of the extent to which teachers are motivated and supported in their learning (Linder, 2011). To the areas mentioned by Smith (2005, see Sect. 3.1), Zaslavsky (2008) adds further requirements for facilitators in mathematics education such as adaptability and conscious selection of methods and media. It should be noted that facilitators are often expected to be very good (or even the best) teachers (as Smith notes in his list above). However, the role of a facilitator can be compared to that of a soccer coach, in that someone who may not be (or have been) the best player or teacher may be able to successfully train others. Carroll and Mumme (2007) also suggested that facilitators should have detailed subject content knowledge, information about the participating teachers as well as the students of those teachers, knowledge of how to teach students and adults, and knowledge of how to use materials for training to create a productive learning environment.

Various studies have set different priorities for the content knowledge that facilitators should have. However, all studies emphasize that facilitators' knowledge must exceed the teachers' knowledge to enable the former to encourage the latter to grow and acquire new knowledge, i.e., fulfilling a similar role that teachers must play for their students (Borko et al., 2014). This goes hand in hand with the 3TM already described (Fig. 3, Prediger et al., 2019).

Although, according to these three levels, facilitators should have extended knowledge compared to teachers, it should be emphasized that there are also knowledge elements that are relevant for teachers but not for facilitators (Beswick & Chapman,
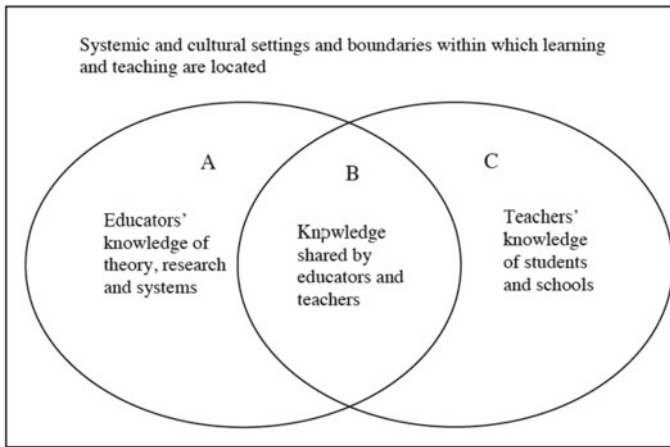
Fig. 5 Knowledge in teacher education (Jaworski, 2008, p. 336)

2015). These include, for example, detailed knowledge of school curricula or background knowledge about individual students. For facilitators, only general knowledge of educational standards and curricula is important, as well as relevant empirical findings from (current) research. This is also expressed in Jaworski's framework of knowledge in teacher education (2008, Fig. 5).

The extended knowledge of facilitators refers not only to new knowledge of mathematical content and the relevant pedagogical aspects aimed at PD-level, but also to their pedagogical knowledge of adult education. This includes, for example, knowledge of teachers' existing practices (Even, 2005; Even et al., 2003) and current views on PD programs in mathematics education (Borko et al., 2014).

In addition to current views on PD programs, Borko and colleagues (2011) also highlighted knowledge about mentoring (i.e., the accompanying support in the implementation of training content) for PD in mathematics. Facilitators should be able to stimulate productive mathematical work in teachers and lead discussions about student reasoning and instructional practices while encouraging reflection, as well as build professional learning communities. In this context, mentoring is seen as a special form of individual support and, unlike coaching, the individual's interests are seen as the absolute priority. Particularly in the second key aspect of mentoring discussed by Borko and colleagues (2011), leading discussions about student reasoning and instructional practices as well as effective use of video-based PD programs can contribute as forms of facilitation (e.g., Ebers, 2020; van Es & Sherin, 2010; van Es et al., 2014; Zhang et al., 2011). In this regard, communication about video cases is an important component in training teachers' awareness and ability to analyze. Content should be purposefully related to the teaching and learning of mathematics, ideally contributing to more reflective classroom practice.

Certainly, several of the interdisciplinary competencies mentioned earlier can be directly applied to the roles of facilitators for mathematics teachers. However, it is

also apparent from the formulation of individual areas of competencies that a domain-specific focus is needed. As Tzur (2001), in describing in his own development as a mathematics facilitator, states, "… a development from a lower to a higher level is not a simple extension, that is, doing more and better of the same thing. On the contrary, development entails a conceptual leap that results from making one's and others' activities and ways of thinking at a lower level the explicit focus of reflection" (Tzur, 2001, p. 275).

As the following example shows, it is not enough to swap out content, rather the entire spectrum of competencies mentioned is needed because they are all systemically interrelated. In one teacher training program on mathematical modeling, the focus is the student task "There is a 3 km traffic jam on the motorway. How many vehicles are caught in this traffic jam?" (see Peter-Koop, 2005, p. 6). It quickly becomes clear that there is no standard procedure or clear solution to this problem. Some of the participating teachers loudly reject the task as irrelevant to mathematics teaching since the task is not in line with their idea of mathematical thinking. The teacher training situation is now challenging for the facilitator in several respects. First, the subject matter "modeling" needs to be taught—meaning that a subject-specific PD program is essential (e.g., Bardy et al., 2021; Dreher et al., 2018). At the same time, it is also important to question participants' skeptical attitude towards the task and to address their basic beliefs about teaching and learning mathematics. At this point, competencies are required of the facilitator that go beyond pure content or pedagogical content knowledge.

The example shows the need for facilitators to have a wide range of competencies: subject-specific competencies at the classroom level and subject-specific competencies regarding mathematics teachers' particular needs, problems, and obstacles as well as social competencies as adult educators. All these must be deployed as needed in a fluid interplay of PD activities. As Koster and colleagues (2005) concluded, references to PD activities should be made in addition to a competency framework. In this article, a competency framework is understood as part of a more comprehensive concept of the professional profile, which is additionally linked to activities. These activities define the purpose of the competencies (see Koster et al., 2005). In other words, a competency framework is a working repertoire of expertise that provides orientation and enables someone to perform professionally. As the previous PD activity example showed, facilitators need different competencies for different PD activities on specific PD topics.

In the field of mathematics education, plenty of studies exist which describe the differentiation of facilitators' knowledge towards teachers in mathematical education (e.g., Ball et al., 2008; Beswick & Chapman, 2015; Beswick & Goos, 2018; Borko et al., 2014; Lesseig et al., 2016; Smith, 2005). However, a systematic description of a framework is still missing. This is essential to adequately support and, potentially, qualify facilitators. As early as 1999, Even stressed the importance of holding frequent planning meetings with facilitators learning a new mathematics PD program to develop their knowledge and leadership skills and to create a professional reference group. She described such meetings as the cornerstone for the "development of a common vision and feeling of shared ownership" (Even, 1999, p. 20).

Reviewing the relevant literature on the generation of competency frameworks in adult education enabled us to identify the relevant categories that underpin these frameworks. Furthermore, we were able to deduce the important stakeholders for this process. With these findings, we laid the foundations for implementing a Delphi study, which involved researchers in mathematics education and key stakeholders (senior administrators). Through cycles of design, evaluation, and redesign, the framework was evaluated for holism, integrity, and practicality. It was important to strike the right balance between general adult education and mathematics education. We briefly report on this process in the following section.

3.4 Facilitators' Competencies Framework

Here, we give a brief insight into the study undertaken and, above all, present the result of the Delphi study. Based on our results, we will once again take up the findings from the literature review. A detailed version of the Delphi study can be found in the PME paper by Peters-Dasdemir and colleagues (2021). As with other reported studies, it was important for us to identify a competency framework designed to fit the present setting, local requirements, and stakeholder acceptance to qualify facilitators in Germany appropriately. For this purpose, we also included the literature review shown above and embedded it in the Delphi study.

The process consisted of three consecutive rounds in which 61 experts with different professional backgrounds participated. In these discourses, the most important stakeholders were invited to evaluate the framework regarding its practical applicability. We involved 33 researchers, 28 stakeholders, and several teachers with experience in CPD. We completed three cycles of further development. All researchers involved were experts in the field of CPD in mathematics education for primary and secondary levels and were asked to use this expertise to point out key competencies for facilitators. The selection of the people involved in the Delphi study was carried out along the 3TM so that all levels involved in the facilitator activities were included. This was in line with the basic idea of a Delphi study which should include all experts with different backgrounds. The results of the Delphi study showed that experts from different fields were able to develop a common understanding of the competencies necessary for the qualification of facilitators who are responsible for the CPD of mathematics teachers. It leads to the DZLM framework covering four areas, which are concretized from the perspective of mathematics education (see Fig. 6): (1) Professional Values and Beliefs, (2) Professional Self-Monitoring, (3) Competencies at the Professional Development Level, and (4) Competencies at the Classroom Level.

Like the GRETA framework, we have chosen to structure these aspects and the related competencies in a circle format to symbolize the dynamic fluidity and interconnection between all competency domains in the inner ring. Besides the *Competencies on the Classroom Level* with PK-C, CK-C, and PCK-C, there is an extra level for *Competencies on the Professional Development Level.* All surveys focused on

Fig. 6 Competency framework of facilitators in mathematics education

the question of the differences and similarities that exist between the competencies of teachers at the classroom level and those of facilitators at the PD level. This is similar to the work of Borko and colleagues (2014), who based their framework on the work of Ball and colleagues (2008) on "Mathematical Knowledge for Teaching (MKT)" analogously for facilitators. There was an intense debate about whether these two aspects should be separated as equal parts or whether competencies at classroom level are an integral part of competencies at PD level. This led to the specification in the two areas on competencies at classroom and PD levels. The final agreement was that content knowledge at PD level (CK-PD) would cover all aspects of teachers' knowledge. This is in line with Beswick and Chapman's (2015) and Jaworski's (2008) similar views but in our case was expanded to include knowledge domains. Looking at the established specifications CK, PK and PCK at the PD level, PK-PD and PCK-PD for facilitators need to be further specified (Wilhelm et al., 2019). Both consider the specific focus on teachers as learners, either from a general adult education perspective (PK-PD) or in a subject-related way (PCK-PD) (Prediger,

2019). PCK-PD encompasses all "skills to engage teachers in focused activities and conversations about these mathematical concepts and relationships and to help them gain a better understanding of how students are likely to approach related tasks" (Jacobs et al., 2017, p. 3). This also includes, for example, the possible learning hurdles when teaching mathematics (Rösken-Winter et al., 2015). Furthermore, the PD and qualification programs developed and implemented within the framework of the DZLM were also examined as examples. A subject-specific view was appropriate for working out specific requirements and then classifying them into a larger framework. As with Borko and colleagues (2014), a PD excerpt for problem-solving in mathematics was chosen to gain insight into the facilitators' concrete tasks. As a result, both perspectives ("tasks" and "activities and competencies") are set in relation to each other.

To cooperate efficiently with higher authorities such as ministries or learning communities, facilitators require competencies similar to the coaching knowledge required of teachers communicating with laypersons. As a result, the clear structure of the framework with four key competency areas (*Competencies on the Classroom Level, Competencies on the PD Level, Professional Values and Beliefs, and Professional Self-Monitoring*) needed to be changed to become five by adding *Professional Social Competencies.* Another reason was that "communication and cooperation" must be considered at all levels and is a competency relevant to all actors (teachers, facilitators, stakeholders).

An intermediate result of the evaluation of an online questionnaire with a response rate of 34% was that the domains of *Communication and Cooperation* as well as *Coaching and Counseling* are often only perceived as a level between facilitators and teachers. This is due to the representation of *Competencies on the PD Level.* Therefore, a small change was made here so that both aspects were placed in the new area of *Professional Social Competencies.* Thus, as in the 3TM, the relationship of these competency aspects to facilitators, to teachers, and among facilitators themselves should be better emphasized. Essentially, no significant discrepancies occurred here. A comparison with the literature on general adult education reveals that the competencies of facilitators of mathematics teachers can also be divided into the four essential areas of social and communicative competencies, personal competencies or self-competencies, values and beliefs, and field competencies. In terms of different levels, however, content knowledge and (content) pedagogical knowledge are subdivided into PD and classroom levels.

