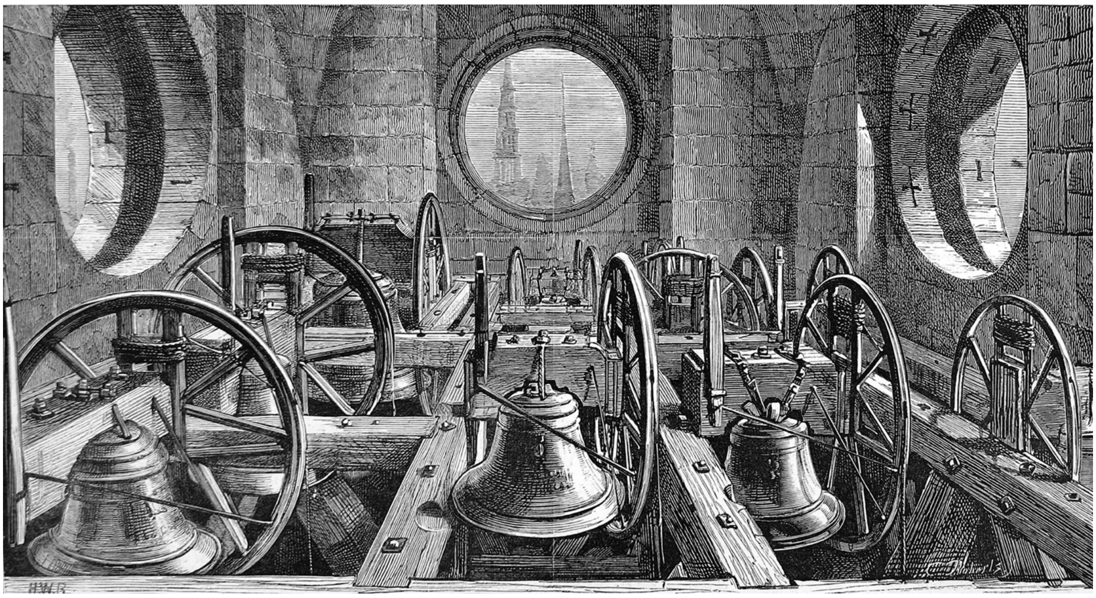
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Department of Mathematics

Bachelor Thesis

Campanology - Ringing the changes



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Introduction

This bachelor thesis is about the mathematics behind the ancient English art of **change ringing**. The science (mathematics) and practice (ringing actual church bells or handbells) of change ringing is concerned with the ringing of distinguished sequences of permutations of a number of bells. First, I would like to write a few words about the history and practice of change ringing. Second, I will present a short overview over the structure of my thesis.

History and practice of change ringing

The origin of change ringing was in the 17th century and primarily due to the invention of full-circle wheels to which the church bells were attached (see picture on the cover page). This invention made it possible to determine the exact moment when the clapper hits the bell. Hence the timing of the ringing of a single bell could be fully controlled. Thus, if the ringers had more than one full-circle bell and if all these bells had different pitches, it became possible to ring them one after another and get an exact sequence of different notes. Bell ringers defined such a sequence of notes as a **change**. Since it would have been boring to keep ringing this exact sequence over and over, the ringers tried changing the position of the bells in the sequence. However, they soon realised that they could only swap adjacent bells (notes) in a change (sequence), the reason being the considerable amount of weight of the full-circle bells. It was not possible to hold a bell in the balance position (see Illustration 1.1) over a longer period of time. Thus, the first rule of change ringing was born: When transitioning from one change to the next, each bell may not move more than one position in its sequence of ringing. Ringers became interested in ringing all possible changes one after another until they returned to the change they have started with. In other words, they wanted to ring an **extent**.

Once the basic rules of change ringing had been established, bell ringers began to study ringing patterns. The word **Campanology** is often used to describe the study of ringing patterns. It is made up of the Latin word *campana*, which means "bell", and the Greek ending *-logia*, which stands for "the study of something". In the beginning, most campanologists were people who actually practised the art of change ringing. They made use of group-theoretic ideas long before groups and their properties were formally introduced by mathematicians. One of those campanologists was Fabian Stedman. His book "Campanalogia" ([S66]) is considered as one of the earliest works about campanology. It was not until the 20th century that mathematicians became interested in the study of ringing patterns. But since then, they have found answers to unsolved questions and proven various theorems concerning this subject.

Although the ringing of changes is nowadays practised not just in England but all over the world, full-circle bells hung for change ringing beyond the British Isles are rare. The installation and maintenance of a ring of full-circle bells in a church tower is quite costly. This is one of the reason why practising ringers sometimes use handbells instead of church bells. An additional reason for the use of handbells is of course the simplicity of carrying them from one place to another. That way, ringers are not limited to performing in one particular place like for example in a church tower. For a more detailed description of the (geographical) history and practice of change ringing, the reader may consult Johnston's essay "A most public of musical performances: the English art of change-ringing" ([J06]).

Overview of my bachelor thesis

Chapter 1 of this bachelor thesis will acquaint the reader with common change ringing terminologies, which build the foundation of all subsequent chapters. Some extents on a small number of bells, for example the plain course of the Erin principle on four bells, will be shown in chapter 2. Chapter 3 will introduce Cayley color graphs and their connection to change ringing. Furthermore, it will be proven that there exists an extent for any arbitrary number of given bells. Chapter 4 will be concerned with right and left cosets and with their use for the construction of extents. In chapter 5 the reader will get to know the so-called Stedman principle and Colin Wyld's answer to an old problem concerning Stedman Triples. The non-existence of Grandsire Doubles or Triples using only plain and bob leads will be proven in chapter 6. In chapter 7 we will present Rankin's campanological theorem and apply it to the Grandsire method on seven bells. Consequently, chapter 6 and chapter 7 will be closely related.

This bachelor thesis is based on many different books, journals, papers and web pages, which are all listed in the bibliography. If, however, a whole chapter or a subsection of my work is based predominantly on a few specific papers, then I mention them at the very beginning of these sections. One of the most extensive analysis of the subject of campanology has been done by the mathematician Arthur

T. White (see, for example, [W85], [W87], [W,89]). Thus, his work is, in addition to Burkhard Polster's book [P03] and Gary McGuire's paper [McG12], the main source for my thesis.

My concluding remarks concern the list of illustrations. If a table or a figure in my thesis is not listed in the list of illustrations, I have created them without any templates. The ones I have done by using available templates and making small alterations (like, for example, using colors in graphs or adding lettering in figures) are all listed in the list of illustrations. For the creation of all figures I used a drawing editor called Ipe.

1. Common change ringing terms

1.1 Basic definitions

Suppose there are n bells given, identified by the natural numbers $1, 2, 3, \ldots, n$ and arranged in order of descending pitch. In change ringing, bells are rung at a constant beat, one bell at a time.

Definition 1.1.1. The bell with number 1, being the highest pitched bell, is called the **treble**. The lowest pitched bell is known as the **tenor**. A **change** is the ringing of the n bells in a particular order or arrangement. In other words, a change corresponds to the ringing of a permutation of $[n]$, the ordered set containing all integers from 1 to n in their natural order. The special change $1\,2\,3\,\ldots\,n$, in which the bells are rung in order of descending pitch, is called **rounds**. The first bell of a change is defined as the **lead**.

Remark 1.1.2. Some literature suggests that a change should be defined as a **row**.

Definition 1.1.3. An n-bell **extent** is a list of $n! + 1$ successive changes, satisfying the following rules:
- (i) The first and the last change must be rounds.
- (ii) No change apart from the two rounds is repeated. (Thus in case of a list of $n!+1$ successive changes, each possible change with the exception of rounds is rung exactly once.)
- (iii) When transitioning from one change to the next, each bell may not move more than one position in its order of ringing. Unless $n = 2$, the bells 1 and n are not adjacent and according to that their position cannot be interchanged.
- (iv) No bell occupies the same position for more than two successive changes.
- (v) Apart from the treble, each bell does exactly the same amount of work (hunting, dodging, etc.).
- (vi) The list of transitions is palindromic, meaning that it is self-reversing. (see subsection 1.2.1 called "Basic operations")

Remark 1.1.4. Condition (iv) and especially conditions (v) & (vi) are sometimes omitted in practice, since in spite of their desirability they are not seen as necessary. Therefore a list of $n!+1$ changes fulfilling the requirements (i)-(iv) or sometimes even just (i)-(iii) is often defined as being an extent. Furthermore, if in a list of changes of arbitrary length no change apart from maybe rounds is repeated and if condition (iii) is satisfied, then we call this list of successive changes a **ringing sequence**.

Remark 1.1.5. While the origin of rule (i) is musical, rule (ii) is there for thoroughness, given that it ensures some sort of perfection. Condition (iii) exists because of mechanical limitations, in particular because of how the bells are hung and rung in church towers (see Illustration 1.1). The aim of requirements (iv) and (v) is to keep the ringing performance interesting. The purpose of rule (vi) is to help the ringers memorize the list of changes.

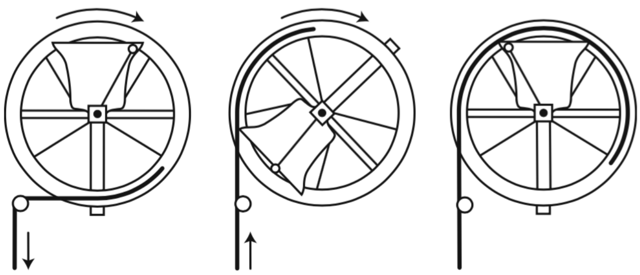


Illustration 1.1: A bell swings nearly a full circle. The clapper strikes the bell on its way up to the balance position (depicted in the right-hand diagram) and hence the bell rings. If ringers want to interchange the position of two adjacent bells in a change, the bell that comes first in the change has to be swung all the way up and held in the balance position. This until the proximate bell, which does not reach the balance position since it's been given less momentum, is again pulled down. Keeping a bell in the balance position is quite hard. Thus, balancing a bell over a longer period of time (longer than the time needed to swap adjacent bells in a change) is practically impossible.

In most practical applications, the number of bells n ranges between 3 and 12. In change ringing, the maximum number of bells that has ever been used seems to be 16. The names of odd-bell extents correspond to the maximum number of pairs of bells that can be interchanged when transitioning from one change to the next. (see Table 1.1)

number of bells	name	changes in extent	time required to ring extent
3	Singles	7	14 seconds
4	Minimus	25	50 seconds
5	Doubles	121	4 minutes
6	Minor	721	24 minutes
7	Triples	5'041	2 hours 48 minutes
8	Major	40'321	22 hours 24 minutes
9	Caters	362'881	8 days 10 hours
10	Royal	3'628'801	84 days
11	Cinques	39'916'801	2 years 194 days
12	Maximus	479'001'601	30 years 138 days
16		20'922'789'888'001	1'326'914 years

Table 1.1: Assuming that 30 changes can be rung per minute, we obtain in the rightmost column the time required to ring a given extent.

Ringing a Major extent seems to lie on the verge of human endurability. Through the course of history this physical and intellectual achievement has been rung on tower bells just once without relays. Thus, when ringing more than seven bells, the main goal for ringers is not to ring a full extent but to ring a so-called peal.

Definition 1.1.6. Let $n > 7$. A **peal** is composed of at least 5000 successive changes satisfying conditions (i)-(iii) of Definition 1.1.3. Hence, a peal is nothing more than a partial n - bell extent.

1.2 Concept of transition sequences (words)

This subsection follows mainly Burkhard Polster's book [P03].

Definition 1.2.1. For $k \geq 1$ and pairwise disjoint numbers $i_1, \ldots, i_\mathrm{k} \in \{1, \ldots, n\}$, the permutation $(i_1 \ldots i_\mathrm{k})$ in the symmetric group S_n with $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1$ is called a k-cycle. A 2-cycle is called a **transposition** and an **involution** is a product of disjoint transpositions.

Remark 1.2.2. Rule (iii) of Definition 1.1.3 implies that each transition from change to change is represented by an involution in S_n, namely by a product of disjoint transpositions of consecutive numbers.

Definition 1.2.3. Any n - bell ringing sequence of length $k+1$ can be rewritten as an n - bell **transition sequence** of length k which describes the transitions from one change to the next. We write such a transition sequence as a **word** of length k (as a string of k letters), in which every letter describes a particular kind of transition.

Remark 1.2.4. An n - bell transition sequence is nothing else than a product of elements of the symmetric group S_n. Such a product is always evaluated **from left to right**.

Remark 1.2.5. Let c_1 be a change on n bells and let c_2 be the change which we receive by applying a transition T to c_1. As mentioned in Definition 1.1.1, we can consider the changes c_1 and c_2 as elements of the symmetric group S_n and thus $c_1, c_2, T \in S_n$. Therefore c_2 is exactly the product $c_1 T$ evaluated from left to right.

1.2.1 Basic operations

Let $t = T_1 T_2 \cdots T_k$ be a transition sequence of length k corresponding to an n - bell ringing sequence r.

Definition 1.2.6. A transition sequence t is called **palindromic** if $T_1 = T_{k-1}$ (but not $T_1 = T_k$), $T_2 = T_{k-2}, \ldots$ and so on.

Definition 1.2.7. The **inverse** of r is the ringing sequence that starts with the last change of r and corresponds to the transition sequence $t_{inv} = T_k T_{k-1} \cdots T_1$.

Definition 1.2.8. The **cyclic shift** of r is the ringing sequence that starts with the first change of r and corresponds to the transition sequence $t_{cyc} = T_k T_1 T_2 \cdots T_{k-1}$.

Definition 1.2.9. A **ringing array** is a ringing sequence written down in the traditional notation used by bell ringers such that two successive changes are displayed one below the other, the sole exception being the start of a new column in case of space limitations. If the length of an n - bell ringing sequence r is $k+1$, then the corresponding ringing array is the $(k+1) \times n$ array (matrix) whose jth row is the jth change of the ringing sequence.

Definition 1.2.10. By replacing every array entry j of the ringing array of r by the number $(n+1-j)$ and afterwards reflecting the resulting array at the centred vertical axis we get a new ringing array, which corresponds to the **reverse** ringing sequence of r.

Remark 1.2.11. The inverse of the inverse of the ringing sequence r and the reverse of the reverse of r coincide with r itself. If r is, in addition to being a ringing sequence, also an n - bell extent (r has length $n!+1$ and fulfills at least conditions (i)-(iii) of Definition 1.1.3), then the inverse, the cyclic shift and the reverse of r are also extents on n bells.

Definition 1.2.12. Let m be a positive integer. By affixing $n+1\; n+2\; \ldots\; n+m$ after each row of the ringing array of r we receive a new ringing array, which corresponds to a $(n+m)$ - bell ringing sequence called the **right vertical m - shift**.

Definition 1.2.13. Let m be a positive integer. By adding m to every array entry of the ringing array of r and by prefixing $1\; 2\; \ldots\; m$ before each row of the resulting array we receive a new ringing array, which corresponds to a $(n+m)$ - bell ringing sequence called the **left vertical m - shift**.

1.3 Call change ringing vs. method ringing

In call change ringing the conductor calls every single change. In other words, he tells the other ringers how to change the position of their bells from change to change. The most used transitions in call change ringing are done by calling up and calling down.

Definition 1.3.1. Let x and y be two bells.

(i) In **calling up**, the positions of the two bells mentioned have to be right next to each other in the change, with the first-named ringing prior to the second-named. The command "x to y" results in the swap of the position of these two bells. In short, the first-named bell **moves up** and thus further away from the lead.

(ii) In **calling down** "x to y", the first-named bell is ordered to move one position towards the lead. In other words x **moves down**. The second-named bell does not shift its place in the change, whereas the bell that swaps its position with x is not itself named in the call.

Example 1.3.2.

change	conductor's intent	successive change	call, if calling up	call, if calling down
$1\,2\,3\,\ldots\,n$	swap bells 2 and 3	$1\,3\,2\,4\,\ldots\,n$	"2 to 3"	"3 to treble" or "3 to 1"
$1\,2\,3\,\ldots\,n$	swap bells 1 and 2	$2\,1\,3\,\ldots\,n$	"1 to 2"	"2 to lead"

Table 1.2: Calling up and calling down on rounds.

In contrast to the work of the conductor (number of calls) in call change ringing, the ringers receive almost no or just very few instructions of him in method ringing. For the most part, ringers perform ringing sequences that are based on easy and elementary algorithms referred to as methods and principles because of their simplicity of being memorized.

1.3.1 Difference between method and principle

Mainly, this subsection follows again Burkhard Polster's book [P03].

Definition 1.3.3. Let G be a group. The **order** of an element $g \in G$ is the smallest $d \geq 1$ such that $g^{\mathrm{d}} = 1_{\mathrm{G}}$.

Let t be an n - bell transition sequence of length k with $n \geq 3$ and let $\tilde{t}$ be the permutation we get by multiplying all letters in t in the order, in which they occur (from left to right). It's obvious that $\tilde{t}$ has to be an element of S_n. Let d be the order of $\tilde{t} \in S_n$.

Definition 1.3.4. Suppose we start with rounds as our initial change. If t^d is the n - bell transition sequence of a ringing sequence s, then we define s as the **plain course of t**. Since s has length $kd+1$ (rounds count twice), the plain course can be divided into d blocks such that the first $d-1$ blocks are of length k and the last block has length $k+1$. Neglecting the last change of the last block, which is the second occurrence of rounds, each block is a so-called **plain lead**. The entries of the **lead ringing array** are the changes of the plain course such that its d columns are the d blocks.

Remark 1.3.5. Suppose we do not start with rounds as our initial change in Definition 1.3.4. Since it is not permitted to count a change apart from rounds twice, the plain course of t would have length kd instead of length $kd+1$ (the second appearance of the initial change does not count). As before, the plain course could be divided into d blocks, but this time all blocks would be of length k. Hence, in this case, each block is exactly a plain lead.

Remark 1.3.6. If it is clear from the context that we talk about a plain lead, then the word "plain" is not mentioned each time and we often call a plain lead just a lead.

Definition 1.3.7. One says that a bell is **not working** if this bell is fixed by the permutation $\tilde{t}$. Similarly, the bells which are not fixed by $\tilde{t}$ are said to be **working**.

Definition 1.3.8. The transition sequence t is a **principle** if $\tilde{t}$ is a n-cycle. Hence the order of the permutation $\tilde{t}$ is $d = n$. This is equivalent to the declaration that all bells are working.

Remark 1.3.9. Let $\tilde{t}^{-1}$ be the inverse of the permutation $\tilde{t}$. In a principle all n bells do the same work in the sense that the path that bell $j \in \{1, \ldots, n\}$ follows through the plain lead $i \in \{1, \ldots, d\}$ of the plain course is exactly followed by bell $\tilde{t}^{-1}(j)$ in the plain lead $i+1$ for $i+1 \leq d$ and in the plain lead 1 for $i+1 > d$. (for an example see Erin principle in subsection 2.2)

Definition 1.3.10. Vaguely speaking, a bell **hunts** if it waves back and forth across the ringing array of the plain course of t. (see subsection 4.3.1 called "Most popular ringing method: Plain Bob using Plain Hunt")

Definition 1.3.11. Suppose the treble and perhaps a few other bells do not work. Let w be the number of working bells. Then the transition sequence t is a **method** if $\tilde{t}$ is a w-cycle and the not working bells hunt.

Remark 1.3.12. As in principles, all bells that are working in a method do precisely the same amount of work (campare Remark 1.3.9). A hunting treble helps the ringers keep track of which change they are ringing in a given sequence and which change comes next.

Definition 1.3.13. A **cover** is a bell ringing at the very end of each change of a ringing sequence while the other bells ring a method.

Definition 1.3.14. Let t be a principle or a method. The transition sequence of an ***n*-bell extent** (not necessarily fulfilling conditions (iv)-(vi) of Definition 1.1.3) **of *t*** is a transition sequence $t_{ext} = t_1 t_2 \cdots t_l$ such that $\forall i \in \{1, \ldots, l\}$: t_i is a transition sequence of the same length as t and it differs from t in one letter of the word. Such different letters are called **calls**. In rare cases it may happen that t_i differs from t in more than one letter of the word.

Remark 1.3.15. Since the main goal is to ring an extent, ringers ring t again and again until they either arrive back at rounds, in which case the plain course of t is exactly an extent, or the conductor orders a call. If the plain course of t starting with rounds has length p, then there are at least $\frac{n!}{p-1} - 1$ calls needed to ring an extent.

