STABILITY AND INSTABILITY IN THE KINETIC THEORY OF PLASMAS

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Abstract

A plasma is a gas of charged particles, say of electrons and ions. In the tail of a comet, for instance, the dominant force on the particles is the electromagnetic force. In kinetic theory the velocity is treated as an independent variable, which leads to the Vlasov equation. In the absence of collisions, entropy plays no special role. There are many equilibria, some of which are stable and some unstable. We consider three classes of equilibria, the homogeneous ones, the electric BGK equilibria, and the magnetic equilibria. In the second and third classes some of the particles are trapped by the field and there is no exact dispersion relation. Until two years ago very little was known about their stability properties. We discuss several results asserting their stability or instability.

1. Introduction

A gas may be modeled in three fundamentally different ways. In a Particle Model, each of the N molecules satisfies a differential equation $m\ddot{x} = force + collisions$. Unfortunately N typically has the order 10^{25}. In a Fluids Model, the velocity v, spatial density ρ, etc. are functions of time t and space x. This kind of modeling leads to the Euler equations, Navier-Stokes equations, etc. In a Kinetic Model, the density $f(t, x, v)$ of particles in phase space (x, v) plays the central role. The velocity v is an independent variable. The passage from a Particle to a Kinetic Model is the subject of Statistical Mechanics. The passage from a Kinetic to a Fluids Model has also been much studied, for instance by Hilbert in 1912.

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In this lecture I will talk exclusively about Kinetic Theory. If the individual particles (of unit mass) are acted upon by a force K, and the phase-space density of particles is $f(t, x, v)$, then the basic dynamical equation is

$$\partial_t f + \ v \cdot \partial_x f \ + \ K \cdot \partial_v f \ = \ Q(f)$$

where K is the force and Q is the collision operator. The characteristics of the PDE are the paths of the physical particles.

A collision between particles is governed by the conservation of momentum $u + v = u' + v'$ and energy $|u|^2 + |v|^2 = |u'|^2 + |v'|^2$. It occurs with a probability $\sigma(|u - v|, \omega)$ that depends on the particular physical situation. The collision operator then takes the form

$$Q(f)(v) = \int \sigma[f(u')f(v') - f(u)f(v)]$$

where the integral is taken over all possibilities. The Boltzmann equation $\partial_t f + v \cdot \partial_x f = Q(f)$ governs a pure gas of uncharged particles that undergo collisions. The total mass $\int f$, momentum $\int vf$ and energy $\int |v|^2 f$ are conserved. Furthermore, the entropy is increasing: $\frac{d}{dt} \int f \log f \ \leq \ 0$. (The entropy is the negative of this integral.) It is natural to expect that the entropy is driven to a maximum. The critical points of the entropy subject to constant mass, momentum and energy are the distributions $\mu = \exp(a + b \cdot v - c|v|^2)$, called the maxwellia. Ukai [U] was the first to prove that under some reasonable assumptions the equilibria μ are asymptotically stable. This means that if f is initially near μ then $f \to \mu$ as $t \to +\infty$.

In most of this lecture we will be concerned with a plasma, a large collection of *charged* particles. Examples include the stellar interior, interstellar dust, a fluorescent bulb, the solar wind, the magnetosphere, the tail of a comet, a particle accelerator, and a fusion reactor such as a tokamak.

We will assume that collisions between the particles are negligible, that the plasma is relativistic, that both the mass and the charge of an individual particle are 1 and that the speed of light is $c = 1$. We consider a collection of electrons

($-$) and ions ($+$). Let

$f_\pm(t,x,v)$ = density of ions, electrons; $E(t,x)\,[B(t,x)]$ = electric [magnetic] field

The momentum of a particle is v, the velocity $\hat{v} = v/\sqrt{1+v^2}$, the energy $\langle v\rangle = \sqrt{1+v^2}$ and the charge $e = \pm 1$. A single particle satisfies

$$\dot{x} = \hat{v}, \quad \dot{v} = e(E + \hat{v}\times B) = force.$$

The particle density and field satisfy the Vlasov-Maxwell system (here in its relativistic form)

$$\{\partial_t + \hat{v}\cdot\nabla_x \pm (E + \hat{v}\times B)\cdot\nabla_v\}f_\pm = 0,$$

$$\partial_t E = \nabla\times B - j, \qquad \nabla\cdot E = \rho, \qquad \rho = \int (f_+ - f_-)\, dv,$$

$$\partial_t B = -\nabla\times E, \qquad \nabla\cdot B = 0, \qquad j = \int \hat{v}\,(f_+ - f_-)\, dv.$$

2. One-dimensional case

In one-dimension all the variables are scalars and the system reduces to

$$\begin{cases} \partial_t f_\pm \;+\; \hat{v}\;\partial_x f_\pm \;\pm\; E\;\partial_v f_\pm = 0 \\ \partial_t E \;=\; -j, \qquad \partial_x E \;=\; \rho \end{cases}$$

This system has been proven to be well-posed. Notice that the last two (Maxwell) equations are compatible because $\partial_t\rho + \partial_x j = 0$ from the first (Vlasov) equation. A similar system describes a continuous distribution of particles under gravity (e.g. stars in a galaxy).

