Collecting lightly ramified L-functions

David P. Roberts
University of Minnesota, Morris

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Sections of the talk

1. Warm up by looking at very familiar objects having L-functions. Define *analytic conductor* $A = RN$ for general rank n L-functions. Work also with the *analytic root conductor* $\alpha = A^{1/n}$. Call an L-function *lightly ramified* if $A < 1$ or equivalently $\alpha < 1$. Speculatively conjecture that there are only finitely many lightly ramified L-functions.

2ABCD. Collect lightly ramified L-functions from four less familiar sources, in roughly increasing ranks n. Observe a general increase in the smallest α encountered as n increases.

3. Discuss how the Guinand-Weil-Mestre explicit formula gives a special role to $\alpha = 1$, and gives some theoretical plausibility to the finiteness conjecture.

1. Conductors as measures of complexity

It is standard to order number fields of a given type by increasing absolute discriminant.

Real quadratic fields:	$5, 8, 12, 13, 17, 21, 24, 28, 29, 33 \ldots$
Imag. quadratic fields:	$3, 4, 7, 8, 11, 15, 19, 20, 23, 24 \ldots$
Totally real cubic fields:	$\mathit{49}, \mathit{81}, 148, \mathit{169}, 229, 257, 316, 321, \mathit{361}, 404 \ldots$
Remaining cubic fields:	$23, 31, 44, 59, 76, 83, 87, 104, 107, 108 \ldots$

In fact, one can order all number fields this way, in the sense that there are only finitely many number fields with discriminant less than any given bound.

Absolute discriminants are a special type of conductor and one can similarly order other objects by conductor, e.g.,

Isogeny classes of elliptic curves:	$11, 14, 15, 17, 19, 20, 21, 24, 26, 26, \ldots$

Conductors as an insufficient measure

The number of newforms of weight k on $\Gamma_0(N)$ is

$k \setminus N$	1	2	3	4	5	6	7	8	9	10	11	12
2											1	
4						1	1	1	1	1	2	1
6			1	1	1	1	3	1	1	3	4	
8		1	1		3	1	3	2	3	1	6	2
10		1	2	1	3	1	5	2	3	3	8	1
12	1		1	1	3	3	5	3	4	5	8	2

For general (k, N), the number of newforms is approximately $(k-1)\psi(N)/12$ where $\psi(N)$ is a simple function agreeing with $\phi(N)$ for N square-free.

So while conductor has the "height property" in each row, it doesn't have this property overall. So one would like to incorporate k somehow into the measure of overall complexity.

The uniform context of L-functions

Let $\mathcal{L}_n$ be the set of standard L-functions $L(s, \pi)$ associated to unitary cuspidal automorphic representations π of the adelic group $GL_n(\mathbb{A}_\mathbb{Q})$. Put $\mathcal{L} = \cup_{n=1}^{\infty} \mathcal{L}_n$. An L-function $L(s, \pi) \in \mathcal{L}$ comes with an *infinity factor* $L_\infty(s)$ and a *conductor* $N \in \mathbb{Z}_{\geq 1}$.

Let

$$\Lambda(s, \pi) = N^{s/2} L_\infty(s) L(s, \pi)$$

be the *completed L-function*. One has the *functional equation*

$$\Lambda(s, \pi) = \epsilon \overline{\Lambda}(1 - s, \pi).$$

for some ϵ on the unit circle.

Objects on the previous slides give rise to L-functions of rank 1 or 2. *We will assume throughout this talk the general expectation that all irreducible rank n motives likewise give rise to L-functions in $\mathcal{L}_n$.*

