Fun With Nonstandard Models

A self-contained sequel to:

2500 Years of very Small Numbers

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Outline of Talk:

Slides 1..3 Random Blather

Slides 4..8 Math 455 in 5 minutes (or less)

Slides 9..14 Nonstandard analysis

Slides 15..17 Infinitesimals and infinite numbers

Slides 18..19 Hyperfinite sets

Slide 20 (and blackboard) Applications

The Three Ages of Mathematics

(A Mathematical Logician's Point of View)

Ancient ($-\infty$–1829) Mathematics is about *objects* (numbers; geometric objects;...)

(Nikolai Ivanovich Lobachevsky: Non-Euclidean geometry; Announced Feb. 23, 1826, published 1829)

Modern (1829–1939) Mathematics is about *statements* about objects

(1939: Gödel, Turing, Church; also Löwenheim 1915, Skolem 1920, Hilbert 1921)

Postmodern (1939–present) Mathematics is about statements about mathematics

Typical mathematical statements:

1. $(\forall\epsilon \in \mathbb{R}^+)(\exists\delta \in \mathbb{R}^+)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon]$
2. $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(\forall z \in \mathbb{N})[(x + y) + z = x + (y + z)]$
3. $(\forall x \in G)(\exists y \in G)[(x \cdot y) = e \ \& \ (y \cdot x) = e]$

What are these in English?

Note:

- Literally: *first-order sentences with bounded quantifiers*; c.f. Math 455
- All non-common symbols come from the structure we are considering
- Quantification is over elements, not sets or functions

The following are **not** mathematical statements in this sense:

- $\mathbb{R}$ has an uncountable subset
- X is infinite

However: If the following statements are all true in X, then X *is* infinite:

$$(\exists x_1 \in X)(\exists x_2 \in X)[x_1 \neq x_2]$$
$$(\exists x_1 \in X)(\exists x_2 \in X)(\exists x_3 \in X)[(x_1 \neq x_2)\&(x_1 \neq x_3)\&(x_2 \neq x_3)]$$
$$(\exists x_1 \in X)(\exists x_2 \in X)(\exists x_3 \in X)(\exists x_4 \in X)[(x_1 \neq x_2)\&(x_1 \neq x_3)\&(x_1 \neq x_4)\&\cdots\&(x_3 \neq x_4)]$$
$$\vdots$$

So infinitude of a set X can be characterized by a countable collection of mathematical statements.

A **MODEL** is just a mathematical structure where we specify the functions, relations, constants of interest (i.e., the *language* of our model).

Example: $(\mathbb{Q}, <)$

(where $\mathbb{Q}$ =the rational numbers, $<$ is the usual *order relation* on $\mathbb{Q}$)

This model happens to satisfies the following statements:

- "$<$ is a linear order."
- "There are no endpoints."
- "The ordering is dense."

(For *example*, the *second condition* is: $(\forall x)(\exists y)(\exists z)[(z < x) \;\&\; (x < y)]$)

We might call $\mathbb{Q}$ the "standard" *countable model of a dense linear order without endpoints.*

Note: Any countable *order* $(X, \prec)$ satisfying these statements is isomorphic to $(\mathbb{Q}, <)$

(i.e., there is an *order-preserving* bijection between X and $\mathbb{Q}$)

Proof: nontrivial Math 321-level *exercise*

But: $(\mathbb{R}, <)$ is *also* a dense linear order without endpoints.

In fact, it satisfies *every* mathematical statement that $(\mathbb{Q}, <)$ *does*!*

However, they are <u>not</u> isomorphic (cardinalities differ.)

***Note:** For this example, our language consists only of the "less than" symbol. Every statement *about orders* that is true in $\mathbb{Q}$ is true in $\mathbb{R}$ and vice versa. For a bigger language, this would no longer hold.

For example, if we had $+$, $\cdot$, and the number 1 in it, then this would no longer be true:

$$(\exists x)[x \cdot x = 1 + 1]$$

is true in $\mathbb{R}$, not in $\mathbb{Q}$.

Moral: Always specify your language.

Example: The model $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, s)$ is the **standard model of arithmetic** (where s is the function $s(x) = x + 1$).