Definition 1.3.16. The call **single** is a transition that interrupts the regular work of an even number of bells. The call **bob** is a transition which pauses the usual work of an odd number of bells.

Remark 1.3.17. If t_i in Definition 1.3.14 coincides with t, then it has length k and we call the sequence of the first k changes corresponding to t_i a **plain lead** (recall also Definition 1.3.4). If t_i in Definition 1.3.14 contains the call single, then it has length k and we call the sequence of the first k changes corresponding to t_i a **single lead**. Similarly, if t_i contains the call bob, then the associated sequence of the first k changes is referred to as a **bob lead**.

Definition 1.3.18. The first change of a (plain, bob or single) lead is called a **lead head**, while the last change of a lead is called a **lead end**.

2. n - bell extents for $n \leq 4$

Definition 2.0.1. The sequence of positions a chosen bell rings in a ringing array is called a **line**.

Claim 2.0.2. *The 1 - bell and the 2 - bell extents are unique.*

Proof. Looking at the according ringing arrays in Table 2.1, the statement follows straight-forward. ☐

2.1 3 - bell extents

Claim 2.1.1. *There exist solely two extents on three bells.*

Proof. $A = (1\,2)$ and $B = (2\,3)$ are the only transitions that can be used to construct a 3 - bell extent. Hence the transition sequence $ABABAB = (AB)^3$, which corresponds to the depicted ringing array in Table 2.1, describes an extent on three bells. The sole other 3 - bell extent is its inverse. ☐

1	1 2	1 2 3
1	2 1	2 1 3
	1 2	2 3 1
		3 2 1
		3 1 2
		1 3 2
		1 2 3

Table 2.1: Ringing arrays of extents on one, two and three bells. The red dashed line in the 3 - bell extent describes the position sequence of the treble.

Definition 2.1.2. The 3 - bell extent described by the transition sequence $(AB)^3$ is called **quick six** and the one described by $(BA)^3$ is called **slow six**.

2.2 4 - bell extents

Burkard Polster writes in [P03] that the mathematician Alexander E. Holroyd said that there exist exactly 10792 extents on 4 bells (162 up to inversion, cyclic shifts and reversals). Because there are this many extents we concentrate on the Erin, Reverse Erin, Stanton and Reverse Stanton principle and on eleven 4 - bell methods. The possible transitions on 4 bells are $A = (12)(34)$, $B = (23)$, $C = (34)$ and $D = (12)$.

2.2.1 Erin principle & Reverse Erin

The Erin principle on 4 bells has transition sequence $t = (DB)^2DA$. The associated permutation $\tilde{t}$ is given by $\tilde{t} = (1243)$ and its inverse is $\tilde{t}^{-1} = (3421)$. Looking at Table 2.2 and using Remark 1.3.9, one can recognize that the red dashed path of the treble in lead 1 is precisely mirrored by the paths of bells $\tilde{t}^{-1}(1) = 3$ in lead 2 (blue dashed), $\tilde{t}^{-1}(3) = 4$ in lead 3 (orange dashed) and $\tilde{t}^{-1}(4) = 2$ in lead 4 (green dashed). Thus all bells do exactly the same amount of work.
The Reverse Erin principle on 4 bells has transition sequence $t = (CB)^2CA$, permutation $\tilde{t} = (1243)$ and $\tilde{t}^{-1} = (3421)$. As before, all 4 bells are working the same amount.

1 2 3 4	3 1 4 2	4 3 2 1	2 4 1 3	1 2 3 4	3 1 4 2	4 3 2 1	2 4 1 3
2 1 3 4	1 3 4 2	3 4 2 1	4 2 1 3	1 2 4 3	3 1 2 4	4 3 1 2	2 4 3 1
2 3 1 4	1 4 3 2	3 2 4 1	4 1 2 3	1 4 2 3	3 2 1 4	4 1 3 2	2 3 4 1
3 2 1 4	4 1 3 2	2 3 4 1	1 4 2 3	1 4 3 2	3 2 4 1	4 1 2 3	2 3 1 4
3 1 2 4	4 3 1 2	2 4 3 1	1 2 4 3	1 3 4 2	3 4 2 1	4 2 1 3	2 1 3 4
1 3 2 4	3 4 1 2	4 2 3 1	2 1 4 3	1 3 2 4	3 4 1 2	4 2 3 1	2 1 4 3
			1 2 3 4				1 2 3 4

Table 2.2: The lead ringing arrays of the priciples Erin Minimus (on the left of the vertical line) and Reverse Erin Minimus (on the right of the vertical line).

As we can see in Table 2.2, all requirements of Definition 1.1.3 except (iv) are fulfilled. With the reasoning in Remark 1.1.4 we receive that the plain courses of the Erin and Reverse Erin principles starting with rounds themselves are 4 - bell extents. Further, we can check that if we replace every entry j of the ringing array of Erin Minimus by $(5 - j)$ and afterwards reflect the resulting array at the centred vertical axis, we get exactly the ringing array of Reverse Erin Minimus (recall Definition 1.2.10).

Remark 2.2.1. Let $t = T_1 T_2 \cdots T_k$ be an n-bell transition sequence of a principle with

$$\forall i \in \{1, \ldots, k\}: \quad T_i = (x_{1,i},\, x_{2,i})(x_{3,i},\, x_{4,i}) \cdots .$$

Then the transition sequence corresponding to the revere of the plain course of t is $\hat{t} = \widehat{T_1}\widehat{T_2} \cdots \widehat{T_k}$, where

$$\forall i \in \{1, \ldots, k\}: \quad \widehat{T_i} = (n+1-x_{1,i},\, n+1-x_{2,i})(n+1-x_{3,i},\, n+1-x_{4,i}) \cdots .$$

2.2.2 Stanton principle & Reverse Stanton

Like the Erin principle, the Stanton and Reverse Stanton principles are generated by three transitions each. The 4-bell principle Stanton Minimus has transition sequence $t = DBCBDA$ and the 4-bell principle Reverse Stanton Minimus has transition sequence $t = CBDBCA$.

1 2 3 4	2 4 1 3	4 3 2 1	3 1 4 2
2 1 3 4	4 2 1 3	3 4 2 1	1 3 4 2
2 3 1 4	4 1 2 3	3 2 4 1	1 4 3 2
2 3 4 1	4 1 3 2	3 2 1 4	1 4 2 3
2 4 3 1	4 3 1 2	3 1 2 4	1 2 4 3
4 2 3 1	3 4 1 2	1 3 2 4	2 1 4 3
			1 2 3 4

Table 2.3: The lead ringing array of the Stanton Minimus principle.

Once again, one can see that the plain course of the Stanton principle starting with rounds is a 4-bell extent. Using Remark 1.2.11, we see that the same holds for the Reverse Stanton principle. Condition (iv) of Definition 1.1.3 is still not fulfilled, but because of Remark 1.1.4, this does not bother us any further.

2.2.3 Eleven methods on four bells

There are eleven 4-bell methods for which the plain course of the transition sequence t starting with rounds coincides with being an extent (but only the first three listed satisfy condition (iv) of Definition 1.1.3, see [W85, p. 267]):

name of the method	transition sequence t	permutation $\tilde{t}$
Plain Bob	$(AB)^3AC$	(243)
Reverse Bob	$ABAD(AB)^2$	(243)
Double Bob	$ABADABAC$	(234)
Canterbury	$ABCDCBAB$	(243)
Reverse Canterbury	$DB(AB)^2DC$	(234)
Double Canterbury	$DBCDCBDC$	(243)
Single Court	$DB(AB)^2DB$	(243)
Reverse Court	$AB(CB)^2AB$	(234)
Double Court	$DB(CB)^2DB$	(234)
St. Nicholas	$DBADABDC$	(243)
Reverse St. Nicholas	$ABCDCBAC$	(234)

Table 2.4: By multiplying each letter in t from left to right we receive the permutation $\tilde{t}$. Since the order of $\tilde{t}$ is 3 and $\tilde{t}$ fixes the treble for each transition sequence, the treble is the only bell that hunts. Using Definition 1.3.11 we can see that these eleven transition sequences are indeed methods.

Remark 2.2.2. The transition sequences of the 4-bell extents of these eleven methods ($t_{ext} = t^3$) are all palindromic because every t is itself palindromic (recall Definition 1.2.6).

3. Existence of an extent

For the main part, this chapter follows the work of Arthur T. White ([W83] and [W85]). In addition, I used [P03] in subsections 3.1 & 3.2 and W. Cherowitzo's work [Che] for some proofs in subsection 3.3.

3.1 Construction of ringing sequences via elevation

There exists a simple procedure on how to turn an n - bell ringing sequence r starting and ending with rounds into an $(n+1)$ - bell ringing sequence r_{elev} that starts and ends with rounds.

Definition 3.1.1. The sequence r_{elev} that we will construct in this subsection is commonly known as the **elevation** of r.

Let $k+1$ be the length of the ringing sequence r, which starts and ends with rounds.

Definition 3.1.2. The **auxiliary array** is constructed in two stages: Firstly, we build an $((n+1)k+1)\times n$ array, where the first $n+1$ rows are the first change of r, the second $n+1$ rows the second change of r, and so forth. The last row of the array coincides with the first change of r. Secondly, we add 1 to every entry of the received array and as a result we get the so-called auxiliary array.

Given the auxiliary array of the ringing sequence r, one has to squeeze the number 1 into each row of the array such that this 1 is placed in positions

$$1, 2, \ldots, n, n+1, n+1, n, \ldots, 2, 1, 1, 2, \ldots, n, n+1, n+1, n, \ldots, 2, 1, 1, \ldots$$

in rows $1, 2, 3, 4, \ldots, ((n+1)k+1)$. The resulting array is the ringing array of the $(n+1)$ - bell ringing sequence r_{elev}.

Example 3.1.3. The following table shows how to turn a 3 - bell extent into a 4 - bell extent.

3 - bell extent r	auxiliary array			r_{elev}		
1 2 3	2 3 4	3 4 2	4 2 3	1 2 3 4	1 3 4 2	1 4 2 3
2 1 3	2 3 4	3 4 2	4 2 3	2 1 3 4	3 1 4 2	4 1 2 3
2 3 1	2 3 4	3 4 2	4 2 3	2 3 1 4	3 4 1 2	4 2 1 3
3 2 1	2 3 4	3 4 2	4 2 3	2 3 4 1	3 4 2 1	4 2 3 1
3 1 2	3 2 4	4 3 2	2 4 3	3 2 4 1	4 3 2 1	2 4 3 1
1 3 2	3 2 4	4 3 2	2 4 3	3 2 1 4	4 3 1 2	2 4 1 3
1 2 3	3 2 4	4 3 2	2 4 3	3 1 2 4	4 1 3 2	2 1 4 3
	3 2 4	4 3 2	2 4 3	1 3 2 4	1 4 3 2	1 2 4 3
			2 3 4			1 2 3 4

Table 3.1: The elevation of the ringing sequence of the 3 - bell extent is exactly the extent named Double Canterbury Minimus. The red and black colors I used in the auxiliary array do not have any special meaning other than pointing out the structure of the array.

Proposition 3.1.4. *r_{elev} is an $(n+1)$ - bell extent (without fulfilling condition (iv) of Definition 1.1.3) iff r is an n - bell extent.*

Proof. This follows directly from the construction of r_{elev}. ☐

Corollary 3.1.5. *For any integer $n \geq 1$, there exists an n - bell extent satisfying requirements (i)-(iii) of Definition 1.1.3.*

Proof. (by induction)
Base case: We know that there exists a unique 1 - bell and a unique 2 - bell extent.
Induction hypothesis: We assume that the statement of the corollary holds for n.
Induction step: Constructing the elevation of the n - bell extent and using Proposition 3.1.4 we receive the desired conclusion of our proof. ☐

3.2 Cayley color graph

Definition 3.2.1. Using initial condition $F_0 = F_1 = 1$, the ***n*th Fibonacci number** F_{n} is defined by the following recurrence relation:

$$F_{\text{n}} = F_{\text{n-1}} + F_{\text{n-2}}.$$

Theorem 3.2.2. *Let $n \geq 2$. The number of possible transitions for n bells $t(n)$ is given by*

$$t(n) = F_n - 1.$$

Proof. (by induction)
Base case: $t(2) = F_2 - 1 = 2 - 1 = 1$ and $t(3) = F_3 - 1 = 3 - 1 = 2$
Induction hypothesis: We assume that the statement holds for all $k < n$: $t(k) = F_{\mathrm{k}} - 1$.
Induction step: Considering $t(n)$, there are $t(n-1)$ admissible transitions fixing position n. In addition to the transposition $(n-1\,n)$, we have $t(n-2)$ further possible transitions interchanging positions n and $n-1$. Thus by using the induction hypothesis and Definition 3.2.1 we receive:

$$\begin{aligned} t(n) &= t(n-1) + 1 + t(n-2) \\ &= F_{\text{n-1}} - 1 + 1 + F_{\text{n-2}} - 1 \\ &= F_{\text{n-1}} + F_{\text{n-2}} - 1 \\ &= F_{\text{n}} - 1. \end{aligned}$$

□

Definition 3.2.3. Let $S \subseteq G$ be a subset of a group G. We say S is a **generating set** of this group if there does not exist a proper subgroup of G containing S.

Definition 3.2.4. A **directed graph** Γ (sometimes also called **digraph**) consists of a vertex set $V(\Gamma)$, an edge set $E(\Gamma) = V \times V$ and a function, which assigns an ordered pair of vertices (v_1, v_2) to each edge such that v_1 is the **tail** and v_2 is the **head** of this edge.

Definition 3.2.5. Let G be a group with generating set S. The **Cayley color graph** of G with respect to S, denoted by $C_S(G)$, is a colored directed graph such that the following three conditions hold:

(i) Every vertex of $C_S(G)$ corresponds to an element of the group G.
(ii) Every element $s \in S$ is assigned to a color c_s.
(iii) $\forall g_1 \in G$, $\forall s \in S$ the vertex corresponding to g_1 is connected with the vertex corresponding to $g_2 = g_1 s$ by a directed edge of color c_s.

Remark 3.2.6. If the set S in Definition 3.2.5 wouldn't be a generating set, then the Cayley color graph $C_S(G)$ wouldn't be a connected graph. Thus, $C_S(G)$ is connected iff the group G is generated by S.

Definition 3.2.7. An **automorphism of $C_S(G)$** is a permutation ϕ of the vertex set $V(C_S(G))$ such that $\forall g_1,\ g_2 \in G$ and $\forall s \in S$ we have $g_2 = g_1 s$ iff $\phi(g_2) = \phi(g_1)s$. The union of all automorphisms of $C_S(G)$ forms the so-called **automorphism group of $C_S(G)$**.

Remark 3.2.8. The automorphism group of $C_S(G)$ acts transitively on the vertices $V(C_S(G))$. Hence, if we do not label the vertices in the graph $C_S(G)$, we do not lose any information about the graph itself.

Definition 3.2.9. The graph underlying the Cayley color graph $C_S(G)$ satisfying the following three requirements:

(i) The identity element 1_G is not contained in S.
(ii) If $s \in S$ then, unless $s^2 = 1_G$, the inverse element s^{-1} is not contained in S.
(iii) If $s \in S$ and $s^2 = 1_G$, then every pair of directed edges (g, gs) and (gs, s) is merged into one single undirected edge $\{g, gs\}$.

is called a **Cayley graph** and is denoted by $Cay_S(G)$.

Definition 3.2.10. A **path** in a graph Γ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k \in V(\Gamma)$ and a sequence of edges $(v_i, v_{i+1}) \in E(\Gamma)$. If $k \geq 3$ and $(v_k, v_1) \in E(\Gamma)$, then the path according to the sequence $v_1, v_2, \ldots, v_k, v_1$ is called a **cycle**.

Definition 3.2.11. A path in a graph Γ that visits each vertex of $V(\Gamma)$ exactly once is called a **Hamiltonian path**. If in addition the Hamiltonian path is a cycle, then we call it a **Hamiltonian cycle**. A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

Theorem 3.2.12. *Let $n \geq 3$ and let S be an arbitrary collection of n - bell transitions (products of disjoint transpositions of consecutive numbers) which generates S_n. An n - bell extent, fulfilling conditions (i)-(iii) (and not necessarily (iv)-(vi)) of Definition 1.1.3 and using transitions from S only, can be rung iff $C_S(S_n)$ is Hamiltonian.*

Proof. The statement is apparent following the above definitions. For instance, condition (ii) is fulfilled because S is a generating set of S_n and because there exists a Hamiltonian cycle in $C_S(S_n)$. Condition (iii) results directly from the definition of the set S. Furthermore, the assumption $n \geq 3$ is essential since otherwise the definition of what it means for a path to be a cycle would not make sense (recall Definition 3.2.10). □

Remark 3.2.13. Let S be defined as in Theorem 3.2.12. Every Hamiltonian cycle in the underlying Cayley graph $Cay_S(S_n)$ corresponds to two oriented Hamiltonian cycles in $C_S(S_n)$. The ringing sequences corresponding to these two Hamiltonian cycles in $C_S(S_n)$ are inverses of each other. Thus, if $Cay_S(S_n)$ is Hamiltonian then $C_S(S_n)$ is Hamiltonian as well.

Remark 3.2.14. If S, defined as in Theorem 3.2.12, contains all possible transitions for n bells, then there are exactly $t(n)$ edges incident to every vertex of $C_S(S_n)$ (or to every vertex of $Cay_S(S_n)$). Recall Theorem 3.2.2 concerning the number $t(n)$.

Example 3.2.15. Let $n = 3$ and let $S = \{A = (12), B = (23)\}$. By labelling the lower-left vertex of the 3 - bell Cayley graph $Cay_S(S_3)$ in Figure 3.1 by rounds (or rather by 1_{S_3}), we get exactly one Hamiltonian cycle in this Cayley graph that starts at this vertex. The transition sequences corresponding to the two Hamiltonian cycles in $C_S(S_3)$ are $(AB)^3$ (describing the 3 - bell extent in Table 2.1) and $(BA)^3$ (describing its inverse).

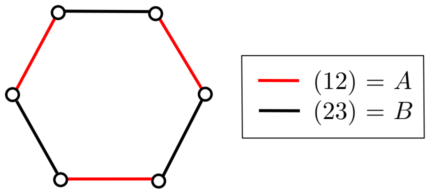


Figure 3.1: The 3 - bell Cayley graph $Cay_S(S_3)$.

3.2.1 Detailed example for 4 bells

Let $S = \{A = (12)(34), B = (23), C = (34), D = (12)\}$.

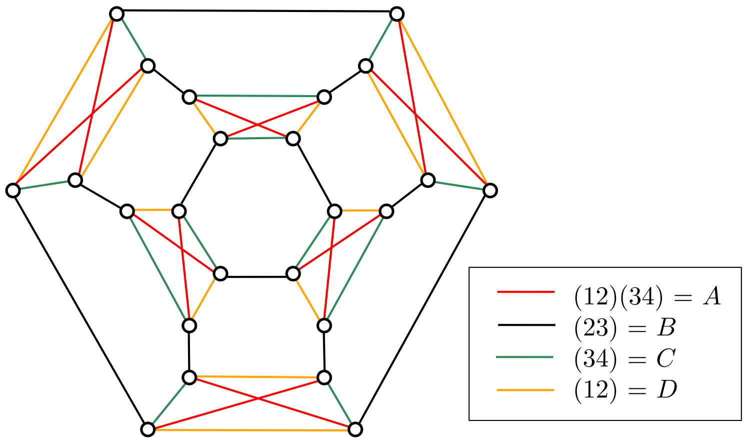


Figure 3.2: The 4 - bell Cayley graph $Cay_S(S_4)$.

Definition 3.2.16. The **order of rotational symmetry** is the number of times a shape or figure can be rotated and still look the same as it did before the first rotation.

Remark 3.2.17. The 4 - bell Cayley graph $Cay_S(S_4)$ has rotational symmetry of order three.

Remark 3.2.18. The spacial rendering of $Cay_S(S_4)$ resembles a truncated octahedron with diagonals across its square faces.

Definition 3.2.19. The **crossing number of a graph** Γ, denoted by $cr(\Gamma)$, is the minimum number of edge crossings of a plane drawing of Γ. For instance, a **planar** graph Γ has crossing number $cr(\Gamma) = 0$. Thus if Γ is **non-planar**, then its crossing number is bigger than or equal to 1.

Remark 3.2.20. Figure 3.2 shows that the crossing number $cr(Cay_S(S_4))$ is at most six.

