we can write (since $z_{22} = -z_{11}$)

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} i\alpha_1 & \beta_1 \\ \bar{\beta}_1 & -i\alpha_1 \end{pmatrix} + i\begin{pmatrix} i\alpha_2 & \beta_2 \\ \bar{\beta}_2 & -i\alpha_2 \end{pmatrix}$$

for $\alpha_1, \alpha_2 \in \boldsymbol{R}$, $\beta_1, \beta_2 \in \boldsymbol{C}$.

(iii) The Lie algebra $\mathfrak{su}(2)$ of skew-Hermitian matrices of trace 0,

$$X = \begin{pmatrix} i\alpha & \beta \\ -\bar{\beta} & -i\alpha \end{pmatrix} \qquad \alpha \in \boldsymbol{R},\ \beta \in \boldsymbol{C}$$

is obviously a real form of $\mathfrak{sl}(2, \boldsymbol{C})$. Since the Killing form of a real form is in general obtained by restriction we see from (4) §3-1 that

$$B(X, X) = 4 \operatorname{Trace}(XX) = -8(\alpha^2 + |\beta|^2)$$

so $\mathfrak{su}(2)$ is a compact real form of $\mathfrak{sl}(2, \boldsymbol{C})$.

The following two results are of fundamental importance.

Theorem 2.2. Every semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{C}$ contains a Cartan subalgebra $\mathfrak{h}$.

Theorem 2.3. Every semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{C}$ has a real form $\mathfrak{u}$ which is compact.

Ordinarily Theorem 2.2 is proved first using theorems on solvable Lie algebras (Lie's theorem that a solvable Lie algebra of complex matrices has a common eigenvector). The simultaneous diagonalization of the endomorphisms ad $\mathfrak{h}$ leads to a detailed structure theory for $\mathfrak{g}$ by which the compact real form $\mathfrak{u}$ is constructed. The details are as follows:

Assume $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Given a linear form $\alpha \neq 0$ on $\mathfrak{h}$ let

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \operatorname{ad} H(X) = \alpha(H)X \qquad \text{for all } H \in \mathfrak{h}\}$$

This linear form α is called a *root* if $\mathfrak{g}^\alpha \neq \{0\}$. Let Δ denote the set of all roots. Then

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha \qquad \text{(direct sum)} \tag{1}$$

and it can be proved that

$$\dim \mathfrak{g}^\alpha = 1 \qquad (\alpha \in \Delta) \tag{2}$$

Let $\mathfrak{h}^*$ denote the subset (real-linear subspace) of $\mathfrak{h}$, where all the roots have real values. Then for a suitable choice of vectors $X_\alpha \in \mathfrak{g}^\alpha$ the set

$$\mathfrak{u} = i\mathfrak{h}^* + \sum_{\alpha \in \Delta} \boldsymbol{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha \in \Delta} \boldsymbol{R}(i(X_\alpha + X_{-\alpha})) \tag{3}$$

is a compact real form of $\mathfrak{g}$.

Example

Consider again the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$ and its Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of trace 0. Let again E_{ij} denote the matrix

$$(\delta_{ai}\ \delta_{bj})_{1 \leqslant a,\, b \leqslant n}$$

and for each $H \in \mathfrak{h}$ let $e_i(H)$ denote the ith diagonal element in H. Then

$$[H, E_{ij}] = (e_i(H) - e_j(H))E_{ij}$$

for all $H \in \mathfrak{h}$ so the linear form $\alpha_{ij}(H) = e_i(H) - e_j(H)$ is a root for $i \neq j$ and by (1) this does give all the roots. The space $\mathfrak{h}^*$ consists of all real diagonal matrices of trace 0. Let us put $X_{\alpha_{ij}} = E_{ij}\ (i \neq j)$. Then it is easily seen that the space (3) is the set $\mathfrak{su}(n)$ of all skew-Hermitian $n \times n$ matrices, which is indeed a compact real form of $\mathfrak{sl}\,(n, \boldsymbol{C})$ (*cf.* example above).

It is tempting to try to prove Theorem 2.3 directly, because then Theorem 2.2 would be an immediate corollary. In fact, for each $X \in \mathfrak{u}$, ad X can be diagonalized, so if $\mathfrak{t} \subset \mathfrak{u}$ is any maximal Abelian subalgebra, the space $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

A direct and elementary proof of Theorem 2.3 (without the use of Theorem 2.2) does not seem to be available. However, Cartan has proposed an idea for this purpose (*J. Math. Pures Appl.* **8** (1929), p. 23), which I shall describe here.

Since the Killing form of $\mathfrak{g}$ is nondegenerate, there exists a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ such that

$$B(Z, Z) = -\sum_1^n z_i^{\,2} \qquad \text{if} \quad Z = \sum_1^n z_i\, e_i \tag{4}$$

Let the structural constants $c_{ijk} \in \boldsymbol{C}$ be determined by

$$[e_i, e_j] = \sum_1^n c_{ijk}\, e_k$$

Then

$$B(Z, Z) = \operatorname{Tr}\,(\operatorname{ad} Z \operatorname{ad} Z) = \sum_{i,j}\Bigl(\sum_{h,k} c_{ikh}\, c_{jhk}\Bigr) z_i\, z_j$$

so by (4)

$$\sum_{h,k} c_{ikh}\, c_{jhk} = -\delta_{ij} \tag{5}$$

Also,

$$B([X_i, X_j], X_k) + B(X_j, [X_i, X_k]) = 0$$

so

$$c_{ijk} + c_{ikj} = 0$$

and by (5)

$$\sum_{i,h,k} c_{ihk}^2 = n$$

The space

$$\mathfrak{u} = \sum_1^n \boldsymbol{R} e_i$$

is a real form of $\mathfrak{g}$ if and only if all the c_{ijk} are real.

