**International Journal of Algebra, Vol. 7, 2013, no. 1, 17 - 24**
**HIKARI Ltd, www.m-hikari.com**

# Multiplication Graded Modules

**A. Khaksari and F. Rasti Jahromi**

Department of Mathematics,Payame Noor University
P.O. Box:19395-3697,Tehran, Iran
A_khaksari@pnu.ac.ir
Frzr282@gmail.com



**Abstract**

Let $G$ be a multiplicative group and $R$ be a $G$-graded commutative ring and $M$ a $G$-graded $R$-module. Various properties of multiplicative ideals in a graded ring are discussed and we extend this to graded modules over graded rings. We have also discussed the set of $P$-primary ideals and modules of $R$ when $P$ is a graded multiplication prime ideals and modules.

**Keywords**: graded rings, graded modules, graded multiplication Modules.

## 1. Introduction

Let $G$ be a group. A ring $R$ is called $G$-graded ring if there exist a family $\{R_g\}_{g\in G}$ of additive subgroup of $R$ such that $R = \underset{g\in G}{\oplus} R_g$ that $R_g R_h \subset R_{gh}$ for each $g,h \in G$. A $R$- module $M$ is called $R$-graded Module over $G$ if $M = \underset{g\in G}{\oplus} M_g$ and $R_g M_h \subset M_{gh}$ for all $g,h \in G$. Thus each $M_g$ submodule of $M$ is $R = R_g$ - module. An element of a graded ring $R$ is called homogeneous if it belongs to $\bigcup_{g\in G} M_g$. If an element $m \in M$ is belongs to $\bigcup_{g\in G} M_g$, then $m$ is called homogeneous element and the set of all homogeneous elements of $M$ is denoted by $H(M)$ (for a ring $R$ is denoted by $H(R)$). A graded submodule $N$ of a graded $R$ -Module $M$ ($R$ is a graded ring) is a submodule such that $N = \underset{g\in G}{\oplus}(M_g \cap N) = \underset{g\in G}{\oplus} N_g$. Equivalently, $N$ is graded in $M$ if and only if

$N$ has a homogeneous set of generators. If $R = \oplus_{g\in G} R_g$ and $R' = \oplus_{g\in G} R'_g$ are two graded ring, then the mapping $\phi : R \to R'$ whit $\phi(1_R) = 1_{R'}$ is called graded homomorphism if $\phi(R_g) \subset R'_g$, for all $g \in G$.

If $M = \oplus_{g\in G} M_g$ and $M' = \oplus_{g\in G} M'_g$ are two graded $R$-modules ($R$ is a graded ring), the mapping $\lambda : M \to M'$ is called graded homomorphism if $\lambda(M_g) \subset M'_{g'}$ for all $g \in G$. A graded ideal $P$ of a graded ring $R$ is called gr-prime if whenever $x, y \in H(R)$ with $xy \in P$ then $x \in P$ or $y \in P$. And a graded submodule of a graded module $M$ over a graded ring $R$ is called gr-prime submodule if $rx \in N$, for $r \in H(R)$ and $x \in H(M)$, then $x \in N$ or $rM \subset N$. A graded ideal $m$ of a graded ring $R$ is called gr-maximal if it is maximal in the lattice of graded ideals of $R$. (similarly we have for $R$-modules). A graded ring $R$ is called a gr-local ring if it has unique gr-maximal ideal. Let $R$ be a graded ring and let $S \subset H(R)$ be a multiplicatively closed subset of $R$. Then the ring of fraction $S^{-1}R$ is a graded ring which is called a gr-ring of fractions. Indeed, $S^{-1}R = \oplus_{g\in G} (S^{-1}R)_g$ where $(S^{-1}R)_g = \left\{ \frac{r}{s}, r \in R, s \in S, g = \frac{\deg(r)}{\deg(s)} \right\}$. And $S^{-1}M = \oplus_{g\in G} (S^{-1}M)_g$ where $(S^{-1}M)_g = \left\{ \frac{m}{s}, m \in M, s \in S, g = \frac{\deg(M)}{\deg(s)} \right\}$. Consider the ring gr-homomorphism $\pi : R \to S^{-1}R$ defined by $\pi(r) = \frac{r}{1}$. And $\pi : M \to S^{-1}M$ is called gr-homomorphism if $\pi(m) = \frac{m}{1}$. Let $P$ be any gr-prime ideal of a graded ring $R$ and consider the multiplicatively closed subset $S = H(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of $R$ by $R_p^g$ and we call it the gr-localization of $R$. This is a gr-local with the unique gr-maximal ideal $S^{-1}P$ which will be denoted by $PR_p^g$. Let $I$ be a graded ideal in a graded ring $R$. The graded radical of $I$ (gr-rad($I$)) is defined the set of all $x_g \in R$ such that for each $g \in G$, there exists $n_g > 0$ such that $x_g^{\,n_g} \in I$. A graded radical submodule $N$ of a graded $R$-module $M$ ($R$ is a graded ring) is the intersections of graded prime submodules of $M$ such that containing $N$ as a submodules. A submodule $N$ of on $R$-module $M$ is called multiplication if $N = IM$, for some gr-ideal $I$ of $R$. If each sub module of $M$ is gr-multiplication, $M$ is called gr-multiplication $R$-module.