Unlike the Boltzmann case there are many equilibria. Any equilibrium must satisfy $(\hat{v}\partial_x \pm E\partial_v)f_\pm = 0$. Because $f_\pm$ must be constant on the characteristics, the typical equilibrium takes the form

$$\begin{cases} f_\pm \;=\; \mu_\pm(\langle v\rangle \mp \Phi(x)) \\ E \;=\; \Phi'(x) \end{cases}$$

for some functions μ_+, μ_- and Φ where

$$\frac{d^2\Phi}{dx^2} = \int_{-\infty}^{\infty} [\mu_+(\langle v\rangle - \Phi(x)) - \mu_-(\langle v\rangle + \Phi(x))]\, dv = -H'(\Phi).$$

Such solutions are called *BGK equilibria* after the authors of the original paper [BGK]. There are hundreds of physics papers about them. Here we assume neutrality:

$$\int_{-\infty}^{\infty} [\mu_+(\langle v\rangle) - \mu_-(\langle v\rangle)]\, dv = 0.$$

For example, if $H'(0) = 0$ and $H''(0) > 0$, then the origin is a center for the second-order ODE. Thus there is a family of small periodic solutions. Although such equilibria are quite simple, initial perturbations in the PDE can induce *very complicated* spatial and temporal behavior!

Theorem 1.[GS1,GS2] *If $H'(0) = 0$ and $H''(0) > 0$ together with some technical assumptions, then a small periodic solution of period P is unstable (both linearly and nonlinearly) with respect to perturbations of the initial data of period $2P$. More precisely, there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ there exists a solution $f = f^\delta$ such that*

$$\| f(0) - \mu \|_{W^{1,1}} + ||E(0) - \Phi_x||_{W^{1,1}} < \delta$$

but

$$\sup_{0\le t\le C|\log\delta|} ||f(t) - \mu||_{L^1} + ||E(t) - \Phi_x||_{L^1} \ge \epsilon_0.$$

The strategy of the proof is (I) to linearize and (II) to pass from the linear to the nonlinear system. For (I) it has to be proven that there exist exponentially growing solutions of the linearized system.

3. Proof of linear instability

The linearized system around the inhomogeneous equilibrium $[\mu_+, \mu_-, \Phi]$ is

$$[\partial_t + \hat{v}\partial_x \pm \Phi_x\partial_v]\, g_\pm = -E\ \partial_v\mu_\pm(\langle v\rangle \mp \Phi(x))$$

$$\partial_x E = \int_{-\infty}^{\infty} (g_+ - g_-)\, dv, \qquad \partial_t E = -\int_{-\infty}^{\infty} \hat{v}\ (g_+ - g_-)\, dv.$$

This problem has a lot of marginal, continuous spectrum! However, we will look for some unstable point spectrum $E = e^{-i\omega t}\tilde{E}(x)$ with $\Im\omega > 0$. The characteristics (particle paths) are given by $\dot{x} = \hat{v},\ \dot{v} = \pm\Phi_x$ with a phase portrait just like a pendulum that includes trapped particles (periodic orbits). Thus by inverting the operator $[\partial_t + \hat{v}\partial_x \pm \Phi_x\partial_v]$, we get an equation like

$$g_\pm = \int_{characteristics} E\ \partial_v\mu_\pm\ .$$

Then we look for solutions $E = e^{-i\omega t}\ \tilde{E}(x)$ and plug them into the Poisson equation. We obtain

$$\partial_x\tilde{E}(x) = \int_{-\infty}^{\infty} k(x,x',\omega)\ \tilde{E}(x')\ dx'\ ,$$

where $k = k^+ - k^-$ and

$$k^\pm(x,x',\omega) = -\int_0^\infty\int_{-\infty}^{\infty}\delta(x - X^\pm(t;0,x',v'))\ \partial_{v'}\mu_\pm(\langle v\rangle \mp \Phi(x'))\ e^{+i\omega t}dv'dt.$$

Integrating from $-\infty$ to x, we obtain an equation in the abstract form

$$\tilde{E} = \mathcal{C}(\omega,\Phi)\ \tilde{E}.$$

We prove three statements about this operator:

$$\begin{aligned} &\omega \longrightarrow \mathcal{C}(\omega,\Phi) \quad \text{is analytic in } \{\Im\omega > 0\}.\\ &\Phi \longrightarrow \mathcal{C}(\omega,\Phi) \quad \text{is continuous in a certain sense near } \Phi = 0.\\ &\tilde{E} \longrightarrow \mathcal{C}(\omega,\Phi)\tilde{E} \quad \text{is a compact operator in } L^1.\end{aligned}$$

The last statement is the Main Lemma. Under these conditions it is well-known by general operator theory that the poles of $[I - \mathcal{C}(\omega,\Phi)]^{-1}$ vary continuously as functions of Φ. Therefore the problem is reduced to the case of $\Phi \equiv 0$.