The aspects of *Professional Values and Beliefs* and *Professional Self-Monitoring* were essentially retained but required partial restructuring due to specific characteristics of facilitators of mathematics. *Respect of Professional Values and Beliefs* can be taken as an example: it may be useful to consider beliefs about teaching and learning mathematics (Grigutsch et al., 1998) in all PD courses. However, if a facilitator is mainly concerned with the subject of language sensitization, then their beliefs on language sensitization would be important or, in the case of the subject of digitization, beliefs on use of digital media would matter. In the same course, changes in the focus on self-efficacy beliefs or in the knowledge of frequent teacher problems occur when delivering different PD content, and it is therefore not possible to present

competency facets in great detail. The areas of *Professional Values and Beliefs* and *Professional Self-Monitoring* have a strong interplay. Above all, the dual role and one's own understanding of one's role as a facilitator has an increased influence. Thus, *Role Identity* was included alongside *Professional Beliefs* and *Professional Ethics*. The concept of motivational orientation was replaced by *Self-efficacy Beliefs*, as this aspect of PD is more relevant to mathematics (Bandura, 1999; Thurm & Barzel, 2020). At times, the question arose in the discussions as to whether one's own experience should be included as a competency domain. For the stakeholders, it was important to include *Professional Experience* to explicitly promote appreciation of teaching practice. In addition to formal learning pathways, it is often the informal paths (practice experiences) that strengthen facilitators' competencies (Zaslavsky & Leikin, 2004).

4 Outlook

Teacher PD plays an important role in the continuous development of mathematics teaching and learning. Medley (1987) articulates this in the context of teachers' pre-existing characteristics (Type F) and their required competencies (Type E). In this chapter, we focused on facilitators as core actors and on professionalizing mathematics teachers (Type J). One can also note that while Medley ends with instructors as teachers, it is clear to see that the chain of effects extends beyond this point. To strengthen teacher competencies, we need to start one level higher (e.g., Lipowsky, 2014; Prediger et al., 2019). As Medley states, teacher competencies need to be strengthened, and by extension, research on the competency development of facilitators needs to be undertaken in the field of teacher education. This is necessary to achieve real improvement in the quality of PD programs and their implementation and thus for Type J to positively impact on Type E. The framework developed within this study is designed to be used for this purpose.

There are many competency frameworks for adult educators or facilitators generally, but they are not always usable in specific cases, either because they lack focus or because they are too non-specific. In such cases what is needed is adaptation; for our purposes, a framework that is tailored to the teaching profession and which focuses on the didactic perspective of mathematics. But what is new about this competency framework compared to existing frameworks as reported in the overview by Rossman and Bunning (1978), Wahlgren (2016) or the framework by Bernhardsson and Lattke (2011)? The challenge was to use these rather general frameworks as a starting point to develop a specific competency framework for facilitators in mathematics education, and even more specifically to the context of the DZLM, and to develop such a framework in cooperation with key stakeholders in school administration to ensure a systemic strategy.

First, the professional field of teaching must be considered as a specific feature here. The general competency frameworks that have emerged from adult education are too broad in their orientation for this and do not emphasize aspects relevant to the

teaching profession or inadequately differentiate individual competency facets (see GRETA framework; Wahlgren, 2016). However, this differentiation is necessary to address the relevant domains.

A further specification that had to be considered here was the discipline of mathematics because even within the teaching profession, there are significant differences across subjects. In addition to the differences between scientific disciplines such as mathematics and physics, professional cultures in the field (teachers, engineers, etc.) also vary considerably. Therefore, specific areas have emerged, which also account for attitudes towards the subject.

Lattke and Zhu (2010) drew attention to another reason for developing a specific framework for mathematics facilitators: cultural context is key. Cross-cultural studies show that cultural norms significantly influence views of what constitutes "good" mathematics teaching (Dreher et al., 2021). Of course, a different focus can be applied in terms of cultural context (whether this is regional, national, or global). However, in a field where close cooperation with local authorities and schools is required, local challenges must be considered. Close dialogue with stakeholders in the local system should therefore largely determine which components are included in such a framework. Local, cultural anchoring is always present when such a framework is developed with stakeholders on the ground—and the resulting competency framework would in all probability look different if it were to be developed elsewhere.

The continuous changes taking place in the school system require teachers to act dynamically and to respond in a differentiated way to changing needs. Facilitators, through the relationship between the field of practice and the teacher education system, can build an important bridge here by considering social changes and responding to them within the educational system and, accordingly, by responding to these innovations in PD programs. To do so they need to be competent and perform their role properly. As we have seen, facilitators in mathematics education have professional status and must therefore have a wide range of competencies.

The developed competency framework provides a research-based systematic overview for research and development of what should be kept in mind, especially the affective and self-regulatory competencies, in addition to the central competencies PK-PD, PCK-PD, and CK-PD. For communication at educational administration level, the competency framework is useful to sensitize facilitators to the fact that performing their role effectively requires not only being well versed in CK, but a multitude of other facets as well. Alternatively, the competency framework can be helpful for quality assurance at the level of educational administration.

The need for the whole range of competencies in the frame of PD programs became obvious when (for example) we looked at integrating technology in mathematics classrooms (Barzel & Biehler, 2020; Thurm et al., accepted), possibly due to the fact that PD aims relating to technology are manifold. Teachers must familiarize themselves with the latest technology and consequently rethink their tasks, teaching routines, and practices. All these aspects also touch on underlying beliefs about mathematics and teaching mathematics with technology (Clark-Wilson & Hoyles, 2019; Thurm & Barzel, 2020).

Covering all these areas requires facilitators to have not only mathematical competencies at classroom level but also competencies at the PD level. For example, regarding attitudes and beliefs, awareness of the latest research on teachers' attitudes and beliefs about teaching with technology is relevant to ensure facilitators are adequately prepared to address teachers' diverse knowledge and beliefs on the matter in a PD program. Especially for teachers with a more traditional, instructor-centered teaching approach, teaching with technology often means a greater challenge and loss of control than for those teachers who are more used to managing more open-ended approaches (Simonsen & Dick, 1997; Zbiek & Hollebrands, 2008, p. 291). Therefore, research results highlighting the importance of fostering self-efficacy to be able to teach mathematics with technology are not at all surprising (Thurm, 2020). Specific activities such as classroom trials are suitable as the basis for reflection-in-action (Schön, 1983), a promising method at PD level for the implementation of new teaching routines and innovations (cf. Arsal, 2014; Hattie, 2009; Lipowsky & Rzejak, 2015; Thurm, 2020). This has been identified as an especially important element in PD concerning technology in mathematics (Thurm, 2020). Besides general design principles for effective PD (Barzel & Selter, 2015), all these aspects demand highly developed moderation skills (PK-PD) to deal with disruptive situations in PD, self-regulation to organize the different fields of requirements, and strong professional ethics to ensure they always consider teachers in their thinking. The fluidity of all these facets in the competency framework is essential for facilitators to achieve their aims when enabling teachers to integrate technology into their everyday classrooms.

The success of PD from the facilitator to the student outcomes could not be identified in various research studies and have not been the subject of investigation. What can be assumed, however, is that the levels of success of PD as shown in Fig. 1 can be extended upwards and can lead to a change in student outcomes if facilitators train teachers adequately.

The presented competency framework offers a starting point for making the competency level of facilitators measurable. In a further step, instruments need to be developed to measure competencies. The competency framework could be a good way to map and check which competencies are addressed in a qualification program. For example, there are already approaches to designing PD for mathematics education with technology to foster teacher and facilitator noticing (see Thurm et al., accepted), which could be researched in more detail regarding implementation, where the interplay of competencies could also be considered.

It has been shown that this field of research, "mathematics teacher training and experience" (Type J), is still relatively young area of mathematics didactics, and it can be assumed that research in this area will continue to sharpen. Already in the various mathematics conferences, as described above, there are more and more workshop groups discussing this topic. In different countries with differing structures, the need for quality standards for facilitators is still present and qualification programs must be specifically developed and implemented to ensure the nationwide success of PD programs. Our findings show that facilitators are a central factor in such success and that their own training must therefore not be neglected, as they play a crucial role in the PD process for mathematics teachers.

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Continuing Evolution of Research on Teaching and Learning: Exploring Emerging Methods for Unpacking Research on Teachers, Teaching, and Learning

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1 Introduction

In 1987, Medley offered an explanation of the "present state of the art of research in teaching" (p. 105). By doing so, he outlined the categories of variables that could be studied and provided strong guidance for high-quality research on these variables. His guidance suggested that research should seek to find out why teaching quality varies widely and to do that, one must have a conceptualization of what good teaching is, an instrument that is valid for distinguishing good teaching from poor teaching, and a plan for collecting accurate data and for analyzing that data. While aspects of research on the relationship between all of the variables that shape the complex act of teaching students in a formal learning environment have remained unchanged since 1987, much has changed in educational research including the emergence of new methodologies and the increasing presence of technology both as a tool for teaching and learning and as a tool for research, making research on the connections between and among the variables both richer and more flexible than Medley originally suggested.

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In this chapter, our goal is to provide some clear examples of the ways in which the field of mathematics education has been able to pursue the interactions put forth by Medley (1987) thanks to the emergence of new theories, research methods, and technologies. We discuss some of the critical ways in which the research landscape has changed since the original article was published. First, we look at methodologies that have become more widespread since Medley introduced his framework. Second, we explore some examples of research methods that offer new ways of thinking about research questions related to teaching and learning in formal environments. These methods are important because they have opened opportunities to look across the variables in ways that were unavailable to researchers until recently. We finish by considering the opportunities that have been created by technology. These have changed the ways we collect data and the data we collect. We close by briefly discussing how the changes discussed in this chapter specifically relate to presage-process–product research (Medley, 1987). Our goal in this chapter is not to present a comprehensive review of the literature, rather, we seek to highlight both where the field is now in terms of research methods and tools as well as to provide examples of the ways in which Medley's framework is being pursued in newer research. We have chosen to rely on Fig. 3 in Chap. 1 (Manizade et al., 2023) rather than the original Medley model for the purposes of our discussion, except where noted.

2 Changing Methodologies

Since Medley's framework was originally proposed, the field has seen the emergence of new research methodologies (i.e., scientific frameworks), methods (i.e., specific approaches), and tools (i.e., instruments) that allow innovative lenses with which to make sense of the multi-faceted enterprise that is teaching and learning. In this chapter, we offer brief overviews of just a few of the methods currently available for answering questions related to the presage-process–product model. These tools sometimes allow us ways to look at connections between more than two variable categories (e.g., cultural-historical activity theory) or allow us to conceive of research as a web of interconnected studies all serving to develop a larger theory (e.g., Design-Based Research). Below, we talk about the emergence of Qualitative and Mixed Methods methodologies as well as emerging psychometric models, then we introduce several methods that provide new ways of thinking about the interconnected nature of the variables for teaching and learning. These include teaching experiments, Design-Based Research, cultural-historical activity theory (CHAT), and quantitative ethnography.

3   Qualitative and Mixed Methods Research Methodologies

In 1987, qualitative research was not often used in education. While there were certainly some examples of qualitative research emerging (e.g., Erlwanger, 1973), those studies were not as widely accepted as quantitative studies. However, with the shifts in ontology and increased acceptance of new methods, the field of mathematics education research became more open to—and, indeed, primarily focused on—qualitative research. The reason is simple: quantitative research methodologies are particularly appropriate for a model of teaching and learning that relies on transfer of knowledge from the expert to the novice. As we moved toward theories of learning that were more grounded in constructivism, socio-cultural theories, and critical theories, new questions were being asked. As an array of new learning theories emerged, the definitions of teaching and learning became more diverse and even questions could be pursued. Rather than asking if teachers who took teaching methods courses in mathematics or science are more effective than those who did not, researchers began wondering in what ways particular backgrounds might shape learning experiences (Type E—teacher's competencies, knowledge and skills—and Type F—pre-existing teacher characteristics) and their interaction on Type A (student learning outcome) variables (e.g., Manizate et al., 2023; Medley, 1987)) and how teachers conceive of making content learnable for children (Type D variables interacting with Type B and A variables (e.g., Manizade et al., 2023; Medley, 1987)). Further, with the emergence of new theories of learning, the definitions of what constitutes learning, and, thus, how learning is measured, also changed. Consider, for example, Wenger's theory of situated cognition (Wenger, 1998). Within this theory, "learning" is defined as a change in participation, because learning is viewed as becoming a member of a community of practice. Thus, as one learns, one becomes a fully-integrated member of the community of practice. If participation is the goal, a written assessment of content knowledge is no longer an appropriate instrument for measuring learning and new approaches need to be developed. Thus, it is consistent with the rise of the cognitive, socio-cultural, and critical theories that qualitative research would become a critical tool for research.

