Probability Comprehensive Exam, January 2006

There are 4 questions, 100 points total. All random variables are definied on a common probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ with expectation operator $\mathbb{E}$.

1. $\boxed{\text{25 points}}$ Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras. Suppose that $\mathscr{C}_1$ and $\mathscr{C}_2$ are π-systems with

$$\sigma(\mathscr{C}_1) = \mathscr{F}_1 \quad \text{and} \qquad \sigma(\mathscr{C}_2) = \mathscr{F}_2,$$

and such that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for any $A \in \mathscr{C}_1$, $B \in \mathscr{C}_2$. Use the $\pi - \lambda$-theorem to prove that $\mathscr{F}_1$ and $\mathscr{F}_2$ are independent.

2. (a) $\boxed{\text{10 points}}$ Prove the second Borel-Cantelli lemma: If $\{A_n\}_1^\infty$ is a sequence of independent events and $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\overline{\lim}_n A_n) = 1$.

   (b) $\boxed{\text{15 points}}$ (Durrett) Give an example of a sequence $\{X_k\}_1^\infty$ of $\{0,1\}$-valued random variables such that $X_k \to 0$ in probability, but such that for almost every $\omega \in \Omega$, there is an increasing sequence $\{N_n(\omega)\}_1^\infty$ of integers such that $X_{N_n(\omega)}(\omega) \to 1$.

3. $\boxed{\text{25 points}}$ Let $\{X_k\}_1^\infty$ be a sequence of mutually independent random variables such that $X_k = \pm 1$ with probability $\frac{1}{2}(1-k^{-2})$ and $X_k = \pm k$ with probability $\frac{1}{2}k^{-2}$. If

$$S_n \overset{\text{def}}{=} X_1 + \cdots + X_n,$$

show that $\text{Var}[S_n/\sqrt{n}] \to 2$ and also that $S_n/\sqrt{n}$ converges weakly to a standard normal random variable. *Hint: compare* X_k *to* $X'_k \overset{\text{def}}{=} X_k/|X_k|$.

4. $\boxed{\text{25 points}}$ Let $X_1, X_2, \ldots$ be a sequence of independent nonnegative random variables, each with mean 1. Set $M_0 = 1$, and for $n = 1, 2, \ldots$, let

$$M_n \overset{\text{def}}{=} X_1 X_2 \cdots X_n.$$

   (a) $\boxed{\text{10 points}}$ Let $\mathscr{F}_0 \overset{\text{def}}{=} \{\emptyset, \Omega\}$, and $\mathscr{F}_n \overset{\text{def}}{=} \sigma\{X_1, \ldots, X_0\}$. Show that M_n is a martingale with respect to $\{\mathscr{F}_n\}_1^\infty$ and that $M_\infty \overset{\text{def}}{=} \lim_{n\to\infty} M_n$ exists $\mathbb{P}$-a.s.

   (b) $\boxed{\text{15 points}}$ If $\prod_{n=1}^\infty a_n > 0$, where $0 < a_n \overset{\text{def}}{=} \mathbb{E}[X_n^{1/2}] \leq 1$, show that $\mathbb{E}[M_\infty] = 1$.