R as a twin to N

Let $L \in \mathcal{L}$ and assume that $L(1/2) \neq 0$. Then $L(1/2)$ is mysterious and $L'(1/2)$ is more mysterious still. But the real part of their ratio is simple!:

$$\begin{aligned}
\Lambda(s) &= \epsilon\overline{\Lambda}(1-s) \\
N^{s/2}L_\infty(s)L(s) &= \epsilon N^{(1-s)/2}\overline{L}_\infty(1-s)\overline{L}(1-s) \\
\frac{1}{2}\log(N) + \frac{L'_\infty(s)}{L_\infty(s)} + \frac{L'(s)}{L(s)} &= -\frac{1}{2}\log(N) - \frac{\overline{L}'_\infty(1-s)}{\overline{L}_\infty(1-s)} - \frac{\overline{L}'(1-s)}{\overline{L}(1-s)} \\
2\mathrm{Re}\left(\frac{L'(1/2)}{L(1/2)}\right) &= -\log(N) - 2\mathrm{Re}\left(\frac{L'_\infty(1/2)}{L_\infty(1/2)}\right).
\end{aligned}$$

For general $L(s)$, we define its *archimedean conductor* to be

$$\boxed{R = \exp\left(2\mathrm{Re}\left(\frac{L'_\infty(1/2)}{L_\infty(1/2)}\right)\right).}$$

This is a variant of the original [Iwaniec-Sarnak] notion.

Hodge numbers and signature

Corresponding to normalizing L-functions to have central point $1/2$, it is best to write standard Hodge numbers of motives via single-indexing: $h^{p-q} := h^{p,q}$. Moreover, it is convenient to package the h^j into a *Hodge vector*,

$$h = (h^{-w}, h^{-w+2}, \ldots, h^{w-2}, h^w).$$

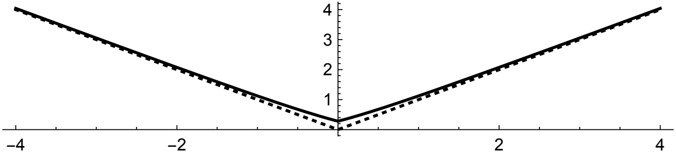
Familiar examples:

$$\begin{array}{rcl} \text{Rank } n \text{ Artin } L\text{-function:} & h = & (n), \\ H^1 \text{ of genus } g \text{ curve:} & h = & (g, g), \\ H^2_{\mathrm{trans}} \text{ of } K3 \text{ surface:} & h = & (1, a, 1). \end{array}$$

Also important is the decomposition $h^0 = h^0_+ + h^0_-$ according to the eigenvalues of complex conjugation. The difference $\sigma := h^0_+ - h^0_-$ is called the *signature*.

Formula for R in the motivic case

For $j + it \in \mathbb{C}$ define $||j + it|| = 2\exp\left(\mathrm{Re}\frac{\Gamma'((1+|j|+it)/2)}{\Gamma((1+|j|+it)/2)}\right)$. The function $||\cdot||$ is asymptotic to the absolute value function $|\cdot|$. On $\mathbb{R}$:



Some special values: $||0|| = e^{-\gamma}/2 \approx 0.28$ and $||1|| = 2e^{-\gamma} \approx 1.12$.

From the recipe for $L_\infty(s)$ as a product of shifted Gamma functions, the archimedean conductor of an L-function coming from a motive is

$$\boxed{R = \frac{\prod ||j||^{h^j}}{(4\pi)^n} e^{-\sigma\pi/2}.}$$

For transcendental L-functions, there is a similar formula.

$A = RN$ as a height function

Let $\mathcal{L}_{n,B}$ be the set of rank n L-functions with analytic conductor at most B. An important justification of the notion of analytic conductor is the following:

Theorem. [Brumley]. $|\mathcal{L}_{n,B}|$ *is always finite.*

Follow-up work [Brumley-Milicevic] has made substantial progress towards estimating the size of $|\mathcal{L}_{n,B}|$ as B tends to ∞.

There are different ways to measure complexity when one changes ranks. Today we will focus not on conductors but their corresponding *root conductors*, $(\alpha, \rho, \nu) = (A^{1/n}, R^{1/n}, N^{1/n})$. Let $\mathcal{L}(\beta)$ be the set of all L-functions with analytic root conductor at most β.

Conjecture. $|\mathcal{L}(1)|$ *is finite.*

Note that from towers of number fields [Hajir-Maire], $|\mathcal{L}(1.84)|$ is infinite.

Lightly ramified L-functions

Say that an L-function is *lightly ramified* if its conductor A or equivalently its root conductor α is less than 1.

