Homological Dimension

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1 Introduction

In this note, we explore the notion of homological dimension. After introducing the basic concepts, our two main goals are to give a proof of the Hilbert syzygy theorem and to apply the theory of homological dimension to the study of local rings.

2 Elementary results from homological algebra

Assume throughout that R is a commutative ring with identity. We assume some familiarity with the basic concepts from homological algebra, including but not limited to the following: chain complexes and double complexes, projective modules and resolutions, Tor, and Ext. We state a number of elementary results without proof, all of which can be found in [4], that will be useful in the duration.

The first result gives a method for constructing projective resolutions:

Lemma 2.1. *Let*

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ \cdots & \longrightarrow & P_1' & \longrightarrow & P_0' & \longrightarrow & M' & \longrightarrow & 0 \\ & & & & & & {\scriptstyle i}\downarrow & & \\ & & & & & & M & & \\ & & & & & & {\scriptstyle \pi}\downarrow & & \\ \cdots & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \longrightarrow & M'' & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

be a diagram of R-modules where the column is exact and the rows are projective resolutions. We can complete this diagram to give the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1^{'} & \longrightarrow & P_0^{'} & \longrightarrow & M^{'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} & & \\
\cdots & \longrightarrow & P_1^{''} & \longrightarrow & P_0^{''} & \longrightarrow & M^{''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $P_i = P_i^{'} \oplus P_i^{''}$ give a projective resolution of M and the columns are exact with the canonical inclusion and projection maps.

Our next result, which follows directly from the long exact sequence for $\mathrm{Ext}_R^i(M, -)$ associated to a short exact sequence of R-modules, characterizes projective R-modules.

Proposition 2.2. *Let M be an R-module, then the following are equivalent:*

1. *M is projective.*
2. *$\mathrm{Ext}_R^i(M, N) = 0$ for all $i > 0$ and all R-modules N.*
3. *$\mathrm{Ext}_R^1(M, N) = 0$ for all R-modules N.*

3 Basic concepts

Let M be an R-module. The **projective dimension** of M, denoted $\mathrm{pd}_R(M)$ is the smallest natural number n so that there exists an R-module projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If M does not admit a projective resolution of finite length, then we set $\mathrm{pd}_R(M) = \infty$.

Our first result gives several useful conditions characterizing projective dimension:

Lemma 3.1. *Let M be an R-module, then the following are equivalent:*

1. $\mathrm{pd}_R(M) \leq n$.

2. $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i > n$ *and all* R*-modules* N.

3. $\mathrm{Ext}^{n+1}_R(M,N) = 0$ *for all* R*-modules* N.

4. *Whenever*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence with each P_i *projective,* K *is projective.*

Proof. Since $\mathrm{Ext}^i_R(-,N)$ is the right derived functor of $Hom_R(-,N)$, we have that item 4 implies item 1 implies item 2 implies item 3.

Now, if

$$0 \longrightarrow A \longrightarrow P \longrightarrow B \longrightarrow 0$$

is an exact sequence with P projective, the long exact sequence for $\mathrm{Ext}^i_R(-,N)$ gives that

$$\mathrm{Ext}^i_R(A,N) \cong \mathrm{Ext}^{i+1}_R(B,N)$$

for $i \geq 1$. By interlacing long exact sequences with short exact sequences, this in turn implies that if

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact with each P_i projective, then

$$\mathrm{Ext}^{n+1}_R(M,N) \cong \mathrm{Ext}^1_R(K,N).$$

Thus, if item 3 holds then $\mathrm{Ext}^1_R(K,N) = 0$ for all R-modules N. Proposition 2.2 then gives that K is projective. □

In particular, item 2 gives that if $\mathrm{pd}_R(M) < \infty$ then

$$\mathrm{pd}_R(M) = \min\{i : \mathrm{Ext}^{i+1}_R(M,N) = 0 \text{ for all } R\text{-modules } N\}.$$

Proposition 3.1 now gives that for a ring R the following numbers are the same

1. $\sup\{i : \mathrm{Ext}^i_R(M,N) \neq 0 \text{ for some } R\text{-modules } M \text{ and } N\}$

2. $\sup\{\mathrm{pd}_R(M) : M \in R\text{-mod}\}$

Definition 3.2. *This common number is the* ***global dimension*** *of the ring* R*, denoted* $\mathrm{gldim}(R)$.

4 The Hilbert syzygy theorem

Our goal in this section is to give a proof of the Hilbert syzygy theorem. The strongest form of this statement is the following:

Theorem 4.1. *Let k be a field and suppose that M is a finitely generated $k[x_1, \ldots, x_n]$-module. If*

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \tag{4.1}$$

is an exact sequence with each F_i free and finitely generated, then K is free.