The following figure shows the Hamiltonian cycles in $Cay_S(S_4)$ corresponding to the plain courses of the Erin (tranistion sequence $(DB)^2DA$), Reverse Erin ($(CB)^2CA$), Stanton ($DBCBDA$) and Reverse Stanton ($CBDBCA$) principle on 4 bells.

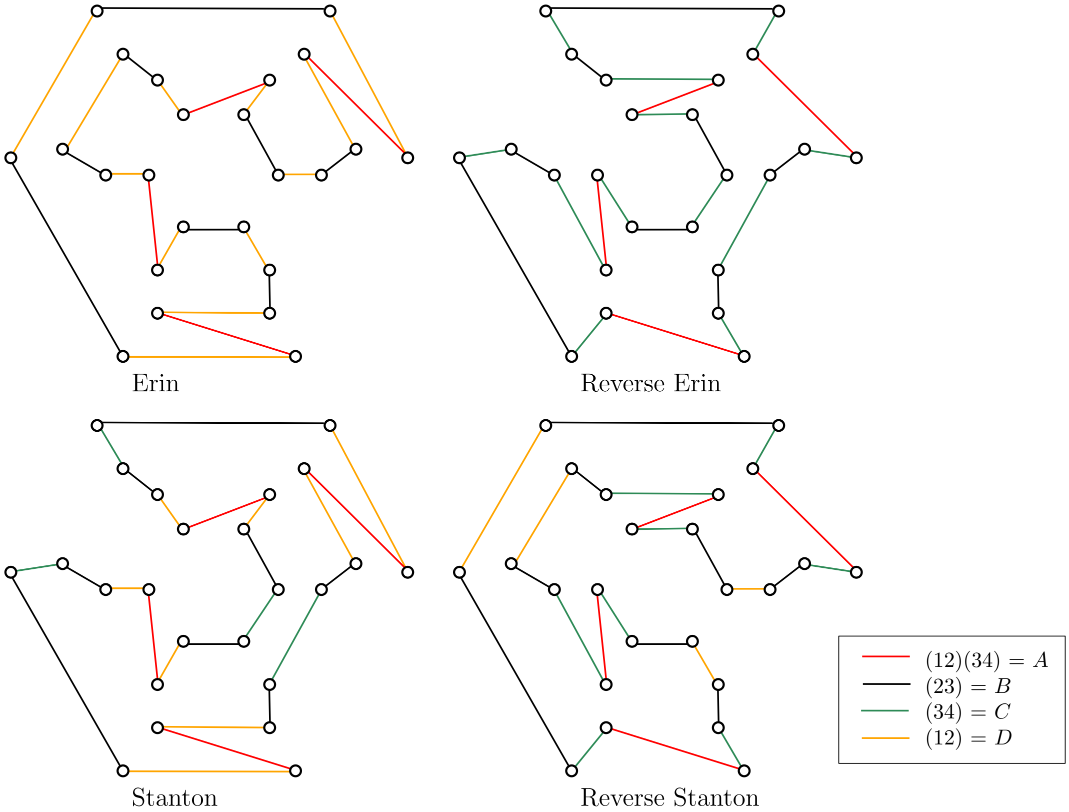
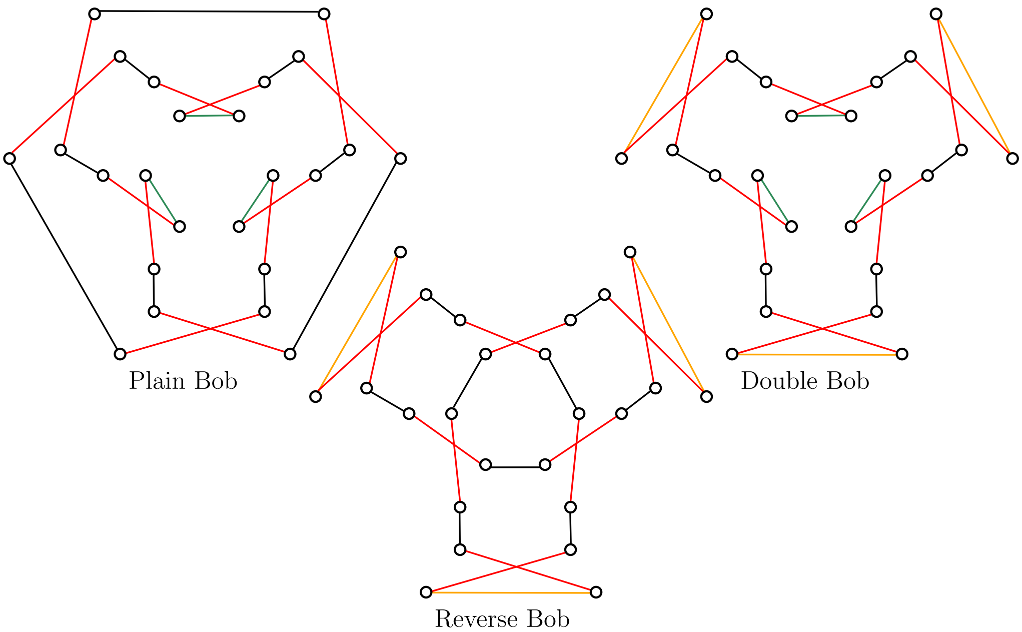


Figure 3.3: We label the upper-left vertex of the inner hexagon of $Cay_S(S_4)$ in Figure 3.2 by rounds (or rather by 1_{S_4}), which means it's the starting vertex for all four Hamiltonian cycles that correspond to Erin Minimus $(((DB)^2DA)^4)$, Reverse Erin Minimus $(((CB)^2CA)^4)$, Stanton Minimus $((DBCBDA)^4)$ and to Reverse Stanton Minimus $((CBDBCA)^4)$ respectively.

In the following figure, we display the Hamiltonian cycles in $Cay_S(S_4)$ corresponding to the plain courses of the eleven 4-bell methods mentioned in Table 2.4. All eleven extents on 4 bells have divisions into 3 leads of 8 changes each. These symmetries of order three of the extents are converted into rotational symmetries of order three of the according Hamiltonian cycles. Remark 2.2.2 even implies that the cycles have to be mirror symmetric.



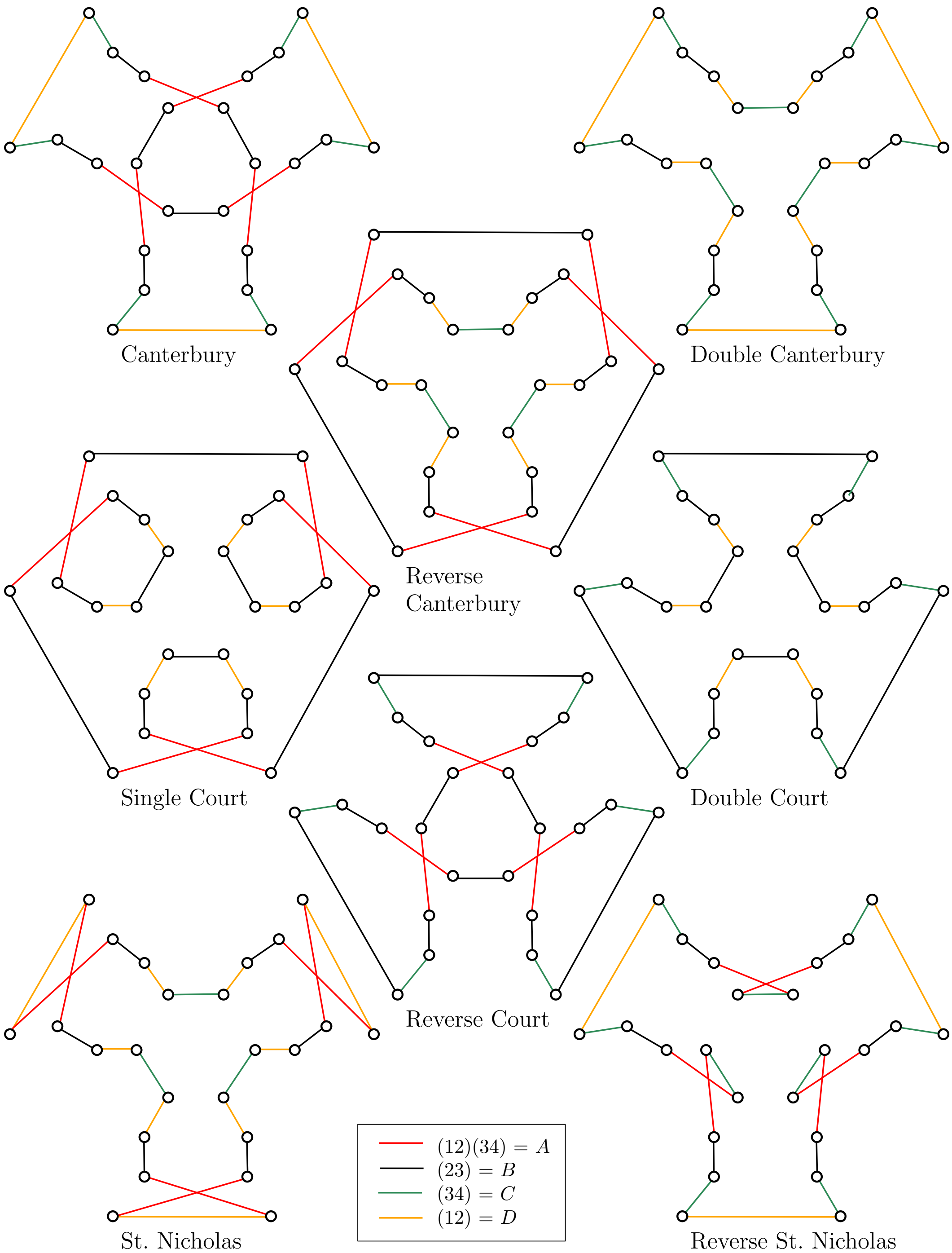


Figure 3.4: We label the upper-left vertex of the inner hexagon of $Cay_S(S_4)$ in Figure 3.2 by rounds (or rather by 1_{S_4}), which means it represents the starting vertex for all eleven Hamiltonian cycles that correspond to Plain Bob Minimus $(((AB)^3AC)^3)$, Reverse Bob Minimus $((ABAD(AB)^2)^3)$, Double Bob Minimus $((ABADABAC)^3)$, Canterbury Minimus $((ABCDCBAB)^3)$, Reverse Canterbury Minimus $((DB(AB)^2DC)^3)$, Double Canterbury Minimus $((DBCDCBDC)^3)$, Single Court Minimus $((DB(AB)^2DB)^3)$, Reverse Court Minimus $((AB(CB)^2AB)^3)$, Double Court Minimus $((DB(CB)^2DB)^3)$, St. Nicholas Minimus $((DBADABDC)^3)$ and to Reverse St. Nicholas Minimus $((ABCDCBAC)^3)$ respectively.

3.3 Rapaport's construction: Existence of an extent using only three different transitions

Definition 3.3.1. Given a graph Γ and a vertex $v \in V(\Gamma)$, we define the **degree** of v to be the number of neighbours of v. A vertex $v \in V(\Gamma)$ is **isolated** if its degree is zero.

Definition 3.3.2. Let d be a natural number. A graph is called **regular of degree *d*** if every vertex has d neighbouring vertices, in other words, if the degree of every vertex is d.

Elvira Strasser Rapaport based her proof on the following Lemma:

Lemma 3.3.3. *A connected regular graph of degree three, denoted by Γ, has a Hamiltonian cycle if there exists a set of polygons P and a set of quadrilaterals Q such that both P and Q partition the vertex set of the graph and no $p \in P$ contains all vertices of some $q \in Q$.*

Proof. (case by case)
Assume we are given the sets P and Q as defined in the statement of the Lemma. Firstly, we define the following operation:

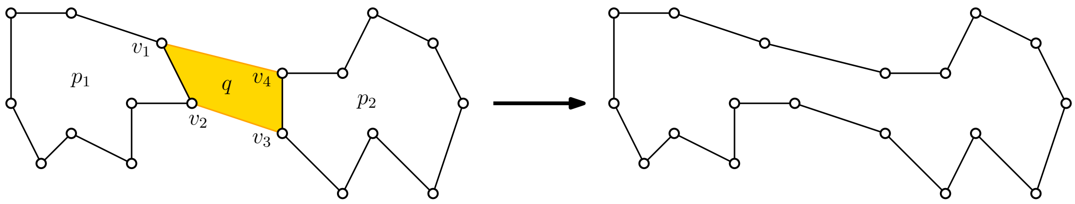


Figure 3.5: Let $p_1, p_2 \in P$. If there exists a quadrilateral $q \in Q$ which ties p_1 and p_2 together, then by removing the edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ we get a new polygon.

By repeating the above operation until it is not longer possible, we receive a new set of polygons, denoted by $\widehat{P}$. Like P, $\widehat{P}$ is still a disjoint union of polygons that covers the vertex set $V(\Gamma)$. If $\widehat{P}$ contains just a single polygon $\widehat{p}$, then the sequence of all vertices and all edges on the outline of $\widehat{p}$ is a Hamiltonian cycle and hence the proof would be finished.
If $\widehat{P}$ contains more than one polygon, then due to the connectedness of Γ there exist $\widehat{p_1}, \widehat{p_2} \in \widehat{P}$ and an edge $e = \{v_5, v_6\} \in E(\Gamma)$ with $v_5 \in \widehat{p_1}$ and $v_6 \in \widehat{p_2}$ such that the two polygons $\widehat{p_1}$ and $\widehat{p_2}$ are joined together by e.
Based on our precondition that Γ is a regular graph of degree three, we know that there are exactly two other edges besides e which are incident to the vertex v_5. If e is in some $q_e \in Q$, then the only two other incident edges to v_5, defined as $\{v_5, v_{\widehat{p_1}}\}$ and $\{v_5, \widetilde{v_{\widehat{p_1}}}\}$, have to be in $\widehat{p_1}$ and one of them, say for instance $\{v_5, v_{\widehat{p_1}}\}$, is contained in q_e. Similarly, there is an edge $\{v_6, v_{\widehat{p_2}}\} \in q_e$. Therefore, one could apply the operation of Figure 3.5 again, but this is a direct contradiction to the definition of $\widehat{P}$. Thus e is in no $q_e \in Q$.
Based on the assumption that the elements of Q form a partition of $V(\Gamma)$, we know that v_5 is contained in some $q \in Q$. Hence, by using the facts that every vertex is of degree three and e cannot be in q, the two edges $\{v_5, v_{\widehat{p_1}}\}$ and $\{v_5, \widetilde{v_{\widehat{p_1}}}\}$ have to be in q. That is why $\widehat{p_1}$ contains three vertices and two consecutive edges of q. These two consecutive edges must have been in some $p_i \in P$ from the beginning because they could not have been arisen from the operation of Figure 3.5. The forth vertex of q, defined as v, cannot be contained in this p_i due to our hypothesis that no polygon contains all vertices of some quadrilateral. Further, v is not in any other polygon $p_j \in P \setminus \{p_i\}$. If v would be part of a polygon $p_j \in P \setminus \{p_i\}$, then the vertex would have to be connected to two vertices of p_j plus to two vertices of p_i. However, since the degree of v has to be three, the two polygons p_i and p_j are forced to have a common vertex. This is in direct contradiction to the definition of the set P and proves that $\widehat{P}$ contains only one polygon. ☐

Remark 3.3.4. A polygon $p \in P$ with k vertices (a k-polygon) can be seen as a k-cycle in Γ. Thus, a quadrilateral $q \in Q$ is the same as a 4-cycle.

Lemma 3.3.5. *For $n \geq 2$, the symmetric group S_n is generated by its transposition.*

Proof. Clearly, this holds for $n = 2$. For $n \geq 3$, we see that $(1) = (12)^2$ and every k-cycle with $k > 2$ is a product of transpositions:

$$(i_1 \dots i_k) = (i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i_k).$$

Using the facts that cycles generate the symmetric group S_n and all cycles are products of transpositions, we conclude that S_n is generated by transpositions. ☐

The following Lemma shows that two n-bell transitions do not suffice to ring an n-bell extent for n bigger than three.

Lemma 3.3.6. *For $n \geq 4$, let F, G be two involutions in S_n such that both involutions are products of disjoint transpositions of consecutive numbers, meaning they are n-bell transitions. Then the group generated by the two involutions F and G is never the whole symmetric group S_n.*

Proof. The proof needs the following claim:

Claim 3.3.7. *Any set of transpositions which generates the symmetric group S_n has to contain at least $n-1$ transpositions.*

Proof. Let $\alpha_1, \ldots, \alpha_k$ be distinct transpositions in S_n such that $\langle\{\alpha_1, \ldots, \alpha_k\}\rangle = S_n$. It remains to show $k \geq n-1$. To accomplish that we'll use graph theory. We create a graph Γ with vertex set $V(\Gamma) = \{1, \ldots, n\}$ and edge $\{i, j\} \in E(\Gamma)$ if $(ij) \in \{\alpha_1, \ldots, \alpha_k\}$. Hence our graph consists of n vertices and k edges. Given two vertices i and j of Γ, we say there is an element in $\langle\{\alpha_1, \ldots, \alpha_k\}\rangle$ which sends i to j iff there exists a path in Γ starting at vertex i and ending at vertex j. Γ is connected as a result of our assumption $\langle\{\alpha_1, \ldots, \alpha_k\}\rangle = S_n$. A connected graph with n vertices has to contain at least $n-1$ edges, thus $k \geq n-1$. $\square_{\text{Claim}}$

We assume that S_n is generated by only two n-bell transitions. Using Claim 3.3.7, we realise that $F = (12)(34)(56)\cdots$ and $G = (23)(45)(67)\cdots$ are the only involutions in S_n that possibly could do exactly that, since they are the only two n-bell transitions that together are created from $n-1$ different transpositions. The next claim proves that for n bigger than three, S_n is not generated by F and G and as a result is not generated by two n-bell transitions.

Claim 3.3.8. $\forall n \geq 4: \quad |\langle\{F, G\}\rangle| < |S_n|$

Proof. The products of FG and GF can be written as

$$FG = \begin{cases} (1\,3\,5\,7\,\ldots\,n-2\,n\,n-1\,n-3\,\ldots\,6\,4\,2), & \text{if } n \text{ is odd} \\ (1\,3\,5\,7\,\ldots\,n-3\,n-1\,n\,n-2\,\ldots\,6\,4\,2), & \text{if } n \text{ is even} \end{cases} \quad \text{and}$$

$$GF = \begin{cases} (2\,4\,6\,8\,\ldots\,n-1\,n\,n-2\,n-4\,\ldots\,5\,3\,1), & \text{if } n \text{ is odd} \\ (2\,4\,6\,8\,\ldots\,n-2\,n\,n-1\,n-3\,\ldots\,5\,3\,1), & \text{if } n \text{ is even.} \end{cases}$$

Since $F^{-1} = F$, $G^{-1} = G$ and $(FG)^n = 1_{S_n} = (GF)^n$, the set generated by F and G is given by

$$\begin{aligned} \langle\{F, G\}\rangle = \{&\overbrace{1_{S_n}, F, G,}^{3 \text{ elements}} \overbrace{FG, FGF, (FG)^2, (FG)^2F, \ldots, (FG)^{(n-1)}, (FG)^{(n-1)}F,}^{2n-2 \text{ elements}} \\ &\underbrace{GF, GFG, (GF)^2, (GF)^2G, \ldots, (GF)^{(n-1)}, (GF)^{(n-1)}G}_{2n-2 \text{ elements}}\}. \end{aligned} \tag{1}$$

It is possible that two or more elements of the set, which is displayed on the right hand side of equation (1), coincide. (If, for example, n is equal 4, then we have $(FG)^2 = (14)(23) = (GF)^2$.) Thus, equation (1) yields the following relation:

$$|\langle\{F, G\}\rangle| \leq 3 + (2n-2) + (2n-2) = 4n - 1.$$

Claim 3.3.9. $\forall n \geq 4: \quad 4n - 1 < n!$

Proof. (by induction)
Base case: $n = 4: \quad 15 = 4 \cdot 4 - 1 < 24 = 4!$
Induction hypothesis: We assume that the statement holds for all integers smaller than n, in particular, $4(n-1) - 1 < (n-1)!$.
Induction step: Adding 4 to both sides of the induction hypothesis gives us the following inequality:

$$4n - 1 < (n-1)! + 4. \tag{2}$$

Based on our assumption that $(n-1) \geq 4$, it follows immediately that $4 < 24 = 4! \leq (n-1)!$. Taking the right hand side of inequality (2) and doing this exact estimation leads us to the following inequality:

$$(n-1)! + 4 < (n-1)! + (n-1)! = 2(n-1)! < n(n-1)! = n!. \tag{3}$$

The combination of inequalities (2) and (3) yields the desired result:

$$4n - 1 < (n-1)! + 4 < n!.$$

$\square_{\text{Claim 3.3.9}}$

Claim 3.3.9 implies that for $n \geq 4$ we have $|\langle\{F, G\}\rangle| < |S_n|$. Consequently, we have finished the proof of Claim 3.3.8. $\square_{\text{Claim 3.3.8}}$

$\square$

Lemma 3.3.10. *For $n \geq 2$, the symmetric group S_n is generated by the following three permutations:*

$$E = (12), \quad F = (12)(34)(56)\cdots, \quad G = (23)(45)(67)\cdots.$$

Proof. The statement is clear for $n = 2$. Therefore, we assume that $n \geq 3$. The product of EFG is given by

$$EFG = \begin{cases} (2\,3\,5\,7\ \ldots\ n-2\ n\ n-1\ n-3\ \ldots\ 6\,4), & \text{if } n \text{ is odd} \\ (2\,3\,5\,7\ \ldots\ n-3\ n-1\ n\ n-2\ \ldots\ 6\,4), & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

Let x be an integer in $\{2, 3, \ldots, n\}$. We choose $k \in \{1, 2, 3, \ldots, n\}$ such that $(EFG)^k$ maps 2 to x. The existence of such an integer k follows directly from equation (4). As a result, we can write the transposition $(1x)$ in the following way:

$$(EFG)^{-k}E(EFG)^k = (1x).$$

Moreover, it is possible to write any transposition (xy) with $x, y \neq 1$ as $(xy) = (1x)(1y)(1x)$. It is for this reason and because of Lemma 3.3.5 that the symmetric group S_n is generated by E, F and G. $\square$

Theorem 3.3.11. *(Rapaport)*
For any integer $n \geq 4$, the Cayley color graph $C_S(S_n)$ with

$$S = \{E = (12),\ F = (12)(34)(56)\cdots,\ G = (23)(45)(67)\cdots\}$$

is Hamiltonian.

