MA-231 Topology

9. Convergence — Inadquacy of sequences, Nets[1])

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(1862-1932)

9.1. a). (Topology of first-countable spaces) Let X and Y be first-countable topological spaces. Then

(1) A subset $U \subseteq X$ is open if and only if whenever a sequence (x_n) in X converges to a point $x \in U$, then there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

(2) A subset $F \subseteq X$ is closed if and only if whenever a sequence (x_n) is contained in F and x_n converges to a point $x \in X$, then $x \in F$.

(3) A map $f : X \to Y$ is continuous if and only if whenever a sequence (x_n) converges to x in X, then the image sequence $f(x_n)$ converges to $f(x)$ in Y.

b). Which of the above properties ain the above part a) hold for an uncountable set X with the cofinite topology ?

9.2. a). In $\mathbb{R}^{\mathbb{R}}$, let $E = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(x) \in \{0, 1\}$ and $f(x) = 0$ only finitely often$\}$ and let $g \in \mathbb{R}^{\mathbb{R}}$ be the constant function 0. Then in the product topology on $\mathbb{R}^{\mathbb{R}}$, $g \in \overline{E}$. Construct a net (f_λ) in E which converges to g.

b). Let (X, d) be a mtric space and let $x_0 \in X$. Define $\leq$ on $X \setminus \{x_0\}$ by $y \leq z$ if and only if $d(y, x_0) \geq d(z, x_0)$. With this relation $X \setminus \{x_0\}$ is a directed set and hence any map $f : X \to Y$ into another metric space Y defines (by restricting f to $X \setminus \{x_0\}$) a net in Y. Show that this net converges to a point $y_0 \in Y$ if and only if $\lim_{x \to x_0} f(x) = y_0$.

9.3. (Cluster points) Let X be a topological space and let $x \in X$. We say that x is a cluster point of a net $(x_\lambda)_{\lambda \in \Lambda}$ in X if for each nhood U of x and for each $\lambda_0 \in \Lambda$, there exists $\lambda \in \Lambda$ with $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in a topological space X.

a). Show that $(x_\lambda)_{\lambda \in \Lambda}$ has a cluster point $x \in X$ if and only if $(x_\lambda)_{\lambda \in \Lambda}$ has a subnet which converges to x. (**Proof :** Suppose that x is a cluster point of the net (x_λ). Let $M := \{(\lambda, U) \mid \lambda \in \Lambda, U \in \mathfrak{U}_x$ with $x_\lambda \in U\}$. The set M is directed by the relation : $(\lambda_1, U_1) \leq (\lambda_2, U_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $U_2 \subseteq U_1$. The map $\varphi : M \to \Lambda$ defined by $\varphi((\lambda, U)) = \lambda$ is increasing and cofinal in Λ and hence defines a subnet of (x_λ) which converges to x. Conversely, suppose that $\varphi : M \to \Lambda$ defines a subset of (x_λ) which converges to x. Then for each nhood U of x, there is some $\mu_U \in M$ such that $\mu \geq \mu_U$ implies that $x_{\varphi(\mu)} \in U$. Now, suppose that a nhood U of x and a point $\lambda_0 \in \Lambda$ are given. Since $\varphi(M)$ is cofinal in Λ, there is some $\mu_0 \in M$ such that $\varphi(\mu_0) \geq \lambda_0$. But there is also $\mu_U \in M$ such that $\mu \geq \mu_U$ implies that $x_{\varphi(\mu)} \in U$. Choose $\mu^* \in M$ such that $\mu^* \geq \mu_0$ and $\mu^* \geq \mu_U$. Then $\varphi(\mu^*) = \lambda^* \geq \lambda_0$, since $\varphi(\mu^*) \geq \varphi(\mu_0)$, and $x_{\lambda^*} = x_{\varphi(\mu^*)} \in U$, since $\mu^* \geq \mu_U$. Therefore for any nhood U of x and any $\lambda_0 \in \Lambda$, there is some $\lambda^* \geq \lambda_0$ with $x_{\lambda^*} \in U$ and hence it follows that x is a cluster point of the net (x_λ).)

[1]) E. H. Moore (1862-1932) and later R. L. Moore and H. L. Smith developed the general theory of convergence motivated by the considerations of lower and upper Riemann sums of a real valued functions on a closed interval. Subnets were introduced by Moore and develpoed by John L. Kelly who there coined the word "net".

b). For $\lambda_0 \in \Lambda$, let $T_\lambda := \{x_\mu \mid \mu \geq \lambda\}$. Then a point $y \in X$, is a cluster point of the net $(x_\lambda)_{\lambda\in\Lambda}$ if and only if $y \in \overline{T_\lambda}$ for each $\lambda \in \Lambda$.

c). If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in the product space $\prod_{i\in I} X_i$ having a cluster point $x \in \prod_{i\in I} X_i$, then for each $i \in I$, $\pi_i(x)$ is a cluster point of the image net $(\pi_i(x_\lambda))_{\lambda\in\Lambda}$. The converse fails, even in $\mathbb{R} \times \mathbb{R}$.