Once qualitative methodologies were established as a norm within the field, it was natural for some researchers to use qualitative and quantitative methods together to better understand a phenomenon. Thus, mixed methods approaches have been used by some researchers to understand the interactions inherent in the learning environment. Grounded in pragmatism (Johnson et al., 2007), mixed methods research is a methodology that combines qualitative and quantitative methods to produce "defensible and usable research findings" (p. 129). For example, a researcher may conduct a survey (quantitative), then conduct interviews with a subset of participants (qualitative) to more deeply understand the findings of the survey (Creswell, 2014). For example, in one recent study (Starrett et al., 2021), researchers used surveys and interviews with teachers and students to understand how teacher's proximity to their school impacted their use of place-based education, thus using mixed

methods to connect Type I (external context) variables to Type D (pre- and post-active mathematics teacher activities), Type C (interactive teacher activities), and Type B (student learning activities). Another approach to mixed methods analysis is to use the various analyses to dig into specific aspects of the data. For example, an approach the first author of this chapter has used (e.g., Izsák et al., 2010; Orrill & Cohen, 2016), included mixture Rasch analysis of an assessment of teacher's math knowledge. Our goals was to identify specific mathematics tasks that teachers found difficult or with which different groups of teachers had different experience. From those data, we were able to identify specific items on which to focus in the qualitative analysis of the interviews. Using this approach, we were able to not only see how teachers performed on the assessment, but also to generate assertions about why they performed in these ways, thus providing us with additional information for designing effective instruction. These methods allowed us to more thoroughly understanding Type E variables.

For understanding the ways in which teachers, their experiences, and their actions intersect with student learning, access to a wide array of methodologies is crucial. While quantitative research is still used, studies using qualitative and mixed methods approaches are as accepted in most venues as quantitative research. The critical factor in high-quality research is not the methodologies and methods used, but rather the alignment of the methods and methodologies to the research questions.

4   Continuing to Develop Quantitatively: Emerging Psychometric Models

Methods for conducting quantitative research have also continued to develop since 1987. While quantitative research has remained theoretically grounded in positivism and still adheres to the methodological frameworks that were in place in the 1980s, quantitative research has benefitted tremendously from increased access to computers and the emergence of new models that has been possible because of computers. Now that nearly everyone has access to extremely powerful computing technology, quantitative analysis can be more robust and more accessible than ever. Particularly important for presage-process–product research are the myriad statistical and psychometric models that have emerged in the past few decades. In this section, we briefly introduce four such models that have played a role in our own research on teacher knowledge and student learning: Item Response Theory (IRT), mixture Rasch Models, Diagnostic Classification Models, and Topic Models. We have selected these four models because each provides researchers with different information about learning. Further, we included IRT because of its widespread use. Each of the four offers a way to better connect the variables highlighted by Medley. We do not, however, intend this as an exhaustive list.

Item Response Theory (Baker, 2001; Baker & Kim, 2004) is probably the most influential psychometric model in widespread use as it has largely replaced classic test

theory for scoring standardized tests. Rather than simply assigning a score indicating how many items are correct or incorrect, IRT provides various kinds of information. First, IRT provides a score that expresses a participant's performance in terms of the number of standard deviations above or below the mean of participant scores. The second piece of data provided by IRT is information regarding the difficulty of an item, where difficulty is reported as the probability that someone scoring at the identified difficulty level would have a 50% chance of answering the item correctly. IRT provides researchers with information that allows them to consider not just whether "learning" has occurred, but where there may be deficits in aspects of content knowledge as well as a relative weighting of participant's performances. Because of the information that it can return, IRT is currently a critical component in the development of learning trajectories (e.g., Clements et al., 2011; Confrey et al., 2017), which sit at the intersection of student learning (Type A), teacher planning (Type D), and teacher knowledge of students (Types G & H).

Building from IRT, mixture Rasch models (Izsák & Templin, 2016; Rost, 1990) look for latent trends in the patterns of responses among participants to determine whether all the participants should be placed along the same continuum or whether there are groups within the data that performed in ways different from others. This approach has been used to identify patterns of reasoning among teachers. These patterns highlight that performance on an assessment can be tied to patterns in teacher's reasoning about the content (e.g., Izsák et al., 2010; Orrill & Cohen, 2016). The data can also be used to capture a different kind of "learning". Rather than focusing only on whether participants have improved their scores on an assessment, researchers can also determine whether participants have changed latent classes. Such change would indicate a fundamental shift in the ways the participants are reasoning about the mathematics items on the assessment (Izsák et al., 2010). While mixture Rasch has primarily been used for in-depth consideration of Type E variables, we assert that it could be used as a lens for understanding the relationship between Type E and Types C and D (how teachers plan and implement instruction). It could also be readily used to look at connections between Type A and Type C and D variables if an assessment were given to students and correlated to observations of classroom practice.

Another emerging family of models is Diagnostic Classification Models (DCMs; Bradshaw et al., 2014; Izsák & Templin, 2016; Rupp et al., 2010). DCMs require an a priori defining of the specific attributes each item of an assessment measures (e.g., Tatsuoka et al., 2016). From that mapping, analysis is done on participant's performance, and results are reported as the probability that the participant has mastered each individual attribute. For example, in Bradshaw et al. (2014), the authors identified four attributes being measured by the assessment of fractions: referent unit understanding, partitioning and iterating, appropriateness, and multiplicative comparison. The attribute inventories that are returned in place of traditional test scores can provide insights into specific aspects of understanding demonstrated by a given sample, thus providing data that can shape the instructional experiences for participants. As with mixture Rasch models, DCMs provide an opportunity to connect teacher or student

understanding of mathematics (Type A or E) to the activity in the classroom (Type B or C).

One final emergent psychometric model is Topic Modeling (e.g., Blei, 2012). Topic Models allow a statistical analysis of qualitative data to show a change in patterns of language usage. Topic models rely on looking for particular words in natural text or natural speech to see their patterns of co-occurrence. From those patterns, groupings are created that separate participants. For example, in Kim et al. (2017), the researchers found three main themes in their analysis of student's responses on a science assessment. The first theme featured answers that included appropriate technical terms for middle grades science students (e.g., change, variable, dependent). The second theme was discipline-specific terms (e.g., energy, population, kinetic). The third theme focused on everyday language (e.g., put, stronger, big, think). Across four assessments, the participating middle school students shifted from using the everyday language topic to the other two topics (Linking Type B to Type C variables). These results suggested that students were learning about the discipline. Topic Modeling is particularly important for measuring learning through a socio-cultural lens.

New psychometric models do not fundamentally change the design and limitations of quantitative research. Thus, they have some limitations outlined by Medley in terms of looking at relationships between the variables. However, the new models allow measurement of different kinds of learning (e.g., change in participation or in natural language use, rather than acquisition of knowledge), and they open opportunities for mixed methods approaches such as those described in the discussion of mixed methods above. So, even within the realm of quantitative research, there are more tools available to support asking questions in new ways and looking at relationships through different theoretical lenses than was possible in 1987.

5   Emerging Research Approaches

In this section, we introduce four of the approaches to research that have changed the ways in which we can answer questions about teaching and learning in formal (and informal) contexts. As with the discussion above, we do not assert that these are the only approaches appropriate for research in the presage-process–product framework. Rather, these are tools that have been used, or are emerging in use, in mathematics education and the learning sciences to answer research questions related to the variables in Medley's framework and the connections between them. Certainly, there are many other approaches that could also be used for this purpose. Below, we discuss: Teacher Experiments, Design-Based Research, Cultural-Historical Activity Theory (CHAT), and Quantitative Ethnography. Teaching experiments are featured because of their prominence in mathematics education research over the past three decades, while the other approaches were selected because they offer robust and diverse pathways for making sense of the complexity of learning environments through their analytical lenses or through iterative implementation. For each, we describe what

it is, some benefits and limitations for the approach, and some examples of studies done with the approach.

6 Teaching Experiments

6.1 What is It?

Teaching experiments stem from Piaget's clinical interviews (Steffe & Thompson, 2000) and have roots in Russian education research (e.g., Davydov, 1975). Teaching experiments are fundamentally constructivist and have been used with a wide variety of constructivist perspectives, ranging from radical constructivism to social constructivism (Cobb, 2000). In a teaching experiment, the researcher serves in the role of a teacher and conducts a series of teaching episodes, usually working with a small group of students or one individual (Cobb & Steffe, 1983). The key goal is to develop a "living model of student's mathematics" (Steffe & Thompson, 2000, p. 284), testing and revising instructional activities designed to support student learning (McClain, 2002).

Teaching experiments go beyond the scope of a clinical interview by aiming to help the researcher understand the change and progress of a student's mind rather than just the current state of the mind. The teacher-researcher constructs a conjecture about student's mathematical knowledge, then tests the conjecture with teaching episodes designed to move the student's understanding forward, reformulating the conjecture after each episode (Cobb & Steffe, 1983; Steffe & Thompson, 2000). Initial hypotheses can be abandoned based on student's responses as it is vital that the teacher-researcher allows the student's contributions to guide the trajectory of the teaching episode.

Typically, a teaching experiment consists of the teacher-researcher, the student(s), and an observer. The role of the observer is to witness and document student's reaction and behavior. The teacher-researcher, constantly interacting with the students and instantaneously reacting to the students, may not be able to capture all relevant observations (Cobb, 2000). The teacher-researcher engages the students in instructional tasks or activities to observe and promote mathematical learning and reasoning by posing tasks and asking follow-up questions (Steffe & Thompson, 2000). The data that is collected is qualitative and meant to record the models of student's mathematical understanding. The teacher-researcher uses this data to revise conjectures as teaching episodes progress and to revise the activities in the episodes. Ultimately, a model of student thinking about the specific content or topic is generated by the researcher, with a variety of qualitative data to support the model (Cobb & Steffe, 1983). From the perspective of the framework of research on teaching mathematics adapted from Medley (Manizade et al., 2023), teacher experiments allow conjectures to be made about how student mathematics learning outcomes (Type A) are shaped by the interactive mathematics teacher activities (Type C) that are pre-active

activities (Type D) from considering student's engagement in previous mathematics learning activities (Type B). In a sense, while the outcome of the teaching experiment is a theory about student's learning, the experiment itself is an iteration of the relationships of Type A, B, C, and D variables conducted by a person with competency, knowledge, and skill (Type E) in the mathematics, in student learning, and in designing instructional interventions.

6.2 Benefits and Limitations

One clear benefit of teaching experiments is the insight they provide about how to support a student or small group of students to move forward in their understanding of specific concepts. A unique feature of teaching experiments is that the researcher is directly involved with the teaching. Therefore, the researcher should have teaching experience and the ability to interact and engage with students (Steffe & Thompson, 2000). The goal is to elicit and support thinking during these interactions; teacher-researchers should be cognizant of how their actions and language are perceived by students (Tallman & Weber, 2015).

Teaching experiments are powerful tools for understanding learning, however, they are also very challenging. One clear challenge of this method, particularly for inexperienced researchers, is that data collection and data analysis are simultaneous during the series of episodes (Tallman & Weber, 2015). Additionally, to demonstrate the evolution of the conjectures and model building, the researcher-teacher needs to maintain on-going documentation of the reasoning for decisions and the interpretation of student's thinking (McClain, 2002). Self-reflexivity becomes a key assumption, where the researcher-teacher acknowledges that he/she is an active participant of the student's constructions (Steffe & Thompson, 2000; Tallman & Weber, 2015). A common data analysis method after the series of teaching experiments is retrospective analysis, changing and revising the hypothesized model (Cobb & Steffe, 1983).