In very low ranks, it is easy to find lightly ramified L-functions. The most extreme case is the Riemann zeta function $\zeta(s)$ which has

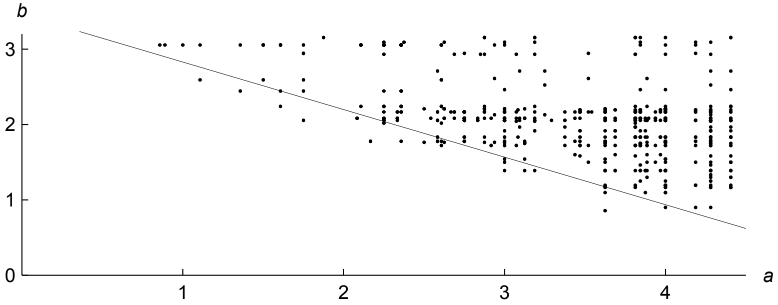
$$A = \alpha = 1/(8\pi e^{\gamma+\pi/2}) \approx 1/215.333 \approx 0.004644.$$

In higher ranks, it gets harder. For example, for $\mathrm{Sym}^j H^1(X_0(11))$:

j	h	σ	ρ	ν	$=$	α
1	$(1,1)$	0	0.089	$11^{1/2}$	$\approx$	0.296
2	$(1,1,1)$	-1	0.143	$11^{2/3}$	$\approx$	0.708
3	$(1,1,1,1)$	0	0.147	$11^{3/4}$	$\approx$	0.890
4	$(1,1,1,1,1)$	1	0.106	$11^{4/5}$	$\approx$	0.719
5	$(1,1,1,1,1,1)$	0	0.206	$11^{5/6}$	$\approx$	1.517
6	$(1,1,1,1,1,1,1)$	-1	0.255	$11^{6/7}$	$\approx$	1.989

The case of large degree number fields

For a number field with no real places, the archimedean root conductor is $1/\Omega$, where $\Omega = 8\pi e^{\gamma} \approx 44.7632$ is the famous Odylzko-Serre constant. Nonsolvable Galois fields on [Jones-R 2014] with root discriminant $2^a 3^b$ correspond to points in this plane:



The diagonal line is $\alpha = 1$. Putting this into our context, lightly ramified large rank Artin L-functions [Jones-R 2017] are also rare.

2A. Rank four symplectic motives

By working with the universal rank four system $V = H^1(A)$ over the moduli space A_2 and decomposing cohomology via Hecke operators, [Bergström-Faber-van der Geer] found many motives with

$$h = (1, \overbrace{0, \ldots, 0}^{k-3}, 1, \overbrace{0, \ldots, 0}^{j}, 1, \overbrace{0, \ldots, 0}^{k-3}, 1),$$

Sato-Tate group all of Sp_4, and conductor $N = 1$.

For those motives appearing in isolation, [B-F-vdG] give 210 terms of the L-function. This allows many high precision analytic calculations, using the *Magma* implementation of [Dokchitser].

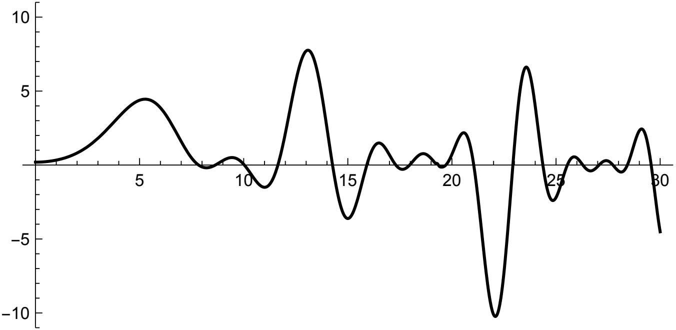
The smallest analytic root conductor is $\alpha \approx 0.5130$ from $(j, k) = (0, 20)$ and thus Hodge vector

$$(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1).$$

Its L-function is

$$
\begin{aligned}
L(s,M) &= \frac{1}{1^s} - \frac{840960}{2^{18.5+s}} + \frac{346935960}{3^{18.5+s}} - \frac{5232247240500}{4^{18.5+s}} - \cdots \\
&\approx \frac{1}{1^s} - \frac{2.27}{2^s} + \frac{0.52}{3^s} + \frac{2.31}{4^s} + \frac{0.61}{5^s} - \frac{1.17}{6^s} + \frac{0.61}{7^s} - \cdots
\end{aligned}
$$

