From the Inside, the Unique Non-Computable Computably Enumerable Set

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$\mathcal{E} = \{W \subseteq \omega; W \text{ c.e}\}$
For A c.e:
$\mathcal{E}(A) = \{W \subseteq A; W \text{ c.e}\}$
$\mathcal{R}(A) = \{R \subseteq A; R \text{ computable}\}$

Theorem (Soare)

The structures $< \mathcal{E}(A), \subseteq, \mathcal{R}(A) >$, for A non-computable c.e, are all isomorphic.

The c.e. sets under inclusion

$\mathcal{E} = \{W \subseteq \omega; W \text{ is c.e }\}$.

The study of the c.e sets as sets is the study of the structure $< \mathcal{E}. \subseteq>$:
This ignores the dynamic features of c.e. sets, like the computable enumerations $W = \cup_s W_s$,
and only looks at static features of c.e sets.

Soare changed the nature of this study with his discover of a technique for building non-trivial automorphisms.
The key theorem was Soare's extension theorem, which,
given an effective enough isomorphism from the "outside" of a c.e. set A,
extended the isomorphism to an automorhism on all of $\mathcal{E}$
by effectively producing the part on the "inside" of A,
namely on $\mathcal{E}(A) = \{W \subseteq A; W \in \mathcal{E}\}$.

As part of this Soare encountered what seems to be the key to understanding the relationship between the "outside" of a c.e. set A and its "inside" ($\mathcal{E}(A)$),
namely the computable subsets of A

$$\mathcal{R}(A) = \{R \subseteq A; R \text{ is computable }\}.$$

And Soare discovered the remarkable theorem:

Theorem (Soare)

The structures $< \mathcal{E}(A), \subseteq, \mathcal{R}(A) >$, *for A non-computable c.e, are all isomorphic.*

This talk will be organized around the question:
what is the isomorphism type of this structure

$< \mathcal{E}(A), \subseteq, \mathcal{R}(A) >$?

This talk will in no way come close to answering this question; but this will allow for a presentation of some of the developements steming from Soare's automorphism technique. In particular, these will include why $\mathcal{R}(A)$ is the key to the connection between the "outside" and "inside" of a c.e. set A.

The structures $<\mathcal{E}(A), \subseteq>$ for A an infinite c.e. set are all easily isomorphic, in particular they are isomorphic to $<\mathcal{E}(\omega), \subseteq>$ or $<\mathcal{E}, \subseteq>$.
So our isomorphism type

$$<\mathcal{E}(A), \subseteq, \mathcal{R}(A)>$$

has representatives of the form

$$<\mathcal{E}, \subseteq, \mathcal{R}>$$

where $\mathcal{R}$ must be a collection of computable sets.

Looked at this way, we are now looking at a c.e. set (ω) that has no outside, and trying to ask what plays the counterpart to its "recursive" subsets, where "recursive" refers to having a complement all the way through that "outside" (an "outside" that is no longer there).

It is known (Cholak-H) that the structure $< \mathcal{E}, \subseteq >$ has complicated isomorphism type, having Scott rank as high as possible.
So let's reformulate our question by concentrating on the $\mathcal{R}(A)$ part:

What can be said about:

$\mathcal{F} =$
$\{\mathcal{R}$; such that there is an isomorphism between
$< \mathcal{E}, \subseteq, \mathcal{R} >$ and $< \mathcal{E}(A), \subseteq, \mathcal{R}(A) >$
(for any non-computable c.e. A)$\}$

The current state of knowledge about automorphisms gives:

(Lachlan) if $\mathcal{R} \in \mathcal{F}$, then
$\mathcal{R}$ is a Σ^0_3 set of recursive sets

(Cholak-H) if $\mathcal{R} \in \mathcal{F}$, then there is an arithmetic (Δ^0_6) isomorphism between $< \mathcal{E}, \subseteq, \mathcal{R} >$ and $< \mathcal{E}(A), \subseteq, \mathcal{R}(A) >$ (for any non-computable c.e. A)

(Cholak-H) there is an infinitary ($\mathcal{L}_{\omega_1,\omega}$) sentence φ of rank $< \omega + \omega$ such that
$\mathcal{R} \in \mathcal{F}$ iff $< \mathcal{E}, \subseteq, \mathcal{R} >$ satisfies φ

Comment:
These results fall far short of actually understanding $\mathcal{F}$ for what it is, since it is an orbit under $Aut(<\mathcal{E},\subseteq>)$.
The remarkable thing about Soare's Theorem is that it shows that many apparently disparate things (the non-computable c.e. sets A) are seen as the same when we ignore their "outsides", retaining only their essential "inside" connection to their "outsides".
Yet, by Soare, they're all the same.
The orbit $\mathcal{F}$ is just a way of pointing out the remarkableness of all this: when, for each non-computable c.e. set A, we look at $\mathcal{R}(A)$, there is an essential reference to the "outside" of A, and that reference points to something that depends on what A ("inside" and "outside") actual is, as distinct from the other possible A's.
And yet they are the same.

This situation seems related to a poem that Ted Slaman once shared:

Flower in the crannied wall

By Alfred, Lord Tennyson

Flower in the crannied wall,
I pluck you out of the crannies;
Hold you here, root and all, in my hand,
Little flower- but if I could understand
What you are, root and all, and all in all,
I should know what God and man is.

This talk (so far) has been an attempt to take what Soare plucked and hold it in the hand.