6.3 Examples

Many researchers used teaching experiments as exploratory tools, usually as part of larger projects. Simon and colleagues (2018) conducted a teaching experiment with a single student as part of the Measurement Approach to Rational Number (MARN) Project. Their goal was to understand how instruction could promote student's construction of the concept of multiplication with whole number and fractions, and to develop a hypothetical learning trajectory based on their analysis. The participant was a fifth-grade student, Kylie, that the research group had been working with for two years, conducting various clinical interviews and teaching experiments. One of the teaching experiments involved the use of a computer application called Java Bars as

the instructional tool. Simon served as the teacher-researcher and posed several multiplication tasks designed to explore Kylie's changing conceptions about the meaning of the multiplier. The research group initially used the concept of generalizing assimilation, as defined by Piaget (1952) as the theoretical base for their hypothetical learning trajectory. They hypothesized that the progression of their instructional tasks would stimulate changes in the assimilatory structure of the student. However, during the initial teaching episodes, the researchers were not seeing any evidence of conceptual change and modified the teaching tasks. Using retrospective analysis of the data collected, Simon and colleagues revised their hypothetical learning trajectory conjecture to rather be stimulated by reflective abstraction, also a construct defined by Piaget (1952). When reporting their findings, the authors included a detailed description of the progression and rationale of the learning trajectory. The research group went on to conduct more teaching experiments with the revised instructional tasks and hypothetical learning trajectory (Simon et al., 2018). This research is focused on the interaction of student mathematics learning activities and student mathematical outcomes (Type B and Type A)—that is, how does an instructional intervention affect learning. Because it was a teaching experiment, though, it extended to Type C, interactive mathematics teacher activities, because one of the outcomes of this work was a hypothetical learning trajectory which could be used to guide other teachers for supporting student learning. Finally, consistent with the Framework of research on teaching mathematics as shown in Fig. 3 of Manizade and colleagues (Manizade et al., 2023), Type G research (individual student characteristics, abilities, and personal qualities) is also tacitly happening as the teacher-researcher is consistently assessing the student's abilities and understandings to make the instructional decisions that lead to particular learning activities.

Teaching experiments can also be done with larger groups of students. A study conducted with 299 undergraduate calculus students by Wagner and colleagues (2017) consisted of eight teaching episodes which were designed to study the change in student's ability to generate examples for the purpose of understanding novel concepts. The researchers formulated a hypothesized learning sequence and developed an instructional sequence of tasks and questions. The learning progression was broken down into intended student's awareness and behavior on specific skills and views. Over the course of eight teaching episodes, the teacher-researcher introduced the tasks, which progressed from more rigid to more open-ended to allow students to show their views and ability to generate examples. Data analysis was done using emergent codes from the participant's words and phrases from the reflections and written tasks. The evidence showed a progression of positive changes in the student's views of generating examples. The researchers revised their proposed learning sequence for the third iteration of the teaching experiment, where new students were chosen who had not yet learned calculus material, to test their revisions.

As with the Simon et al. (2018) example, this research approach follows a Type C-B-A flow, moving from considering interactive mathematics teacher activities to student mathematics learning activities to student mathematics learning outcomes. The researcher plays the role of a teacher, so data can be gathered on Type C, interactive mathematics teacher activities. Second, the research group designs a set of

activities based on data from Type G (individual students characteristics, abilities, and personal qualities) and Type A (student learning outcomes) to simulate student learning experiences that could occur in the classroom, allowing the study of Type B (student mathematics learning activities) variables. Finally, the researchers analyze data and make assertions on how the activities impact student learning. Both groups used iterative design to develop a learning theory, ultimately laying out a trajectory of activities for teacher to implement and student to experience (Type C and Type B). Simon et al. (2018) implicitly also study the individual student Kylie, providing details about her abilities and personal qualities (Type G). In summary, teaching experiments are an extension of clinical interviews and align to constructivist learning theories. The researcher becomes a teacher in this methodology where he/she formulates, tests, and revises a hypothesis of a model for a change in student thinking as a response to some instructional sequence (Cobb, 2000; Steffe & Thompson, 2000).

7 Design-Based Research

7.1 What is It?

Design-Based Research (DBR) approaches originated from a desire to pursue research questions that cannot be answered in a laboratory setting (e.g., Brown, 1992; Collins, 1992). Underlying the development of DBR was a desire to develop an approach that overcame the issues in attempting to apply results from education laboratory studies into actual classrooms (Cobb et al., 2003; McKenney & Reeves, 2013). Over time, Brown's and Collin's notions of "design experiments" matured into an approach known by many names, that we refer to as Design-Based Research (DBR; Design-Based Research Collective (DBRC), 2003). DBR focuses on the development and refinement of theory along with the development and refinement of innovations that embody that theory (e.g., Barab & Squire, 2004; Cobb et al., 2003; DBRC, 2003; McKenney & Reeves, 2013). DBR is an approach to research that relies on a trajectory of inter-connected studies conducted, often over several years, rather than a single study (Cobb et al., 2003). It is inherently grounded in partnerships between researchers and practitioners.

A unique feature of DBR is the dual goal of generating a theory and developing and refining a particular intervention that embodies that theory (McKenney & Reeves, 2013; Sandoval, 2014). Researchers focus on both problematizing the context and on using theories to generate usable knowledge (DBRC, 2003). The development of such theories is a key component of DBR (Cobb et al., 2003). Through the iterative processes, conjectures are made, tested in the natural setting, revised based on the outcome, and tested again. The theory becomes emergent through this process and is refined at the end of the project (Barab & Squire, 2004). Because of this focus on developing theory and innovation together, DBR projects tend to include serious consideration of student learning outcomes (Type A), student mathematics learning

activities (Type B), interactive mathematics teacher activities (Type C) and pre or post-active mathematics teacher activities (Type D). Many also include consideration of teacher competencies, knowledge, and skills (Type E); external context variables (Type I); and teacher development and experiences (Type J).

DBR projects usually span several years. The reason for this is twofold. First, DBR focuses on iterative in design (Cobb et al., 2003; DBRC, 2003), which involves multiple implementations, data collection and analysis cycles. Second, to understand an innovation and the theory it embodies in a way that moves toward generalizable knowledge, different grain sizes must be considered. Studies may focus on a single tool with a few students, then that tool used in a classroom, then that tool used in the context of the delivery of a piece of curriculum, etc. Because the studies focus on a series of related questions of different grain sizes, they often benefit from mixed methods approaches across the lifespan of the research (DBRC, 2003). Rather than having confined control variables, multiple dependent variables such as classroom environment and learning outcomes are examined to generate a deep understanding of the issues and the effect of the intervention (Cobb et al., 2003).

In DBR, researchers partner with various stakeholders to achieve the goals of refining theory and refining the intervention. Interventions can be educational products, policies, or programs (McKenney & Reeves, 2013). Examples of stakeholders are teachers, school leaders, coaches, and subject matter experts. These participants become an integral part of the development and implementation of the design, sharing their expertise to collaboratively work through the project (DBRC, 2003). Much of the work is conducted in a natural authentic setting, such as schools and classrooms; the context is problematized and studied as a vital part of understanding the learning and teaching that occurs (Barab & Squire, 2004).

The overall structure of DBR is flexible and iterative, but also systematic. It is a sequence of approaches rather than just one approach (Barab & Squire, 2004). Several models of approaches have been offered by researchers (e.g., Eljersbo et al., 2008; McKenney & Reeves, 2012; Reeves, 2011). Most of these models include the initial phase of exploring and analyzing a problem, followed by the construction of a design and then reflection and evaluation. Since the entire process is iterative and usually non-linear, most researchers using DBR work back and forth through those phases. Theories are developed and tested throughout the process until enough evidence and data is gathered for a mature theory and usable knowledge. Usable knowledge can be declarative knowledge, such as describing a certain phenomenon or prescriptive knowledge, such as ways to facilitate learning with a certain intervention (McKenney & Reeves, 2019). In the initial phase, researchers study a setting and develop testable conjectures about how to address an educational problem or how to influence a change in students learning (Cobb et al., 2003). Data analysis becomes an ongoing process as both researchers and practitioners aim to deepen their understanding of phenomenon that occur in the natural setting (Barab & Squire, 2004). This collective partnership and iterative design process can be seen as unique features of DBR whose purpose is to close the gap between educational research and classroom practice and to further theoretical knowledge that can influence change in settings facing similar problems (DBRC, 2003; McKenney & Reeves, 2019). The DBR

approach has been applied in various sectors of education such as learning sciences, curriculum development, instructional design and teacher professional development (See special issues of journals such as *Educational Researcher (*2003, 31(1)), *Journal of the Learning Science* (2004, 13(1)) and *Educational Psychologist* (2004, 39(4)).

7.2 Benefits and Limitations

One of the key benefits of DBR is its ability to inform the development of a usable intervention while also yielding a generalizable theory. This two-faceted benefit ensures that both the immediate outcome of the project (the intervention) has educational merit while also ensuring that there is something beyond a single application of the theory that can support teaching and learning. This ensures that the theory can continue to inform practice beyond the lifespan of the intervention.

Due to its multi-faceted design, DBR can be a challenging approach even for experienced researchers. The role of the researcher is less defined and more fluid; she can be the designer and the implementor, which can introduce threats to validity and objectivity (Barab & Squire, 2004). Furthermore, the researcher needs to anticipate and communicate means of support for the various groups of people involved in the project, who often can have different opinions and perspectives on educational issues (Cobb et al., 2003). Time and personal commitment are devoted to maintaining close and respectful relationship with partnerships (Cobb et al., 2003). Another source of difficulty arises from the various sources of data and the extended period of collection time (DBRC, 2003). Various techniques for data collection and analysis are often required along with an appropriate balance between rich data and a surplus of data to ensure validity (McKenney & Reeves, 2013) and often retrospective analysis is needed for theory development (Cobb et al., 2003). Despite these limitations and challenges, researchers have found DBR to be useful for a wide range of studies. A few such studies are highlighted in the next section.

7.3 Examples

Barab and colleagues (e.g., Barab et al., 2010) combined DBR and socially responsible design to create an intervention that would help students develop their identity both as individuals and members of their community along being educated to be knowledgeable citizen of the world. The project spanned over five years and included several iterative components, ultimately designing a video game that became known as Quest Atlantis (QA), with teachers, students, community members and web designers as part of the research partnership. Key to the DBR methodology, the research group developed a theory about transformational play: that video games can serve as effective mediums for deep and sustained learning by providing engagement not possible in the classroom (Barab et al., 2010). This theory has also been used

by other researchers (e.g., de Sousa et al., 2018) as a framework to their work. One project that grew from the theory developed by Barab and colleagues is the extension of the *Adventures of Jasper Woodbury* mathematics curriculum work that has been undertaken by Gresalfi and her colleagues (e.g., Gresalfi & Barnes, 2012). Like the development of Quest Atlantis and the development of the original *Adventure of Jasper Woodbury* series (e.g., CTGV 1992; 1994), Gresalfi has undertaken the new work related to the *Adventures of Jasper Woodbury* by engaging in a DBR approach that looks at the relationship of student mathematics learning activities (Type B) and student mathematics learning outcomes (Type A), but also incorporates variables of Types C (interactive mathematics teacher activities) and G (individual student characteristics, abilities, and personal qualities). Her group designed the Boone's Meador mission as an activity that provides insight into Type B variables (student mathematics learning activities). In Boone's Meadow, students are tasked with making calculations and decisions regarding how to reach a destination in order to save an endangered eagle. The game measures student learning outcomes based on student's responses and decisions made throughout the activity, shedding insight into Type A variables (student mathematics learning outcomes). The game includes feedback that is meant to reflect the actions of a teacher, thus including Type C variables (interactive mathematics teacher activities), as well as how that feedback affects the activity of the game (Type B—student learning activities) and student learning (Type A—student learning outcomes). Gresalfi and Barnes (2012) did two iterations of DBR to design and explore the effect of consequential feedback, which is feedback that is embedded in context and requires students to evaluate their mathematical reasoning based on the outcome of their decisions. The two rounds of implementations spanned across two years; data sources included videotapes of discussions and student work. Several rounds of data analysis were done using both a priori and emergent codes. The team saw an increase in the use of mathematical justification and critical engagement when feedback was embedded in context and given prior to the end of the game.

An extension of DBR that arose in the early 2010s is Design-Based Implementation research (DBIR), in which implementation becomes the vital focus of theory development and analysis (Penuel et al., 2011). DBIR often includes the combination of learning sciences research and policy research. One such example of DBIR is the work of Cobb and colleagues (e.g., Cobb et al., 2013), who partnered with four urban schools to improve the quality of mathematics instruction with an 8-year project titled Middle School Mathematics and the Instructional Setting of Teaching (MIST). The focus of improving mathematical instruction was broken down into increasing the learning of conceptual understanding, justifying solutions, and explicit connection between multiple representations. The researchers believed that a reorganization of teacher's instructional practice was necessary for these improvements to occur.