Our approach is as follows: We will first deduce that

$$\mathrm{gldim}(k[x_1, \ldots, x_n]) = n. \tag{4.2}$$

Since each F_i is projective, Lemma 3.1 implies that K is projective, and K is finitely generated as $k[x_1, \ldots, x_n]$ is noetherian. The result then follows from the Quillen-Suslin theorem [3, Chapter XXI, §4, Theorem 3.7]:

Theorem 4.2. *Any finitely generated projective $k[x_1, \ldots, x_n]$-module is free.*

We now aim to show (4.2). We follow the presentation given in [4, Section 4.3], elaborating on the proofs given there. The result will follow from a sequence of "change of rings" results. In order to prove the first of these results, we need the following preparatory lemma.

Lemma 4.3. *Let $\{M_i\}_{i \in I}$ be a collection of R-modules, then*

$$\mathrm{pd}_R\left(\bigoplus_{i \in I} M_i\right) = \sup_{i \in I}\left\{\mathrm{pd}_R(M_i)\right\}.$$

Proof. Since the direct sum of a family of projective modules is projective, given projective resolutions $P_{*,i} \twoheadrightarrow M_i$ for each M_i we have a projective resolution $\oplus P_{*,i} \twoheadrightarrow \oplus M_i$ which gives

$$\mathrm{pd}_R\left(\bigoplus_{i \in I} M_i\right) \leq \sup_{i \in I}\left\{\mathrm{pd}_R(M_i)\right\}.$$

Next, if $\mathrm{pd}_R(\oplus M_i) = \infty$ then the result follows, so suppose $\mathrm{pd}_R(\oplus M_i) = n < \infty$. For each i, consider an exact sequence

$$0 \longrightarrow K_i \longrightarrow P_{n-1,i} \longrightarrow \cdots \longrightarrow P_{0,i} \longrightarrow M_i \longrightarrow 0$$

with each $P_{m,i}$ projective. This gives an exact sequence

$$0 \longrightarrow \bigoplus_{i \in I} K_i \longrightarrow \bigoplus_{i \in I} P_{n-1,i} \longrightarrow \cdots \longrightarrow \bigoplus_{i \in I} P_{0,i} \longrightarrow \bigoplus_{i \in I} M_i \longrightarrow 0$$

and Lemma 3.1 gives that $\oplus K_i$ is projective. Since direct summands of projective modules are projective, it follows that each K_i is projective. Lemma 3.1 then gives $\mathrm{pd}_R(M_i) \leq n$ and the result follows. $\square$

Given a morphism of rings $R \to S$, the first result relates the projective dimension of an S-module M over S to its projective dimension over R.

Theorem 4.4. *If $R \to S$ is a morphism of rings and M is an S-module, then*

$$\mathrm{pd}_R(M) \leq \mathrm{pd}_R(S) + \mathrm{pd}_S(M).$$

Proof. We can assume that $\mathrm{pd}_S(M) = n < \infty$ and $\mathrm{pd}_R(S) = d < \infty$ or else there is nothing to prove. Let

$$0 \longrightarrow Q_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M \longrightarrow 0$$

be an S-module projective resolution.

Choose R-module projective resolutions of $M = \mathrm{im}\, f_0$ and $\ker f_0$ and use Lemma 2.1 to construct an R-module projective resolution $P'_{*,0}$ of Q_0. Now construct R-module projective resolutions for each Q_i via the same procedure, using the resolution of $\mathrm{im}\, f_i = \ker f_{i-1}$ used to construct the projective resolution of Q_{i-1} and choosing a projective resolution of $\ker f_i$. We can then patch these short exact sequences of projective resolutions together to obtain the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & Q_n & \longleftarrow & P'_{0,n} & \longleftarrow & P'_{1,n} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & Q_1 & \longleftarrow & P'_{0,1} & \longleftarrow & P'_{1,1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & Q_0 & \longleftarrow & P'_{0,0} & \longleftarrow & P'_{1,0} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the columns are chain complexes and the rows are R-module projective resolutions.