9.4. (Ultranets) A net (x_λ) in a set X is called an untranet if for each subset $E \subseteq X$, either a tail of (x_λ) is contained in E or a tail of (x_λ) is contained in $X \setminus E$. For any directed set Λ, the constant map $\lambda \to X$, $\lambda \mapsto x \in X$, is an ultranet and is called the trivial ultranet.[2]) Let $(x_\lambda)_{\lambda\in\Lambda}$ be an ultranet in a topological space X. Show that

a). $(x_\lambda)_{\lambda\in\Lambda}$ must converge to each of its cluster points. (see Exercise 9.2)

b). For any map $f : X \to Y$ be a map, the image $(f(x_\lambda))_{\lambda\in\Lambda}$ is an ultranet in Y.

c). Every subnet of an ultranet is an ultranet.

d). Every net has a subnet which is an ultranet.

9.5. (Nets describes topology) Let X be a topological space.

a). Nets in X have the following four properties : Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a topological space X and let $x \in X$.

(i) If $x_\lambda = x$ for each $\lambda \in \Lambda$, then $x_\lambda \to x$.

(ii) If $x_\lambda \to x$, then every subnet of $(x_\lambda)_{\lambda\in\Lambda}$ converges to x.

(iii) If every subnet of $(x_\lambda)_{\lambda\in\Lambda}$ has a subnet converging to x, then the net $(x_\lambda)_{\lambda\in\Lambda}$ converges to x.

(iv) If $x_\lambda \to x$. and for each $\lambda \in \Lambda$, a net $\left(x^\lambda_u\right)_{u\in M_\lambda}$ converges to x_λ, then there exists a diagonal net converging to x_λ; i.e., the net $\left(x^\lambda_\mu\right)_{\lambda\in\Lambda\,,\,\mu\in M_\lambda}$ (ordered lexicographically by Λ and then by M_λ) has a subnet which converges to x.

b). Conversely, suppose that in a set X a notion of a net convergence has been specified (telling what nets converge to what points) satisfying (i), (ii), (iii) and (iv) of the part a). If the closure of a subset $E \subseteq X$ is *defined* by $\overline{E} = \{x \in X \mid x_\lambda \to x$ for some net (x_λ) contained in $E\}$, the result is a topological space in which the notion of net convergence is as originaly specified.

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† **Eliakim Hastings Moore (1862-1932)** Eliakim Hastings Moore was born on 26 Jan 1862 in Marietta, Ohio, USA and died on 30 Dec 1932 in Chicago, Illinois, USA. Eliakim Moore's father was David Hastings Moore while his mother was Julia Sophia Carpenter. David Moore was a Methodist minister and he and his wife gave their son Eliakim an excellent education. As a child Eliakim played with Martha Morris Young and they would later marry. He attended Woodward High School from 1876 to 1879 and there he prepared for his studies at Yale University. It was at this time that he became interested in mathematics and this happened through a summer job which he took. One summer he worked as an assistant to Ormond Stone, who was the director of the Cincinnati Observatory, and from this time on Eliakim knew that he wanted to study mathematics and astronomy at university.

In 1879 Moore entered Yale University and took, as he had planned, courses in mathematics and astronomy. His interests became more clearly in the area of mathematics and he received his B.A. in 1883. He remained at Yale to study for his Ph.D. under the supervision of Hubert Anson Newton. Moore's doctoral dissertation was entitled Extensions of Certain Theorems of Clifford and Cayley in the Geometry of n Dimensions and this led to the award of his doctorate in 1885. Newton encouraged Moore to go to Europe for a year and helped to finance the trip. Moore spent the year in Germany, going first to Göttingen where he spent the summer of 1885 studying the German language, but spending most of the academic year 1885-86 attending lectures by Kronecker and Weierstrass at the University of Berlin. Parshall writes : *While direct influences of this German study tour on Moore's subsequent mathematical career are difficult to isolate, it is undeniable that Moore returned to the United States with a sense of the importance and desirability of the active and sustained pursuit of original research.*

It is worth adding that American academics were really forced to train in Europe in Moore's day but when six years later he set up his research school in Chicago it provided for the first time the opportunity for American mathematicians to train in a research-intensive environment in the United States.