Theorem: (Skolem,1933) There is a countable nonstandard model of arithmetic extending $\mathfrak{N}$

What would such a model look like?

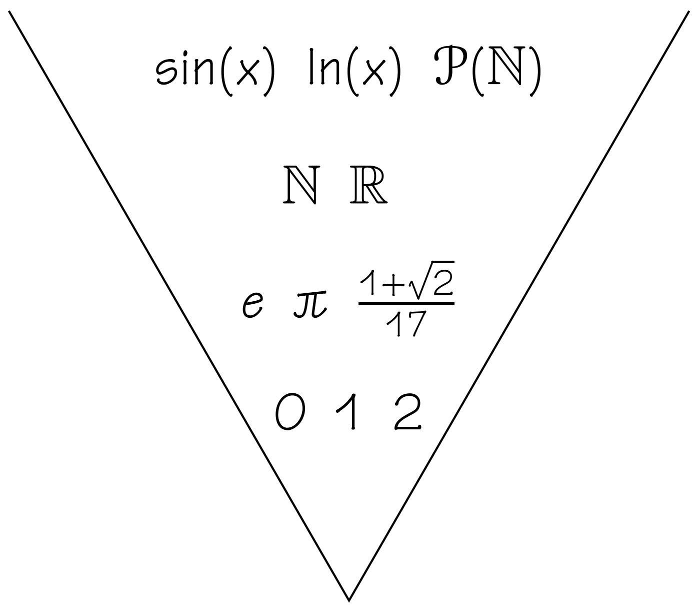
$$\underbrace{0\ 1\ 2\ 3\ \cdots\ n\ \cdots}_{\text{copy of } \mathbb{N}}\underbrace{\cdots H-2\ H-1\ H\ H+1\ H+2\ \cdots}_{\text{(many copies of } \mathbb{Z})}$$

(Henkin, 1950)

A nonstandard model for all of mathematics

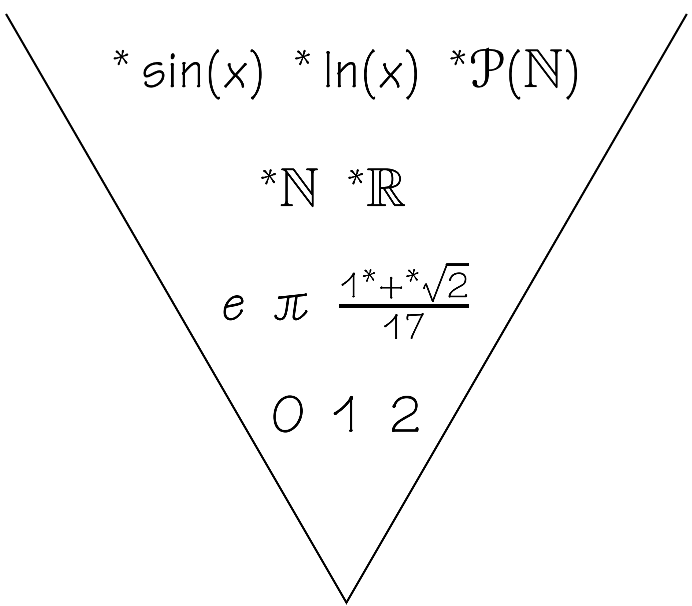
Start with a mathematical universe (*superstructure*) V, containing:

- All natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \phi, \ldots$; *etc.*
- The *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; *etc.*
- Every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions
- Your favorite groups, Banach spaces, etc
- Every other mathematical *object* we might want to talk about
- We call the elements of this mathematical universe *standard*.



Extend to a *nonstandard* mathematical universe $^*V$:

- For *every object* A in V, there is a *corresponding object* $^*A$ in $^*V$
- EG, $^*V$ has *objects* $^*\mathbb{N}$, $^*\mathbb{R}$, $^*\sin(x)$, *etc.*
- (For simplicitry, *we drop* the stars from simple *objects* like numbers: 12 instead of ^{*12} *etc*)
- There may (generally will) be many more *objects* in $^*V$ than in V
- An *element* of $^*V$ that is **not** in V is called *nonstandard*.