The research partnership consisted of school and district leaders, math coaches, teachers, and researchers. The iterative design process consisted of yearly cycles of data collection, analysis and feedback: they documented district's improvement strategies, collected and analyzed data to assess the implementation of the strategies, and recommended revisions of strategies for the following year. Additionally, a secondary level of focus was on gathering data to test and revise conjectures about

supports and accountability measures that the research group had generated from literature. Examples of data collection methods include audio-recorded interviews, district organizational schedules, evaluation forms, online surveys, classroom observations, and student achievement data. Cobb's team (Cobb et al., 2003) analyzed their recommendations for each district, looking patterns and similarities to find potential generality. After the third year of data collection, retrospective analysis was conducted to provide evidence for conjectures about major components of their emerging theory of action for instructional improvement in mathematics.

As part of the theory development, the researchers designed, tested and modified conjectures about instructional improvement, more specifically on methods of both supporting and holding teachers accountable for reorganization of practices. They developed an interpretative framework that captured four general supports that the districts used in the improvement strategies: new positions, learning events, organizational routines and tools. The research on the ways that mathematical instruction improved map to Medley's Type E variables, focusing on teacher competency, knowledge, and skills. The researchers also attended to the ways in which a change in teacher's instructional practice, including variable D, pre and post-active mathematics teacher activities, and variable C, interactive mathematics teacher activities, was tied to mathematics teacher's competence, knowledge, and skills (Type E). Further, the MIST research team was able to provide recommendations for the district on how to support teachers. The four recommendation areas focused on variables of Type I (external context variables) and J (mathematics teacher development and experience). Therefore, MIST was able to study the relationship of variables of Type E (teacher's competence, knowledge and skills), D (pre and post-active mathematics teacher activities), and C (interactive mathematics teacher activities) by surveying and observing teachers. Recommendations are also focused on variables Type I (external context variables) and J (mathematics teacher training and experience) as the research team partnered with the school leaders to influence those categories outside of teacher control. MIST has continued to work with schools as partnerships in implementing strategies to improve math instruction and teacher practices (see Cobb et al., 2018 for more detail).

8   Cultural Historical Activity Theory

8.1   What is It?

Cultural-Historical Activity Theory (CHAT) is a theoretical framework for conducting sociocultural research. CHAT supports the analysis of human interaction while considering how an individual or group of individuals and their interactions with the environment affect their activities (Kuutti, 1996; Cole & Engeström, 1993; Engeström, 1993). The basic idea of CHAT is that humans should not be separated from their participation in various activities; therefore, rather than focusing on

the individual as the unit of analysis, CHAT instead focuses on the activity in which people participate (Cole & Engeström, 1993; Engeström, 2004). Thus, the unit of analysis includes both the setting and individuals. The activity system refers to a collective concept—it is object-oriented, designed to think about the phenomenon in terms of the inner relations of activity and collaborative relationship between people (Roth, 2012). For the purposes of this chapter, we are focusing solely on the psychological aspects of CHAT and not on economic or materialist aspects. We assert that is most productive for this chapter's focus.

CHAT was initiated and developed by Russian theorists who saw behaviorism and analytical psychology lacking in its ability to describe cultural realities. The pedigree of CHAT can be traced back to dialectical materialism, and then to Lev Vygotsky who founded the first-generation of activity theory in the 1920s, centering it around his core idea: cultural mediation that is graphically expressed as a triangle with subject, object, and mediating artifact/tool comprising the vertices (Cole, 1998). The basic elements could be described as:

- Subject—The individual or subgroup involved in the activity.
- Object—The problem space or recipient of action to which the activity is directed to be molded or transformed in reaching the outcome that is sought.
- Mediating Artifacts/Tools—Internal mental signs and external physical objects that facilitate and support thinking processes and regulate interaction between the individual and the world. The artifact is "an aspect of the material world that has been modified over the history of its incorporation into goal-directed human action" (Cole, 1996, p.117)

Beyond the prevailing behaviorist theories about the stimulus–response association at that time, Vygotsky's mediation triangle, as a semiotic process between subject, mediating artifact, and the object of an activity, was a revolutionary way individual make meaning of the world (Cole, 1996; Cole & Engeström, 1993; Yamagata-Lynch, 2007).

The first-generation theory was critiqued, because the unit of analysis still focused on individuals. To overcome it, Alexei Leont'ev (1981), Vygotsky's colleague and disciple, along with his colleagues, created a second generation of CHAT, which took into account inter-relationships between the individual and the community, history, context, and interaction of the situation and activity. According to Leont'ev (1974): "activity is…a system possessing structure, inner transformations, conversations, and development" (p. 10). Thus, the consequences of events and activities that occur during the activity can qualitatively change the participants, the participant's participation purpose and motivation, the social environment of the activity, and the activity itself (Rogoff, 2008; Rozin, 2004; Yamagata-Lynch, 2010).

According to Engeström (2004), Leont'ev never graphically expanded Vygotsky's original model to illustrate a collective activity system. In addition, Leont'ev and his colleagues did not adequately address the methodological challenges for capturing, analyzing, and presenting activity-based data. To address these shortcomings,

Fig. 1 Expanded mediational triangle (EMT) (Engeström & Cole, 1993)
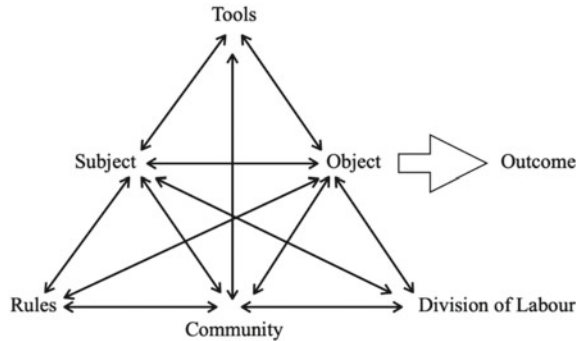
Engeström created the third generation of CHAT, offering a foundation for understanding and designing learning as a transformation of human activities and organizations. Engeström and his colleagues developed CHAT as an analytical framework by introducing a descriptive model of activity, which can be used in analyses of complex qualitative data. Compared to other sociocultural learning theories, Engeström's theory of expansive learning puts the primacy "on communities as learners, on transformation and creation of culture, on horizontal movement and hybridization, and on the formation of theoretical concepts" (Engeström, 2010, p.2). Cole and Engeström (1993) further detailed the representation of modeling human activity as a system form in the diagram of expanded mediational triangle, shown as Fig. 1. This is the triangle that typifies CHAT research.

The triangle provides an organizer to support researchers in mapping complex human interactions that take place in collective settings. The uppermost sub-triangle is identical to Vygotsky's basic structure of mediated action. In addition to the basic components of Subject, Tools and Object presented in the basic first-generation triangle, the expanded mediational model also includes the following three elements:

- Rules-norms, regulations, convention and gnorms, regulations, convention and guidelines that afford or constrain action and interaction within an activity system.
- Community-multiple individuals and subgroups involved in an activity.
- Division of Labor-distribution of work and responsibilities between members of the community.

The Rules, Community, and Division of Labor in the bottom portion of the triangle model add the sociohistorical collective nature of mediation that was not addressed by Vygotsky (Engeström, 1999a, 1999b). The outcome is the results or consequences that the subject finds once the activity is completed (Engeström, 1993, 1999a). Engeström (1999a) explained that the relationship between components of an activity system is two-way as people not only use instruments, but also renew them, they not only use rules, but also reformulate them.

The interactions among the components of the triangle model highlight tensions that are inherent in human activities; researchers find tensions in activity systems when elements from one or more components pull participants away from achieving

the purpose of the activity thus cause changes in activities, so tensions may either promote or hinder human activities. (Cole & Engeström, 1993; Engeström, 1993, 2004; Yamagata-Lynch, 2003). We argue that from the perspective of Medley's (1987) variables, CHAT is appropriate for looking at relationships between any subset of the variables, depending on the data to be collected. Because CHAT was developed to consider complex systems, it is particularly suited to the task Medley was conceptualizing in the development of the presage-product-process perspective (Medley, 1987).

8.2 Benefits and Limitations

The primary benefit of CHAT is its inherent ability to make sense of a complex system in a way that accounts for the actors and mediators at work in that system (Yamagata-Lynch, 2010). As exemplified in the examples below, CHAT provides a means for making sense of external context variables and the effects they have on instructional activities for teachers and students. This is important if the field wants to extend beyond Mendeley's (1987) original assertion that only two adjacent levels of variables can be considered at one time.

The limitations of CHAT are important considerations. First, it is not appropriate for considering human thinking, as it relies on observable activity (Yamagata-Lynch, 2010). This has implications for the kinds of growth that can be considered, how knowledge is characterized, and other elements of consideration that can only be observed by proxy. Further, the triangle model, while supporting sense-making about human activity systems, also oversimplifies those systems (Yamagata-Lynch, 2003, 2010). That is, complex human interactions are summarized to the point that they are "…not as rich and complex as real experiences" (Yamagata-Lynch, 2010, p. 33). Finally, CHAT is complicated to learn. This is because it requires the researcher to be proficient in qualitative methods, to understand and honor the complexity of collecting trustworthy data, and the ability to bring all of that together within a very specific theory (Yamagata-Lynch, 2010).

8.3 Examples

One of the challenges of STEM education is to integrate activities, content, and tools in a meaningful in-class activity. Using CHAT models as a basis for analyzing learning, teaching, and in-class interactions between different subjects calls for the transformation of authentic scientific/mathematics practices into classroom activity systems. Here we provide two examples that have relied on Engeström's theorizing of CHAT. We also invite the reader to look at the work of Schmittau, who used cultural-historical theory as put forward by Vygotsky to make sense of student's mathematical learning (e.g., Schmittau, 2004, 2005, 2011).

CHAT has become an important lens in mathematics education research because it has "power to deal with complexity in educational systems" (Jaworski & Potari 2009, p. 222). To build on early research with use of activity theory in mathematics teaching–learning, as well as with a focus on the classroom tasks and their related macro-social setting, Jaworski and Potari (2009) considered teaching as activity in their study in the 10th grade classroom of a UK secondary school where students in this grade group are considered "lower achievers". They used CHAT to consider the role of the social framework within which classroom teaching is situated. They had two primary goals. The first was to understand the relationship between teacher-student interactions and the ways in which cognition is evident in classroom dialogue. The second was to analyze the relationships between classroom interaction and cognition within the broader cultural context in which learning occurs. They employed triangles from EMT to characterize the "subject" to be any teacher or pupil learning in this setting, each with their goal or object for their activity.

Specifically, Jaworski and Potari (2009) analyzed teacher-student interactions through classroom dialogue, which they viewed as a micro-analysis. In an episode offered by the authors, the teacher, Sam, had planned a didactical inquiry including in-class activities and relevant resources. In the implementation of this plan, Sam met with some "tensions" (p. 229). For example, students who had not done their homework completely derailed Sam's lesson plan. Jaworski and Potari suggested explanations behind the homework issue. They talked about the task that teacher assigned to students: from a teacher's perspective, the activity seemed "logico-mathematical" and reasonable in "didactical communities"; however, for student peer and family communities, it is "strange and unreasonable".

The representation of the application of CHAT to allow analysis from both teacher's and student's perspective. For example, through their analysis, Jaworski and Potari (2009) determined: teacher's object could be "understanding of basic statistical terms and associated concepts," while pupil's object would be "classroom survival"; teacher's rules could be "teacher/student authority structures," while pupil's rules are "homework expectations within the school"; and the outcomes for teachers are "Non achievement of object due to pupils not taking the required responsibility, tension in the classroom", for pupils are "survival by ignoring terms of homework, contravening rules and contributing to classroom tension" (p. 231). By illustrating the descriptive power of CHAT for making sense of the observable activities in mathematics classrooms, the researchers framed teacher's mathematics teaching as inconsistent with their socio-cultural histories. Further, they found that the teaching did not match non-dominant student's learning. This highlighted, for the researchers, the lack of opportunities for mathematics teachers to challenge privilege-oriented activities. Without these opportunities, many well-intentioned mathematics teachers may unconsciously continue to perceive, explain, and respond to the classroom activities and specific learners through the dominant discourse system, which triggers the equality that they originally desired to abolish. (Jaworski & Potari, 2009).