Now, since each Q_i is a projective S-module, it follows that for each i there exists an S-module N_i so that $Q_i \oplus N_i$ is a free S-module $F_i(S)$. Lemma 4.3 then implies that

$$\mathrm{pd}_R(Q_i) \leq \mathrm{pd}_R(F_i(S)) = \mathrm{pd}_R(S) = d.$$

We hence may construct the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & Q_n & \longleftarrow & P_{0,n} & \longleftarrow & \cdots & \longleftarrow & P_{d,n} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & Q_0 & \longleftarrow & P_{0,0} & \longleftarrow & \cdots & \longleftarrow & P_{d,0} & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{4.3}$$

where the columns are chain complexes and the rows are R-module projective resolutions by taking $P_{m,i} = P'_{m,i}$ for $m \neq d$ and $P_{d,i} = P'_{d,i}/\ker(P'_{d,i} \to P'_{d-1,i})$ and using Lemma 3.1.

A standard argument now gives that the complexes Q_* and $\mathrm{Tot}(P_{*,*})_*$, the total complex associated the double complex $P_{*,*}$ obtained from truncating the Q_i's in the rows of (4.3), are quasi-isomorphic (i.e. have the same homology). Since $\mathrm{Tot}(P_{*,*})_*$ is a complex of projective R-modules of length at most $n+d$, the result follows. $\square$

The proofs of the next two change of rings theorems proceed via induction on $\mathrm{pd}_R(M)$, so the following lemma aids in these considerations.

Lemma 4.5. *If the sequence of R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then $\mathrm{pd}_R(B) \leq \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(C)\}$. Moreover, if we have strict inequality, then $\mathrm{pd}_R(C) = \mathrm{pd}_R(A) + 1$.

Proof. The first statement follows from Lemma 2.1. The second follows considering the characterization provided in Lemma 3.1. Indeed suppose that $\mathrm{pd}_B(B) = n < \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(C)\}$. The long exact sequence for $\mathrm{Ext}^i_R(-, N)$ gives that

$$\mathrm{Ext}^i_R(A, N) \cong \mathrm{Ext}^{i+1}_R(C, N)$$

for $i \geq n+1$ and all R-modules N. This gives the result provided $\mathrm{pd}_R(A) \geq n+1$ or $\mathrm{pd}_R(C) \geq n+2$. The remaining case is when $\mathrm{pd}_R(C) = n+1$ and $\mathrm{pd}_R(A) < n+1$. If $\mathrm{pd}_R(A) = n$ then the result holds and otherwise the $\mathrm{Ext}^i_R(-, N)$ long exact sequence implies that $\mathrm{Ext}^{n+1}_R(C, N) = 0$ for all R-modules N, contradicting the fact that $\mathrm{pd}_R(C) = n+1$. $\square$

Given a ring R and an element $r \in R$ that is not a zero divisor, consider an R/rR-module M. Such a module is an R-module that is annihilated by r. The next result gives the projective dimension of M over R/rR in terms of the projective dimension of M over R. This is particularly relevant to our present consideration, since taking $R = S[x]$ for some ring S and $r = x$, we will be able to deduce information about $\mathrm{gldim}(S[x])$ from $\mathrm{gldim}(S)$.

Theorem 4.6. *Let $r \in R$ be a non-zerodivisor, M be a non-zero R/rR-module, and suppose that $\mathrm{pd}_{R/rR}(M) < \infty$, then*

$$\mathrm{pd}_R(M) = 1 + \mathrm{pd}_{R/rR}(M).$$

Proof. First, note that $\mathrm{pd}_R(M) \neq 0$. Indeed, this would mean that M is a projective R-module, implying that M is a direct summand of a free R-module and contradicting the fact that M is annihilated by r. We hence have that $\mathrm{pd}_R(M) \geq 1$.

Next, suppose that M is a projective R/rR-module, then since

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/rR \longrightarrow 0 \tag{4.4}$$

gives a projective resolution of R/rR of minimal length, Theorem 4.4 gives

$$\mathrm{pd}_R(M) \leq \mathrm{pd}_R(R/rR) + \mathrm{pd}_{R/rR}(M) = 1$$

so the result holds.

It now suffices to consider the case when $\mathrm{pd}_R(M), \mathrm{pd}_{R/rR}(M) \geq 1$ and we proceed via induction on $n = \mathrm{pd}_{R/rR}(M) < \infty$. Consider an R/rR-module projective resolution

$$0 \longrightarrow P_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \longrightarrow 0$$

of minimal length. Taking $K = \ker f_0$, we have that

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is exact and $\mathrm{pd}_{R/rR}(M) = 1 + \mathrm{pd}_{R/rR}(K)$. Lemma 4.5 implies that either

$$1 = \mathrm{pd}_R(P_0) = max\{\mathrm{pd}_R(K), \mathrm{pd}_R(M)\} \tag{4.5}$$

or

$$\mathrm{pd}_R(M) = 1 + \mathrm{pd}_R(K). \tag{4.6}$$

Since the induction hypothesis gives that

$$\mathrm{pd}_R(K) = 1 + \mathrm{pd}_{R/rR}(K)$$

provided $\mathrm{pd}_R(K) \geq 1$, we are done if (4.6) holds or if (4.5) holds and $\mathrm{pd}_R(K) = 0$, since $\mathrm{pd}_R(M) \geq 1$.