In this paper, we study some properties of gr-multiplication submodules in a graded multiplication R-module $M$, when $M$ is gr-module over gr-ring $R$. And give a characterization of finitely generated gr-multiplication submodules of a gr-multiplication $M$ over a gr-ring $R$.

**Definition 1** Let $R$ be a graded ring over the group $G$ and $M$ an $R$-graded module. A graded submodule $N$ of $M$ is called graded multiplication . If $K \underset{g}{<} N$ then there is an gr-ideal of $R$ such that $K = NI$ .

**Definition 2** A graded $R$-module $M$ is called gr-multiplication module if every gr-submodule of $M$ is gr-multiplication .

**Definition 3** A graded ideal $Q$ of a graded ring $R$ is called gr-primary if $Q \neq R$ and whenever, $a,b \in H(R)$ whit $ab \in Q$ , then $a \in Q$ or $b^n \in Q$ .If $Q$ is gr-primary ideal of $R$ and gr-rad $(Q) = P$ , we say that $Q$ is gr-p-primary.

**Definition 4** An gr-submodule $N$ of graded $R$-module $M$ is called gr-primary, if $a \in H(R)$, $b \in H(M)$ and $ab \in N$ ,then $b \in N$ or $a^n M \subset N$ for some integer $n \geq 0$.

Recall that if $N, K$ are two gr-submodules of a graded R-module $M$ , then $(N : K) = \{ r \in R | rK \subset N \}$ is a graded ideal of $R$.

**Lemma 1** Let $I$ be a graded ideal in a graded ring $R$ then $I$ is multiplication if $I \cap J = I(J:I)$ for gr-ideal $J \subset I$ .

**Proof.** Suppose that $J \subset I$ for some gr- ideal $J$ of $R$.Then $J = I \cap J = I(J:I)$. Hence $J$ is gr-multiplication ideal of $I$ .

Conversely, Let $I$ be a graded multiplication ideal in $R$, Let $J$ be any graded ideal of $R$ Then $I \cap J \subset I$ , so there is a graded ideal $K$ of $R$ such that $I \cap J = IK$ .Therefore $K \subseteq ((I \cap J):I) \subseteq (J:I)$, and then $I \cap J = IK \subset I(J:I)$. On the other hand, clearly $I(J:I) \subset I \cap J$. Hence $J = I \cap J = (J:I)I$ .

**Proposition 1** Let $M$ be a graded $R$-module ($R$ is a graded ring). Then $M$ is gr-multiplication if for every gr-submodule $N$ of $M$ , $N = [N : M]M$ .

**Proof.** Let $M$ be gr-multiplication $R$-module, and $N$ a gr-submodule of $M$ , then there is an gr-ideal $I$ of $R$ such that $N = IM$ , as $IM \subset N$ we have $I \subset [N : M]$ and $N = IM \subset M[N : M]$. Since $[N : M]M \subset N$ , so $N = [N : M]M$ . Conversely it is clearly. Recall that Graded $R$ -module $M$ is called graded cyclic if $M = Rx$ , for some $x \in H(M)$.