The extension should satisfy two important properties:

Transfer If S is a statement about objects in V, then S is true in V if and only if it true in $^*V$

For example, since the following is true in V:

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x + y = y + x]$$

then in $^*V$ it follows:

$$(\forall x \in {}^*\mathbb{R})(\forall y \in {}^*\mathbb{R})[x {}^*{+} y = y {}^*{+} x]$$

In other words, since "addition is commutative in $\mathbb{R}$", by transfer "*addition is commutative in $^*\mathbb{R}$"

As another example, since 12 is an element of $\mathbb{N}$, ^{*12} is an element of $^*\mathbb{N}$.

Since we can think of the basic elements (like ^{*12}) of $^*V$ as just being the same as their counterparts (like 12) in V, $^*\mathbb{N}$ is a superset of $\mathbb{N}$.

Similarly, for *any* standard set A which is an object of V, the set $^*A$ in $^*V$ extends the set A.

Remark: We might imagine a standard mathematician living in universe V, and a nonstandard mathematician living in $^*V$. The transfer principle says that both these mathematicians experience exactly the same true statements. The reason this is possible is that they both speak the same language - the language of V. In particular, the 'nonstandard' mathematician will not be able to refer to any particular element of $^*V$ that is not the $*$—image of an element of V.

Saturation Suppose that $\mathcal{S}$ is a "small" infinite collection of statements about an object X, and that for *every* finite subcollection of $\mathcal{S}$ there is an object in $^*V$ for which they hold; then there is an object in $^*V$ for which **all** the statements in $\mathcal{S}$ hold ***at the same time***.

Roughly means: Anything that *can* happen in $^*V$, *does* happen.

"Small" here is a technical condition meaning either:

- Small in the sense of cardinality (smaller than some fixed infinite cardinal κ); *or*
- Small in the sense of the standard universe (can be indexed by a standard set)

I have not explained how such a nonstandard model is created. We now know many ways to construct such models; all employ straightforward techniques from mathematical logic, and none is especially difficult.

Example: Consider the statements:

$$x \text{ is a real number}$$

$$x > 0$$

$$x < 1$$

$$x < 1/2$$

$$x < 1/3$$

$$x < 1/4$$

$$\vdots$$

Any finite set of these statements refers to a smallest fraction $1/N$; but then, $x = \frac{1}{N+1}$ satisfies this finite set of statements.

It follows that there is a an element of $^*\mathbb{R}$, call it ϵ, such that

$$\epsilon > 0$$

and, for every (standard) natural number N,

$$\epsilon < 1/N$$

We have proved that ${}^*\mathbb{R}$ contains nonzero infinitesimals, where

Definition: An *infinitesimal* is an element ϵ of ${}^*\mathbb{R}$ such that

$$|\epsilon| < 1/N$$

for *every* natural number N in $\mathbb{N}$

How does this avoid internal contradictions?

For example, the following statement, called the Archimedean Property, is true for the usual real numbers:

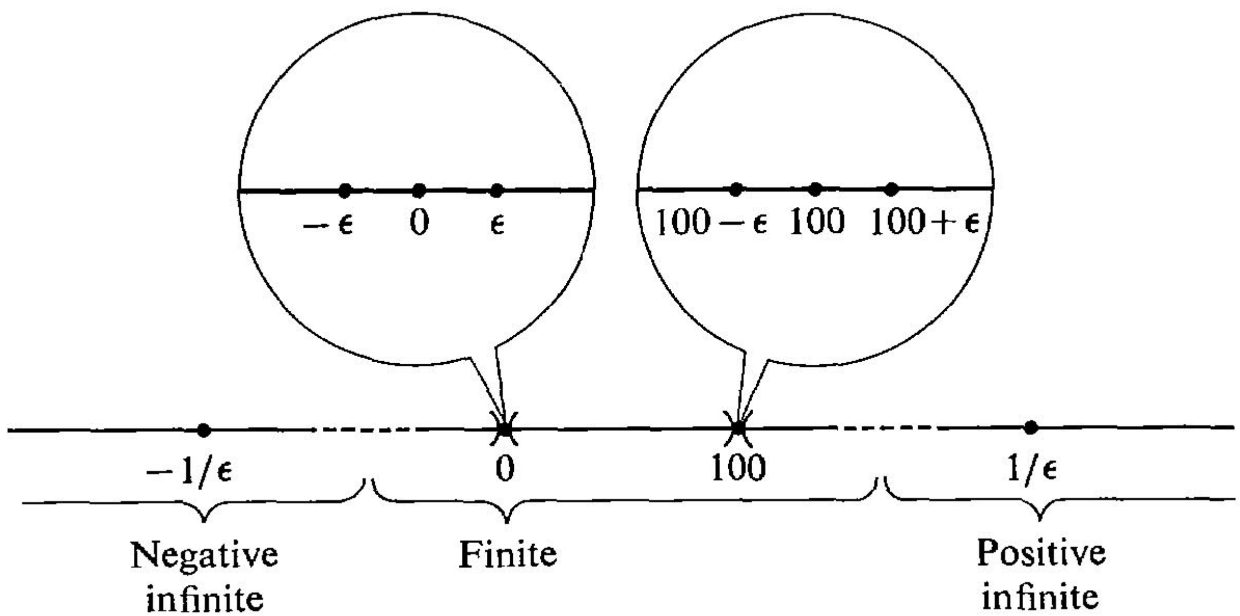
$$\boxed{\text{For every positive real number } x \text{ there is an } N \text{ in } \mathbb{N} \text{ such that } Nx > 1.}$$

By the transfer property,

$$\boxed{\text{For every positive } x \text{ in } {}^*\mathbb{R} \text{ there is an } N \text{ in } {}^*\mathbb{N} \text{ such that } Nx > 1.}$$

Note that this is true for our ϵ as well; while $N\epsilon < 1$ for every N in $\mathbb{N}$, there *will be* elements of ${}^*\mathbb{N}$ which are greater than than $1/\epsilon$!

Since ${}^*\mathbb{R}$ (sometimes called the set of *"hyperreal numbers"*) is, like the usual set of real numbers, closed under the basic arithmetic operations, it also contains negative infinitesimals (like $-\epsilon$), infinite numbers (like $1/\epsilon$), and many other objects:



In particular, as we have seen there are elements of ${}^*\mathbb{N}$ which are bigger than every element of $\mathbb{N}$; in other words, there are *infinite integers*.

(In fact, $({}^*\mathbb{N}, {}^*+, {}^*\cdot, 0, {}^*s)$ is a nonstandard model of arithmetic in the sense of Skolem.)

Properties: Some notation and arithmetic rules for manipulating infinitesimals (and infinite real numbers) will be useful later; they are *easy* to prove:

If two numbers x and y differ by an infinitesimal, *write* $x \approx y$

The set of infinitesimals is therefore $\{x \text{ in } {}^*\mathbb{R} \ : \ x \approx 0\}$.

x is *finite* if $|x| < M$ for some standard $M \in \mathbb{R}^+$, otherwise it is *infinite*.

No standard nonzero real number is infinitesimal; *every* standard real number is finite.

The product of an infinitesimal *and* a finite number is infinitesimal.

The quotient *of* a finite number *and* an infinite number is an infinitesimal.

The sum *of* a finite number *of* infinitesimals is an infinitesimal.

Every finite hyperreal s *differs* infinitesimally from some unique standard real r; call r the *standard part of* s.

For more properties see H.J. Keisler, *Elementary Calculus: An Approach Using Infinitesimals* (available free online)

Hyperfinite sets

The nonstandard model has turned out to be a surprisingly useful construct in many areas of pure and applied mathematics. The general practice of using these large models for work in mathematics has come to be called "nonstandard analysis" (term coined by creator Abraham Robinson)

Many applications are based on the ubiquity of "hyperfinite sets"

Definition: A set E in $^*V$ is *hyperfinite* if there is a *one-to-one correspondence between E and $\{0, 1, 2, \ldots, H\}$ for some H in $^*\mathbb{N}$. Equivalently, if the mathematical statement "E is finite" holds in $^*V$.

Examples: 1. Every finite set is hyperfinite.