It remains to consider the case when $\mathrm{pd}_R(M) = 1 = \mathrm{pd}_{R/rR}(M)$ and we shall now see that this cannot happen. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an R-module projective resolution of M. Applying the functor $- \otimes_R R/rR$ gives the exact sequence

$$0 \longrightarrow Tor_1^R(M, R/rR) \longrightarrow P_1 \otimes_R R/rR \longrightarrow P_0 \otimes_R R/rR \longrightarrow M \longrightarrow 0$$

of R/rR-modules. Since $P_1 \otimes_R R/rR$ and $P_0 \otimes_R R/rR$ are projective R/rR-modules, Lemma 3.1 implies that $Tor_1^R(M, R/rR)$ is projective.

Computing $Tor_1^R(M, R/rR)$ by applying $M \otimes_R -$ to the R-module projective resolution (4.4), we find that

$$Tor_1^R(M, R/rR) \cong M$$

which implies $\mathrm{pd}_{R/rR}(M) = 0$, a contradiction. □

The next result examines the case where $r \in R$ is neither a zero divisor on R nor on M.

Theorem 4.7. *Let M be an R-module and $r \in R$ be neither a zero divisor on R nor on M, then*

$$\mathrm{pd}_{R/rR}(M/rM) \leq \mathrm{pd}_R(M).$$

Proof. Since the result is trivial if $\mathrm{pd}_R(M) = \infty$, we proceed via induction on $\mathrm{pd}_R(M) < \infty$. If $\mathrm{pd}_R(M) = 0$ then M is projective, hence $M/rM \cong R/rR \otimes_R M$ is projective, i.e. $\mathrm{pd}_{R/rR}(M/rM) = 0$ and the result holds.

Now suppose $n = \mathrm{pd}_R(M) \geq 1$ and let

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0 \qquad (4.7)$$

be an exact sequence of R-modules with F free. Lemma 3.1 gives that $\mathrm{pd}_R(K) = n - 1$ and the induction hypothesis gives

$$\mathrm{pd}_{R/rR}(K/rK) \leq n - 1.$$

Applying $- \otimes_R R/rR$ to (4.7) gives

$$0 \longrightarrow Tor_1^R(M, R/rR) \longrightarrow K/rK \longrightarrow F/rF \longrightarrow M/rM \longrightarrow 0$$

and a calculation gives that $Tor_1^R(M, R/rR) = 0$, so

$$0 \longrightarrow K/rK \longrightarrow F/rF \longrightarrow M/rM \longrightarrow 0$$

is exact. Lemma 4.5 now gives that either

$$0 = \mathrm{pd}_{R/rR}(F/rF) = max\{\mathrm{pd}_{R/rR}(K/rK), \mathrm{pd}_{R/rR}(M/rM)\}$$

so $\mathrm{pd}_{R/rR}(M/rM) = 0$ or that

$$\begin{aligned}\mathrm{pd}_{R/rR}(M/rM) &= \mathrm{pd}_{R/rR}(K/rK) + 1 \\ &\leq (n-1) + 1 = n\end{aligned}$$

which gives the result. □

Corollary 4.8. *Let M be an R-module, then*

$$\mathrm{pd}_R(M) = \mathrm{pd}_{R[x]}(R[x] \otimes_R M).$$

Proof. Since x is a non-zerodivisor on $R[x]$ and $R[x] \otimes_R M$, Theorem 4.7 gives

$$\mathrm{pd}_R(M) \leq \mathrm{pd}_{R[x]}(R[x] \otimes_R M).$$

Now, if $P_* \twoheadrightarrow M$ is an R-module projective resolution, then applying $R[x] \otimes_R -$ gives an $R[x]$-module projective resolution $R[x] \otimes_R P_* \twoheadrightarrow R[x] \otimes_R M$ since $R[x]$ is a flat R-module. This gives

$$\mathrm{pd}_R(M) \geq \mathrm{pd}_{R[x]}(R[x] \otimes_R M).$$

□

We now arrive at the main theorem in this section.

Theorem 4.9. *Let R be a ring, then* $\mathrm{gldim}(R[x]) = \mathrm{gldim}(R) + 1$.