On his return to the United States, Moore was appointed as an instructor at Northwestern University for the year 1886-87, then he spent two years as a mathematics tutor at Yale before spending the years 1889 to 1892 back at Northwestern University. During this second spell at Northwestern Moore was approached by William Rainey Harper and offered a post at Chautauqua in New York. Moore refused, knowing that this would mean taking up a post which involved a lot of low level teaching, while he wanted to concentrate on a research. Perhaps it was a fortunate episode since, a year later in 1891, Harper was President-designate of the University of Chicago and wanted to staff the mathematics department with research active staff. Now he could offer Moore the post he wanted to satisfy his research ambitions and Moore quickly accepted.

Moore was appointed professor and acting head of the mathematics department at Chicago when the university first opened in 1892. Prior to this he had married Martha Morris Young on 21 June 1893 in Columbus, Ohio; they had been friends from their childhood days. The Moores had two sons, only one of whom reached adulthood. In 1896 Moore became head of the mathematics department at Chicago, a post he retained until 1931.

When he was appointed at Chicago, Moore persuaded the university authorities to appoint two young German mathematicians Bolza and Maschke to his department. Archibald describes this Chicago mathematics team : *These three men supplemented one another remarkably. Moore was a fiery enthusiast, brilliant, and keenly interested in the popular mathematical research movements of the day; Bolza, a product of the meticulous German school of analysis led by Weierstrass, was an able, and widely read research scholar; Maschke was more deliberate than the other two, sagacious, brilliant in research, and a most delightful lecturer in geometry. During the period 1892-1908 the University of Chicago was unsurpassed in America as an institution for the study of higher mathematics.*

Among Moore's Ph.D. students at Chicago were Dickson, Veblen, Anna Pell Wheeler and G D Birkhoff. Although Robert Moore had Veblen as his supervisor in Chicago he worked with, and was strongly influenced by, Eliakim Moore.

Moore's first main areas of research, which he studied from about 1892 to 1900, were algebra and groups where he proved in 1893 that every finite field is a Galois field. He also studied infinite series of finite simple groups. In his work on the foundations of geometry begun around 1900 Moore examined the independence of Hilbert's axioms. He reformulated these in terms of points as the only undefined quantities, rather than points, lines and planes as Hilbert had done. His 1902 paper On the projective axioms of geometry showed that Hilbert's axiom system contained redundant axioms.

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[2]) Nontrivial ultranets can be proved to exist (relying on the axiom of choice) but none has ever been explicitly constructed. Most facts about untranets are best developed using *filters* and *ultrafilters* as a vehicle.

Moore is described as follows: *Although a gentle man, he sometimes displayed impatience as he strove for excellence in his classes.* Parshall writes in a similar vein: Moore's style of teaching - characterised by a quick-paced presentation of new research ideas and what could be stinging impatience with those who failed to follow - certainly proved effective, at least for the most talented. Moore was a man of high intellectual and academic standards; he expected much from himself, his students, and his colleagues.

The third interest that Moore took up after 1906 was on the foundations of analysis : He brought to culmination the study of improper integrals before the appearance of the more effective integration theories of Borel and Lebesgue . He diligently advanced general analysis, which for him meant the development of a theory of classes of functions on a general range. ... Throughout his work in general analysis, Moore stressed fundamentals, as he sought to strengthen the foundations of mathematics. Moore brought precision and rigour to all the fields he studied. Other topics he worked on include algebraic geometry, number theory and integral equations. In 1893 Moore was one of the main organisers of the first international mathematical congress to be held in the United States. He then approached the New York Mathematical Society with a view to publishing the Proceedings of the congress. In the process he persuaded this society to take a more national role and to change their name to the American Mathematical Society . Setting up a Chicago branch of the Society, which he led, Moore helped in the having the Society reach across the United States. He became a strong supporter of the American Mathematical Society being Vice-President from 1898 to 1900, President from 1901 to 1902 and Colloquium Lecturer in 1906. He acted as an editor of the Transactions of the American Mathematical Society from 1899 to 1907.

His contribution is summed up as follows: Moore was an extraordinary genius, vivid, imaginative, sympathetic, foremost leader in freeing American mathematicians from dependence on foreign universities, and in building up a vigorous American School, drawing unto itself workers from all parts of the world.

Moore received many honours from around the world for his contributions. He was elected to the National Academy of Sciences, the American Academy of Arts and Sciences , and the American Philosophical Society. He received honorary degrees from Göttingen, Yale, Clark, Toronto, Kansas, and Northwestern.