When we place this study in Manizade et al.'s (2023) adaptation of Medley's (1987) framework, we can see the role of external context variables (Type I), where the teacher's preparation of instructional materials (e.g., homework designed for

students), school rules about homework, and student's parent's supportive attitudes toward homework are all factors that contribute to student's responses to homework. In Sam's example above, his preparation for class belongs to Type D (mathematics teacher's competencies, knowledge, and skills), including his requirement for students to do pre-work before class, and his design of in-class activities based on student's pre-work. The relationship of variables of Type I (external context variables) to Type D (pre- and post-active teacher activities) show up as tensions between *tool* and *community* in EMT framework. Then, in the analysis of the student's and teacher's perspectives, we can see interactions between variables of Type C (interactive teacher activities) and Type B (student learning activities), but unlike other research in which the influence is only considered in one direction (e.g., from Type C to Type B), CHAT allowed the researchers to understand the relationship in both directions—that is, the student's perspectives on the learning activities and the teacher's perspectives on the student's characteristics, abilities, and personal qualities (Type G) through the lens of their interaction with the learning activities. In summary, this example of mathematics teaching–learning interaction in the CHAT framework shows the power of this new method for addressing the interactions between and among the presage-process–product variables in Medley's framework.

In a separate study, Black et al. (2010) offered new insights into student's identity development by exploring an implicit mediation: they drew on Leont'ev's approach to gain an understanding of "self" related to mathematics. Mediation has a complex and abstract nature, studying an unintentional and less obvious object, like identity development or mental functioning, could be implicit. Driven by the interest in student's perception of themselves in relation to future aspirations, particularly their mathematical identity shifts, Black et al. (2010) conducted post-observation interviews with Mary and Lee (aged 16–17 years), two students studying advanced-subsidiary level (AS level) mathematics in England, to explore the relationship between learner's identity and mathematics. Black and colleagues (Black et al., 2010) adopted the methodological tool "leading activity" adopted from Leont'ev, which framed their understanding that activities become leading and can trigger a new activity when new motives are generated that surpass the original motive. In this work, the researchers found that satisfying mathematics-learning experiences implicitly mediated a "leading identity" that affected student's career choice, for example, in Mary's case, her identity also represented as her motive for studying mathematics is 'vocational (get a good job)'; however, in Lee's case, his focus on study as an activity is mediated by both his identity shifting and motive for attending a university. As such, Black et al. (2010) built on CHAT theories by presenting a relationship between self-identity and one's motive to engage in activity.

Considering the Black et al. paper (2010) from the perspective of Medley's framework, we can see the interaction of variables of Type G (individual student characteristics, abilities, and personal qualities) with Type A (student learning outcomes). In Mary's case, her engineering project experience as a leading activity significantly drove her to her future potentiality. Meanwhile, her self-awareness of the needs as "I like hands-on stuff" with some other positive aspects in her personality contribute to her motivation to become an engineer. In contrast, Lee did not value mathematics

as much as Mary did, so his purpose for studying mathematics and engagement with the subject was less meaningful than Mary's (Black et al., 2010). This analysis highlights how Type G variables (individual student characteristics, abilities, and personal qualities) may impact learning outcomes.

9 Quantitative Ethnography

9.1 What is It?

Quantitative ethnography (QE) is an emerging approach to research that attempts to bring quantitative and qualitative analysis of data into the same conceptual framework (Shaffer, 2018a, 2018b). That is, QE draws from the tools, perspectives, and approaches of both qualitative traditions and quantitative traditions to create a mixed methods approach that is philosophically consistent with both. This is a research approach that builds from the emergence of Big Data, which has allowed the collection of data that can be simultaneously as rich as traditional ethnographic data while being collected in quantities previously reserved for only the largest studies (Shaffer, 2017, 2018a; Wooldridge et al., 2018). While many approaches to working with big data have focused on statistical analysis, QE offers a different approach.

QE has been developed grounded in the belief that learning is an interpersonal activity. Learning is conceived of as making meaning of the world in a way that is consistent with how a particular group makes meaning of the world (Shaffer, 2018a). That is, learning is about induction into a community of practice (e.g., Lave & Wenger, 1991; Wenger, 1998) and, thus, it is about learning the Discourse of that community or culture. Discourse, in this case, refers to Gee's (2014) notions of "Big D" Discourse, which is any culture's way of being, including people's ways of talking, listening, interacting, believing, valuing, and feeling. In this case "small d" discourse becomes the observable behaviors through which researchers can gain insight into Discourse, as Discourse cannot be readily observed. "Small d" discourse is what people actually say and do. Thus, when we use QE to assess and understand learning, we are looking for the ways in which participants express their changes as they are inducted into a community of practice. This focus on induction as the outcome of learning makes QE particularly appropriate for considering student learning outcomes (Type A) in light of the instructional environment variables including student activities, interactive teacher activities, and pre and post-active teacher activities (Types B, C, and D) while considering many contextual variables, including Type H (internal context variables), Type I (external context variables) and Type J (mathematics teacher development and experience).

In quantitative ethnography, the research process is distinctly and necessarily mixed methods (Wooldridge et al., 2018). The data collected are rich in nature, just as they would be in traditional ethnography. They are collected using rigorous qualitative methods and may include traditional qualitative data such as observations

and interviews, or newer forms of data collection such as data collected by the computer as students work together in a virtual environment. Such data could include clickstream data as well as full transcripts of interactions. The analysis of the data is where we start to see the mixed methods nature of the approaches. For example, in epistemic network analysis (ENA; Shaffer et al., 2009, 2016; Shaffer & Ruiz, 2017; Shaffer, 2018b), the data are coded using frameworks from discourse analysis (Gee, 2014), which structures analysis by breaking data into segments that are typically a single utterance and joining those segments into logical chunks called stanzas. Then, ENA draws from traditional qualitative research, particularly grounded theory (e.g., Charmaz, 2014) to code data using approaches such as those used in grounded theory or inductive analysis (e.g., Maxwell, 2013) to create a coding scheme which is then applied to every segment. Once this is done, ENA draws from social network analysis (e.g., Robins, 2015) to mathematically create a visual display of the interactions between codes. The visual display (e.g., Fig. 2), shows the prevalence of single code (represented by a node) through the size of the node, and it shows the strength of the connections between nodes through line thickness. In this way, the visual shows those ideas (codes) that co-occurred in a single statement, which is a proxy measure for the codes having some kind of connection to each other for the person speaking. From this visual, additional statistical analysis, such as t-tests to determine whether particular groups are significantly different, or additional qualitative analysis, such as looking at all of the instances in the transcripts captured by particular node connections can be pursued.

As an example, we present two ENA maps from the first author's dataset in Fig. 2. This data was collected as part of a larger study focused on how middle school mathematics teachers understand proportional reasoning. The two teachers featured here (Autumn and Patricia—all names are pseudonyms) are representatives of the larger pool of 32 teachers. Each teacher responded to a number of items related to
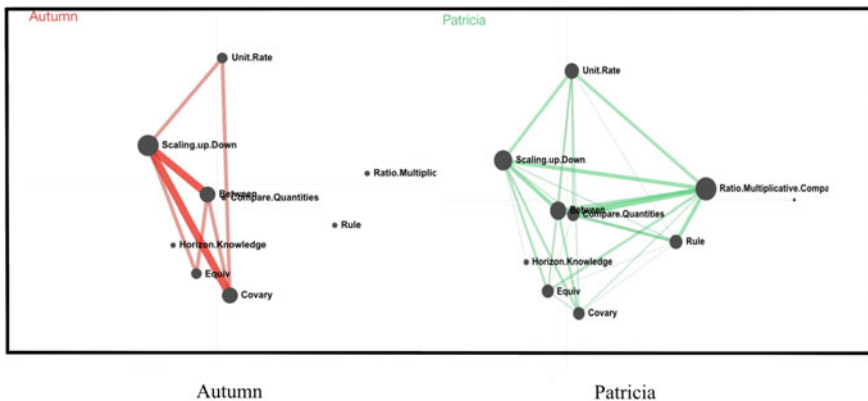

Fig. 2 Two teacher's ENA plots showing the knowledge resources they used for reasoning about proportional situations

proportions that were designed to help us understand how they reason about proportional situations. Part of the data was collected using a face-to-face clinical interview (Ginsburg, 1997) and part was collected using a think-aloud protocol mailed to the participants for which they used a Livescribe pen that captured their talking and their writing to create a record of their thinking. The coding scheme was developed using a grounded theory approach (Weiland et al., 2020). The figure shows the ENA mapping of our analysis of Autumn and Patricia's responses to the items. In these maps, each node shows a particular mathematical understanding that was included in their response, and the lines connecting the nodes show where they discussed those mathematical ideas in the same utterance. For example, in Autumn's map, we can see that Comparing Quantities, Scaling Up and Down, and Covary were the most commonly used knowledge resources because those nodes are largest. Further, we can see that she often talked about Covary and Scaling together and she talked about Scaling and Comparing Quantities together frequently. In contrast, we can see that Patricia relied more on Ratio as a Multiplicative Comparison and Scaling Up and Down. However, she did not demonstrate strong connections between the knowledge resources that were as frequent as Autumn demonstrated. By using ENA, we can see different patterns among teacher's data, which helps us understand what knowledge they access while solving problems and where there may be opportunities for professional learning. While this may appear to be only focused on variable Type E (mathematics teacher's competencies, knowledge, and skills), we would argue that it is also capitalizing on Type C (interactive mathematics teacher activities) and Type D (pre and post-active mathematics teacher activities) because we have found that situating conversations of teacher knowledge in the context of the decisions teachers make about students and instruction provides additional insights into the teacher's knowledge of the content as it relates to their teaching. Thus, doing this kind of research relies on the interaction between Types C, D, and E to understand teacher knowledge in context.

9.2   Benefits and Limitations

Quantitative ethnography is unique in its approach to using large amounts of data to create thick, rich accounts of the situation. From the perspective of Medley's (1987) framework, this method allows us to look explicitly at the interactions within a single element or to look at interactions across elements depending on the framing of the research questions. In fact, using epistemic frame theory (Shaffer, 2004, 2006, 2009, 2018b), which is one set of axioms that can form the basis for the framing of the study and the analysis of data, one would specifically consider how the elements of learning or teaching are situated within the culture of the classroom over time. Learning would only be conceived of as interpersonal, meaning that the interactions between teachers and students would be one site in which one would look for changes in the nature of discourse. QE allows the collection of large amounts of rich data that can be analyzed in ways that capitalize on both statistics and qualitative approaches.

Despite its origins as an assessment tool for learning in simulation environments, QE has evolved to be useful for other kinds of analyses, such as the analysis of teacher knowledge shown above. Thus, while it is grounded in a clear theory of learning, the methods can be used in other ways. This is consistent with other methods as well, including grounded theory.

Depending on the research question and focus for coding, there are tools that can help with the initial coding of data for QE. If the data being analyzed can be coded by the computer, that can save considerable time as coding a large body of data can be very slow. Studies that adhere more closely to the ideas of measuring discourse as a means for understanding Discourse, for example, could include coding of keywords and concepts that could be captured through computerized coding. In contrast, work like that done by the first author cannot benefit from computerized coding, because interviewees do not necessarily use consistent language to express certain understandings and because some keywords are used in a variety of ways ranging from ways that indicate strong understanding to ways that do not. Thus, for some research, QE can be very time intensive, while for other research it is less so.