Proof. If $\mathrm{gldim}(R) = \infty$ then Corollary 4.8 implies that $\mathrm{gldim}(R[x]) = \infty$ and the result holds.

We now assume that $\mathrm{gldim}(R) = n < \infty$. If N is an R-module, then N is an $R[x]$-module (let x act trivially) and Theorem 4.6 gives

$$\mathrm{pd}_{R[x]}(N) = 1 + \mathrm{pd}_R(N)$$

and hence $\mathrm{gldim}(R[x]) \geq n + 1$.

Now, let M be an $R[x]$-module. A computation shows that the sequence of $R[x]$-modules

$$0 \longrightarrow R[x] \otimes_R M \xrightarrow{\varphi} R[x] \otimes_R M \xrightarrow{\psi} M \longrightarrow 0$$

with

$$\begin{aligned}\varphi(f \otimes m) &= xf \otimes m - f \otimes xm \\ \psi(f \otimes m) &= fm\end{aligned}$$

is exact. Lemma 4.5 then gives that either $\mathrm{pd}_{R[x]}(M) = 1 + \mathrm{pd}_{R[x]}(R[x] \otimes_R M)$ or $\mathrm{pd}_{R[x]}(M) \leq \mathrm{pd}_{R[x]}(R[x] \otimes_R M)$. This gives

$$\begin{aligned} \mathrm{pd}_{R[x]}(M) &\leq 1 + \mathrm{pd}_{R[x]}(R[x] \otimes_R M) \\ &= 1 + \mathrm{pd}_R(M) \\ &\leq 1 + n \end{aligned}$$

where we have used Corollary 4.8. This implies $\mathrm{gldim}(R[x]) \leq n + 1$ which completes the proof. □

This result immediately implies the following corollary.

Corollary 4.10. *Let R be a ring, then* $\mathrm{gldim}(R[x_1, \ldots, x_n]) = \mathrm{gldim}(R) + n$.

Our proof of the Hilbert syzygy theorem follows directly from this result, noting that if k is a field, then $\mathrm{gldim}(k) = 0$ since all k-modules are free.

5 Local Rings

For the duration, we consider a noetherian local ring R with maximal ideal $\mathfrak{m}$. Recall that such a ring is called a **regular local ring** provided

$$\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$$

where $k = R/\mathfrak{m}$ is the residue field, $\dim(-)$ denotes Krull dimension, and $\dim_k(-)$ denotes vector space dimension. Such rings arise naturally in algebraic geometry, where they characterize nonsingularity of varieties. It is a standard fact that if R is a noetherian local ring then $\dim(R)$ is finite and in fact $\dim(R) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ [1, Chapter 11].

In this section we aim to prove the following result [4, Corollary 4.4.18]:

Theorem 5.1. *If R is a regular local ring and $\mathfrak{p} \subseteq R$ is a prime ideal, then $R_{\mathfrak{p}}$ is a regular local ring.*

It is interesting to note that from the statement of this theorem, one would not suspect that homological algebra would play any role in its proof. As in the last section, we will prove this result using a sequence of preliminary results. We again follow the presentation given in [4, Section 4.4].

We begin with an additional change of rings theorem, a strengthening of Theorem 4.7 in the case that M is finitely generated.

Theorem 5.2. *Let R a noetherian local ring with maximal ideal $\mathfrak{m}$. If M is a finitely generated R-module and $r \in \mathfrak{m}$ is neither a zero divisor on R nor on M, then*

$$\mathrm{pd}_{R/rR}(M/rM) = \mathrm{pd}_R(M).$$

Proof. We have $\mathrm{pd}_{R/rR}(M/rM) \le \mathrm{pd}_R(M)$ by Theorem 4.7 so we are done if $\mathrm{pd}_{R/rR}(M/rM) = \infty$. Otherwise, we may proceed via induction on $n = \mathrm{pd}_{R/rR}(M/rM)$.

If $n = 0$ then M/rM is projective and it follows that M/rM is free since R/rR is local and projective modules over local rings are free. We now claim that this implies that M is free, i.e. $\mathrm{pd}_R(M) = 0$.

Indeed, let $x_1, \ldots, x_m \in M$ be elements mapping to a basis for M/rM and let $(x_1, \ldots, x_m)$ denote the R-submodule they generate. We have

$$\begin{aligned} M &= (x_1, \ldots, x_m) + rM \\ &= (x_1, \ldots, x_m) + \mathfrak{m}M \end{aligned}$$

so Nakayama's lemma implies $(x_1, \ldots, x_m) = M$.