Proof. Let the set S be given as described in the theorem. Then the Cayley color graph $C_S(S_n)$ is regular of degree three, and by Remark 3.2.6 and Lemma 3.3.10 $C_S(S_n)$ is connected. It remains to show that the other assumptions of Lemma 3.3.3 are satisfied. We start at an arbitrary vertex in $C_S(S_n)$ and follow the path that corresponds to the transition sequence $EFEF$. The product $EFEF$ is exactly the identity transformation 1_{S_n} and as a consequence a quadrilateral is described by the above-mentioned path. If two quadrilaterals generated in such a way have a common vertex, then they must be precisely identical. Let Q be the set of all $\frac{n!}{4}$ quadrilaterals that are formed in this way. The vertex set $V(C_S(S_n))$ is partitioned by Q. We divide the rest of the proof into two cases:

$n > 4$: Since $(EG)^6 = 1_{S_n}$, let P be the set of all 12 - polygons that are determined from the transition sequence $(EG)^6$. There exist $\frac{n!}{12}$ disjoint 12 - polygons in P. We know that F cannot be equal to a product of alternating E and G by considering what happens, for example, to the integers 4 and 5. Consequently, no 12 - polygon $p \in P$ is able to contain three vertices of some quadrilateral $q \in Q$. In other words, no $p \in P$ contains all vertices of some $q \in Q$.

$n = 4$: Since $(EG)^3 = 1_{S_n}$, replace the 12 - polygons of P by the 6 - polygons that are formed by the transition sequence $(EG)^3$. There exist $\frac{n!}{6}$ such disjoint 6 - polygons in P. As before, F cannot be equal to a product of alternating E and G since each such product does not map the integer 3 to the number 4. Thus, no 6 - polygon $p \in P$ contains three (or accordingly all) vertices of some quadrilateral $q \in Q$.

Therefore, all assumptions of Lemma 3.3.3 are fulfilled and hence there exists a Hamiltonian cycle in $C_S(S_n)$. $\square$

Corollary 3.3.12. *For any integer $n \geq 2$, there exists an n - bell extent satisfying requirements (i)-(iii) of Definition 1.1.3 that uses only the following three transitions:*

$$E = (12), \quad F = (12)(34)(56)\cdots, \quad G = (23)(45)(67)\cdots.$$

Proof. The cases $n = 2$ and $n = 3$ clearly hold as is visible in Table 2.1. For $n \geq 4$, theorems 3.3.11 and 3.2.12 yield the requested result. $\square$

4. Composing extents using group theory

As I briefly mentioned in my preface, bell ringers made use of group-theoretic ideas, such as decomposing groups into cosets of other groups, long before (more precisely 200 years before) groups and their properties as for example Lagrange's theorem were formally introduced. As a matter of fact, Stedman's book "Campanalogia" was first published in 1677 and later revised and republished in 1766 ([S66]). Another relatively early work (though more than 100 years after Stedman) that contains examples of ringing sequences is Shipway's book "The Campanalogia: or, Universal Instructor in the Art of Ringing: in three parts" ([S16]), which was published in 1816. The following subsection is a short repetition of cosets. I primarily used the handwritten lecture notes I took when attending E. Kowalski's lecture "Algebra I & II" at ETH. Unfortunately, there does not exist an online or published version I could note in my bibliography. Please take into consideration that in some literature the definitions of left and right cosets are precisely reversed. I present them the way I was taught by professor Kowalski.

4.1 Recall: Cosets and Lagrange's theorem

Let G be a group.

Definition 4.1.1. Let $g \in G$ be an arbitrary element. Further, let H be a subgroup of G.

(i) The **left H-coset** of g is $Hg = \{y \in G \mid \exists h \in H : y = hg\} \subset G$.
(ii) A subset $X \subset G$ is called a **left H-coset** if $X = Hg$ for some element $g \in G$.
(iii) The **right H-coset** of g is $gH = \{y \in G \mid \exists h \in H : y = gh\} \subset G$.
(iv) A subset $X \subset G$ is called a **right H-coset** if $X = gH$ for some element $g \in G$.

Notation 4.1.2. If H is a subgroup of G, then we denote this by writing $H < G$.

Proposition 4.1.3. *If $H < G$ is given, then the left H-cosets (and accordingly the right H-cosets) form a partition of G.*

Proof. We only show the statement for the left H-cosets since it is nearly the same for the right H-cosets. The proof contains the following three parts:

(i) A left H-coset Hg is not empty because it contains $g = 1_{\mathrm{H}}g$.
(ii) For all $g \in G$, $g \in Hg$ and thus every g is contained in some H-coset. $\Rightarrow \bigcup\limits_{g \in G} Hg = G$
(iii) Suppose X_1, X_2 are two left H-cosets that intersect: $X_1 \cap X_2 \neq \emptyset$. Hence $X_1 = Hg_1$ and $X_2 = Hg_2$ for some $g_1, g_2 \in G$. Let y be in the intersection $X_1 \cap X_2$, then there exist $h_1 \in H$ such that $y = h_1 g_1$ and $h_2 \in H$ such that $y = h_2 g_2$.
Claim 4.1.4. *$X_1 \subset X_2$*
Proof. Let $x \in X_1$, so $x = hg_1$ for some $h \in H$. We obtain $x = (hg_1 g_2^{-1})g_2 = (hh_1^{-1}h_2)g_2 \in X_2$ since $h_1 g_1 = h_2 g_2 \Leftrightarrow g_1 g_2^{-1} = h_1^{-1} h_2$ and $hh_1^{-1}h_2 \in H$. $\square_{\text{Claim}}$
In a similar way, one can check that $X_2 \subset X_1$, so that $X_1 = X_2$. In conclusion, we get that if $X_1 \neq X_2$, then $X_1 \cap X_2 = \emptyset$.

$\square$

Definition 4.1.5. Let $H < G$. The **equivalence relation** $g \,{}^{\mathrm{H}}\!\!\sim y$ on G defined by $g \,{}^{\mathrm{H}}\!\!\sim y :\Leftrightarrow y \in Hg$ gives us a **quotient set**, which is denoted by ${}_H\backslash^G$. In a similar way, the equivalence relation $g \sim^{\mathrm{H}} y$ on G defined by $g \sim^{\mathrm{H}} y :\Leftrightarrow y \in gH$ gives us the quotient set ${}^G/_H$.

Theorem 4.1.6. *(Lagrange)*
Let G be finite and $H < G$. Then $|H|$ divides $|G|$ and $\frac{|G|}{|H|} = |{}_H\backslash^G|$ (and accordingly $\frac{|G|}{|H|} = |{}^G/_H|$).

Proof. Once again, we only show the equality for the set of left cosets since it can be proven similarly for the set of right cosets. ${}_H\backslash^G$ is a finite set since it is a subset of $\mathcal{P}(G)$. Let $d = |{}_H\backslash^G|$ and let $y_1, \ldots, y_{\mathrm{d}}$ be elements in distinct left H-cosets such that ${}_H\backslash^G = \{Hy_1, \ldots, Hy_{\mathrm{d}}\}$. With Proposition 4.1.3 we get:

$$|G| = |Hy_1| + \ldots + |Hy_{\mathrm{d}}|$$

Claim 4.1.7. $\forall y \in G : |Hy| = |H|$

Proof. Let us construct a bijection $f\colon H \to Hy$, $h \mapsto hy$. It remains to show the well-definedness of f. The definition of Hy implies that f is surjective. Let $h_1, h_2 \in H$ be such that $f(h_1) = f(h_2)$. Then $h_1 y = h_2 y \Rightarrow h_1 = h_2$ and thus f is injective. This concludes the proof of the claim. $\square_{\text{Claim}}$

Assuming Claim 4.1.7, we receive $|G| = |H| \cdot d$ and therefore we have finished the proof of Lagrange's theorem. $\square$

Definition 4.1.8. $[G : H] \coloneqq |{}_H\backslash^G|$ (and accordingly $[G : H] \coloneqq |{}^G/_H|$) is called the **index of H in G**.

4.2 Explanation on Plain Bob Minimus

In this small subsection, I mainly used A. T. White's work, which can be found in [W87], and Burkhard Polster's book [P03].

Recall the transition sequence of the 4-bell Plain Bob method in Table 2.4: $t = (AB)^3AC$ where $A = (12)(34)$, $B = (23)$ and $C = (34)$. The product of this transition sequence gives us the associated permutation $\tilde{t} = (243)$. The lead ringing array according to this method is displayed in the following table:

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

Table 4.1: The lead ringing array of Plain Bob Minimus.

Definition 4.2.1. For $n < \infty$, the **dihedral group $\boldsymbol{D_n}$** of degree n has order $2n$ and is defined as:

$$D_n \coloneqq \langle x, y \mid x^n = y^2 = 1,\ yxy^{-1} = x^{-1} \rangle.$$

In the special case of $n = \infty$, we define the dihedral group of infinite degree as follows:

$$D_\infty \coloneqq \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

Remark 4.2.2. $\forall n < \infty : \quad D_n \cong \langle y, z \mid y^2 = z^2 = (yz)^n = 1 \rangle$

The elements of the symmetric group S_4 that correspond to the changes in the first lead (first column) in Table 4.1 form a group, which is isomorphic to the dihedral group D_4 of degree 4 and order 8 because of Remark 4.2.2 and $(AB)^4 = (1342)^4 = 1_{S_4}$.

$$D_4 \cong \{1_{S_4}, A, AB, ABA, (AB)^2, (AB)^2A, (AB)^3, (AB)^3A\} = \langle \{A, B\} \rangle$$

Note that the elements of the above group are ordered in the way they appear in the ringing sequence. The second lead (second column) of the lead ringing array is given by the right coset

$$\tilde{t}D_4 \cong \{\tilde{t}, \tilde{t}A, \tilde{t}AB, \tilde{t}ABA, \tilde{t}(AB)^2, \tilde{t}(AB)^2A, \tilde{t}(AB)^3, \tilde{t}(AB)^3A\},$$

and the third lead (third column) by the right coset

$$\tilde{t}^2D_4 \cong \{\tilde{t}^2, \tilde{t}^2A, \tilde{t}^2AB, \tilde{t}^2ABA, \tilde{t}^2(AB)^2, \tilde{t}^2(AB)^2A, \tilde{t}^2(AB)^3, \tilde{t}^2(AB)^3A\}.$$

As a consequence, we have a decomposition of S_4 into the three aforementioned right cosets. Since $(\tilde{t}^2(AB)^3A)C = \tilde{t}^2((AB)^3AC) = \tilde{t}^3 = 1_{S_4}$, the plain course of t starting with rounds coincides with being an extent.

Instead of focusing on the columns of the array in Table 4.1, we can also look at the rows of the lead ringing array. The elements of S_4 that correspond to the first row form a cyclic group $H = \{1_{S_4}, \tilde{t}, \tilde{t}^2\}$. Because of Lagrange's theorem 4.1.6, the index of H in S_4 is equal $[S_4 : H] = |{}_H\backslash^{S_4}| = \frac{|S_4|}{|H|} = \frac{4!}{3} = 8$. Thus, the following seven disjoint left H-cosets together with H itself form a partition of S_4:

$$\begin{aligned} HA &= \{A, \tilde{t}A, \tilde{t}^2A\} \\ HAB &= \{AB, \tilde{t}AB, \tilde{t}^2AB\} \\ HABA &= \{ABA, \tilde{t}ABA, \tilde{t}^2ABA\} \\ H(AB)^2 &= \{(AB)^2, \tilde{t}(AB)^2, \tilde{t}^2(AB)^2\} \\ H(AB)^2A &= \{(AB)^2A, \tilde{t}(AB)^2A, \tilde{t}^2(AB)^2A\} \\ H(AB)^3 &= \{(AB)^3, \tilde{t}(AB)^3, \tilde{t}^2(AB)^3\} \\ H(AB)^3A &= \{(AB)^3A, \tilde{t}(AB)^3A, \tilde{t}^2(AB)^3A\}. \end{aligned}$$

There is another decomposition into left cosets which is crucial to Plain Bob in particular. Let the subgroup $F_4 \cong S_3$ of S_4 consist of all permutations stabilizing 1:

$$F_4 = \{1_{S_4}, (23), (243), (24), (234), (34)\}.$$

These permutations correspond to rows one and eight in Table 4.1. The other three left F_4-cosets in S_4 correspond to rows $\pm i \mod 9$, where $i = 2, 3, 4$:

$$\begin{aligned} F_4A &= \{(12)(34), (2431), (123), (2341), (241), (12)\} \\ F_4AB &= \{(1342), (24)(31), (13), (341), (2413), (132)\} \\ F_4ABA &= \{(14), (23)(14), (1432), (421), (2314), (143)\} \end{aligned}$$

Accordingly, each half lead (consisting of the first four rows or the second four rows of a plain lead) intersects each of the four left F_4-coset in S_4 in exactly one element. In other words, each half lead is a left transversal of F_4 in S_4.

Another rather important decomposition for bell ringers is the partitioning into the alternating group and its complement in the according symmetric group.

Definition 4.2.3. Let σ be a permutation in S_n. A pair (i, j) with $1 \le i < j \le n$ and $\sigma j < \sigma i$ is called an **inversion of $\boldsymbol{\sigma}$**. Let $N(\sigma)$ be the number of inversions of σ. The number

$$\operatorname{sgn}(\sigma) := (-1)^{N(\sigma)}$$

is the sign of the permutation σ. A permutation with $\operatorname{sgn}(\sigma) = 1$ is called **even**, one with $\operatorname{sgn}(\sigma) = -1$ is called **odd**.

Definition 4.2.4. A change is **in-course** if the corresponding permutation in S_n is even. A change is **out-of-course** if the corresponding permutation in S_n is odd.

Remark 4.2.5. The alternating group A_n consists of all permutations that represent the in-course changes.

In our example of Plain Bob Minimus, the changes in rows $1, 2, 5$ and 6 of the lead ringing array in Table 4.1 are in-course. All other changes are out-of-course. Since the index of the alternating group A_n in the symmetric group S_n is always equal to two, the in-course changes and the out-of-course changes provide each a right and a left coset.

4.3 Right cosets and their application in change ringing

This subsection uses the contents of A. T. White's work [W87], Burkhard Polster's book [P03] and G. McGuire's paper [McG12].

Let us describe how right cosets are used for the construction of n-bell extents. Let H be a subgroup of S_n of order d. Using Lagrange's theorem 4.1.6, we know that there exist $m = \frac{n!}{d}$ right H-cosets. Let $T_1T_2 \cdots T_{d-1}$ be a transition sequence such that the permutations of the corresponding sequence of changes are exactly the elements of H. Let $p' \in S_n$ be the permutation we get by multiplying all letters of the transition sequence $T_1T_2 \cdots T_{d-1}$. Then, we attempt to construct a sequence of transitions $\widehat{T_1}\widehat{T_2} \cdots \widehat{T_m}$ such that

$$T_1T_2 \cdots T_{d-1}\widehat{T_1}T_1T_2 \cdots T_{d-1}\widehat{T_2} \cdots T_1T_2 \cdots T_{d-1}\widehat{T_m}$$

is the transition sequence of an n-bell extent. Thereby, it's a necessary condition that all m right H-cosets given by

$$(p'\widehat{T_1})(p'\widehat{T_2}) \cdots (p'\widehat{T_j})H, \text{ where } j \in \{1, \dots, m\},$$

are mutually disjoint and that $(p'\widehat{T_1})(p'\widehat{T_2}) \cdots (p'\widehat{T_m})H$ coincides with H.
In our explanation on Plain Bob Minimus, the subgroup $\langle\{A, B\}\rangle$ of S_4 is isomorphic to D_4. Thus, the order of $\langle\{A, B\}\rangle$ is $d = 8$ and a working transition sequence is given by $T_1T_2 \cdots T_7 = (AB)^3A$. The second sequence of transitions is given by $\widehat{T_1}\widehat{T_2}\widehat{T_3} = CCC$.

From now on we apply the following notation to ensure that our use of variables is as consistent as possible:

Notation 4.3.1. As of now, let p be the permutation we get by multiplying all letters of the transition sequence associated with the plain lead. Let b be the permutation we get by multiplying all letters of the transition sequence corresponding to the bob lead. And finally, let s be the permutation we receive by multiplying all letters of the transition sequence associated with the single lead.

4.3.1 Most popular ringing method: Plain Bob using Plain Hunt

Let $A = (12)(34)(56)\cdots$ and $B = (23)(45)(67)\cdots$ be the given n-bell transitions. Starting with rounds, the Plain Hunt method applies alternately the transitions A and B until we end up back at rounds.

Lemma 4.3.2. $H_n \coloneqq \langle A, B\rangle \cong D_n$

Proof.

$$AB = \begin{cases} (1\,3\,5\,7\,\ldots\,n-2\,n\,n-1\,n-3\,\ldots\,6\,4\,2), & \text{if } n \text{ is odd} \\ (1\,3\,5\,7\,\ldots\,n-3\,n-1\,n\,n-2\,\ldots\,6\,4\,2), & \text{if } n \text{ is even} \end{cases}$$

Since A and B are involutions and since $(AB)^n = 1_{S_n}$, we can use Remark 4.2.2. Thus, it follows that $\langle A, B\rangle \cong D_n$. ☐

Definition 4.3.3. As we have seen in Lemma 4.3.2, the permutations A and B generate a subgroup of S_n of order $2n$, denoted by H_n, which is the so-called **hunting subgroup** of S_n.