9.3 Examples

Much of the initial work with ENA that has led to the development of QE was focused on learning games designed to help learners assimilate into the community of practice relevant to the game. For example, Nephrotex (e.g., Arastopoor et al., 2012) and RescuShell are two simulations that provide engineering students with virtual internships during their first year of an engineering program. In each, students are presented with a design problem that they work to solve. In one recent study of these environments (Chester et al., 2015), the researchers wanted to know what students learn from a course based entirely on working in these two simulations. They collected data from 50 students across the semester. Data collected included pre and post-surveys built into each simulation as well as all of the student's chats, emails, notebook entries, and work products entered into the systems throughout the semester. Data were analyzed using an engineering epistemic frame that had codes in the categories of knowledge, skills, identity, values, and epistemology. The researchers were able to analyze these data using ENA to measure student's development within the engineering epistemic framework. From the analysis, they learned that participating in two virtual internships was more effective than participating in just one. While participation in one simulation led to connection making between skills and knowledge, participation in a second simulation led to additional connections with knowledge of the client and epistemic aspects of engineering, which the authors assert are important aspects of thinking like an engineer. They also found that students were more satisfied with the course at the end of the second simulation than at the end of the first, though student satisfaction was predominantly positive for both. From the perspective of the framework of research on teaching mathematics adapted from Medley (Manizade et al., 2023), this is research focused on Type A variable

(student mathematics learning outcomes). But, it uses Type B (student learning activities) and Type E (teacher's competencies, knowledge, and skills) variables to explore the student learning. Specifically, the researchers used student's evidence from their activities (Type B) to determine the learning outcomes (Type A); and the measure of those learning outcomes was based on how similar the student's connection-making had become to the instructor's (Type E). Because moving students to think in ways that are consistent with the instructor is the explicit goal of these simulations, the planning (Type D—pre- and post-active mathematics teacher activities) and interactive mathematics teacher activities (Type C) are developed as explicit stepping stones connecting teacher knowledge to student knowledge.

While popular in the learning sciences, QE is only beginning to emerge in mathematics education. One example of a mathematics education implementation of QE is from pilot work completed by the first author and her colleagues (e.g., Burke et al., 2012; Orrill & Shaffer, 2012). That research focused on the knowledge resources in-service middle school mathematics teachers exhibited as they reasoned about a number of mathematics tasks. In this work, Knowledge in Pieces (e.g., diSessa, 2018) was used as a conceptual framework to drive the identification of fine-grained understandings being used by the teachers. The focus on this work was determining whether there are differences among the relative connectedness of the knowledge resources for the teachers. The hypothesis being that teachers who exhibit more connections between and among their knowledge resources may be better situated to engage with a wider range of student ideas. The work showed that there were unique patterns of knowledge resource used among the teachers and suggested that areas worthy of further research included consideration of teacher's classroom experience (e.g., the development of pedagogical content knowledge) and the relative strength of teacher's mathematics knowledge. As noted above, this line of research embeds teacher's competencies, knowledge, and skills (Type E) in the work that teachers do, which is interactive, pre-active, and post-active mathematics teacher activity (Type D and Type C) to understand how it impacts student's opportunities to learn.

10 Technology for Research

As we alluded to in the discussion of quantitative methods, technology has revolutionized aspects of the research enterprise. It has changed the kinds of data we can collect, which changes the kinds of questions we can ask. Suddenly, we can access new data through tracking devices (e.g., Lee et al., 2015), uncover thinking in news ways by collecting data using different tools (e.g., Hickman, 2015), and engage in mathematical thinking in different ways as technology allows us to interact in more tangible ways this those ideas (e.g., Hegedus & Roschelle, 2013). While a comprehensive review of the ways in which technology has shaped presage-process–product research is beyond the scope of this chapter, we offer three examples of the ways in which technology has fundamentally shaped the research that can be done. We first look at eye tracking, which allows the capture of data previously unavailable

to researchers, thus being appropriate for questions about the interactions between student learning outcomes and learning activities (Types A and B) with the interactive mathematics teacher activities (Type C). Then, we discuss the use of dynamic geometry software as one tool that is useful for better understanding how people reason about geometric situations as it allows the researcher and participant to move away from discussing a single example, to instead potentially focusing on an entire class of examples. This allows us to consider the interplay of variable Types A, B, C, and D, but even more, it allows us to ask different questions about student learning outcomes (Type A) and teacher's competency, knowledge, and skills (Type E) than we can ask without dynamic environments. We end with a discussion of 360° video, which opens opportunities for both teaching and researching teaching, and has supported researchers in adding in important ways to the literature on teacher noticing. As with CHAT, 360° video opens an array of possibilities for the researcher to examine all the variables acting together to create the learning environment.

11 Eye Tracking

11.1 What is It?

Eye-tracking technology has made it possible to track and record the eye movement of people looking at screens or paper, which provides data focused on what the person is attending to on the screen. Eye tracking was initially used primarily in reading research but has been gaining popularity in the field of mathematics, particularly being used to analyze multimedia learning processes. Multimedia learning can be referred to as creating mental models from resources that contain both verbal, both spoken and written, and pictorial representations, such as graphs, animations, or tables (Mayer, 2005). Eye trackers can either be attached to a computer monitor or to a head mount wore by the participant. In a recent review of 161 eye tracking studies in mathematics education research (Strohmaier et al., 2020), almost all of the studies used a computer monitor attachment. The data provided from eye trackers is usually in the form of coordinates, which are then categorized into groups of events using automated or manual algorithms (Strohmaier et al., 2020). The information gained from eye tracking can be used, for example, to improve the design of instructional material or answer questions about the differences between novice and experts. Eye tracking provides insight into the relationship between variables of Type B (student learning activities) and Type A (student learning outcomes). More specifically, it allows researchers to better understand which aspects of the screen (instructional activity) students focus on as they complete their work. Assertions can be made about the design of the activity and how it influences student learning. Implicitly, researchers can study individual student characteristics, abilities, and personal qualities (Type G) and internal context variables (Type H) such as patterns in where student attention is given.

11.2　Benefits and Limitations

Eye tracking has allowed researchers to gain insight into visual attention, which is often done too quickly and even subconsciously for participants to register and report on; researchers now have access to data that is not observable to people. This technology provides objective and numerical data that can be used both in qualitative and quantitative research; this unique information on what is being attended to, for how long, and in what order can be used in numerous ways to answer questions that were unable to be addresses previously (van Gog & Scheiter, 2010).

With any use of technology comes limitations. There can be data loss, particularly in very young or old participants; about 10% of data can be blinks and saccades, which provide no valuable information. Additionally, accuracy of the eye tracking device can be hindered with head-mounted devices (Strohmaier et al., 2020). It is noteworthy for researchers to be aware that eye tracking only reports data on what the participant is attending to; there is no data on that can give any explanation as to *why* the participant is looking at certain places. Therefore, researchers must rely on making inferences about any cognitive processes underlying the movement.

11.3　Examples

Eye tracking can be used to provide data for numerous research purposes. For example, it can be used to study how participants split their attention when presented with texts and diagrams. For example, Andra et al. (2015) investigated difference between how students look at formulas and graphs of linear equations, thus linking variables of Type A to Type C, with implicit attention to Type B. The review mentioned previously (Strohmaier et al., 2020) found that a majority of the mathematics studies covered the topic of numbers and arithmetic, studying, for example, how participants represent and process numbers, calculations, and equations. The topic of geometry, particularly shapes and form, was another common topic for researchers to investigate (See Strohmaier et al., 2020 and the special issue of *Learning and Instruction* (2010, 20(2) for mathematics examples).

12　Dynamic Geometry Software

12.1　What is It?

To enhance twenty-first century student's learning process and academic performance, a pioneering technology development, dynamic geometry software (DGS) has become a main feature that acknowledges the idea of 'interpretative flexibility' (Ruthven, 2018). By borrowing the idea of creating dynamic rather than static

graphics from contemporary drawing software, it is possible to drag the objects such as points, line segments, or circles of the graphics while retaining the defined properties, and the dynamic feature can be reflected in some "transformation" manipulation, such as translations, reflections, rotations, and dilations, with the help of mouse or tracker-ball on a laptop. DGS has been regularly used worldwide for teaching and learning geometry, with software like GeoGebra, Geometer's SketchPad, and Cabri Géomètre being common in many mathematics classrooms. More and more, it is being used to uncover understandings about mathematics concepts in ways that attend to transformations, thus allowing the researcher and participant to have something visual to discuss as they consider the mathematical ideas. DGS can be used to better understand student learning outcomes (Type A) as well as the interaction between Type B (student mathematics learning activities) and Type A variables. Research on teacher knowledge, such as that discussed below, can also focus on Type E (mathematics teacher's competencies, knowledge, and skills) and, if a researcher wanted to understand the ways in which DGS can be used to promote learning, a design that connects variable Types E (teacher competency, knowledge and skills), D (pre and post-active teacher activities), C (interactive teacher activities), B (student learning activities), and A (student learning outcomes) could be developed.

12.2 Benefits and Limitations

In Geometry class, DGS could support children's learning transition from "because it looks correct" or "because it works in these situations" to robust mathematical understanding of the geometric situation (Jones, 2000). To be specific, applying DGS can provide opportunities for students to find patterns in abstract geometrical graphics, so they can conceptualize mathematical ideas, such as invariance, or perceive mathematics rules, such as the relationship of the leg lengths in triangles, with less vagueness. For example, purposive manipulation like dragging along a circle can help make a defined property—the unchanging measure of an angle of circumference—comprehensible and convincing to students. Despite the benefit of visualization and facilitation, teachers hold different perspectives on the efficiency issue of students using software in class. In some situations, only teachers use software for in-class presenting because they concern that students would invest in-class time to get familiar with software operation (Ruthven, 2018).

As a research tool, DGS opens new ways to engage teachers and students in explaining how they understand geometric concepts. Rather than being limited to *describing* a phenomenon based on a drawing on paper, the interviewer and interviewee can engage in *showing* each other what they mean. This opens a pathway for richer understanding of participant's knowledge.

12.3　Examples

Martinovic and Manizade (2020) explored teacher's thinking through the use of DGS by examination of both the teacher's written work and GeoGebra sketches from 23 in-service secondary school teachers in the USA. How these teachers visualized and verified the trapezoid area formula conjectures in GeoGebra is quantitatively as well as qualitatively analyzed as "empirical proofs" (p. 3) of their strategies and connection to teaching. From the qualitative analysis, the authors identified four distinct strategies: eyeballing, measurement, constructions, and written statements, and they found that the teachers used a combination of these four strategies. They also found out teacher's "misconceptions" (p. 16) that were magnified in the process of using technology, and some operation failure of that some teachers may treat DGS as a paint software. Overall, this study contributes on the teacher's strategies of visualizing and verifying the trapezoid area formula conjectures, also widen the scope of potential research on teacher's knowledge in the context of geometry class. This study focused on Type E (teacher competencies, knowledge, and skills), with implications for improving and modifying Type J (mathematics teacher development and experiences).

Nagar (2019 ; Nagar et al., 2022) found four categories of invariance that teachers were able to identify by engaging them with a series of four DGE protocols. He found that teachers did not discuss invariance at all without prompting, but when prompted, they were able to use the DGE to highlight important aspects of the geometry. This was particularly interesting given the notoriously difficult task of uncovering invariance in other work (e.g., Laborde, 2005). Consistent with Nagar's speculations that this work will inform how we teach students to better understand geometry, we would consider it research focused on variable Type E (mathematics teacher competencies, knowledge, and skills) with implications for variable Types D (pre- and post-active teacher activities), C (interactive teacher activities), and B (student learning activities).

12.4　360° Video and Other Full-Room Video Capture

12.4.1　What is it?

The emergence of affordable video tools and better computer programs for controlling video has opened opportunities for research classrooms to be built that are designed for researchers to capture the entire experience of the classroom. These rooms can include multiple video cameras and multiple microphones. Sometimes, they include a control room from which a researcher can control the data collection. The goal of these rooms is to capture as much data as possible in real time.

At the same time as these teaching and research labs are emerging, new technologies are making it possible to collect full-room video in other ways, too. For example,

some researchers are using video cameras that are designed to capture 360° views rather than just the normal framing of video. Other researchers are using tools that allow you to control a tablet computer to follow a speaker and record that person as they teach or present. This can allow remote data collection as well as collection of data using a number of devices in a single setting. Similarly, some researchers have used wearable video cameras, such as GoPros, to see what each participant in a study can see (e.g., Sherin et al., 2008). Like CHAT, this kind of research is rich in the research opportunities it opens. We would argue that any variables, except Type F (pre-existing mathematics teacher characteristics) and Type I (external context variables) could be studied using this technology depending on the design of the study.

12.5 Benefits and Limitations

While the benefits of capturing classroom activity this way are myriad, it is not true that the data is unbiased. As with any data, there is always bias in video data because a human has made a set of decisions based on a set of criteria for data being collected in the space (e.g., Hall, 2000). However, these whole-room approaches to video capture allow something closer to unbiased capture of the experience to happen. While dedicated video suites remain relatively rare because they require dedicated space, the other options (e.g., 360° cameras, GoPros, etc.) are relatively inexpensive and easy to set up in a variety of settings. Clearly, capturing the volume of data made available through this application requires careful attention to research design to ensure that the studies resulting from high volumes of data are doable.