Its Z-function $Z(t) = \overline{\text{phase}(L(\frac{1}{2}+it, M))} L(\frac{1}{2}+it, M)$ graphs to



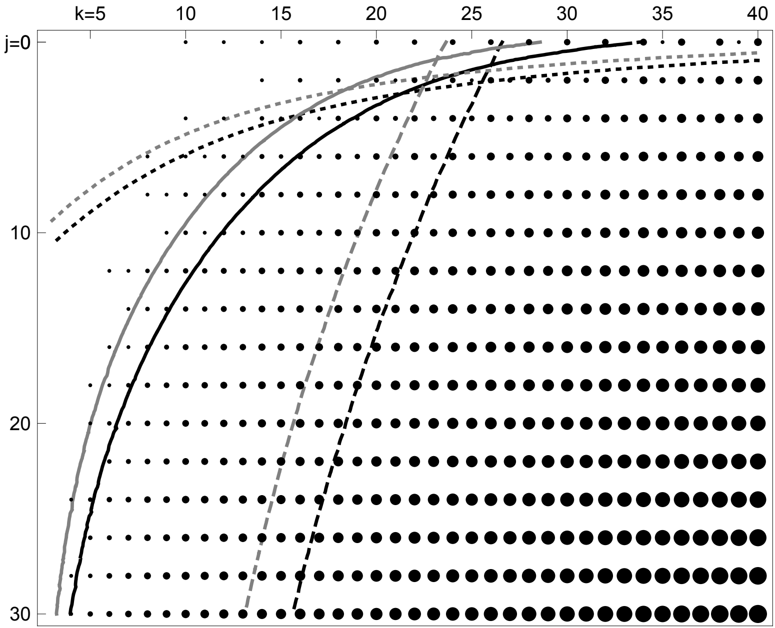
The number of Sp_4 motives with a given h and $N = 1$ is indicated on the next slide. Here

- The number of motives at (j, k) is proportional to the square of the area of the printed disk.
- Three curves $\alpha_n = 1$ are given in black

$n = 4:$	(dotted)	Symplectic L-function	$L(s, M)$
$n = 5:$	(dashed)	Orthogonal L-function	$L(s, (\Lambda^2 M)')$
$n = 10:$	(solid)	Adjoint L-function	$L(s, \mathrm{Sym}^2 M)$

- Three corresponding curves $\alpha_n = 0.9$ are given in gray.

(A few dots are present because I forgot to discard lifts. These include the dots at $(0, k)$ for $k < 20$.)


k=5
10
15
20
25
30
35
40
j=0
10
20
30

By adding a full-level 2 structure and working over $A_2[2]$ instead, [B,F,vdG] get similar data for conductors $N|16$. For $N = 1, 2, 4, 8, 16$, the smallest α appearing are 0.5130, 0.5401, 0.5199, 0.5849, 0.6084 from $(0, j)$ with $j = 20, 16, 11, 10$, and 8.

For various h, the known Sp_4 motives with smallest conductor N and their α from [LMFDB] and [Ibukiyama-Kitayama] are

h	N	α
$(2, 2)$	249	0.3550
$(1, 1, 1, 1)$	61	0.4117
$(1, 0, 1, 1, 0, 1)$	31	0.4464
$(1, 0, 0, 1, 1, 0, 0, 1)$	19	0.4666

For smaller Sato-Tate groups, one can get slightly smaller α. E.g. the four-dimensional Artin representation from $\mathbb{Q}[x]/(x^5 - 2)$ has $h = (4)$, $\sigma = 0$, and $N = 50000$ for $\alpha \approx 0.3341$.

The comparability of the various α gives a sense that the concept of analytic root conductor properly captures both the archimedean and ultrametric contributions to complexity.