Definition

For $A \in \mathcal{E}$,
$\mathcal{S}(A) = \{S \in \mathcal{E}(A); (A \sim S) \in \mathcal{E}(A)\}$

Definition (Lachlan)

For A, B in $\mathcal{E}$, $B \subseteq A$,
B is a major subset of A iff $R \in \mathcal{R}(A)$ implies $R \subseteq^* B$.
B is a small subset of A iff $\mathcal{E}(B) \cap \mathcal{S}(A) \subseteq \mathcal{R}(A)$.

For B a small major subset of a non-computable c.e. set A, $\mathcal{R}(A)$ is B-definable over $< \mathcal{E}(A), \subseteq >$:

$R \in \mathcal{R}(A)$ iff $R \in \mathcal{S}(A)$ and $R \subseteq^* B$.

The same then holds for an $\mathcal{R} \in \mathcal{F}$:
for some B in $\mathcal{E} = \mathcal{E}(\omega)$

$R \in \mathcal{R}$ iff $R \in \mathcal{S}(\omega)$ and $R \subseteq^* B$.

Given B, this is a Σ^0_3 property.

Let $\mathcal{R}_B$ be this collection.

Then $\mathcal{R} \in \mathcal{F}$ iff for some $B \in \mathcal{E}$
$\mathcal{R} = \mathcal{R}_B$ and there is an isomorphism of $< \mathcal{E}. \subseteq >$ to $< \mathcal{E}(A), \subseteq >$ (for some non-computable c.e. A) such that B is sent to a small major subset of A.

This is now easily is a question about automorphism of $< \mathcal{E}, \subseteq >$ itself (since $\mathcal{E}(A)$ is easily identifyable with $\mathcal{E}$), and so falls under known results.

Definition

$B \in \mathcal{E}$ is simple iff there is no infinite c.e. set disjoint from B.

Theorem (Cholak-H)

For B simple, the $Aut(\mathcal{E})$ orbit of B is Σ^0_7.

In fact, for $B_1, \ldots, B_n$ all simple, the orbit of $B_1, \ldots, B_n$ under $Aut(\mathcal{E})$ is Σ^0_7.

And similarly there are $\mathcal{L}_{\omega_1,\omega}$ descriptions of orbits.
For B simple, (or for $B_1, \ldots, B_n$ all simple) there is a rank $< \omega + \omega$ formula describing the orbit.

The static way of viewing the flow from “outside”:

Definition

For A non-computable c.e., $\Lambda(A) = \{W \in \mathcal{E}; (W \cap A) \in \mathcal{S}(A)\}$

$\mathcal{E}$ is a distributive lattice with reduction
(for A, V there are $A_0 \subseteq A$, $V_0 \subseteq V$ such that $A_0 \cap V_0$ is empty,
and $(A \cup V) = (A_0 \cup V_0)$)
So from $\Lambda(A)$ and $\mathcal{E}(A)$, one can recover $\mathcal{E}$
($V \in \mathcal{E}$ is recovered from V_0 (which reduction ensures is in $\mathcal{S}(A)$,
and from $(W \cap A)$ which is in $\mathcal{E}(A)$.)

But one does not need all of $\Lambda(A)$ for this. The common part between $\Lambda(A)$ and $\mathcal{E}(A)$ is $\mathcal{S}(A)$. This is a boolean algebra and $\mathcal{R}(A)$ is an ideal of this boolean algebra.
$\mathcal{E}$ can be recovered from $\Lambda(A)/\mathcal{R}(A)$ and $\mathcal{E}(A)$.

Theorem (Cholak-H)

For B simple,
for F an automorphism of $< \mathcal{E}, \subseteq >$,
F restricted to $\Lambda(B)/\mathcal{R}(B)$ is Δ^0_6

Definition

For $B \in \mathcal{E}$, $\mathcal{L}(B) = \{W \in \mathcal{E}; B \subseteq W\}$

Theorem

For B, C in $\mathcal{E}$,
If F is an isomorphism between
$\Lambda(B)/\mathcal{R}(B)$ and $\Lambda(C)/\mathcal{R}(C)$
then F restricted to $\mathcal{L}(B)$ extends to an automorphism of $\mathcal{E}$.

Theorem

For B simple,
for F an automorphism of $< \mathcal{E}, \subseteq >$,
F restricted to $\Lambda(B)/\mathcal{R}(B)$ is Δ^0_6.

If F is an isomorphism between
$\Lambda(B)/\mathcal{R}(B)$ and $\Lambda(C)/\mathcal{R}(C)$
then F restricted to $\mathcal{L}(B)$ extends to an automorphism of $\mathcal{E}$.

About the orbits of tuples $B_1, \ldots, B_n$ where all B_i are simple:

By the reduction property of $\mathcal{E}$, there is a simple B such that all the B_i are in $\Lambda(B)$.

Theorem

For B, C *in* $\mathcal{E}$,
If F *is an isomorphism between*
$\Lambda(B)/\mathcal{R}(B)$ *and* $\Lambda(C)/\mathcal{R}(C)$
then F *restricted to* $\mathcal{L}(B)$ *extends to an automorphism of* $\mathcal{E}$.

This Theorem's proof is essentially the Soare Extension Theorem, with a needed connection provided by

Theorem (Cholal-H)

For F *an automorphism of* $\mathcal{E}$, *for* $B \in \mathcal{E}$
F *restricted to* $\mathcal{S}(A)/\mathcal{R}(A)$ *is* Δ^0_3