To see that these elements are independent, suppose there exist $s_1, \ldots, s_m \in R$ so that $\sum s_i x_i = 0$. This gives that $\sum s_i \bar{x_i} = 0$ in M/rM where $\bar{x_i}$ denotes the image of x_i under the canonical quotient map. For each i we have, $s_i = r \cdot s_i'$, so $r \cdot (\sum s_i' x_i) = 0$ and since r is not a zero divisor on M this implies $\sum s_i' x_i = 0$. We may iterate this procedure and obtain an increasing sequence of ideals

$$s_i R \subseteq s_i' R \subseteq \cdots \subseteq s_i^{(j)} R \subseteq \cdots$$

where $s_i^{(j)} = r s_i^{(j+1)}$, which must stabilize since R is noetherian. This gives that $s_i^{(l)} R = s_i^{(l+1)} R$ for some l and hence there exists $a \in R$ so that $ars_i^{(l+1)} = s_i^{(l+1)}$. Rearranging, this gives $(1 - ar)s_i^{(l+1)} = 0$ and since $r \in \mathfrak{m}$ we have that $1 - ar$ is a unit [1, Proposition 1.9]. This implies $s_i^{(l+1)} = 0$ so $s_i = r^{l+1} s_i^{(l+1)} = 0$ for all i.

Having handled the $n = 0$ case, we now easily finish the proof. Let

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules with F free, so $\mathrm{pd}_R(K) = \mathrm{pd}_R(M) - 1$. Applying the functor $- \otimes_R R/R$ gives the exact sequence

$$0 \longrightarrow K/rK \longrightarrow F/rF \longrightarrow M/rM \longrightarrow 0$$

since $\mathrm{Tor}_1^R(M, R/rR) = 0$ if r is not a zero divisor on M. Since F/rF is free, $\mathrm{pd}_{R/rR}(K/rK) = n - 1$, and since M is finitely generated, we can suppose that F is as well. As R is noetherian, this implies that K is finitely generated. The induction hypothesis then gives that $\mathrm{pd}_R(K) = n - 1$ and so $\mathrm{pd}_R(M) = \mathrm{pd}_R(K) + 1 = n$. □

We now need to introduce further machinery for analyzing local rings. Let M be an R-module, then a **regular sequence** on M is a sequence $r_1, \ldots r_d \in \mathfrak{m}$ so that r_1 is not a zero divisor on M and r_i is not a zero divisor on $M/(r_1, \ldots, r_{i-1})M$. We let $\mathrm{G}(M)$ denote the length of the longest regular sequence on M. Now recall the following result from the dimension theory of noetherian local rings [1, Corollary 11.18]:

Proposition 5.3. *Let R be a noetherian local ring and $r \in \mathfrak{m}$ not a zero divisor, then* $\dim R/rR = \dim R - 1$.

This shows that $\mathrm{G}(R) \leq \dim(R)$ and R is called **Cohen-Macaulay** if we have $\mathrm{G}(R) = \dim(R)$.

Proposition 5.4. *A regular local ring R is Cohen-Macaulay. Moreover, any set $r_1, \ldots, r_d \in \mathfrak{m}$ mapping to a basis of $\mathfrak{m}/\mathfrak{m}^2$ is a regular sequence on R.*

Proof. It suffices to show that $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq \mathrm{G}(R)$, and we proceed via induction on $d = \dim(R)$. The case $d = 0$ is trivial, so assume $d \geq 1$ and let $r_1, \ldots, r_d \in \mathfrak{m}$ map to a basis of $\mathfrak{m}/\mathfrak{m}^2$. Since regular local rings are integral domains [1, Lemma 11.23], r_1 is not a zero divisor and hence, by Proposition 5.3, $\dim(R/r_1R) = d - 1$. Let $\bar{r_2}, \ldots, \bar{r_d}$ denote the images in $\bar{R} = R/r_1R$ of $r_2, \ldots, r_d$ and note that $\bar{r_2}, \ldots, \bar{r_d}$ map to a basis of $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ where $\bar{\mathfrak{m}} = \mathfrak{m}\bar{R}$, hence $\bar{R}$ is regular.

The induction hypothesis implies that $\bar{r_2}, \ldots, \bar{r_d}$ is a regular sequence on $\bar{R}$, and thus $r_1, \ldots, r_d$ is a regular sequence on R. □

We now consider a result describing noetherian local rings R with $\mathrm{G}(R) = 0$. Note that this condition is equivalent to all elements of $\mathfrak{m}$ being zerodivisors.