2B. Some exotic rank 7 and 8 motives with $N = 1$

Bergström, Faber, and van der Geer also work with the rank six local system $H^1(A)$ over A_3 to get orthogonal rank eight motives with Hodge vectors depending on three parameters j, k, ℓ. The Sato-Tate group is always contained in the subgroup Spin_7 of SO_8. From one-dimensional spaces, they get 126 motives, each with 26 terms of the L-series. Root analytic conductors:

i	α_i
1	0.3742
2	0.3993
⋮	
10	0.4941
11	0.5247

i	α_i
12	0.5491
13	0.5640
14	0.7616
15	0.7878
⋮	

i	α_i
42	0.9931
43	1.0001
⋮	
125	5.3289
126	5.4482

The first thirteen α_i look strikingly small!

If M has Sato-Tate group $G_2 \subset \mathrm{Spin}_7$, then it is reducible in the form $M_7 \oplus M_1$. A necessary condition for reduction to G_2, satisfied exactly for $i \leq 13$, is that $\ell = j + 4$. In this case the rank 8 Hodge vector is

$$(1, \overbrace{0, \ldots, 0}^{j}, 1, \overbrace{0, \ldots, 0}^{k}, 1, \overbrace{0, \ldots, 0}^{j}, 2, \overbrace{0, \ldots, 0}^{j}, 1, \overbrace{0, \ldots, 0}^{k}, 1, \overbrace{0, \ldots, 0}^{j}, 1).$$

In the case of reduction to G_2, replacing the 2 by a 1 gives the rank 7 Hodge vector.

Another necessary condition for reduction to G_2 is that the local factor at every p has the right shape. This condition is satisfied at $p = 2$ exactly for $i \leq 11$ and $i = 42$.

Analytic computations give strong evidence that indeed the Sato-Tate group is G_2 for all $i \leq 11$. Removing the trivial piece M_1 inflates the previous numbers to analytic root conductors of the expected M_7's: $(\alpha_1, \ldots, \alpha_{10}, \alpha_{11}) = (0.7009, \ldots.., 0.9626, 1.0308)$.

For the least ramified of the presumed G_2 motives, with analytic root conductor $\alpha_1 \approx 0.7009$, the rank seven Hodge vector is

$$(1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1).$$

It would be interesting to compare with G_2 motives in the literature with Hodge vectors $(1, 1, 1, 1, 1, 1, 1, 1)$ and $(2, 3, 2)$, where conductors have not yet been computed.

For the Spin$_7$ motive with analytic root conductor $\alpha_{12} = 0.5491$, the rank eight Hodge vector is

$$(1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1).$$

For the closely related group Sp_6 and Hodge vector $(3, 3)$, the smallest conductor appearing in [Sutherland] is $4727 = 29 \cdot 163$, with corresponding $\alpha = 0.3660$.

2C. Hypergeometric motives

Let A and B be collections of positive integers with

$$\sum_{a \in A} \phi(a) = \sum_{b \in B} \phi(b) =: n.$$

Let $t \in \mathbb{Q}^\times - \{1\}$. Then one has a rank n hypergeometric motive

$$H(A, B; t)$$

almost always with Sato-Tate group Sp_n or O_n. Hodge vectors coming from this setting are extremely varied.

Each family also has a particularly interesting degenerate member $H(A, B; 1)$. Here the central Hodge number decreases by 1 in the orthogonal case and the two central Hodge numbers decrease by 1 in the symplectic case,increasing archimedean root conductors. However ultrametric root conductors are particularly low at $t = 1$.

