Combinatorics Qualifier Exam

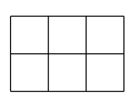
(cross off if you do not want to count it as such)

December 11, 2013

Notation. Let $[n] = \{1, 2, \ldots, n\}$. For a finite set X denote by $\binom{X}{k}$, $\left\{ {X \atop k} \right\}$, $\left[{X \atop k} \right]$, and $X!$ the sets of all subsets of X of size k, partitions of X into k parts, permutations of X into k cycles, and permutations of X, respectively. When $|X| = n$ these sets have cardinalities $\binom{n}{k}$, $\left\{ {n \atop k} \right\}$, $\left[{n \atop k} \right]$, and $n!$, respectively. Define $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

Directions. Solve the 6 problems below, each problem on a separate sheet, with your name and the problem number in the upper right corner. Show your work and explain your reasoning. Ask questions if you find the wording ambiguous or confusing. You may leave your answers in the forms of the notations above without calculating their values or displaying their formulas.

1. Find the number of ways to place n balls into k boxes under the following circumstances.
   (a) the balls are identical, the boxes are distinguishable, and there is at most 1 ball per box.
   (b) the balls are identical, the boxes are distinguishable, and there are at least 3 balls per box.
   (c) the balls are distinguishable, the boxes are identical, and there is at least 1 ball per box.
   (d) the balls are distinguishable, the boxes are distinguishable, and there is at most 1 ball per box.
2. Let $n = 5^3 \cdot 13 \cdot 71^2 \cdot 229$. Write an expression for the number of positive integers that are relatively prime to n. (It is not necessary to calculate its integer value.)
3. Prove for all integers n that $\left[{n \atop n-3} \right] = 6\binom{n}{4} + 20\binom{n}{5} + 15\binom{n}{6}$.
4. Let $f_0 = 4$, $f_1 = 10$, and $f_n = 5f_{n-1} - 4f_{n-2} - 6n + 5$ for all $n \geq 2$. Find the general formula for f_n.
5. Suppose that $a_0 = 1$, $a_1 = 1$, and $a_{n+1} = a_n + na_{n-1}$ for all $n \geq 1$. Let $F(x)$ be the EGF for the sequence $a_0, a_1, \ldots$.
   (a) Prove that $F'(x) = (1+x)F(x)$.
   (b) Use Part (5a) to show that $F(x) = e^{x+x^2/2}$.
6. (a) Write the cycle index polynomial for the group of 3-dimensional symmetries of the panes (faces) of the window figure below.



   (b) Given that the cycle index polynomial for the group of 3-dimensional symmetries of the panes (faces) of the window figure below equals $[z_1^9 + 2z_1^1 z_4^2 + z_1^1 z_2^4 + 4z_1^3 z_2^3]/8$, calculate the number of colorings of the panes that use red twice and blue 7 times.