Lemma 5.5. *Let R be a noetherian local ring with $\mathrm{G}(R) = 0$, then if M is a finitely generated R-module either $\mathrm{pd}_R(M) = 0$ or $\mathrm{pd}_R(M) = \infty$.*

Proof. Suppose $G(R) = 0$ and that $\mathrm{pd}_R(M) = n \neq 0, \infty$. There then exists an exact sequence

$$0 \longrightarrow P_n \xrightarrow{f_n} \cdots \longrightarrow P_0 \xrightarrow{f_0} M \longrightarrow 0$$

with each P_i projective and finitely generated. Let $K = \mathrm{im}\, f_{n-1}$, which is a finitely generated syzygy of M with $\mathrm{pd}_R(K) = 1$. Let $m_1, \ldots, m_t \in M$ be elements which map to a basis for $M/\mathfrak{m}M$ and hence generate M. We now have an exact sequence

$$0 \longrightarrow P \longrightarrow R^t \longrightarrow K \longrightarrow 0$$

where $R^t \to K$ is the obvious map, and Lemma 3.1 gives that P is projective, hence free. Since $\mathfrak{m}R^t$ is the kernel of the map $R^t \to K \to K/\mathfrak{m}K$, it follows that $P \hookrightarrow \mathfrak{m}R^t$.

Now, since $\mathfrak{m}$ consists only of zerodivisors, it is contained in the union of the associated primes of R and as it is maximal, it must be one of the associated primes (by prime avoidance). By definition, this gives that there exists $s \neq 0 \in R$ so that $\mathfrak{m} = \{r \in R | rs = 0\}$. This gives that $sP = 0$, contradicting the fact that P is free. □

We state the next two results, Theorem 4.2.2 and Corollary 4.4.12 from [4] without proof, as the proofs would necessitate the introduction of a number of additional concepts (in particular injective dimension, flat dimension, and Tor-dimension [4, Section 4.1]).

Proposition 5.6. *A ring R is semisimple (i.e. every ideal is a direct summand) iff* $\operatorname{gldim}(R) = 0$.

Proposition 5.7. *Let R be a noetherian local ring with residue field k, then* $\operatorname{gldim}(R) = \operatorname{pd}_R(k)$.

We now arrive at our main theorem, which characterizes regular local rings via their global dimension. Theorem 5.1 will follow as an easy corollary.

Theorem 5.8. *A noetherian local ring R is regular iff* $\operatorname{gldim}(R) \leq \infty$. *Moreover, if R is regular then* $\dim(R) = \operatorname{gldim}(R)$.

Proof. We first assume that R is regular and show $\operatorname{gldim}(R) < \infty$, proceeding via induction on $d = \dim(R)$. If $d = 0$ then $\mathfrak{m} = \mathfrak{m}^2$ so Nakayama's lemma implies that R is a field and the result holds. Now suppose $d > 0$, then Proposition 5.4 gives $\mathrm{G}(R) = d$ so there exists $r \in \mathfrak{m}$ so that r is not a zero divisor on R. Since $\bar{R} = R/rR$ is regular and $\dim(\bar{R}) = d-1$, we have that $\operatorname{gldim}(\bar{R}) = d-1$ by the induction hypothesis. Now, we compute

$$\begin{aligned} \operatorname{gldim}(R) &= \operatorname{pd}_R(R/\mathfrak{m}) \\ &= \operatorname{pd}_R(\bar{R}) + \operatorname{pd}_{\bar{R}}(R/\mathfrak{m}) \\ &= 1 + \operatorname{pd}_{\bar{R}}(\bar{R}/\bar{\mathfrak{m}}) \\ &= 1 + \operatorname{gldim}(\bar{R}) = d \end{aligned}$$

where $\bar{\mathfrak{m}} = \mathfrak{m}\bar{R}$ is the maximal ideal of $\bar{R}$ and we have used Proposition 5.7 and Theorem 4.4.

We now assume that $\operatorname{gldim}(R) < \infty$ and deduce that R is regular; again, we proceed via induction. Suppose that $\operatorname{gldim}(R) = 0$, then R is local and semisimple (by Proposition 5.6) so R is a field, hence regular. Now, let $\operatorname{gldim}(R) \neq 0, \infty$ so by Lemma 5.5 there exists $r \in \mathfrak{m}$ which is not a zerodivisor. Infact, we can even choose $r \in \mathfrak{m} \setminus \mathfrak{m}^2$; indeed, if $\mathfrak{m} \setminus \mathfrak{m}^2$ consists only of zero divisors, then

$$\mathfrak{m} \setminus \mathfrak{m}^2 \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are the associated primes of R (it is a standard fact that the union of these ideals precisely equals the zero divisors in R). This gives

$$\mathfrak{m} \subseteq \mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_m$$

and the strong form of prime avoidance [2, Lemma 3.3] shows that either $\mathfrak{m} \subseteq \mathfrak{p}_j$ for some j or $\mathfrak{m} = 0$. This implies that all elements of $\mathfrak{m}$ are zero divisors.