**Theorem 1** Let $M$ be a gr-multiplication Module over a graded local ring $R$. Then $M$ is gr-multiplication if $M$ is a graded cyclic $R$-module .

**Proof.** If $M=<m>$ for some $m\in H(M)$ then clearly $M$ is gr-multiplication $R$-module .Conversely, Let $M=<m_\alpha \,|\, \alpha\in A>$ where each $M_\alpha$ is a homogeneous element $(m_\alpha\in H(M))$. Since $M$ is gr-multiplication we have $Rm_\alpha=[M_\alpha:M]M$ ,as $M=\sum_{\alpha\in A}Rm_\alpha=\sum_{\alpha\in A}[m_\alpha:M]M=M(\sum_{\alpha\in A}[m_\alpha:M])$. If $\sum_{\alpha\in A}[m_\alpha:M]=R$, then $[m_{\alpha_0}:M]=R$. Since otherwise if $\forall\alpha[m_\alpha:M]\neq R$, then $[m_\alpha:M]\subset J$, where $J$ is the only maximal ideal of $R$, and hence $\sum_{\alpha\in A}[m_\alpha:M]=R\subset J$ that is a contradiction, so $[m_{\alpha_0}:M]=R$ for some $\alpha_\circ\in A$ therefore $<m_{\alpha_\circ}>=[m_{\alpha_\circ}:M]M=M$ Hence $M$ is gr-principal. If $\sum_{\alpha\in A}[m_\alpha:M]\neq R$, then $\sum_{\alpha\in A}[m_\alpha:M]\subset J$, and then $M=\sum_{\alpha\in A}[m_\alpha:M]M\subseteq JM\subset M$ therefore $JM=M$ , hence $M=<0>$.

**Proposition 2** . If $M$ is gr-multiplication $R$-module where $R$ is a graded ring, and $S\subset H(R))$ is a multiplicatively closed subset of $R$. Then $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module.

**Proof.** Let $K$ be a graded $S^{-1}R$-submodule of $S^{-1}M$ . Then $K=S^{-1}N$ for some graded submodule of $N$ of $M$ . Now since $M$ is gr-multiplication $R$-module, then $N=[N:M]M$ so $S^{-1}N=(S^{-1}[N:M])(S^{-1}M)$ Hence $S^{-1}M$ is a gr-multiplication $S^{-1}R$-module.

**Definition 5** A graded submodule $N$ of graded $R$-module $M$ is locally gr-principal if $N\cdot R_p^g$ is gr-principal for every gr-prime ideal $P$ of $R$.

**Proposition 3** Let $R$ be a gr-local ring with graded maximal ideal $J$ and $M$ a graded $R$-module such that $M=<m_1,m_2,\cdots,m_k>$, where $m_1\in H(M)$ for every $1\leq i\leq k$ , then $M=<m_i>$ for some $1\leq i\leq k$ .

**Proof.** Suppose that $M=<a>$ for some $a\in H(M)$ and $M=<a_1,a_2,\cdots,a_k>$, then $a=\sum_{i=1}^{k}r_ia_i$ and each $a_i=s_ia$, so $a=\sum_{i=1}^{k}r_is_ia$

and $a(1-\sum_{i=1}^{k} r_i x_i)=0$ if $1-\sum_{i=1}^{k} r_i s_i$ is a unit in $R$ then $a=0$ since $a_i=s_i a, a_i=0$, for all $1\le i\le k$. and $M=<0>=<a_i>$, for all $i=1,2,\cdots,k$. If $1-\sum_{i=1}^{k} r_i s_i$ is not a unit, then $\sum_{i=1}^{k} r_i s_i \notin J$ and so $\sum_{i=1}^{k} r_i s_i$ is a unit. Therefore, there is an $i\in\{1,2,\cdots,k\}$ such that $r_i s_i$ is a unit. Otherwise since each $r_i s_i$ is not unit then $r_i x_i\in M$, for all $i=1,2,\cdots,k$, hence $\sum_{i=1}^{k} r_i s_i \in M$. That is a contradiction. So $r_i s_i$ is a unit for some $i$, then $s_i$ is a unit. Hence $a=as_i s_i^{-1}=a_i{}^{-1}s\in<a_i>$ Then $M=<a_i>$.