The orthogonal sequence of motives $H([4, 2^{2j-1}], [1^{2j+1}]; 1)$ with rank $2j$, conductor 2^c, and local factor $L_2(s) = f_2(2^{-s})^{-s}$:

j	h	c	$f_2(x)$	ρ	ν	$=$	α
1	$(1,0,1)$	3	1	0.17	2.83	$\approx$	0.47
2	$(1,1,0,1,1)$	9	1	0.23	4.76	$\approx$	1.10
3	$(1,1,1,0,1,1,1)$	9	1	0.29	2.83	$\approx$	0.83
4	$(1,1,1,1,0,1,1,1,1)$	13	1	0.36	3.08	$\approx$	1.10
5	$(1,1,1,1,1,0,1,1,1,1,1)$	11	$1+32x$	0.42	2.14	$\approx$	0.90
6	$\vdots$	19	1	0.48	3.00	$\approx$	1.44
7		19	1	0.54	2.56	$\approx$	1.39

The root conductor ν likely stabilizes, but the archimedean root conductor definitely grows without bound.

In conformity with the Finiteness Conjecture, we have only found lightly ramified L-series in rank ≤ 10. The symplectic rank ten example $H([4,4,4,4,2,2,2,2],[8,8,1,1,1,1];1)$, has Hodge vector $(1,1,2,1,1,2,1,1)$, conductor 2^{18}, and still $\alpha \approx 0.90$.

Small N make computations require few terms and run quickly:

```
H1 := HypergeometricData([4,2,2,2,2,2,2,2,2,2],
            [1,1,1,1,1,1,1,1,1,1,1]);
L1 := LSeries(H1,1: BadPrimes:=[<2,11,1+32*x>],
             Precision:=10);
time <LCfRequired(L1), CFENew(L1), Evaluate(L1,11/2)>;
<1315, 0.0000000000, 0.5444095362>    Time:2.770
```

```
H1 := HypergeometricData([4,4,4,4,2,2,2,2],
            [8,8,1,1,1,1]);
L2 := LSeries(H2,1:BadPrimes:=[<2,18,1+2^2*x+3*2^5*x^2
             + 2^9*x^3 + 2^14*x^4>], Precision:=10);
time <LCfRequired(L2),CFENew(L2), Evaluate(L2,4)>;
<7528, 0.0000000000, 0.0000000000>    Time:40.940
```

The fact that the second L-function vanishes at its central point is unusual given its light ramification.

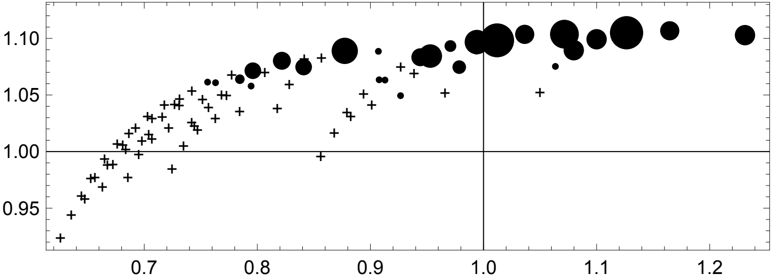
2D. Large rank motives with $N = 1$

A number of recent papers have used the trace formula to count motives with $N = 1$ and other specified invariants. For example [Taïbi] considers, among many other similar cases, motives with Sato-Tate group Sp_{12}, weight ≤ 27, and h consisting of all 1 and 0's with no adjacent 1's except perhaps the middle two. Some counts:

h	α_{12}	#
$(1,0,1,0,1,0,1,0,1,0,1,1,0,1,0,1,0,1,0,1,0,1)$	0.63	0
$(1,0,1,0,1,0,0,0,1,0,0,1,0,1,1,0,1,0,0,1,0,0,0,1,0,1,0,1)$	0.76	1
$(1,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,1)$	0.92	1
$(1,0,1,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,1,0,0,1,0,1,0,1)$	1.01	25
$(1,0,1,0,1,0,1,0,1,0,1,0,0,0,0,0,0,1,0,1,0,1,0,1,0,1,0,1)$	1.23	9

In these cases, one has existence of L-functions, but not yet computation of any coefficients a_n past $a_1 = 1$.

Each Hodge vector h corresponds to a point at $(\alpha_{12}, \alpha_{78})$, with coordinates the analytic root conductor of a standard rank 12 motive M and its the rank 78 adjoint motive $\mathrm{Sym}^2 M$.



An $+$ represents no motives. A $\bullet$ represents from 1 to 25 motives, with the number proportional to the area.