To compose a ringing sequence with more than $2n$ different changes, we have to throw another n-bell transition into the mix. The permutation $C = (34)(56)\cdots$ is used in order to switch into the right H_n-cosets. The Plain Bob method on n bells has transition sequence $t = (AB)^{(n-1)}AC$. The permutation we get by multiplying all letters of t is given by

$$p = B^{-1}C = BC = (23)(45)(67)\cdots(34)(56)\cdots = \begin{cases} (2\,4\,6\,\ldots\,n-1\,n\,n-2\,\ldots\,5\,3), & \text{if } n \text{ is odd} \\ (2\,4\,6\,\ldots\,n-2\,n\,n-1\,\ldots\,5\,3), & \text{if } n \text{ is even} \end{cases}$$

and is therefore of order $n-1$. Hence, the plain course of the Plain Bob method t starting with rounds corresponds to the union of the $n-1$ right H_n-cosets given by $p^j H_n$, where $j \in \{0, 1, \ldots, n-2\}$. As a consequence, the plain course of t consists of $2n\cdot(n-1)+1$ changes. Keep in mind that rounds appear at the start and at the end and thus are counted twice. Only when we have the Plain Bob method on $n = 4$ bells does the following hold: $2n\cdot(n-1)+1 = n!+1$. If $n > 4$, then $2n\cdot(n-1) < n!$ since $2 < (n-2)!$. Consequently, in the case of 4 bells, no call is needed because the plain course of the Plain Bob method starting with rounds coincides with the full 4-bell extent called Plain Bob Minimus. In the case of 5 or more bells, we require the bob $D = (23)(56)(78)\cdots$, which replaces the permutation C in our transition sequence t. In other words, a bob lead is described by the transition sequence $t_b = (AB)^{(n-1)}AD$. The associated permutation is given by

$$b = B^{-1}D = BD = (23)(45)(67)\cdots(23)(56)(78)\cdots = \begin{cases} (4\,6\,\ldots\,n-1\,n\,n-2\,\ldots\,7\,5), & \text{if } n \text{ is odd} \\ (4\,6\,\ldots\,n-2\,n\,n-1\,\ldots\,7\,5), & \text{if } n \text{ is even} \end{cases}$$

and is therefore of order $n-3$. In the case of $n = 5$, no further call besides D is required to ring an extent of the Plain Bob method. Since it holds that $p^3 b = (2453)^3(45) = C^{-1}D$ and since $C^{-1}D = (43)(23) = (423)$ is of order three, the transition sequence of a 5-bell extent of t is given by:

$$t_{ext} = t^3 t_b t^3 t_b t^3 t_b = (((AB)^4AC)^3(AB)^4AD)^3.$$

The following theorem shows us that in the case of $n = 6$ at least one more call is necessary, if we want to ring an extent of Plain Bob. Traditionally, for $n \geq 6$, the single call is given by the transition $E = (56)(78)\cdots$.

Theorem 4.3.4. *There does not exist a 6-bell extent of Plain Bob (satisfying conditions (i)-(iii) of Definition 1.1.3) using only plain leads and bob leads.*

Proof. The number of leads in a 6-bell extent of Plain Bob is $\frac{6!}{|H_6|} = \frac{720}{12} = 60$. Assume that all 60 leads are plain and bob leads. Note that the treble is always in the first position in each lead head and in each lead end. Using Definition 4.2.3, we see that $p = (24653)$, $b = (465)$, $C = (34)(56)$ and $D = (23)(56)$ are even permutations. The fact that p and b are even permutations and the fact that the product of two even permutations is again an even permutation imply that any lead head is an even permutation of 2 3 4 5 6. The lead end before the lead head results by applying either C^{-1} or D^{-1} to the lead head. Since C and D are both even permutations, we see that any lead end has to be an even permutation of 2 3 4 5 6 as well. Thus, if we use only plain leads and bob leads, all 120 changes with the treble in the first position are followed by an even permutation of 2 3 4 5 6. The number of even permutations of 2 3 4 5 6 is equal to the order of the alternating group A_5, which is $|A_5| = \frac{|S_5|}{2} = \frac{120}{2} = 60$. This entails the failing of condition (ii) of Definition 1.1.3 and therefore yields the desired contradiction. In other words, we are not able to ring a 6-bell extent of Plain Bob using only plain and bob leads. ☐

Corollary 4.3.5. *Starting with rounds and using the Plain Bob method on* 6 *bells and only plain and bob leads, it is impossible to exceed the ringing of* 360 *successive changes before returning back to rounds. Furthermore, there exists a* 6 *- bell ringing sequence* r *starting and ending with rounds that uses solely plain and bob leads and has length* 361 *(rounds occur twice and thus are counted twice).*

Proof. The proof of Theorem 4.3.4 implies that Plain Bob with only plain and bob leads has at most $\frac{|A_5|}{2} = \frac{60}{2} = 30$ leads. Each lead has $|H_6| = 2 \cdot 6 = 12$ changes. Therefore, the Plain Bob method on 6 bells using only plain and bob leads has at most $30 \cdot 12 = 360$ different changes. To show that the ringing of 360 different successive changes is indeed possible, we give a corresponding transition sequence:

$$t_{plainbobleads} = t_b t^3 {t_b}^2 t^4 t_b t^3 {t_b}^2 t^4 t_b t^3 {t_b}^2 t^4 = (t_b t^3 {t_b}^2 t^4)^3.$$

If we start with rounds and apply $t_{plainbobleads}$, then the ringing sequence r corresponding to $t_{plainbobleads}$ ends back at rounds, since

$$bp^3b^2p^4 = (465)(24653)^3(465)^2(24653)^4 = (465)(25436)(456)(23564) = (432)$$

is of order three. Thus, the ringing sequence r starting and ending with rounds corresponding to the 6 - bell transition sequence $t_{plainbobleads}$ has length 361. □

Remark 4.3.6. The ringing sequence r of Corollary 4.3.5 satisfies condition (iv) of Definition 1.1.3 because this condition is satisfied for Plain Hunt and thus also for all right cosets of Plain Hunt. Consequently, r fulfills conditions (i)-(iv) of Definition 1.1.3.

4.4 Left cosets and their application in change ringing

As we have seen in the previous subsection, right cosets are extremely valuable for the study of methods (Plain Bob). Left cosets, on the other hand, are first and foremost used for principles. This subsection exploits this latter connection between left cosets and principles and it follows primarily A. T. White's work ([W87] and [W89]) and Burkhard Polster's book [P03].

For Plain Bob Minimus, we have seen that each row, except the last one (which consists of only one change, namely rounds), of the array in Table 4.1 corresponds to a left coset of the cyclic group $H = \langle(243)\rangle \cong Z_3$ in S_4. Our goal in this subsection is the generalization of the use of left cosets of cyclic groups for arbitrary methods and especially for arbitrary principles.
Suppose we start with rounds as our initial change. Let the n - bell transition sequence t be a method or a principle and let p be the permutation we get by multiplying all letters in t. Hence, p is the permutation corresponding to the lead head of the second plain lead. Further, let d be the number of working bells (recall Definition 1.3.7). Then each row of changes, except the last one (which again consists of only one change, namely rounds), of the lead ringing array corresponds to a left coset of $\langle p\rangle \cong Z_d$ in S_n.

Definition 4.4.1. If the plain course of a method or a principle t starting with rounds coincides with an extent, then t is called a **no-call method or principle** because neither bobs nor singles are required for the ringing of this extent.

Example 4.4.2. All eleven 4 - bell methods presented in subsection 2.2.3 are no-call methods.

Remark 4.4.3. If t is a no-call method or principle, then the rows of the associated lead ringing array minus the last row (which again consists of only one rounds) yield a full coset decomposition of S_n. Thus, the $\frac{n!}{d}$ left $\langle p\rangle$-cosets corresponding to those rows form a partition of S_n.

Remark 4.4.4. Every extent of a no-call n - bell principle t automatically satisfies condition (v) of Definition 1.1.3 because of Definition 1.3.8 (i.e. the order of the permutation p is $d = n$) and Remark 1.3.9. Furthermore, if the first plain lead (sometimes also called division) of a no-call principle t is self-reversing (i.e. t is palindromic), then the plain course and thus the extent are self-reversing as well (i.e. condition (vi) of Definition 1.1.3 is fulfilled).

Left cosets are precisely what we need to compose all no-call n - bell principles and thus to construct the n - bell extents of those principles. But first we have to introduce the following definition form algebraic graph theory:

Definition 4.4.5. Let G be a group with generating set S. Let H be a subgroup of G. The **Schreier left coset graph** of ${}_H\backslash^G$ with respect to S, denoted by $\mathfrak{S}_S({}_H\backslash^G)$, is a colored digraph such that the following three conditions hold:
(i) $\mathfrak{S}_S({}_H\backslash^G)$ has the left H-cosets in G as its vertices.

(ii) Each element $s \in S$ is assigned to a color c_s.
(iii) There exists a directed edge (Hg_1, Hg_2), colored with c_s, iff $Hg_1 s = Hg_2$.

Remark 4.4.6. If $H = \{1_G\}$, then the Schreier left coset graph $\mathfrak{S}_S({}_H\backslash^G)$ is exactly the Cayley color graph $C_S(G)$ (as we can see by comparing with Definition 3.2.5).

Remark 4.4.7. Assume $s \in S$ is of order two (i.e. $s^2 = 1_G$). If there exists a directed edge (Hg_1, Hg_2), then by condition (iii) of Definition 4.4.5 we have $Hg_1 s = Hg_2$. Multiplying both sides of the equation by s yields $Hg_1 s^2 = Hg_1 = Hg_2 s$, and hence we have an edge (Hg_2, Hg_1). Consequently, the two directed edges (Hg_1, Hg_2) and (Hg_2, Hg_1) can be merged into one single undirected edge $\{Hg_1, Hg_2\}$.

Theorem 4.4.8. *Let $n \geq 4$ and let S be an arbitrary collection of n - bell transitions which generates S_n. There exists a no-call n - bell principle using only transitions from S iff there is a Hamiltonian cycle in $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n})$ whose word yields an n-cycle p by multiplying all its letters.*

Proof. The assumption $n \geq 4$ is essential since otherwise the definition of what it means for a path to be a cycle would not make sense (recall Definition 3.2.10). If for example $n = 3$, then the Schreier left coset graph $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_3})$ consists of only $\frac{3!}{|\langle p\rangle|} = \frac{3!}{|Z_3|} = \frac{6}{3} = 2$ vertices and hence there could never exist a Hamiltonian cycle in this graph.
"$\Rightarrow$": Let t be a no-call n - bell principle and let p be the permutation we get by multiplying all letters in t. The word t describes a Hamiltonian cycle in $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n})$, if we start at the vertex representing the coset $\langle p\rangle$. Moreover, since t is a principle, we have by Definition 1.3.8 that p is a cycle of order n.
"$\Leftarrow$": Let there be a Hamiltonian cycle in $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n})$ whose word t yields an n-cycle p by multiplying all its letters. The Hamiltonian cycle takes us through $(n-1)!+1$ distinct changes, starting with rounds corresponding to 1_{S_n} and ending in the same coset as rounds with the change corresponding to the permutation p. Using the same succession as before, we visit all left $\langle p\rangle$-coset once again, but this time we ring a different representative of each coset. We repeat this process until there are not any unused representatives left, and thus we end up back at rounds (corresponding to $p^n = 1_{S_n}$) after the ringing of $n! = (n-1)! \cdot n$ distinct changes. Hence conditions (i) and (ii) of Definition 1.1.3 are satisfied. Condition (iii) of Definition 1.1.3 results directly from the definition of the set S and condition (v) from the fact that the order of the permutation p is n (recall Remark 4.4.4). As a result, we've shown that the transition sequence t is indeed a no-call n - bell principle. $\square$

Remark 4.4.9. Let S be defined as in Theorem 4.4.8. Every Hamiltonian cycle in $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n})$ yields because of Remark 4.4.7 not just one but two words corresponding to the two directions in which we can pass through the cycle. However, if one of these transition sequences is a no-call n - bell principle, then the second transition sequence has to be a no-call n - bell principle as well. Furthermore, the extents corresponding to these principles are inverses of each other.

Proposition 4.4.10. *Let S be an arbitrary collection of n - bell transitions which generates S_n. Let the permutations p and σ be two n-cycles in S_n. Then $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n}) \cong \mathfrak{S}_S({}_{\langle \sigma\rangle}\backslash^{S_n})$.*

Proof. p and σ are conjugate because both of them are n-cycles. Hence $\exists \tau \in S_n$ such that $\sigma = \tau p \tau^{-1}$. Let $H_1 = \langle p\rangle$ and $H_2 = \langle \sigma\rangle$. We claim that $\phi : H_1\xi \mapsto H_2\tau\xi$ yields the desired automorphism.

$$\begin{aligned} H_1\xi_1 = H_1\xi_2 &\Leftrightarrow \xi_1\xi_2^{-1} \in H_1 \\ &\Leftrightarrow \xi_1\xi_2^{-1} = p^j \text{ for some } j \in \{0, 1, \ldots, n-1\} \\ &\Leftrightarrow \tau\xi_1\xi_2^{-1}\tau^{-1} = \tau p^j \tau^{-1} = (\tau p \tau^{-1})^j \in H_2 \\ &\Leftrightarrow H_2\tau\xi_1 = H_2\tau\xi_2 \\ &\Leftrightarrow \phi(H_1\xi_1) = \phi(H_1\xi_2) \end{aligned}$$

Hence the function ϕ is well-defined and injective, and consequently ϕ is surjective.
Now we have to check the coloring of the edges. Let there be an edge $\{H_1\xi_1, H_1\xi_2\}$ in $\mathfrak{S}_S({}_{H_1}\backslash^{S_n})$ colored with c_s. This edge is undirected since every $s \in S$ is an involution (recall Remark 4.4.7). Then we have $H_1\xi_1 s = H_1\xi_2 \Leftrightarrow \xi_1 s \xi_2^{-1} \in H_1 \Leftrightarrow \xi_1 s \xi_2^{-1} = p^j$ for some $j \in \{0, 1, \ldots, n-1\}$. It remains to show that there exists an edge $\{\phi(H_1\xi_1), \phi(H_1\xi_2)\}$ which is again colored with c_s. Since $H_2 = \langle\sigma\rangle$, we have:

$$(\tau p \tau^{-1})^j = \tau p^j \tau^{-1} = \tau\xi_1 s \xi_2^{-1}\tau^{-1} \in H_2 \Leftrightarrow H_2\tau\xi_1 s = H_2\tau\xi_2 \Leftrightarrow \phi(H_1\xi_1)s = \phi(H_1\xi_2)$$

and thus the edge $\{\phi(H_1\xi_1), \phi(H_1\xi_2)\}$ has the desired color c_s. Hence the proof of Proposition 4.4.10 is complete. $\square$

Because of Proposition 4.4.10, we can rewrite $\mathfrak{S}_S({}_{\langle p\rangle}\backslash^{S_n})$ in Theorem 4.4.8 as $\mathfrak{S}_S({}_{Z_n}\backslash^{S_n})$, and as a result we receive the following theorem:

Theorem 4.4.11. *Let $n \geq 4$ and let S be an arbitrary collection of n - bell transitions which generates S_n. There exists a no-call n - bell principle using only transitions from S iff there is a Hamiltonian cycle in $\mathfrak{S}_S({}_{Z_n}\backslash^{S_n})$, whose associated word yields a cycle of order n by multiplying all its letters.*

4.4.1 No-call 4 - bell principles for the purpose of exemplification

Let $S = \{A = (12)(34), B = (23), C = (34), D = (12)\}$.

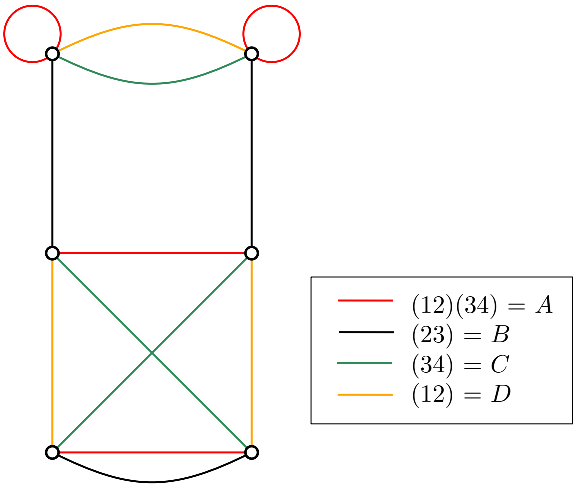


Figure 4.1: The Schreier left coset graph $\mathfrak{S}_S({}_{Z_4}\backslash^{S_4})$.

Starting, without loss of generality, at the vertex in the lower-left corner, we find a total of eight Hamiltonian cycles in $\mathfrak{S}_S({}_{Z_4}\backslash^{S_4})$. We list below one of the two words corresponding to the two directions in which we can pass through each of these Hamiltonian cycles (recall Remark 4.4.9):

$$\begin{aligned}
(DB)^2DA &= (1243) \\
(CB)^2CA &= (1243) \\
DBCBDA &= (1342) \\
CBDBCA &= (1342) \\
(DB)^3 &= 1_{S_4} \\
(CB)^3 &= 1_{S_4} \\
DBCBDB &= (14)(23) \\
CBDBCB &= (14)(23)
\end{aligned}$$

As we can see, the first four listed words yield by multiplying all their letters a cycle of order 4. Hence, Theorem 4.4.11 implies that there are four no-call 4 - bell principles, namely Erin ($(DB)^2DA$), Reverse Erin ($(CB)^2CA$), Stanton ($DBCBDA$), and Reverse Stanton ($CBDBCA$). The remaining no-call 4 - bell principles are exactly the inverses of these four principles. So all in all there are eight no-call principles on 4 bells.

4.5 Unicursal generation

This subsection follows G. McGuire's paper "Bells, Motels and Permutation Groups" ([McG12]).

Definition 4.5.1. Let G be a finite group with order $d = |G| < \infty$. Let S be a subset of G. **S generates G unicursally** if all elements in G can be cyclically ordered $(g_1, \dots, g_d)$ such that the following three conditions are fulfilled:

(i) $\forall h, j$ with $1 \le h < j \le d$: $\quad g_h \ne g_j$

(ii) $\forall h \in \{1, \dots, d-1\}\ \exists s_h \in S$: $\quad g_{h+1} = g_h s_h \quad$ (We say that g_h is **acted on** by s_h.)

(iii) $\exists s_d \in S$: $\quad g_1 = g_d s_d \quad$ (Similarly, we say that g_d is **acted on** by s_d.)

Remark 4.5.2. S being a generating set of G is a necessary but not sufficient condition for G being generated unicursally by S (see Example 7.0.9).

Example 4.5.3. Let G be the symmetric group S_3 and let $S = \{s_1 = (12), s_2 = (23)\}$. The elements in

S_3 can be cyclically ordered $(1_{S_3}, (12), (132), (13), (123), (23))$ such that

$$\begin{aligned} 1_{S_3} s_1 &= (12) \\ (12)s_2 &= (132) \\ (132)s_1 &= (13) \\ (13)s_2 &= (123) \\ (123)s_1 &= (23) \\ (23)s_2 &= 1_{S_3}. \end{aligned}$$

Since all three conditions of Definition 4.5.1 are satisfied, we can conclude that S_3 is generated unicursally by S.

Proposition 4.5.4. *Let G be a finite group. Then G is generated unicursally by a subset S of size one iff G is cyclic.*

Proof. "$\Rightarrow$": Let $d < \infty$ be the order of the group G. Suppose G is generated unicursally by $S = \{s\}$. Assume without loss of generality that $g_1 = 1_G$. Then for every $j \in \{1, \dots, d-1\}$, we have $g_{j+1} = g_j s = s^j$ and thus $G = \langle s \rangle$ is cyclic.
"$\Leftarrow$": If G is cyclic then G is by definition generated by a single element $s \in G$. Thus all elements in G can be cyclically ordered as follows: $(s, s^2, s^3, \dots, 1_G)$. Since conditions (i)-(iii) of Definition 4.5.1 are fulfilled, we draw the conclusion that the cyclic group G is generated unicursally by $S = \{s\}$. ☐

Theorem 4.5.5. *Let $n \geq 3$ and let $S = \{T_1, \dots, T_k\}$ be the allowed n - bell transitions.*

(i) An n - bell extent, fulfilling conditions (i)-(iii) of Definition 1.1.3 and using transitions from S only, exists iff S generates S_n unicursally.

(ii) An n - bell extent, fulfilling conditions (i)-(iv) of Definition 1.1.3 and using transitions from S only, exists iff S generates S_n unicursally with the additional property that for any $h \in \{1, \dots, n! - 1\}$ the transitions $s_h \in S$ and $s_{h+1} \in S$ have no common fixed point and moreover that the transitions $s_{n!} \in S$ and $s_1 \in S$ have no common fixed point.