**Theorem 2** Let $M=<m_1,m_2,\cdots,m_k>$ be a finitely generated graded $R$-module over a graded ring $R$. Then the following are equivalent .

(1) $M$ is gr-multiplication .

(2) $M$ is locally gr-principal .

(3) $\sum_{i=1}^{k}[m_i:M_i]=R$, where $M_i=<a_1,a_2,\cdots,a_{i-1},a_{i+1},\cdots,a_k>$

**Proof.** (1) $\rightarrow$ (2) By Theorem 1 .

(2) $\rightarrow$ (3) Let $M$ be a locally gr-principal. Then for graded prime ideal $P$ of $R$, we have by Proposition 3 $MR_p^g=<\frac{m_1}{1},\frac{m_2}{1},\cdots,\frac{m_k}{1}>=<\frac{m_i}{1}>=<m_1>R_p^g$, for some $i\in\{1,2,\cdots,k\}$}. Hence for any gr-prime ideal $P$ of $R[(m_i)R_p^g:M_iR_p^g]=R_p^g$, where $M_i=<a_1,a_2,\cdots,a_{i-1},a_{i+1},\cdots,a_k>$ and then $(\sum_{i=1}^{k}[(m_i):M_i]R_p^g=\sum_{i=1}^{k}[(m_i)R_p^g:M_iR_p^g]=R_p^g$. Since $M_i$ is finitely generated for each $i$, There for $\sum_{i=1}^{k}((m_i):M_i)=R$.

(3) $\rightarrow$ (2) Suppose that $\sum_{i=1}^{k}((m_i):M_i)=R$. Then for any gr-prime $P$ of $R$ we have $(\sum_{i=1}^{k}[(m_i)R_p^g:MR_p^g]=\sum_{i=1}^{k}([(m_i):M])R_p^g=(\sum_{i=1}^{k}[(m_i):M_i])R_p^g=R_p^g$. Therefore, there is $i\in\{1,2,\cdots,k\}$ such that $((m_i)R_p^g:MR_p^g)=R_p^g$ and then

$MR_p^g \subset (a_i)R_p^g = < \frac{a_i}{1} >$. It follows that $MR_p^g = < \frac{a_i}{1} >$ for each gr-prime ideal $P$ of $R$. Hence $M$ is locally gr-principal. If $M$ is a graded module over the graded ring $R$ we define the $\theta^g(M) = \sum_{x \in (M)} [(x):M]$. It is clear that $\theta^g(M)$ is a graded ideal of $R$.

**Proposition 4** Let $M$ be a graded multiplication module over a graded ring $R$. Then

(1) $M = M\theta^g(M)$

(2) $N = N\theta^g(M)$ for any graded submodule $N$ of $M$.

**Proof.** (1) Let $x \in M$ as $M$ is graded multiplication $R$-module, then $<x> = [(x):M]M$ since

$$M = \sum_{x \in M} <x> = \sum_{x \in M} [(x):M]M = M \sum_{x \in M} [(x):M] = M\theta^g(M).$$

(2) suppose that $N$ is a graded submodule of $M$. Then $N = [N:M]M$, where $[N:M]$ is a graded ideal of $R$. Hence $N = [N:M]M = [N:M]\theta^g(M)M = N\theta^g(M)$.

**Proposition 5** Let N and $K$ be graded submodules of graded multiplication $R$-module $M$ and $S \subset H(R)$ be a multiplicatively closed subset of $R$. Then

(1) $\theta^g(N)\theta^g(K) \subset \theta^g(NK)$

(2) $S^{-1}(\theta^g(N)) \subseteq \theta^g(S^{-1}(N))$

**Proof.** (1) If $M$ is a multiplication $R$-module and $N = IM$ and $K = JM$ we defined $NK = IJM$. If $x \in M$ and $y \in K$, then $xy = \sum_{i=1}^{n} r_i m_i$, where $r_i \in IJ$, for all $i = 1,2,\cdots,n$ and $n \geq 1$. See [2].