The homogeneous case $\Phi \equiv 0$ can be studied easily by Fourier transform in x, or alternatively by looking directly for exponential solutions

$$E = e^{i(kx-\omega t)} \qquad , \qquad g_\pm = e^{i(kx-\omega t)}\tilde{g}_\pm(v),$$

from which we easily get the dispersion relation

$$k^2 = \int_{-\infty}^{\infty}\frac{\partial_v[\mu_+ + \mu_-]}{\hat{v} - z}dv = Z(z)$$

where $z = \omega/k,\ k \neq 0$. We look for real k and complex ω. Since $|e^{-i\omega t}| = e^{t\Im\omega}$, we ask whether there exists $\Im\omega > 0$ or not. Thus the homogeneous equilibrium is linearly unstable if and only if the image of the upper half-plane $\{\Im z > 0\}$ under Z meets the positive real axis. Penrose [P] found a nice necessary and sufficient condition on $\mu_\pm$ for this to be true. In particular, if $\mu_+ + \mu_-$ is a decreasing function of $|v|$, the equilibrium is linearly stable, but if $\mu_+ + \mu_-$ deviates sufficiently from monotonicity, it is linearly unstable. Under the assumption of our theorem, $Z(0) = H''(0) > 0$, which places us in the unstable case.

4. Proof of nonlinear instability

Let us drop the notation $\pm$, and cryptically write the linearized system in the form $(\partial_t + L)\ g = 0$. We have just shown there exists a solution $g = e^{\lambda t}\ R(x,v)$ where $\lambda = -i\omega$ is an eigenvalue of $-L$. The full nonlinear system is

$$(\partial_t + L)(f - \mu) = (E - \Phi') \cdot \partial_v(f - \mu)$$

where we again have simplified the notation. We choose $f(0) = \mu + \delta R$ where δ is a small parameter and $R\, e^{\lambda t}$ has the maximum possible $\Re\lambda$. Next we write the full nonlinear system in the integral form

$$f(t) - \mu = \delta\ Re^{\lambda t} + \int_0^t e^{-L(t-\tau)}\ (E - \Phi_x) \cdot \partial_v(f - \mu)\ d\tau.$$

We will show that the linear term dominates by estimating

$$\|f(t) - \mu - \delta Re^{\lambda t}\|_{L^1} \ \le\ \int_0^t e^{\Re\lambda(t-\tau)}\ \|E - \Phi_x\|_{L^\infty}\ \|\partial_v(f - \mu)\|_{L^1}\ d\tau.$$

We treat the dangerous factor involving the v-derivative of $f - \mu$ by the

Lemma. *If* $\|f - \mu\|_{L^1} = O(e^{\alpha t})$ *and if certain norms of* $f - \mu$ *are bounded, then*

$$\|\partial_v(f - \mu)\|_{L^1} = O(e^{\alpha t}).$$

Then the instability follows with $\alpha = \Re\lambda$. Furthermore we prove that $f_\pm \ge 0$ if for instance $|\mu'| \le C\mu$, so that $f_\pm$ are true densities.

5. Solitary waves and collisionless shocks

Assume now that the distributions $\mu_\pm$ are such that a solitary wave $\Phi(x)$ exists, meaning that the ODE has a homoclinic orbit, and that the linearized system around the homogeneous state $[\mu_\pm(\langle v \rangle),\ E=0]$ has a growing mode.

Theorem 2. [GS3] *The equilibrium* $[\mu_\pm(\langle v \rangle \pm \Phi(x)),\ E=\Phi_x]$ *coming from the solitary wave is linearly and nonlinear unstable with respect to perturbations that vanish as* $x \to \pm\infty$.

Here the new difficulty is that we lose the compactness of the operator $\mathcal{C}(\omega, \Phi)$. Even the unstable spectrum has become continuous! We evade this difficulty by using the causality of the relativistic system. By hypothesis, the linearized system around $\mu(\langle v \rangle)$ has a growing plane wave $\exp i(kx-\omega t)$ of some period $P/2$. Hence the full nonlinear system with boundary conditions of period P has an unstable solution $f_P(t,x,v)$. For the full nonlinear system with boundary conditions for E vanishing as $x \to \pm\infty$, we choose initial data $f(0,x,v) = f_P(0,x,v)$ for $x \in I$, and $f(0,x,v) = \mu(\langle v \rangle \mp \Phi(x))$ for $x \notin I$, where I is a big interval near $-\infty$. Now we use the fact that $\Phi \to 0$ as $x \to -\infty$, together with the causality, to prove that $f_\pm(t,x,v) - \mu_\pm(\langle v \rangle \mp \Phi(x))$ becomes large within the triangle of dependence of I.