12.6 Examples

In one line of research, the 360° video technology is being used to create the research stimuli. Preservice teachers are asked to watch 360° videos of children learning mathematics as part of lessons on teacher noticing (e.g., Kosko et al., 2020; Zolfaghari et al., 2020). The research has focused both on what the preservice teachers notice in the 360° video versus traditional video views as well as how to promote preservice teacher's noticing of student's strategies. Findings in the Kosko et al. (2020) showed that preservice teachers who used the 360° videos were more successful in noticing both reform-oriented and content-specific aspects of the instruction than those who relied on traditional video views. This research considers the connection between Type B (student learning activities) and Type E (teacher's competencies, knowledge, and skills) variables by looking at them through the teacher interactive, pre-active, and post-active activities (Types D and C) that were implemented.

In another line of research related to video technologies is research focused on determining the most effective uses of the technology for a variety of teaching,

learning, and research purposes. For example, van der Kleij and colleagues (2019) undertook a study in Australia to explore the feasibility of using GoPro cameras with iPads to capture student–teacher interactions. Their findings showed that the two technologies used together can be useful and that teachers are able to engage in teacher noticing activities while viewing the videos created with these devices. This was similar to the findings of Sherin et al. a decade earlier (2008), though they only considered wearable cameras and not the addition of the iPad tablets. As with the research above, this study is considering students learning activities (Type B) by watching the planned instruction (Type D—pre-active teacher activities) as it is enacted (Type C—interactive teacher activities).

13   Presage-Process–Product Research in the 21st Century

While this chapter cannot possibly provide an exhaustive discussion of the ways in which research has evolved since the Medley (1987) framework was introduced, we have attempted to offer insights into changes that have shaped the ways in which we think about research, learning, and teaching as well to provide some examples of approaches to research that simply were not available in 1987. Despite the development in methods and tools, it still holds that one cannot study the interactions of the variables without considering the mediating factors (which, often, are other variables from Medley's framework). His assertion that we need to have a clear conception of good teaching, valid instruments, and appropriate data is still at the heart of good research. Perhaps more than in 1987, modern researchers recognize that teaching is multifaceted and there is no single definition of "good teaching"; thus the onus is on the researcher to define the construct and clearly convey the purpose of the research (e.g., Orrill & Cohen, 2016).

Looking back, we can see the emergence of new methods and theories that attend far more to contextual variables and rich details than those commonly in use in 1987. Rather than trying to find particular variables that explain learning or teaching, the field is now more concerned with context and complexity. The research methods and theories that are in use and emerging now reflect that shift. In part, technology is to be thanked for this change as it has made data collection and analysis much easier than it was in 1987.

The advantage of our current research landscape lies in the ways we can challenge Medley's (1987) assertion that "research designed to correlate nonadjacent points is not worth doing" (p. 111). With the tools and approaches we now have available, current researchers have opportunities to think about the relationships between Medley's variables in ways that are more robustly interconnected and less hierarchical. For example, QE and CHAT are both explicitly focused on finding the connections between and among elements within and between the variables. CHAT, particularly, is interested in how teaching, learning, and instruction interact with each other. Similarly, DBR is expressly focused on including the context of the research as part of the consideration of what happened.

At the same time, we challenge Medley's assertion about the necessity of looking only at adjacent variables, as we can now examine the relationships between and among variables in ways Medley could not have imagined. QE, for example, with its ability to draw on large datasets to yield thick, rich description can help us understand connections between the adjacent variables. Instead of being limited to understanding whether Type B student learning activities (Manizade et al., 2023) shape student's Type A learning outcomes, we can now look across large groups of students to find out how those activities shaped learning, how particular groups of students (Type G—individual student characteristics, abilities, and personal qualities and Type H—interactive context variables) interacted with those activities and what was learned, and the influence on individual and group learning outcomes. Teaching experiments offer one model for diving deeply into student outcomes and their relationship to student learning activities by focusing explicitly on student characteristics.

As with QE, DBR also allows us to conduct research on multiple pairs of adjacent variables simultaneously. The unique characteristic of partnering with a variable of professionals allows for each group to provide insight and perspective that can be used to study multiple variables. By partnering with teachers, each DBR project indirectly provides informal development experiences (Type J) that can influence teacher competency, knowledge, and skills (Type E). The initial phase of DBR projects includes exploring and understanding a project in natural context (McKenney & Reeves, 2013), which gives researchers the opportunity to examine internal context variables (Type H) through interviews and questionnaires to better design student mathematics learning activities (Type B).

The examples of DBR studies described in the previous section were able to study multiple pairs of variables. The goal of MIST was to improve interactive mathematics teacher activities (Type C). The research group studied external context variables of the support systems (Type I—external context variables) and teacher pre-active and post-active activities (Type D). They made recommendations for changes in the support system to influence these practices (Type C). Additionally, they observed interactive teacher activities (Type C) to understand how the changes in support impacted what happened in the classroom (student mathematics learning activities, Type B, and student learning outcomes, Type A). They spent several years on investigating variables of individual student characteristics (Type G) and internal context (Type H) variables. The knowledge of these variables directly affected the design of QA, a student learning activity (Type B). Then, the group collected data on the relationship between this activity and learning outcomes.

As shown throughout this chapter, there are explicit and implicit connections between the variables of interest to any research effort focused on the relationship between teaching and learning. When these connections are not explicitly attended to in the research design, the result is research that yields inconclusive or confounded findings. For example, large scale studies of professional development, in an effort to yield clear relationships between teacher development and experience (Type J) and student learning outcomes (Type A) rely on data of Type E (teacher competence, knowledge, and skills) and Type A (student learning outcomes) only without consideration of the steps in between that mediate the effectiveness of the PD (e.g., Wayne

et al., 2008, 2011). It is for this reason that conclusive findings from PD are often elusive (Yoon et al., 2007) or very broad, such as those offered by Garet et al. (2011). Attending to only measures of teacher knowledge and student knowledge can also lead to the appearance that professional development had no significant impact on student learning, when the actual relationship is more complicated than those data would suggest (e.g., Garet et al., 2011). This lack of attention to the "in-between" variables is understandable given the immense complexity of understanding not only whether PD impacted student learning (cf., Banilower et al., 2006, 2007). However, exploring the relationships in-between Type E (teacher's mathematics competencies knowledge and skills) and Type A (student learning outcomes) is critical for understanding how, when, and under what conditions teacher professional development can lead to better student learning. The kinds of methods and technologies discussed in this chapter open opportunities for thinking about these connections in new ways.

Our parting observation is that we believe the presage-process–product framework remains a relevant way to conceptualize research. In this chapter, we have attempted to highlight the ways in which the research field has changed over the three decades since Medley offered his framework. We assert that the evolution of research methodologies, research methods, and available technologies has fundamentally changed the landscape in ways that allow the inclusion of multiple variables, rather than limiting them only to adjacent relationships and has allowed more careful consideration of connections and relationships between and among the variables than was possible with quantitative methods and classic test theory.

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Appendix

Brief, Capsule Definitions of Terms and Documents for Chap. 6

- *Assessment, curriculum and evaluation, and professional standards for school mathematics:* A trilogy of documents that provided a vision for the organization of curriculum reform in the U.S. in the 1990s (National Council of Teachers of Mathematics [NCTM], 1989, 1991, 1995).
- *Behaviors and dispositions*: Identification of student experiences, such as, proficiencies, processes, practices, competencies, and habits of mind (Kobett & Karp, 2020, p. 40) that demonstrate how students develop and show evidence of their mathematical thinking.
- *Cognitive technological (CT) tools:* Consists of tools that support a "synergistic relationship" between technical and conceptual dimensions of mathematical activity in technological environments (Zbiek et al., 2007).
- *Competencies:* Frameworks for knowing and doing mathematics, such as, (1) Denmark's (2003) mathematical competencies that provided evidence of student's "mental or physical processes, activities, and behaviors" (p. 9); (2) Program for International Student Assessment [PISA] (PISA, 2021) assessed *mathematical competencies* as "an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts" (PISA, 2021); and (3) Identified in the New Zealand Curriculum (NZC), competencies "that describe what they [students] will come to know and do" (Ministry of Education, 2015, p. 37).
- *Conceptual understanding:* Student learning is defined as the "comprehension of mathematical concepts, operations, and relationships" (National Research Council [NRC], 2001, p. 116).
- *Direct instruction (DI)*: Traditional, instructional methods where students watch, listen, and take notes about problems that teachers provide procedures and work out for students to follow and use (Kapur, 2014).

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- *Learning goals:* Focus on student "understanding" where students build knowledge; "Explicitly state what students will understand about mathematics as a result of engaging in a particular lesson" (Smith & Sherin, 2019, p. 14).
- *Learning through activity* [LTA]: A research model that examines how learners actively engage in learning activities through a progressive coordination of mathematical concepts (Simon et al., 2016, 2018).
- *Mathematical sense-making:* Student engagement in processes, such as problem solving, to learn mathematics with understanding; one aspect of what it means to know and do mathematics.
- *National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO]:* Authors of the U.S. *Common Core State Standards for Mathematics (CCSSM),* 2010.
- *Organization for Economic Development (OECD) Definition and Selection Competencies (DeSeCo) Project:* Created a framework to guide the development of PISA assessments.
- *Performance goals:* Focus on the end result or product of student's engagement in learning mathematics: "What students will be able to do as a result in engaging in a lesson" (Smith & Sherin, 2019, p. 14).
- *Principles and standards for school mathematics*: Updated U.S. document that provides a vision for curriculum reform at the beginning of the twenty-first century (NCTM, 2000).
- *Problem-solving:* Defined as "the systematic study of what the process of formulating and solving problems entails and the ways to structure problem-solving approaches to learn mathematics" (Santos-Trigo, 2020, p. 687).
- *Process standards:* Five processes that define what mathematicians might do and say when engaged in doing mathematics: Problem solving, communication, representation, making connections, and reasoning and proof (NCTM, 2000).
- *Productive disposition:* An affective construct defined as learners having an "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p. 116).
- *Productive failure:* Student's initial problem-solving attempts are unsuccessful and became productive when supported with appropriate mathematics classroom instruction (Kapur, 2014).
- *Productive struggle:* A student learning behavior that promotes learners making sense of mathematics and is necessary to develop conceptual understanding (Hiebert & Grouws, 2007); "Intellectual effort students expend to make sense of mathematical concepts that are challenging but fall within the student's reasoning capabilities" (Dingman et al., 2019, p. 91)
- *Proficiencies:* Frameworks for student's engagement while learning mathematics, such as, (a) Cognitive and affective proficiencies for five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001); and (b) *Reasoning* as one of the four proficiency strands students engage in when "thinking and doing of mathematics" (Australia Curriculum and Assessment Reporting Authority [ACARA], 2017).

- *Prospective elementary school teachers (PTs) and AHA! Experience*: Students engage in problem solving and experience how "a problem has just been solved, or a new piece of mathematics has been found, and it has happened in a flash of insight, in a moment of illumination" (Liljedahl, 2005, p. 219).
- *Representational fluency:* Within or outside technological environments, "The ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (Zbiek et al., 2007, p. 1192).
- *Research for principles and standards for school mathematics:* Research literature that informed the U.S. vision of school mathematics in the 1990s and 2000 (NCTM, 2003).
- *Scheme:* A cycle of perturbation, action, and reflection in which an individual anticipates, acts and mentally prepares, and assesses the outcome of his or her actions (Hackenberg, 2010; Steffe, 1994; von Glaserfeld, 1989)
- *Standards for Mathematical practice (SMP):* Eight mathematical competencies identified as a national *Common Core State Standards for Mathematics (CCSSM)* in the U.S., 2010.
- *Student learning activities:* "In the classroom… All learning depends on the activity of the learner" (Medley, 1987, p. 105).
- *Student engagement:* Defined as "an interactive relationship students have with the subject matter, as manifested in the moment through expressions of behavior and experiences of emotion and cognitive activity, and is constructed through opportunities to do mathematics" (Jansen, 2020, p. 273).
- *Teaching for robust understanding [TRU] project:* Framework of five dimensions of classroom activity that supports professional development (PD) to engage teachers in creating a classroom student learning environment that facilitates the development of powerful thinkers (Schoenfeld & the TRU project, 2016).