Let $a \in N \cap H(M)$ and $b \in K \cap H(M)$. It is enough to prove that $[(a):N][(b):K] \subseteq [(ab):NK]$. Let $\sum_{i=1}^{n} x_i y_i \in [(a):N][(b):K]$ where $x_i \in [(a):N]$ and $y_i \in [(b):K]$, for $i = 1,2,\cdots,n$. Then $x_i N \subset (a)$ and $y_i K \subset (b)$, for $i = 1,2,\cdots,n$. Hence, $x_i y_i NK \subset (ab)$ and then $x_i y_i \in [(ab):NK]$. Therefore $\sum_{i=1}^{n} x_i y_i \in [(ab):NK]$.

(2)

$$S^{-1}(\theta^{g}(N)) \subseteq S^{-1}\Big(\sum_{x \in N \cap H(M)} [(x):N]\Big] = \sum_{x \in N \cap H(M)} s^{-1}[(x):N] \subseteq \sum_{x \in N \cap H(M)} [(\frac{x}{1}) : s^{-1}N] \subseteq \theta^{g}(s^{-1}N).$$

Recall that a graded module $M$ over graded ring $R$ is called gr-finitely generated if $M$ is generated by a finite set of homogeneous elements .

**Theorem 3** Let $M$ be a graded $R$-module where $R$ is a graded ring. Then $M$ is gr-finitely generated and locally gr-principal if $\theta^{g}(M) = R$ .

**Proof.** Let $J$ be a gr-maximal ideal in $R$. Then $MR_J^g = (x)R_J^g$ for some $x \in H(M)$. Hence, $R_J^g = [(x)R_J^g : MR_J^g] = [(x):M]R_J^g$ since $M$ is gr-finitely generated. Therefore $R_J^g = \theta^{g}(M)R_J^g$ and they by local property $\theta^{g}(M) = R$.

Conversely, suppose $\theta^{g}(M) = R$. Then there exist, $m_1, m_2, \cdots, m_k \in H(N)$ such that $R = \theta^{g}(M) = [(m_1):M] + [(m_2):M] + \cdots + [(m_k):M]$ . Thus

$$M = \theta^{g}(M)M = M[(m_1):M] + M[(m_2):M] + \cdots + M[(m_k):M] \subseteq (m_1) + (m_2) + \cdots + (m_n) \subset M$$

so $M = (m_1, m_2, \cdots m_k)$ is gr-finitely generated. Let $J$ be a gr-maximal ideal of $R$. Since $\theta^{g}(M) = R$, there is $x \in H(M)$, with $[(x):M] \nsubseteq J$. Therefore, there exists $r \in R - J$ with $rM \subseteq (x)$ and then $rMR_J^g = \langle r \rangle R_J^g \cdot MR_J^g = MR_J^g \subseteq (x)R_J^g$. Hence $MR_J^g = (x)R_J^g$, for any gr-maximal ideal $J$ of $R$ and so $M$ is locally gr-principal .

## References

[1] Ameri, R., On The Prime Submodules of Multiplication Modules, Inter. J. of Mathematics and Mathematical Sciences, 27, 1715-1724 (2003).

[2] Anderson, D.D: some remark on multiplication ideal. Math Japonica 25, 463-469,4(1980).

[3] El-Bast, Z. and Smith, P.F., Multiplication Modules, Communications in Algebra, 16(4), 755-779 (1988).

[4] Escoriza, J. and Torrecillas, B., Multiplication Graded Rings, Algebra and Number Theory, 127-136, Dekker, Lecture Notes in Pure and Appl. Math., 208, New York, 2000.

[5] Escoriza, J. and Torrecillas, B., Multiplication Objects in Commutative Grothendieck Categories, Communication in Algebra, 26(6), 1867-1883 (1998).

[6] Nastasescu, C. and Van oystaeyen, F: Graded Ring theory. Mathematical library , North Holand, Amesterdam, 28 (1982).

[7] Refai, M. Al Zoubi, K : On Graded Primary Ideals. Turkish Journal of Mathematics, 28, 217-229 (2004) .

**Received: November, 2012**