The same proof works for a collisionless shock (a kink), which corresponds to a heteroclinic orbit.

6. Stability: the homogeneous state

If a state is stable, all of the spectrum of the linearized problem is marginal ($e^{i\omega t}$ with ω real). Therefore the linear problem *cannot* help determine the stability of the nonlinear problem. Every known proof of nonlinear stability depends primarily on the nonlinear invariants.

Theorem 3. *Consider a homogeneous state* $[\mu_\pm(\langle v \rangle),\ E=0]$ *and consider* *perturbations of a given x-period. If both μ_+ and μ_- are strictly decreasing, then the full nonlinear system is stable with respect to L^2 norms.*

The idea of the proof goes back to [Ga]. We use the nonlinear invariant

$$I(f,E) = \int \frac{1}{2}|E|^2 dx + \int\int [\langle v\rangle (f_+ + f_-) + \gamma_+(f_+) + \gamma_-(f_-)]\, dv\, dx$$

with $\gamma_\pm$ to be chosen. Then

$$I(f(0),E(0)) - I(\mu,0) = I(f(t),E(t)) - I(\mu,0)$$
$$= \int \frac{1}{2}|E(t)|^2 dx + \sum_{\pm} \int\int [\gamma_\pm(f_\pm(t)) - \gamma_\pm(\mu_\pm) + \langle v\rangle (f_\pm(t) - \mu_\pm)]\, dv\, dx.$$

We choose $\gamma_\pm'(\mu_\pm(\langle v\rangle)) = -\langle v\rangle$. Then γ_+ and γ_- are strictly convex, so that $\gamma_\pm'' \geq c > 0$ for bounded arguments. Hence

$$I(f(0),E(0)) - I(\mu,0) \geq \int \frac{1}{2}|E(t)|^2 dx + c\sum_{\pm} \int\int (f_\pm(t) - \mu_\pm)^2\, dx\, dv.$$

If the left side is small, so is the right side for all t.

Here is an important open problem. If there are little bumps in the graphs of μ_+ and μ_- (small deviations from monotonicity), is the homogeneous equilibrium *nonlinearly* stable or unstable?

7. Magnetic equilibria

Of course, a magnetic field can exist only in more than one dimension. The simplest case is the so-called $1\frac{1}{2}D$ Vlasov-Maxwell system with coordinates $(x,0,0), (v_1,v_2,0), (E_1,E_2,0)$ and $(0,0,B)$. Besides the energy $\langle v\rangle \mp \Phi(x)$, where $E_1 = \Phi'(x)$, there is another invariant if $E_2 = 0$, namely $v_2 \pm \Psi(x)$ where $B = \Psi'(x)$. If $E_2 = 0$, the equilibria have the form

$$f_\pm = \mu_\pm(\langle v\rangle \mp \Phi(x), v_2 \pm \Psi(x)), \quad B = \Psi_x, \quad E_1 = \Phi_x, \quad E_2 = 0.$$

Guo and Ragazzo [GR] observe that Φ and Ψ then satisfy the coupled pair of ODE's

$$\Phi_{xx} = \rho = \int (\mu_+ - \mu_-)\, dv, \qquad \Psi_{xx} = -j_2 = -\int \hat{v}_2(\mu_+ - \mu_-)\, dv$$

where $\mu_\pm = \mu_\pm(\langle v \rangle \mp \Phi(x), v_2 \pm \Psi(x))$. In this way they found many interesting magnetic equilibria.

In the saddle-saddle case they found the "flat-tail" equilibria for which the magnetic field is asymptotic to different constants at $+\infty$ and $-\infty$. Under some conditions Guo [G] proves their *stability.* In the center-saddle case they found the "oscillatory-tail" equilibria, for which the electric field is asymptotic to a periodic solution as $x \to -\infty$ and to a constant as $x \to +\infty$ and whose magnetic field is asymptotic to different constants at $+\infty$ and $-\infty$. Under some conditions we [GS4] prove their *instability.*

Major mathematical/scientific problems in the kinetic theory of plasmas include: equilibria (existence and properties), stability and instability, 3D geometry and boundary effects, singularity creation and propagation, numerical computations (particle methods), the interaction of electromagnetic and collisional effects, and fluid limits (MHD, Navier-Stokes, Euler).

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