Proof. Let the set $S = \{T_1, \dots, T_k\}$ be the allowed n-bell transitions. For every $j \in \{1, \dots, k\}$, the transition T_j has to be an involution in S_n, namely a product of disjoint transpositions of consecutive numbers, by condition (iii) of Definition 1.1.3.
The equivalence statement (i) of Theorem 4.5.5 is therefore a straightforward conclusion of the two definitions 1.1.3 and 4.5.1.
No bell occupying the same position for more than two successive changes (condition (iv) of Definition 1.1.3) is equivalent to no two successive transitions having a common fixed point. Hence, by using this fact and part (i) of Theorem 4.5.5, we receive the equivalence statement (ii) of Theorem 4.5.5. ☐

5. Stedman principle

The Stedman principle was invented somewhere around the year 1640. A precondition for the application of the Stedman principle is that the number of given bells has to be odd. In this chapter I will describe the Stedman principle for five and for seven given bells.

5.1 Stedman Doubles: Description and properties

Let $A = (12)(34)$, $B = (23)(45)$, $C = (12)(45)$ and $D = (34)$. The Stedman principle on 5 bells has transition sequence $t = CBA(CB)^2CABCB$. The associated permutation $p = (14325)$ permutes the working bells. Since the order of p is equal to the number of given bells, the transition sequence t describes indeed a principle (recall Definition 1.3.8). The plain course of t starting with rounds consists of $5 \cdot 12 + 1 = 61$ changes and contains five plain leads. Using Remark 1.3.15, we see that we need at least one call to ring an extent. Let this call be the 5 - bell transition D. The transition sequence corresponding to the bob lead is given by $t_b = CBA(CB)^2CABCD$. In other words, the bob D replaces the last B in the transition sequence t.
The transition sequence of a 5 - bell extent of Stedman using plain and bob leads is given by

$$t_{ext} = (t^4 t_b)^2 = ((CBA(CB)^2CABCB)^4 CBA(CB)^2CABCD)^2.$$

We see that in this transition sequence the bob D is used exactly twice.

5.2 Stedman Triples: Description and properties

Let $A = (12)(34)(56)$, $B = (23)(45)(67)$, $C = (12)(45)(67)$, $D = (12)(34)(67)$ and $E = (12)(34)$. The Stedman principle on 7 bells has transition sequence $t = CBA(CB)^2CABCB$. The permutation we get by multiplying all letters in t is $p = (1743256)$. Indeed, by looking at Definition 1.3.8, we see that the transition sequence t describes a principle. The plain course of t starting with rounds consists of $7 \cdot 12 + 1 = 85$ changes. Using Remark 1.3.15, we see that we need at least $\frac{7!}{84} - 1 = 59$ calls to ring an extent. Since our goal is the ringing of a 7 - bell extent of t, we use the bob D and the single E instead of A in either or both of its occurrences in our transition sequence t. Thus, the transition sequence corresponding to a bob lead is given by $t_{b_1} = CBD(CB)^2CABCB$, by $t_{b_2} = CBA(CB)^2CDBCB$ or by $t_{b_3} = CBD(CB)^2CDBCB$. Similarly, the transition sequence corresponding to a single lead is given by $t_{s_1} = CBE(CB)^2CABCB$, by $t_{s_2} = CBA(CB)^2CEBCB$ or by $t_{s_3} = CBE(CB)^2CEBCB$.

plain lead	bob leads			single leads		
	t_{b_1}	t_{b_2}	t_{b_3}	t_{s_1}	t_{s_2}	t_{s_3}
1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7
2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6
2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7
3 2 4 1 6 5 7	3 2 4 1 5 7 6	3 2 4 1 6 5 7	3 2 4 1 5 7 6	3 2 4 1 5 6 7	3 2 4 1 6 5 7	3 2 4 1 5 6 7
2 3 4 6 1 7 5	2 3 4 5 1 6 7	2 3 4 6 1 7 5	2 3 4 5 1 6 7	2 3 4 5 1 7 6	2 3 4 6 1 7 5	2 3 4 5 1 7 6
2 4 3 1 6 5 7	2 4 3 1 5 7 6	2 4 3 1 6 5 7	2 4 3 1 5 7 6	2 4 3 1 5 6 7	2 4 3 1 6 5 7	2 4 3 1 5 6 7
4 2 3 6 1 7 5	4 2 3 5 1 6 7	4 2 3 6 1 7 5	4 2 3 5 1 6 7	4 2 3 5 1 7 6	4 2 3 6 1 7 5	4 2 3 5 1 7 6
4 3 2 1 6 5 7	4 3 2 1 5 7 6	4 3 2 1 6 5 7	4 3 2 1 5 7 6	4 3 2 1 5 6 7	4 3 2 1 6 5 7	4 3 2 1 5 6 7
3 4 2 6 1 7 5	3 4 2 5 1 6 7	3 4 2 6 1 7 5	3 4 2 5 1 6 7	3 4 2 5 1 7 6	3 4 2 6 1 7 5	3 4 2 5 1 7 6
4 3 6 2 7 1 5	4 3 5 2 6 1 7	4 3 6 2 1 5 7	4 3 5 2 1 7 6	4 3 5 2 7 1 6	4 3 6 2 1 7 5	4 3 5 2 1 7 6
4 6 3 7 2 5 1	4 5 3 6 2 7 1	4 6 3 1 2 7 5	4 5 3 1 2 6 7	4 5 3 7 2 6 1	4 6 3 1 2 5 7	4 5 3 1 2 6 7
6 4 3 2 7 1 5	5 4 3 2 6 1 7	6 4 3 2 1 5 7	5 4 3 2 1 7 6	5 4 3 2 7 1 6	6 4 3 2 1 7 5	5 4 3 2 1 7 6
6 3 4 7 2 5 1	5 3 4 6 2 7 1	6 3 4 1 2 7 5	5 3 4 1 2 6 7	5 3 4 7 2 6 1	6 3 4 1 2 5 7	5 3 4 1 2 6 7

Table 5.1: The plain, bob and single lead of the Stedman principle on 7 bells starting with rounds. The changes in the last row of the table do not belong to the depicted leads, but are the lead heads of the subsequent leads.

Question. Does there exist a 7 - bell extent of Stedman using only the bob D and not the single E as a call?

Answer. An answer to this question was non-existent until 1994, and thus this question formed an open problem for many years. In 1994, Colin J. E. Wyld succeeded to prove the existence of such an extent on 7 bells by using a computer. His composition contains 705 bobs as can be detected by looking at the Appendix. For more information on Wyld's composition please consult the Appendix.

6. Grandsire method

In the 1650's, Robert Roan developed the Grandsire method on five bells. Later the method was extended to seven bells. For the majority of proofs in this subsection I will use.

6.1 Grandsire Doubles: Description and properties

Let $A = (12)(34)$, $B = (23)(45)$, $C = (12)(45)$ and $D = (45)$. The Grandsire method on 5 bells has transition sequence $t = CB(AB)^4$. The permutation we get by multiplying all letters in t is $p = (345)$. The hunting subgroup H_5 of S_5 is generated by the permutations A and B (recall Definition 4.3.3). Starting with rounds and executing the transition sequence t is equivalent to saying that we run through the right H_5-coset of C. Afterwards we run through the right coset $(CB(AB)^4C)H_5 = (CAC)H_5$, and then through $(CACAC)H_5$, which returns us back to rounds.

1 2 3 4 5	1 2 5 3 4	1 2 4 5 3
2 1 3 5 4	2 1 5 4 3	2 1 4 3 5
2 3 1 4 5	2 5 1 3 4	2 4 1 5 3
3 2 4 1 5	5 2 3 1 4	4 2 5 1 3
3 4 2 5 1	5 3 2 4 1	4 5 2 3 1
4 3 5 2 1	3 5 4 2 1	5 4 3 2 1
4 5 3 1 2	3 4 5 1 2	5 3 4 1 2
5 4 1 3 2	4 3 1 5 2	3 5 1 4 2
5 1 4 2 3	4 1 3 2 5	3 1 5 2 4
1 5 2 4 3	1 4 2 3 5	1 3 2 5 4
		1 2 3 4 5

Table 6.1: The plain course of t starting with rounds has length $3 \cdot 10 + 1 = 31$ and contains three plain leads.

The transition sequence corresponding to the bob lead is given by $t_b = CB(AB)^3CB$. In other words, the bob C replaces the last A in the transition sequence t. Multiplying all letters of t_b yields the permutation $b = (24)(35)$, which has order two.

Theorem 6.1.1. *There does not exist a 5 - bell extent of Grandsire (satisfying conditions (i)-(iii) of Definition 1.1.3) using only plain leads and bob leads.*

Proof. The 5 - bell transitions A, B and C are even permutations (recall Definition 4.2.3). We know that any product of even permutations is again an even permutation. Hence, the largest possible number of permutations that A, B and C are able to generate is $|A_5| = \frac{5!}{2} = 60$. Thus, there does not exist an extent of Grandsire on five bells using only plain and bob leads. $\square$

Corollary 6.1.2. *Starting with rounds and using the Grandsire method on* 5 *bells and only plain and bob leads, it is impossible to exceed the ringing of* 60 *successive changes before returning to a change that we have already rung. Furthermore, there exists a* 5 *- bell ringing sequence starting with rounds that uses solely plain and bob leads and has length* 60*.*

Proof. In the proof of Theorem 6.1.1, we have seen that the largest possible number of permutations that the 5 - bell transitions A, B and C are able to generate is the order of the alternating group A_5. To show that the ringing of 60 different successive changes is indeed possible, we give a corresponding transition sequence:

$$t_{plainbobleads} = (t_b t^2)^2 = (CB(AB)^3CB(CB(AB)^4)^2)^2.$$

If we start with rounds and apply $t_{plainbobleads}$, then we end up with the change 1 3 4 2 5, since

$$(bp^2)^2 = ((24)(35)(345)^2)^2 = (234)^2 = (243).$$

Since $bpB^{-1} = bpB = (24)(35)(345)(23)(45) = (243)$, the lead end of the first plain lead according to $t_{plainbobleads}$ is also the change 13425. Hence, because of condition (ii) of Definition 1.1.3 we cannot count the appearance of this change twice. Thus, the ringing sequence corresponding to the 5 - bell transition sequence $t_{plainbobleads}$ has length 60. $\square$

To obtain all possible permutations on five bells (all elements of the symmetric group S_5), we need to use the odd permutation D. The transition sequence corresponding to the single lead is given by $t_s = CB(AB)^3CD$. In other words, the call D replaces the last occurrence of B in the transition sequence t_b.

plain lead	bob lead	single lead
1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
2 1 3 5 4	2 1 3 5 4	2 1 3 5 4
2 3 1 4 5	2 3 1 4 5	2 3 1 4 5
3 2 4 1 5	3 2 4 1 5	3 2 4 1 5
3 4 2 5 1	3 4 2 5 1	3 4 2 5 1
4 3 5 2 1	4 3 5 2 1	4 3 5 2 1
4 5 3 1 2	4 5 3 1 2	4 5 3 1 2
5 4 1 3 2	5 4 1 3 2	5 4 1 3 2
5 1 4 2 3	5 1 4 2 3	5 1 4 2 3
1 5 2 4 3	1 5 4 3 2	1 5 4 3 2
1 2 5 3 4	1 4 5 2 3	1 5 4 2 3

Table 6.2: The plain, bob and single lead of the Grandsire method on 5 bells starting with rounds. The changes in the last row of the table do not belong to the depicted leads, but are the lead heads of the subsequent leads.

The transition sequence of a 5 - bell extent of Grandsire using plain, bob and single leads is given by

$$t_{ext} = ((tt_b)^2 tt_s)^2 = ((CB(AB)^4CB(AB)^3CB)^2CB(AB)^4CB(AB)^3CD)^2.$$

6.2 Grandsire Triples: Description and properties

Let $A = (12)(34)(56)$, $B = (23)(45)(67)$, $C = (12)(45)(67)$, $D = (45)(67)$. The Grandsire method on 7 bells has transition sequence $t = CB(AB)^6$. The permutation we get by multiplying all letters in t is $p = (34675)$. The hunting subgroup H_7 of S_7 is generated by the permutations A and B (recall Definition 4.3.3). Starting with rounds and executing t is equivalent to saying that we run through the right coset CH_7. A plain course of t contains five plain leads because the permutation p has order 5. The transition sequence corresponding to the bob lead is given by $t_b = CB(AB)^5CB$. Multiplying all letters of t_b yields the permutation

$$b = (12)(45)(67)(23)(45)(67)(1357642)^5(12)(45)(67)(23)(45)(67) = (247)(365),$$

which has order 3. The transition sequence corresponding to the single lead is given by $t_s = CB(AB)^5CD$.

plain lead	bob lead	single lead
1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7
2 1 3 5 4 7 6	2 1 3 5 4 7 6	2 1 3 5 4 7 6
2 3 1 4 5 6 7	2 3 1 4 5 6 7	2 3 1 4 5 6 7
3 2 4 1 6 5 7	3 2 4 1 6 5 7	3 2 4 1 6 5 7
3 4 2 6 1 7 5	3 4 2 6 1 7 5	3 4 2 6 1 7 5
4 3 6 2 7 1 5	4 3 6 2 7 1 5	4 3 6 2 7 1 5
4 6 3 7 2 5 1	4 6 3 7 2 5 1	4 6 3 7 2 5 1
6 4 7 3 5 2 1	6 4 7 3 5 2 1	6 4 7 3 5 2 1
6 7 4 5 3 1 2	6 7 4 5 3 1 2	6 7 4 5 3 1 2
7 6 5 4 1 3 2	7 6 5 4 1 3 2	7 6 5 4 1 3 2
7 5 6 1 4 2 3	7 5 6 1 4 2 3	7 5 6 1 4 2 3
5 7 1 6 2 4 3	5 7 1 6 2 4 3	5 7 1 6 2 4 3
5 1 7 2 6 3 4	5 1 7 2 6 3 4	5 1 7 2 6 3 4
1 5 2 7 3 6 4	1 5 7 6 2 4 3	1 5 7 6 2 4 3
1 2 5 3 7 4 6	1 7 5 2 6 3 4	1 5 7 2 6 3 4

Table 6.3: The plain, bob and single lead of the Grandsire method on 7 bells starting with rounds. The changes in the last row of the table do not belong to the depicted leads, but are the lead heads of the subsequent leads.

The next two lemmas show us identities, which we will use again later in subsection 7.2 called "Application to Grandsire Triples".

Lemma 6.2.1. *We have* $pb^{-1} = (27643)$.

Proof. The permutation pb^{-1} is determined by direct calculation:

$$pb^{-1} = (34675)((247)(365))^{-1} = (34675)(742)(563) = (27643).$$

$\square$

Notation 6.2.2. For $n \geq 3$, $F_n := \{\sigma \in S_n \mid \sigma(1) = 1\}$ is a subgroup of the symmetric group S_n. Another subgroup of S_n is given by $K_n := A_n \cap F_n$, where A_n is the alternating group in S_n. Thus, K_n is composed of all even permutations of S_n which keep the integer 1 fixed.

Remark 6.2.3. Note that $F_n \cong S_{n-1}$ and $K_n \cong A_{n-1}$.

In Thompson's theorem 6.2.6 we will prove that there does not exist a 7 - bell extent of Grandsire using only plain and bob leads. But for now, suppose that there exists such an extent. This extent would contain $\frac{7!}{|H_7|} = \frac{7!}{14} = \frac{5040}{14} = 360$ lead heads. The fact that p and b are even permutations and the fact that any lead head corresponds to a product of $p's$ and $b's$ imply that any lead head is an even permutation of 2 3 4 5 6 7. Thus, if $\{p, b\}$ would not be a generating set of K_7, then we would have immediately shown that there cannot exist a 7 - bell extent of Grandsire using only plain and bob leads. However, the following lemma proves that $\{p, b\}$ is in fact a generating set of K_7 (see lemma 4.14 of [W87, p. 739]).

Lemma 6.2.4. $K_7 = \langle p, b\rangle$

Proof. We show the following two inclusions to receive the desired equality:
"$\langle p, b\rangle \subset K_7$": This inclusion follows from the fact that p, b and pb^{-1} are even permutations, which fix the treble.
"$K_7 \subset \langle p, b\rangle$": For the proof of this inclusion we need the following claim:

Claim 6.2.5. *For $n \geq 3$, $\{(1j2) \mid j \in \{3, \ldots, n\}\}$ is a generating set of the alternating group A_n.*

Proof. For $n = 3$, the only such 3-cycle is given by (132). Thus, the identity

$$\langle(132)\rangle = \{1_{S_3}, (132), (123)\} = A_3$$

proves the claim for $n = 3$. Now let $n \geq 4$. It is well known (see, for example, Lemma 3.1 of [Con, p. 5]) that for $n \geq 4$, any element of A_n can be written as a product of 3-cycles. Thus, the 3-cycles generate A_n. For any 3-cycle (ijk) not having $i = 1$, we have

$$(ijk) = (1jk)(1ij).$$

$\{(1jk) \mid j, k \in \{2, \ldots, n\}$ and $j \neq k\}$ is therefore a generating set of A_n. Since $(1j2)^{-1} = (12j)$ and

$$(1jk) = (1j2)(1j2)(1k2)(1j2),$$

it follows that the alternating group A_n is generated by 3-cycles of the form $(1j2)$. $\square_{\text{Claim}}$

Remark 6.2.3 and Claim 6.2.5 imply that it suffices to show that $\{(2j3) \mid j \in \{4, \ldots, 7\}\} \subset \langle p, b\rangle$. As a matter of fact, we have

$$\begin{aligned}
p^{-2}b &= (34675)^{-2}(247)(365) = (57643)^2(247)(365) = (243)\\
pb^{-1}p &= (34675)(742)(563)(34675) = (253)\\
(pb^{-1}p)^{-1}p^2b^{-1}p &= (235)p(253) = (235)(34675)(253) = (24673)\\
(pb^{-1}p)^{-1}p^2b^{-1}pp^{-2}b((pb^{-1}p)^{-1}p^2b^{-1}p)^{-1} &= (24673)p^{-2}b(24673)^{-1} = (24673)(243)(37642) = (273)\\
p^{-2}(273)^{-1}p^2 &= (57643)^2(237)(34675)^2 = (263).
\end{aligned}$$

$\square$

6.2.1 Thompson's theorem

In 1886, W. H. Thompson proved with his publication "A Note on Grandsire Triples" that there does not exist an extent of Grandsire Triples using only plain leads and bob leads. According to T. J. Fletcher ([F56, p. 624]), Thompson was neither a mathematician nor a practising ringer. Allegedly he loved solving change ringing problems as a hobby. Instead of using Thompson's original proof, which is outlined in Fletcher's work [F56] and D. J. Dickinson's paper "On Fletcher's Paper" [D57], I will use a proof that is based on an idea of R. A. Rankin. In the next chapter (chapter 7), I intend to present the work of Rankin in a more general context.

Theorem 6.2.6. *(Thompson)*

(i) K_7 is not generated unicursally by the two permutations $p = (34675)$ and $b = (247)(365)$.

(ii) There does not exist a 7 - bell extent of Grandsire using only plain leads and bob leads.

Proof. We have explained the reasons why (ii) is a direct implication of the first statement (recall also Definition 4.5.1). So it remains to show that (i) holds. The proof of statement (i) is shown by contradiction.

Suppose that K_7 is generated unicursally by p and b. By Definition 4.5.1 this means that all elements of K_7 can be cyclically ordered $(g_1, g_2, \ldots, g_{360})$ such that for each $h \in \{1, \ldots, 359\}$ there exists $s_h \in \{p, b\}$ with $g_{h+1} = g_h s_h$. Further, there exists $s_{360} \in \{p, b\}$ such that $g_1 = g_{360}s_{360}$. In other words, every element of K_7 is acted on by either p or b. We will call any sequence of cyclically ordered elements $(g_1, g_2, \ldots, g_m)$ a **chain of length m** if for each $h \in \{1, \ldots, m\}$, g_h is acted on by some $s_h \in \{p, b\}$ (recall Definition 4.5.1). The key part of the proof is to consider right Q-cosets, where $Q = \langle\sigma\rangle$ and $\sigma = bp^{-1} = (27643)^{-1} = (34672)$. For the calculation of σ we used Lemma 6.2.1. W. H. Thompson called these Q-cosets ***Q*-sets**. K_7 can be partitioned into $\frac{360}{5} = 72$ disjoint Q-sets.