Hence let $r \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a non-zerodivisor. Consider $\bar{R} = R/rR$, which we now aim to show is regular. We have exact sequences

$$0 \longrightarrow \mathfrak{m}\bar{R} \longrightarrow \bar{R} \longrightarrow k \longrightarrow 0 \tag{5.1}$$

and

$$0 \longrightarrow rR/r\mathfrak{m} \longrightarrow \mathfrak{m}/r\mathfrak{m} \xrightarrow{\alpha} \mathfrak{m}\bar{R} \longrightarrow 0\,. \tag{5.2}$$

The Tor long exact sequence associated to (5.1) gives that $rR/r\mathfrak{m} \cong \mathrm{Tor}_1^R(R/rR, k)$, and a standard computation shows that $\mathrm{Tor}_1^R(R/rR, k) \cong k$.

Now let $r_2, \ldots, r_n \in \mathfrak{m}$ be such that the images of $r, r_2, \ldots, r_n$ in $\mathfrak{m}/\mathfrak{m}^2$ form a basis (this is possible since $r \notin \mathfrak{m}^2$). Let $\mathfrak{n} = (r_2, \ldots, r_n)R + r\mathfrak{m}$ and note that $\mathfrak{n}/r\mathfrak{m}$ surjects onto $\mathfrak{m}\bar{R}$ via α. Since $\ker \alpha = rR/r\mathfrak{m}$ is isomorphic to a field and contains $r + r\mathfrak{m}$ (which is not in $\mathfrak{n}/r\mathfrak{m}$) we have that

$$(rR/r\mathfrak{m}) \cap (\mathfrak{n}/r\mathfrak{m}) = 0$$

which implies $\mathfrak{n}/r\mathfrak{m} \cong \mathfrak{m}\bar{R}$. It then follows that $\mathfrak{m}/r\mathfrak{m} \cong k \oplus \mathfrak{m}\bar{R}$ as $\bar{R}$-modules.

We now compute

$$\begin{aligned} \mathrm{gldim}(\bar{R}) &= \mathrm{pd}_{\bar{R}}(k) \\ &\leq \mathrm{pd}_{\bar{R}}(\mathfrak{m}/r\mathfrak{m}) \\ &= \mathrm{pd}_R(\mathfrak{m}) \\ &= \mathrm{pd}_R(k) - 1 \\ &= \mathrm{gldim}(R) - 1 \end{aligned}$$

where we have used Proposition 5.7, Lemma 4.3, and Theorem 5.2. The induction hypothesis now implies that $\bar{R}$ is regular.

This in turn implies that R is regular. Indeed, let $d = \dim(R)$ and note that $\dim(\bar{R}) = d - 1$. Since $\bar{R}$ is regular, Proposition 5.4 gives that there exist elements $r_2, \ldots, r_d \in \mathfrak{m}$ whose images $\bar{r_2}, \ldots, \bar{r_d} \in \mathfrak{m}\bar{R}$ generate $\mathfrak{m}\bar{R}$ and give an $\bar{R}$-regular sequence. It follows that $r, r_2, \ldots, r_d$ give an R-regular sequence and generate $\mathfrak{m}$. This shows $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq G(R)$ and since we always have $G(R) \leq \dim(R) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, this implies R is regular. □

We now explain how this result implies Theorem 5.1. Suppose R is a regular local ring and $\mathfrak{p} \subseteq R$ is a prime ideal. Let M be an $R_\mathfrak{p}$ module, then viewing M as an R-module, there is a projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $n \leq \mathrm{gldim}(R) < \infty$. Applying the functor $R_\mathfrak{p} \otimes_R -$ then gives an exact sequence

$$0 \longrightarrow R_\mathfrak{p} \otimes_R P_n \longrightarrow \cdots \longrightarrow R_\mathfrak{p} \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

of $R_\mathfrak{p}$ modules, since $R_\mathfrak{p}$ is a flat R-module and $R_\mathfrak{p} \otimes_R M \cong M$. Each $R_\mathfrak{p} \otimes_R P_i$ is a projective $R_\mathfrak{p}$-module, so this implies $\mathrm{gldim}(R_\mathfrak{p}) \leq \mathrm{gldim}(R) < \infty$. By Theorem 5.8, $R_\mathfrak{p}$ is regular.

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