The figure illustrates typical behavior in two ranges. In modest rank, around 10 to 25, there is a transition from very few motives with $\alpha < 1$ to many motives with $\alpha > 1$. In high rank, there are no motives from discrete series at all with $\alpha < 1$, since already $\rho > 1$.

3. The role of 1 and support for the conjecture

Our heuristic argument for the conjecture is based on the Guinand-Weil-[Mestre] explicit formula, and the optimistic hope that the prime-power terms are too small to matter.

To simplify, we present the argument only for motivic L-functions with $\sigma = 0$.

As test functions we need only a family of Gaussians parameterized by $v \in \mathbb{R}_{>0}$,

$$\begin{aligned} F_v(x) &= \exp(-\pi x^2/v), \\ \hat{F}_v(t) &= v \exp(-\pi v t^2). \end{aligned}$$

As v increases, F_v tends to the constant function 1, and $\hat{F}_v$ tends to the Dirac delta measure supported at 0.

The explicit formula

Let M be an irreducible rank n motive with Hodge vector h, signature $\sigma = 0$, and conductor N. Assume its L-function $L(s, M) = \sum_n a_n n^{-s}$ satisfies the Riemann hypothesis, and let γ run over the ordinates of the zeros, including multiplicities. Define c_{p^e} by

$$-\frac{L'(s,M)}{L(s,M)} = \sum_{p^e} \frac{c_{p^e} \log(p)}{p^{es}}.$$

so that $c_p = a_p$. Then

$$\int_{-\infty}^{\infty} \hat{F}_v(t) \log\left(\frac{N}{(4\pi)^n} \prod_j ||j + it||^{h^j}\right) dt =$$

$$2\pi \sum_{\gamma} \hat{F}_v(\gamma) + \sum_{p^e} \mathrm{Re}(c_{p^e}) \frac{\log(p)}{p^{e/2}} F_v\left(\frac{\log p}{2\pi}\right).$$

Clearly as $v \to \infty$, the left side tends to $\log(A)$. So when $A < 1$ and v is sufficiently large, the left side is **negative**.

Some of the finiteness conjecture from positivity

In the explicit formula,

$$\int_{-\infty}^{\infty} \hat{F}_v(t) \log \left(\frac{N}{(4\pi)^n} \prod_j ||j + it||^{h^j} \right) dt =$$

$$2\pi \sum_{\gamma} \hat{F}_v(\gamma) + \sum_{p^e} \mathrm{Re}(c_{p^e}) \frac{\log(p)}{p^{e/2}} F_v \left(\frac{\log p}{2\pi} \right),$$

the spectral sum is clearly **positive** for all v.

There are many irreducible M for which one has universally $c_{p^e} \geq -1$, namely $M = M_1 \otimes \overline{M}_1 - \mathbb{C}$ for any irreducible non-self-conjugate motive M_1. For these M, the prime power sum is **bounded below**.

[Chenevier] recently proved under GRH that $\alpha \geq 1$ for all but finitely many of these M.

?!A scaling distinction gives sufficient positivity?!

For motives with a fixed $\alpha < 1$, the three parts scale differently for fixed v under the replacement $(n, h, N) \mapsto (kn, kh, N^k)$:

$$\int_{-\infty}^{\infty} \hat{F}_v(t) \log \left(\frac{N}{(4\pi)^n} \prod_j ||j + it||^{h^j} \right) dt =$$

$$2\pi \sum_{\gamma} \hat{F}_v(\gamma) + \sum_{p^e} \mathrm{Re}(c_{p^e}) \frac{\log(p)}{p^{e/2}} F_v \left(\frac{\log p}{2\pi} \right).$$

- The *negative analytic conductor term decreases linearly with k*
- The *positive spectral sum should increase linearly with k:* $(2\pi)^{-1} \log \left(\frac{N}{(4\pi)^n} \prod_j ||j + it||^{h^j} \right)$ is the expected density of γ.
- But the *mixed-sign prime-power sum should behave independently of k*, as the Sato-Tate conjecture says that the c_p always have variance 1. So one always has a statistical version of the key condition $c_{p^e} \geq -1$ used by Chenevier.

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