The proof of part (i) of Theorem 6.2.6 is based on a series of claims:

Claim 6.2.7. *Let xQ be a Q-sets. Then, $\exists s \in \{p, b\}\ \forall y \in xQ :\ y$ is acted on by s.*

Proof. W.l.o.g., suppose $y = x\sigma^i$ is acted on by p. Hence, the next element in the chain after y is $yp = x\sigma^i p$. But we have $yp = x\sigma^{i-1}(bp^{-1})p = x\sigma^{i-1}b$. Since we want to avoid repetition, b cannot act on $x\sigma^{i-1}$. Therefore, p has to act on $x\sigma^{i-1}$. This argument holds for any $i \in \{1, \ldots, 5\}$. $\square_{\text{Claim 6.2.7}}$

For $i \in \{1, \ldots, 5\}$, define $k_i \in \{1, \ldots, 5\}$ in such a way that the next element of xQ in the chain after $x\sigma^i$ is $x\sigma^{k_i}$. This yields a permutation in the symmetric group S_5 for each right coset xQ:

$$\tau(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ k_1 & k_2 & k_3 & k_4 & k_5 \end{pmatrix}.$$

Claim 6.2.8. *$\tau(x)$ is a 5-cycle.*

Proof. Since we have assumed that there exists a chain of length $360 = |K_7|$, we know that $\tau(x)$ has to be a 5-cycle. $\square_{\text{Claim 6.2.8}}$

Now we will execute a rearrangement of this single chain of length 360. Rearranging this chain may lead to several disjoint chains of smaller length than 360. The procedure of the rearrangement is done as follows:

Let xQ be a right coset, whose elements are acted on by b. Such a right coset has to exist because otherwise each element of K_7 would be acted on by p, which is impossible since we have only a single chain. Thus, the next element in the chain after $x\sigma^i$ is $x\sigma^i b$. As a next step, we divide the chain into five segments with respect to the right coset xQ such that each segment is beginning with $x\sigma^i b$ and ending with $x\sigma^{k_i}$:

$$\ldots, x\sigma^i][x\sigma^i b, \ldots, x\sigma^{k_i}][x\sigma^{k_i} b, \ldots \tag{5}$$

The definition of k_i implies that the last element of a segment is the only element in this segment, which is contained in the right coset xQ. Permuting the five segment of (5) in such a way that the segment after $x\sigma^i$ is starting with $x\sigma^{i-1}b$ and ending with $x\sigma^{k_{i-1}}$, leads to:

$$\ldots, x\sigma^i][x\sigma^{i-1} b, \ldots, x\sigma^{k_{i-1}}][x\sigma^{k_{i-1}} b, \ldots \tag{6}$$

By doing this we finish our rearrangement procedure.

Claim 6.2.9. *After the rearrangement, the elements in xQ are acted on by p. Furthermore, the rearrangement does not affect the other Q-cosets apart from xQ, regarding whether their elements are acted on by p or b.*

Proof. Since $x\sigma^{i-1}b = x\sigma^{i-1}bp^{-1}p = x\sigma^i p$, we see by looking at (6) that the element following $x\sigma^i$ is $x\sigma^i p$. This means that $x\sigma^i$ is acted on by p after our rearrangement. Hence, because of Claim 6.2.7, all elements in xQ are acted on by p. The second part of our claim follows directly from the construction of our rearrangement (6). $\square_{\text{Claim 6.2.9}}$

Claim 6.2.10. *After the rearrangement, it is possible that we have more than one chain. Every chain, however, contains an element of xQ.*

Proof. We show the possibility of having more than one chain with the help of an example: Let $k_1 = 4$ and let $k_3 = 2$. After our rearrangement, we have

$$[x\sigma b, \ldots, x\sigma^4][x\sigma^3 b, \ldots, x\sigma^2].$$

These two segments form a chain, while the remaining segments form at least another chain. Thus, in this case, we would have at least two chains.
Now we proceed with the proof of the second part of Claim 6.2.10, which is done by contradiction. Suppose that after the rearrangement there exists a chain that does not contain any element of xQ. Hence, by Claim 6.2.9, this chain must have existed before the rearrangement. But this is contradicted by the fact that there was only one chain before our rearrangement, and this chain contained all elements of xQ. $\square_{\text{Claim 6.2.10}}$

The next element of xQ in the new arrangement after $x\sigma^i$ is $x\sigma^{k_{i-1}}$. This yields a permutation $\pi(x)$ in S_5 similar to $\tau(x)$:

$$\pi(x) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ k_5 & k_1 & k_2 & k_3 & k_4 \end{smallmatrix}\right).$$

Claim 6.2.11. *The number of disjoint cycles (including* 1*-cylces) in* $\pi(x)$ *is equal to the number of chains after our rearrangement.*

Proof. This is a direct consequence of Claim 6.2.10. $\square_{\text{Claim 6.2.11}}$

Claim 6.2.12. $(12345)\pi(x) = \tau(x)$

Proof. The statement of this claim is straight-forward. $\square_{\text{Claim 6.2.12}}$

Claim 6.2.13. *The number of disjoint cycles (including* 1*-cylces) in* $\pi(x)$ *is odd.*

Proof. Suppose $\pi(x)$ has r disjoint cycles. The number of disjoint cycles in $(15)\pi(x)$ is $r-1$ if the integers 1 and 5 are not in the same cycle in $\pi(x)$, and $r+1$ if the integers 1 and 5 are in the same cycle in $\pi(x)$. Therefore, it follows that the number of disjoint cycles in $(14)(15)\pi(x)$ has exactly the same parity as r. Similarly, the number of disjoint cycles in $(12)(13)(14)(15)\pi(x)$ has the same parity as r. Note that $(12)(13)(14)(15)\pi(x) = (12345)\pi(x)$, so Claim 6.2.12 combined with Claim 6.2.8 yields that r is odd. $\square_{\text{Claim 6.2.13}}$

The number of chains after the rearrangement with respect to the right coset xQ is denoted by c_x.

Claim 6.2.14. c_x *is odd.*

Proof. Combining Claim 6.2.11 and Claim 6.2.13 yields the desired result. $\square_{\text{Claim 6.2.14}}$

Now, let us do the same rearrangement as before, but this time with respect to another right coset $\tilde{x}Q$, whose elements are acted on by b. If there does not exist such a right coset, then skip to the end of the proof of Claim 6.2.19. The number of chains after the rearrangement with respect to the right coset $\tilde{x}Q$ is denoted by $c_{\tilde{x}}$. Let $k_i = k_i(\tilde{x})$ and $\tau(\tilde{x})$ be defined in a similar way as before. Claim 6.2.8 becomes

Claim 6.2.15. *The number of disjoint cycles in* $\tau(\tilde{x})$ *is equal to the number of chains that do contain elements of* $\tilde{x}Q$ *before our rearrangement with respect to the right coset* $\tilde{x}Q$*.*

Claim 6.2.9 proves that after the rearrangement with respect to $\tilde{x}Q$, the elements in $\tilde{x}Q$ are acted on by p. Hence, the proof of Claim 6.2.10 leads to the following claim:

Claim 6.2.16. *The number of chains that do not contain any element of* $\tilde{x}Q$ *does not vary during the rearranging process.*

Let $\pi(\tilde{x})$ be defined in a similar way as $\pi(x)$.

Claim 6.2.17. $c_x -$ *(number of disjoint cycles in* $\tau(\tilde{x})$*)* $= c_{\tilde{x}} -$ *(number of disjoint cycles in* $\pi(\tilde{x})$*)*

Proof. Claim 6.2.15 yields that left hand side of the equation is equal to the number of chains that do not contain any element of $\tilde{x}Q$ before our rearrangement with respect to the right coset $\tilde{x}Q$. The right hand side of the equation is equal to the number of chains that do not contain any element of $\tilde{x}Q$ after our rearrangement. Thus, the equality of the left hand side and the right hand side is implied by Claim 6.2.16. $\square_{\text{Claim 6.2.17}}$

Claim 6.2.18. $(12345)\pi(\tilde{x}) = \tau(\tilde{x})$

Proof. The statement of this claim is again straight-forward. $\square_{\text{Claim 6.2.18}}$

Claim 6.2.19. *(number of disjoint cycles in $\pi(\tilde{x})$)* $\equiv$ *(number of disjoint cycles in $\tau(\tilde{x})$)* mod 2

Proof. Doing the same proof as for Claim 6.2.13, but using $\pi(\tilde{x})$ instead of $\pi(x)$ and Claim 6.2.18 instead of Claim 6.2.12, gives us the desired result. $\square_{\text{Claim 6.2.19}}$

Claim 6.2.20. *$c_{\tilde{x}}$ is odd.*

Proof. Claim 6.2.17 and Claim 6.2.19 yield that $c_{\tilde{x}}$ has to have the same parity as c_x. Hence, as a result of Claim 6.2.14, $c_{\tilde{x}}$ must be odd. $\square_{\text{Claim 6.2.20}}$

We repeat the rearranging process with respect to every right Q-coset, whose elements are acted an by b. Once this is done, the elements in all right Q-coset are acted on by p. Thus, every element of K_7 is acted on by p. Consequently, the chains we have must be the right $\langle p \rangle$-cosets. The number of such chains is odd because of Claim 6.2.20. However, since p is a 5-cycle, there exists an even number of $\langle p \rangle$-cosets, namely exactly $[K_7 : \langle p \rangle] = \frac{360}{5} = 72$. This contradiction finally finishes the proof of part (i) of Theorem 6.2.6. $\square$

Corollary 6.2.21. *Starting with rounds and using the Grandsire method on* 7 *bells and only plain and bob leads, it is impossible to exceed the ringing of* 4998 *successive changes before returning to a change that we have already rung.*

Proof. Because of Thompson's theorem 6.2.6 we know that the elements of K_7 cannot be cyclically ordered in a single chain of length 360. Since b has order three, the shortest possible chain is of length three. Consequently, the longest possible chain is of length smaller than or equal to $360 - 3 = 357$. Since a chain of length 357 corresponds to 357 consecutive leads, the largest number of successive changes that can be rung using only plain and bob leads is $357 \cdot |H_7| = 357 \cdot 14 = 4998$ (recall that H_7 is the hunting subgroup of S_7). $\square$

In 1751, John Holt showed that there exists a ringing sequence of length 4998 using the Grandsire method and only plain and bob leads. This proves that the maximum possible length given by Corollary 6.2.21 is in fact achievable.

7. Rankin's theorem

As I mentioned in subsection 6.2.1, R. A. Rankin ([R48] and [R66]) generalised in 1948 Thompson's idea concerning the Grandsire method. As a matter of fact, Rankin's result is even more general and detailed than the version we will follow in this section. In 1999, a paper of Richard G. Swan was published in *The American Mathematical Monthly* ([S99]) in which he rewrote Rankin's end result in a group theoretic language. Thus, our proofs in this chapter will mainly follow Swan's work.

Definition 7.0.1. Let S be a set of permutations of a given finite set I. An ***S*-cycle in *I*** is a cyclically ordered non-empty subset of I, which we denoted by $(i_1, \dots, i_k)$, that satisfies the following three conditions:
(i) $\forall h, j$ with $1 \leq h < j \leq k: \quad i_h \neq i_j$
(ii) $\forall h \in \{1, \dots, k-1\}\ \exists \sigma_h \in S: \quad i_{h+1} = \sigma_h(i_h)$
(iii) $\exists \sigma_k \in S: \quad i_1 = \sigma_k(i_k)$

Lemma 7.0.2. *Let $S = \{\sigma_1, \sigma_2\}$ be a set containing two permutations of a given finite set I such that σ_1 has m_1 orbits and σ_2 has m_2 orbits, respectively. Suppose that the set I can be partitioned into m disjoint S-cycles. Let us further assume that the permutation $\sigma = \sigma_2^{-1}\sigma_1$ has odd order. Then* $m \equiv m_1 \equiv m_2 \mod 2$.

Proof. There exists a one-to-one correspondence between the partitions of I into disjoint S-cycles and the permutations τ of I with the S-cycles being the orbits of τ: Let $(i_{1,1}, \dots, i_{1,k_1}), \dots, (i_{m,1}, \dots, i_{m,k_m})$ be the m disjoint S-cycles that partition the finite set I. By defining the permutation τ of I in the following way:

$$\tau := (i_{1,1}, \dots, i_{1,k_1}) \cdots (i_{m,1}, \dots, i_{m,k_m}),$$

we see that the above one-to-one correspondence actually holds. For $h \in \{1, 2\}$, we define the sets $J_h := \{i \in I \mid \tau(i) = \sigma_h(i)\}$ and we notice that $J_1 \cup J_2 = I$. Let $\pi = \sigma_2^{-1}\tau$. Then the following holds: $\pi|_{J_1} = \sigma|_{J_1}$ and $\pi|_{J_2} = \mathrm{id}_{J_2}$. Since $J_1 \cup J_2 = I$, we see that $J_1 \setminus J_2 = I \setminus J_2$ is stable under π and therefore also stable under σ. In particular, J_1 is stable under π and therefore also an invariant set of σ. Consequently, $\pi|_{J_1} = \sigma|_{J_1}$ has odd order since by assumption σ has odd order. Further, since $\pi|_{I \setminus J_1} = \mathrm{id}_{I \setminus J_1}$, the order of $\pi|_I$ is odd.

Claim 7.0.3. *If a permutation π of a finite set I has odd order, then π must be an even permutation.*

Proof. If π has odd order, then there exists an integer q such that $\pi^{2q+1} = \mathrm{id}_I$. If we write π as a product k transpositions and insert this into $\pi^{2q+1} = \mathrm{id}_I$, then id_I is equal to a product of $k(2q+1)$ transpositions. It is only possible to write id_I as a product of an even number of transpositions. Therefore, $k(2q+1)$ has to be even, which in turn implies that k must be even as well. Definition 4.2.3 explains why π has to be an even permutation. $\square_{\text{Claim}}$

Using Claim 7.0.3 and Definition 4.2.3 we find $\mathrm{sgn}(\sigma_2^{-1}) \cdot \mathrm{sgn}(\tau) = \mathrm{sgn}(\pi) = 1$.

Claim 7.0.4. (see Appendix A of [K55])
Let τ be a permutation of a finite set I having m cycles. Then $\mathrm{sgn}(\tau) = (-1)^{|I|-m}$.

Proof. Write the permutation τ in the following form: $\tau = (i_{1,1}, \dots, i_{1,k_1}) \cdots (i_{m,1}, \dots, i_{m,k_m})$, where $\forall h \in \{1, \dots, m\}$ and $\forall j(h) \in \{1, \dots, k_h\}$ it holds that $i_{h,j(h)} \in I$. Since $\sum_{h=1}^{m} k_h = |I|$ and since every cycle of length k_h can be written as a product of $k_h - 1$ transpositions, it follows by using Definition 4.2.3 that

$$\mathrm{sgn}(\tau) = \prod_{h=1}^{m} (-1)^{k_h - 1} = (-1)^{\sum_{h=1}^{m}(k_h - 1)} = (-1)^{|I|-m}.$$

$\square_{\text{Claim}}$

Using Claim 7.0.4 and $\mathrm{sgn}(\sigma_2^{-1}) \cdot \mathrm{sgn}(\tau) = 1$, we get

$$(-1)^{|I|-m} = \mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma_2^{-1}) = \mathrm{sgn}(\sigma_2) = (-1)^{|I|-m_2}$$

and hence $m \equiv m_2 \mod 2$. Our assumption that $\sigma_2^{-1}\sigma_1$ has odd order implies that $\sigma_1^{-1}\sigma_2 = (\sigma_2^{-1}\sigma_1)^{-1}$ has odd order as well. As a result, we are able to swap the roles of σ_1 and σ_2 in our proof, and by doing so obtain $m \equiv m_1 \mod 2$. This evidently finishes the proof of Lemma 7.0.2. $\square$

Remark 7.0.5. Let $S = \{\sigma_j \mid j = 1, 2\}$ be a set containing two permutations of a given finite set I. The orbit of an element $i \in I$ is exactly the set of points in the cycle that contain i. Thus, saying in Lemma 7.0.2 that the permutation σ_j has m_j orbits is equivalent to saying that σ_j has m_j cycles.

Definition 7.0.6. Let S be a subset of a given finite group G. An ***S*-cycle in *G*** is a cyclically ordered non-empty subset of G, which we denoted by $(g_1, \ldots, g_k)$, that satisfies the following three conditions:

(i) $\forall h, j$ with $1 \leq h < j \leq k : \quad g_h \neq g_j$

(ii) $\forall h \in \{1, \ldots, k-1\} : \quad g_h^{-1} g_{h+1} \in S$

(iii) $g_k^{-1} g_1 \in S$

Theorem 7.0.7. *Let $S = \{s_1, s_2\}$ be a subset of a finite group G. Suppose that G can be partitioned into m disjoint S-cycles and assume that $s_1 s_2^{-1}$ has odd order. Then m is equivalent to the index of $\langle s_1 \rangle$ in G modulo 2 and to the index of $\langle s_2 \rangle$ in G modulo 2, respectively: $m \equiv [G : \langle s_1 \rangle] \equiv [G : \langle s_2 \rangle] \mod 2$.*

Proof. Let I be the finite group G. For $j = 1, 2$, let σ_j be the permutation of G given by $\sigma_j : g \mapsto g s_j$. The orbits of σ_j are exactly the right $\langle s_j \rangle$-cosets, meaning σ_j has $[G : \langle s_j \rangle]$ orbits. Hence, let $m_1 = [G : \langle s_1 \rangle]$ and $m_2 = [G : \langle s_2 \rangle]$. Further, $\sigma_2^{-1} \sigma_1 : g \mapsto g s_1 s_2^{-1}$ has obviously the same order as $s_1 s_2^{-1}$. Since the S-cycles in G correspond to the $\{\sigma_1, \sigma_2\}$-cycles in G, we can apply Lemma 7.0.2 and by doing so we receive the desired result. ☐

The next corollary describes Rankin's conclusion in the most popular manner, meaning in most change ringing literature it can be found written in exactly this wording.

Corollary 7.0.8. *Let $S = \{s_1, s_2\}$ be a generating set of a finite group G. Suppose that $s_1 s_2^{-1}$ has odd order. If G is generated unicursally by S, then the index of $\langle s_1 \rangle$ in G and the index of $\langle s_2 \rangle$ in G are both odd.*

Proof. If S generates G unicursally, then by Definition 4.5.1 we cannot partition G into more than one disjoint S-cycle. Thus, we are able to apply Theorem 7.0.7 with $m = 1$. Since m is odd, we get the desired result. ☐

Example 7.0.9. Let $S = \{s_1 = (123 \ldots n-1), s_2 = (n-1\, n)\}$. The permutation s_1 is a $(n-1)$-cycle, and s_2 is a transposition. One can easily check that S is a generating set of the symmetric group S_n. Suppose that n is odd and that $n > 3$. The order of $s_1 s_2^{-1} = (123 \ldots n-1)(n-1\, n) = (123 \ldots n-3\, n-2\, n\, n-1)$ is n, and thus it follows that $s_1 s_2^{-1}$ has odd order. For $n > 3$, using Theorem 4.1.6 (Lagrange's theorem) yields that the index of $\langle s_2 \rangle$ in S_n is $\frac{n!}{2}$ and hence $[S_n : \langle s_2 \rangle]$ is even. We conclude that S_n cannot be generated unicursally by S by applying Corollary 7.0.8.

7.1 Translation and extension into a graph theoretic result

In this subsection we will translate and extend Rankin's work into a graph theoretic result as it was done by D. Griffiths in her paper "Twin Bob Plan compositions of Stedman Triples" ([G94]). The remaining contents of chapter 7 are all based on this paper.

Definition 7.1.1. A covering C of a digraph Γ is a union of disjoint directed cycles such that every vertex in $V(\Gamma)$ is used exactly once.

Remark 7.1.2. If a covering C of a digraph Γ involves just one cycle, then the cycle is a Hamiltonian cycle and Γ is Hamiltonian (recall Definition 3.2.11).

Definition 7.1.3. Let Γ be a digraph such that every vertex in $V(\Gamma)$ has in-degree two as well as out-degree two. An **alternating 2*m*-gon** is a sequence of edges of Γ of the form $e_1 e_2^{-1} e_3 e_4^{-1} \ldots e_{2m}^{-1} e_1$ where e_j^{-1} signifies the edge e_j used in the opposite direction (see Figure 7.1).