2. If H is an infinite integer, $\{0, 1, 2, \cdots, H\} = \{n \in {}^*\mathbb{N} : n \leq H\}$ is a hyperfinite subset of $^*\mathbb{N}$

3. If H is an infinite integer, $\{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H}, 1\}$is a hyperfinite subset of $^*[0.1]$

Theorem: If A is an infinite set in V then there is a hyperfinite set $\hat{A}$ in $^*V$ such that *every* element of A is in $\hat{A}$

Proof: Consider the statements: (i) X is finite; (ii) $a \in X$ (one such statement for every element a of A)

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of A are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle there is therefore a set X in $^*V$ satisfying all the statements simultaneously; let $\hat{A}$ be this X.

Corollary: There is a hyperfinite set containing $\mathbb{R}$.

This gives another way of proving that $^*\mathbb{R}$ has infinitesimals. If $\hat{\mathbb{R}}$ is a hyperfinite set extending $\mathbb{R}$, then the least element of $\hat{\mathbb{R}} \cap {}^*(0, \infty)$ is a positive infinitesimal.

> "Nonstandard analysis is the art of making infinite sets finite by extending them." —M. Richter

$\boxed{\textbf{Applications}}$ (proofs on the blackboard)

Theorem: (Maximum Value Theorem) A continuous function on $[0,1]$ attains its maximum on the interval.

Theorem: Let E be an infinite set of natural numbers. Then there is an infinite subset E' of E satisfying exactly one of the following conditions:

1. Each element of E' divides every larger element of E';
2. No element of E' divides any other element of E'

Theorem: (Sierpinski) If $r, a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ then the equation

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} + \cdots + \frac{a_n}{x_n} = r$$

has at most finitely many nonnegative integer solutions $(p_1, p_2, \ldots, p_n) \in \mathbb{N}^n$

Theorem: (Infinite 4-color Theorem) An infinite planar map can be colored with 4 colors in such a way that no two adjacent countries get the same color.

Theorem: Let E be an infinite set of natural numbers. Then there is an infinite subset E' of E satisfying eactly one of the following conditions:

1. Each element of E' divides every larger element of E';
2. No element of E' divides any other element of E'

Proof: Write $m|n$ for "m divides n".

Let H be an infinite integer. The set E can be written as the union of two disjoint sets:

$$A := \{p \in E : p^*|H\} \qquad B := \{p \in E : p^* \nmid H\}$$

Note that these sets are standard subsets of the standard integers, even though the defining condition is nonstandard!

Since E is infinite, either A or B is infinite (possibly both).

Case 1) A is infinite. Define a sequence $\langle a_n \rangle_{n=0}^{\infty}$ in A inductively:

(i) $a_0 :=$ any element of A

(ii) Given $a_n \in A$, note that the statement

$$(\exists x \in {}^*A)[(x > a_n) \ \& \ (a_n|x)]$$

is true in ${}^*V$, so the corresponding statement

$$(\exists x \in A)[(x > a_n) \ \& \ (a_n|x)]$$

is true in the standard world; let $a_{n+1} \in A$ be this element.

Now, let $E' := \{a_n : n \in \mathbb{N}\}$, it is clear from the construction that

$$a_0 | a_1 | a_2 | a_3 | \cdots$$

which is condition (2)

Case 2) B is infinite. Define a sequence $\langle b_n \rangle_{n=0}^{\infty}$ in B inductively:

(i) $b_0 :=$ any element of B

(ii) Given $b_n \in B$, note that the statement

$$(\exists x \in {}^*B)[(x > b_n) \ \& \ (b_0 \nmid x) \ \& \ (b_1 \nmid x) \ \& \cdots \& \ (b_n \nmid x)]$$

is true in ${}^*V$, so the corresponding statement

$$(\exists x \in B)[(x > b_n) \ \& \ (b_0 \nmid x) \ \& \ (b_1 \nmid x) \ \& \cdots \& \ (b_n \nmid x)]$$

is true in the standard world; let $b_{n+1} \in B$ be this element.

Now, let $E' := \{b_n : n \in \mathbb{N}\}$, it is clear from the construction that

$$b_i \nmid b_j \qquad \text{when } i \neq j$$

which is condition (1)