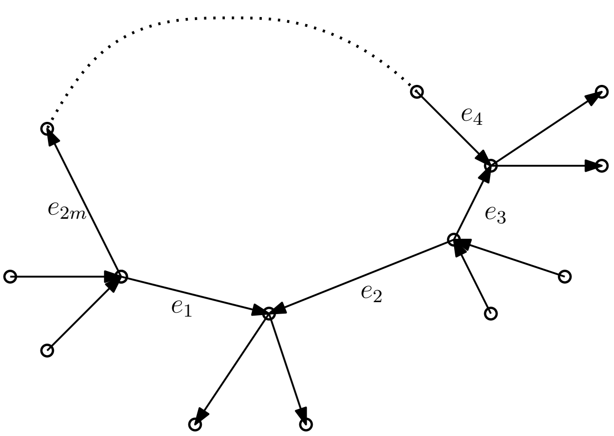


Figure 7.1: Alternating $\boldsymbol{2m}$-gon.

Definition 7.1.4. Let Γ be a digraph such that every vertex in $V(\Gamma)$ has in-degree two as well as out-degree two. Let C_1 be a covering of Γ such that the edge e_1 is part of C_1. The edge e_1 belongs to a unique alternating $2m$-gon $e_1e_2^{-1}e_3e_4^{-1}\dots e_{2m}^{-1}e_1$. Let C_2 be a covering of Γ such that the edge e_1 is not part of C_2. The procedure of transforming C_1 into C_2 is called a **one-step transformation of C_1** which uses the alternating $2m$-gon $e_1e_2^{-1}e_3e_4^{-1}\dots e_{2m}^{-1}e_1$.

Remark 7.1.5. As a direct consequence of the three above-mentioned definitions, we have the following two properties:

(i) $\forall j \in \{1,\dots,m\}$: $e_{2j-1} \in C_1$ and $e_{2j} \notin C_1$

(ii) $\forall j \in \{1,\dots,m\}$: $e_{2j-1} \notin C_2$ and $e_{2j} \in C_2$

A one-step transformation of C_1 which uses the alternating $2m$-gon $e_1e_2^{-1}e_3e_4^{-1}\dots e_{2m}^{-1}e_1$ does not affect the segments of cycles in C_1 that do not contain the edges $e_1, e_2, \dots, e_{2m}$.

Theorem 7.1.6. *Let Γ be a digraph such that every vertex in $V(\Gamma)$ has in-degree two as well as out-degree two. Let C_1 be a covering of Γ. Let C_2 be the covering we get by doing a one-step transformation of C_1 which uses the alternating $2m$-gon $e_1e_2^{-1}e_3e_4^{-1}\dots e_{2m}^{-1}e_1$. Let m_1 and m_2 be the numbers of cycles in C_1 and C_2, respectively. If m is odd, then $m_2 \equiv m_1 \mod 2$. If m is even, then $m_2 \not\equiv m_1 \mod 2$.*

Proof. Let $e_1 \in E(\Gamma)$ be used in one of the m_1 cycles of the covering C_1. It follows by Remark 7.1.5 (i) that the edges $e_1, e_3, \dots, e_{2m-1}$ are contained in C_1, and by Remark 7.1.5 (ii) that the edges $e_2, e_4, \dots, e_{2m}$ are not contained in the cycles of C_1. Let the permutation $\tau_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ k_1 & k_2 & k_3 & \cdots & k_{m-1} & k_m \end{smallmatrix}\right)$ describe the succession in which the edges $e_1, e_3, \dots, e_{2m-1}$ appear in C_1 in such a way that for any $j \in \{1,\dots,m\}$ the first edge of $e_1, e_3, \dots, e_{2m-1}$ following e_{2j-1} is e_{2k_j-1}. It is possible that $k_j = j$, namely if there exists a cycle of C_1 that contains only the edge e_{2j-1} of the edges $e_1, e_3, \dots, e_{2m-1}$. Let m_{τ_1} be the number of cycles in C_1 involving the edges $e_1, e_3, \dots, e_{2m-1}$. Thus, the number of cycles in C_1 not involving the edges $e_1, e_3, \dots, e_{2m-1}$ is $m_1 - m_{\tau_1}$. Our specification of the permutation τ_1 implies that τ_1 has to have m_{τ_1} disjoint cycles.

Now we are doing a one-step transformation of C_1 which uses the alternating $2m$-gon $e_1e_2^{-1}e_3e_4^{-1}\dots e_{2m}^{-1}e_1$: By removing the edges $e_1, e_3, \dots, e_{2m-1}$ from the covering C_1, we get segments of the m_{τ_1} cycles in C_1. These segments are uniquely joined together by the edges $e_2, e_4, \dots, e_{2m}$ such that we receive the covering C_2. Studying the following figure, we can see that the uniqueness of the "joining together-step" is actually ensured.

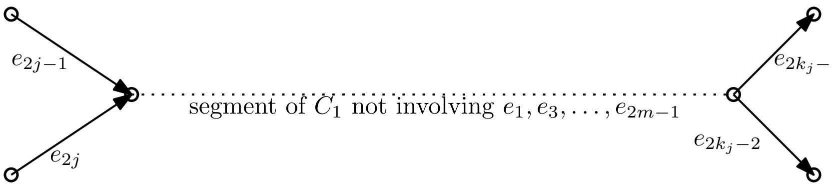


Figure 7.2: For any $j \in \{1,\dots,m\}$, removing the edges e_{2j-1} and e_{2k_j-1} from a cycle of the covering C_1, leaves us with a segment of C_1 which does not involve $e_1, e_3, \dots, e_{2m-1}$. Furthermore, this segment clearly does not involve the edges $e_2, e_4, \dots, e_{2m}$ because of Remark 7.1.5. Since the graph Γ has in-degree two, there exists a unique edge apart from e_{2j-1} with its head adjacent to the start of the segment, namely e_{2j}. Similarly, since the graph Γ has out-degree two, there exists a unique edge apart from e_{2k_j-1} with its tail adjacent to the end of this segment, namely e_{2k_j-2}. Therefore, the edge e_{2j}, this segment of C_1 and the edge e_{2k_j-2} form together a segment of a cycle of the covering C_2. Hence, because this holds for any $j \in \{1,\dots,m\}$, the uniqueness of the "joining together-step" is given.

Let m_{τ_2} be the number of cycles in the covering C_2 involving the edges $e_2, e_4, \dots, e_{2m}$. Then there are $m_2 - m_{\tau_2} = m_1 - m_{\tau_1}$ cycles in C_2 which do not involve the edges $e_2, e_4, \dots, e_{2m}$. Let the permutation $\tau_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ \widehat{k_1} & \widehat{k_2} & \widehat{k_3} & \cdots & \widehat{k_{m-1}} & \widehat{k_m} \end{smallmatrix}\right)$ describe the succession in which the edges $e_2, e_4, \dots, e_{2m}$ appear in C_2 in such a way that for any $j \in \{1,\dots,m\}$ the first edge of $e_2, e_4, \dots, e_{2m}$ following e_{2j} is $e_{2\widehat{k_j}}$. Our specification of the permutation τ_2 implies that τ_2 has to have m_{τ_2} disjoint cycles. Studying once again Figure 7.2, we see that if $\tau_1(j) = k_j \neq 1$ then $\tau_2(j) = k_j - 1$, and if $\tau_1(j) = 1$ then $\tau_2(j) = m$. Let the permutation π be m-cycle $\pi = (12\dots m)$. Then we have $\tau_2 = \tau_1\pi^{-1}$. For $h \in \{2,\dots,m\}$, let π_h be the transposition $\pi_h := (h1)$. The number of cycles (including 1-cycles) in $\tau_1\pi_h$ is $m_{\tau_1}+1$ when h and 1 occur in the same cycle of τ_1, and it is $m_{\tau_1}-1$ when h and 1 occur in different cycles of τ_1. Since the permutation π^{-1} can be written as product of $m-1$ transpositions

$$\pi^{-1} = (m\dots 21) = (m1)(m-1\,1)\cdots(31)(21) = \pi_m\pi_{m-1}\cdots\pi_3\pi_2,$$

we have (see Lemma 2 of [R48, p. 20])

$$m_{\tau_2} - m_{\tau_1} \equiv m - 1 \mod 2.$$

If m is odd, then $m_{\tau_2} \equiv m_{\tau_1} \mod 2$ and hence $m_2 \equiv m_1 \mod 2$. If m is even, then $m_{\tau_2} \not\equiv m_{\tau_1} \mod 2$ and hence $m_2 \not\equiv m_1 \mod 2$. □

Corollary 7.1.7. *Let Γ be a digraph such that every vertex in $V(\Gamma)$ has in-degree two as well as out-degree two. Let C_1 and C_2 be arbitrary coverings of Γ. Let m_1 and m_2 be the numbers of cycles in C_1 and C_2, respectively. If m is odd for all alternating $2m$ - gons, then $m_2 \equiv m_1 \mod 2$. If m is even for all alternating $2m$ - gons and if C_1 is transformed into C_2 by k one-step transformations, then the following holds:*

(i) If k is even, then $m_2 \equiv m_1 \mod 2$.

(ii) If k is odd, then $m_2 \not\equiv m_1 \mod 2$.

Proof. Let C_1 and C_2 be arbitrary coverings of Γ. First we explain why it is possible to transform C_1 into C_2 by a finite number of one-step transformations. Let $C_{old} := C_1$.
As a first step we order the edges used in the covering C_{old}. Then we take the first edge used in C_{old} which is not contained in any cycle of the covering C_2. The application of the one-step transformation of C_{old} which uses the unique alternating $2m$ - gon containing this exact edge yields a new covering C_{new} of Γ. All edges that are changed by this procedure are used in the covering C_2. All edges that coincided with edges used in C_2 prior to the application of this procedure will still coincide with C_2 after the one-step transformation. Redefine $C_{old} := C_{new}$.
Iterate the above described process (above paragraph) until the covering C_{new} coincides with the covering C_2. Hence, it is possible to transform C_1 into C_2 using a finite number of one-step transformations, and consequently the statement of the corollary is a direct conclusion of Theorem 7.1.6. □

Corollary 7.1.8. *Let $S = \{s_1, s_2\}$ be a generating set of a finite group G. The Cayley color graph of G with respect to S as defined in Definition 3.2.5 has in-degree two as well as out-degree two. Let d be the order of $s_1 s_2^{-1}$. Suppose that d is odd. If C_1 is a covering of $C_S(G)$ that involves just one cycle, meaning if $C_S(G)$ is Hamiltonian, then any arbitrary covering of $C_S(G)$ consists of an odd number of cycles.*

Proof. Studying the following figure, we can see that every edge of $C_S(G)$ is part of an alternating $2d$ - gon.

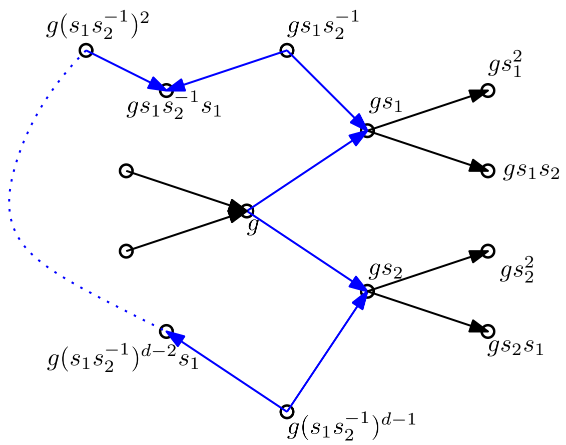


Figure 7.3: Let g be an arbitrary element of the group G. Then the edge (g, gs_1) is part of the unique $2d$ - gon

$$(g, gs_1)(gs_1, gs_1s_2^{-1}) \cdots (g(s_1s_2^{-1})^{d-1}, g(s_1s_2^{-1})^{d-1}s_1)(g(s_1s_2^{-1})^{d-1}s_1, g(s_1s_2^{-1})^d)(g, gs_1)$$

(represented by the blue part of the figure). Our assumption that $s_1s_2^{-1}$ has order d implies that $s_2s_1^{-1} = (s_1s_2^{-1})^{-1}$ has order d as well. Thus, similarly as before, the edge (g, gs_2) is part of the unique $2d$ - gon

$$(g, gs_2)(gs_2, gs_2s_1^{-1}) \cdots (g(s_2s_1^{-1})^{d-1}, g(s_2s_1^{-1})^{d-1}s_2)(g(s_2s_1^{-1})^{d-1}s_2, g(s_2s_1^{-1})^d)(g, gs_2).$$

Since g was arbitrarily chosen, we can see that every edge of $C_S(G)$ is part of an alternating $2d$ - gon.

Let C_1 be a covering of $C_S(G)$ that involves just one cycle, meaning the number of cycles in C_1 is $m_1 = 1$. Let C_2 be an arbitrary covering of $C_S(G)$ and let m_2 be the number of cycles in C_2. Using Corollary 7.1.7 and the assumption that d is odd, we get $m_2 \equiv 1 \mod 2$ and thus the number of cycles in C_2 has to be odd. □

7.2 Application to Grandsire Triples

In subsection 6.2.1 we have shown a very long and complicated proof of Thompson's theorem 6.2.6, which was based on a series of fourteen different claims. Now that we have gotten to know Rankin's work in a more general context, we will use the above derived corollaries to present two shorter, alternative proofs of Thompson's theorem. More specifically, we will prove the theorem by using Corollary 7.0.8 and then by using Corollary 7.1.8.

Using Corollary 7.0.8: We know by Lemma 6.2.4 that $\{p, b\}$ is a generating set of the group K_7. Furthermore, by using Lemma 6.2.1, we see that $pb^{-1} = (27643)$ has odd order, namely order five. The index of $\langle b \rangle$ in K_7 is given by

$$[K_7 : \langle b \rangle] = \frac{|K_7|}{|\langle b \rangle|} = \frac{|A_6|}{|\langle(247)(365)\rangle|} = \frac{\frac{720}{2}}{3} = \frac{360}{3} = 120,$$

and hence it is even. We conclude that K_7 cannot be generated unicursally by $\{p, b\}$ by applying Corollary 7.0.8. This means that part (i) of Thompson's theorem 6.2.6 holds true. Since we know that part (ii) of the theorem is a direct consequence of part (i), we see that there does not exist a 7 - bell extent of Grandsire using only plain leads and bob leads. □

Using Corollary 7.1.8: We know by Lemma 6.2.4 that $\{p, b\}$ is a generating set of the group K_7. Furthermore, by using Lemma 6.2.1, we see that $pb^{-1} = (27643)$ has odd order, namely order five. The Cayley color graph of K_7 with respect to $\{p, b\}$, denoted by $C_{\{p,b\}}(K_7)$, has the elements of K_7 as its vertices (recall Definition 3.2.5). By Definition 3.2.5, $C_{\{p,b\}}(K_7)$ has in-degree two as well as out-degree two. Based on the permutation $p = (34675)$, we are able to cover $C_{\{p,b\}}(K_7)$ by directed cycles of length five. These cycles correspond to the plain courses of the Grandsire method (recall Definition 1.3.4 and Remark 1.3.5). Hence, $C_{\{p,b\}}(K_7)$ can be covered by an even number of directed cycles, namely by $\frac{|K_7|}{5} = \frac{360}{5} = 72$ disjoint directed cycles. We conclude that $C_{\{p,b\}}(K_7)$ cannot be Hamiltonian by applying Corollary 7.1.8. In other words, we can never get a single directed cycle covering $C_{\{p,b\}}(K_7)$. Consequently, K_7 cannot be generated unicursally by $\{p, b\}$ (recall Definition 4.5.1). Thus, since part (i) of Thompson's theorem 6.2.6 holds true, part (ii) follows immediately. □

Appendix

Colin J. E. Wyld's composition of Stedman Triples

2314567	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0
2561743	-		-			-	-		-		-	-	-	-	-	-	-	-	-	
5361742	-	-	-	-		-	-	-	-	-	-	-	-	-			-	-	-	-
4365712	-	-	-	-	-		-	-	-		-	-	-	-	-	-	-	-	-	
4163275	-	-	-	-		-	-	-	-	-	-	-	-		-	-		-	-	-
4165237	-	-	-	-	-	-		-	-	-	-	-	-	-	-	-		-	-	-
4172365	-	-	-		-	-	-	-	-	-	-		-	-	-	-		-	-	-
2146357	-	-	-	-	-	-		-	-	-	-	-		-	-		-	-		-
2517364	-	-	-	-	-	-	-		-	-	-	-	-	-	-	-		-		-
2547316	-	-	-	-	-	-	-	-		-	-	-	-	-	-	-	-	-		-
2571634	-	-	-	-	-	-		-	-	-			-	-	-	-	-	-	-	-
2541673	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-	-	-	-
6541723		-	-	-	-	-	-	-	-	-		-			-	-	-	-	-	-
6751234	-	-	-			-	-	-	-	-	-	-	-	-		-	-	-		-
6451273	-	-	-	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-
7453612	-	-		-	-	-	-	-	-	-	-	-			-			-	-	
7613524	-		-		-	-	-	-	-	-	-	-	-			-	-	-	-	-
7413562	-	-	-	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-
2473561	-	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-		-
2413576	-	-	-	-	-	-	-	-		-	-	-	-	-	-	-	-	-		-
7215643	-	-			-	-	-		-	-	-	-	-	-	-	-		-	-	-
1253674	-		-	-	-	-		-	-	-	-	-	-	-	-		-	-		-
1243657	-	-	-	-	-	-	-	-		-	-	-	-	-	-	-	-	-		-
5213764	-	-	-	-	-		-	-	-	-	-	-	-		-	-	-	-		-
5123746	-	-	-	-	-	-	-	-		-	-	-	-	-		-	-			-
5143627	-	-	-	-	-	-	-	-		-		-	-	-	-	-	-	-	-	-
5643217		-	-	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-
6123754	-	-	-	-		-	-	-	-	-		-	-	-	-		-	-		-
6425731	-	-	-	-	-	-		-	-	-	-	-	-	-		-	-	-	-	-
6124735	-	-	-	-		-	-	-	-	-	-	-	-	-		-		-	-	-
6143725	-	-	-	-	-	-		-		-	-	-	-	-	-	-	-	-		-
6543712	-	-	-	-		-	-	-	-	-	-	-	-	-		-	-	-	-	-
5163724	-	-	-	-			-	-	-	-	-	-	-	-	-		-	-		-
5276134	-	-	-	-		-	-	-		-	-	-	-		-	-		-		-
4356172	-	-		-	-		-	-		-	-	-	-	-	-	-	-		-	-
4367152	-	-	-	-	-	-		-		-	-	-	-	-	-	-	-	-		-
4257163	-	-	-	-	-	-	-	-		-	-	-	-	-		-	-	-	-	-
4657321	-	-	-	-		-	-	-	-	-		-	-	-	-	-	-	-	-	-
4352167		-	-	-	-	-	-	-	-	-		-	-	-		-		-	-	-
5312674			-	-	-	-	-	-	-	-	-			-	-	-	-	-	-	-
7214563	-	-		-	-	-		-	-	-		-	-	-		-	-	-	-	-
7624513	-	-	-	-		-	-	-		-	-	-	-	-	-	-	-	-		-
2314567	-	-	-	-	-	-	-	-		-	-	-	-	-			-	-	-	-

Published in *A Collection of Compositions of Stedman Triples and Erin Triples* (1999).

Wyld's composition does not start with rounds. Instead it begins and thus ends with the change 2314567. This change is exactly the third change of the plain lead starting with rounds, which can be detected by looking at Table 5.1. The fact that Wyld's composition starts with 2 3 1 4 5 6 7 does not have any important consequences other than the failing of condition (i) in Definition 1.1.3. For $j \in \{0, \ldots, 42\}$, every $(1 + j \cdot 120)$th change of the extent is displayed in the leftmost column. Every entry in the above (42×20) - array, containing either a line or an empty space, corresponds to a block of six successive changes. We call such a block of six successive changes a Six. The first change of a Six is called a Six Head, while the last change of a Six is a so-called Six End. Each empty space in the array symbolizes that the new Six Head results from applying $A = (12)(34)(56)$ to the previous Six End. Similarly, each line in the array symbolizes that the Six End to Six Head transition results from the bob $D = (12)(34)(67)$. We can count 705 lines in the above array. Thus, Wyld's composition contains 705 bobs.

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Titlepage: Engraving of the bells in St. Paul's Cathedral in London. Mary Evans Picture Library. http://www.maryevans.com/archiveBlog/wp-content/uploads/2016/09/bells-1.jpg (April 14, 2017).

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Appendix: A Collection of Compositions of Stedman Triples and Erin Triples, 1999. http://www.ringing.org/peals/triples/stedman/#1839 (July 7, 2017).