Consider now the set $\mathfrak{F}$ of all bases $(e_1, \ldots, e_n)$ of $\mathfrak{g}$ such that (4) holds. Consider the function f on $\mathfrak{F}$ given by

$$f(e_1, \ldots, e_n) = \sum_{i,j,k} |c_{ijk}|^2$$

Then we have seen that

$$\sum_{i,j,k} |c_{ijk}|^2 \geqslant \left| \sum_{i,j,k} c_{ijk}^2 \right| = \sum_{i,j,k} c_{ijk}^2 = n \tag{6}$$

and the equality sign holds if and only if all the c_{ijk} are real, that is, if and only if

$$\mathfrak{u} = \sum_1^n \boldsymbol{R} e_i$$

is a real form. In this case it is a compact real form in view of (4) and Prop. 2.1.

Thus Theorem 2.3 follows if one can prove: (I) The function f on $\mathfrak{F}$ has a minimum value; and (II) this minimum value is attained at a point $(e_1{}^0, \ldots, e_n{}^0) \in \mathfrak{F}$ for which the structural constants are real. Note that (II) is equivalent to (II′): The minimum of f is n.

3-3 Cartan Decompositions

We now go back to considering a semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ and as usual we denote by B the Killing form of $\mathfrak{g}$. There are of course many possible ways to find a direct vector space decomposition $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ such that B is positive definite on $\mathfrak{g}^+$ and negative definite on $\mathfrak{g}^-$. However, we should like to find a decomposition which is directly related to the Lie algebra structure of $\mathfrak{g}$.

Definition. A *Cartan decomposition* of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that (i) $B < 0$ on $\mathfrak{k}$, $B > 0$ on $\mathfrak{p}$; and (ii) The mapping $\theta : T + X \to T - X$ ($T \in \mathfrak{k}$, $X \in \mathfrak{p}$) is an automorphism of $\mathfrak{g}$.

In this case θ is called a *Cartan involution* of $\mathfrak{g}$ and the positive definite bilinear form $(X, Y) \to -B(X, \theta Y)$ is denoted by B_θ. We shall now establish the existence of Cartan decompositions, using compact real forms for semisimple Lie algebras over $\boldsymbol{C}$.

Theorem 3.1. Suppose θ is a Cartan involution of a semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ and σ an arbitrary involutive automorphism of $\mathfrak{g}$. There then exists an automorphism ϕ of $\mathfrak{g}$ such that the Cartan involution $\phi\theta\phi^{-1}$ commutes with σ.

PROOF. The product $N = \sigma\theta$ is an automorphism of $\mathfrak{g}$ and if $X, Y \in \mathfrak{g}$,

$$-B_\theta(NX, Y) = B(NX, \theta Y) = B(X, N^{-1}\theta Y) = B(X, \theta N Y)$$

so

$$B_\theta(NX, Y) = B_\theta(X, NY)$$

that is, N is symmetric with respect to the positive definite bilinear form B_θ. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ diagonalizing N. Then $P = N^2$ has a positive diagonal, say, with elements $\lambda_1, \dots, \lambda_n$. Take P^t ($t \in \boldsymbol{R}$) with diagonal elements $\lambda_1{}^t, \dots, \lambda_n{}^t$ and define the structural constants c_{ijk} by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

Since P is an automorphism, we conclude

$$\lambda_i \lambda_j c_{ijk} = \lambda_k c_{ijk}$$

which implies

$$\lambda_i{}^t \lambda_j{}^t c_{ijk} = \lambda_k{}^t c_{ijk} \qquad (t \in \boldsymbol{R})$$

so P^t is an automorphism. Put $\theta_t = P^t\theta P^{-t}$. Since $\theta N \theta^{-1} = N^{-1}$, we have $\theta P \theta^{-1} = P^{-1}$, that is $\theta P = P^{-1}\theta$. In matrix terms (using still the basis $X_1, \dots, X_n$) this means (since θ is symmetric with respect to B_θ)

$$\theta_{ij}\lambda_j = \lambda_i^{-1}\theta_{ij}$$

so

$$\theta_{ij}\lambda_j^t = \lambda_i^{-t}\theta_{ij}$$

that is, $\theta P^t \theta^{-1} = P^{-t}$. Hence,

$$\sigma\theta_t = \sigma P^t \theta P^{-t} = \sigma\theta P^{-2t} = NP^{-2t}$$

$$\theta_t \sigma = (\sigma\theta_t)^{-1} = P^{2t}N^{-1} = N^{-1}P^{2t}$$

so it suffices to put $\phi = P^{1/4}$ $(= \sqrt{\sigma\theta})$. (*cf.* [3], p. 100, [31], p. 156, [47], p. 884). The following result is given in Mostow [54].

Corollary 3.2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\boldsymbol{R}$, $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$ its complexification, $\mathfrak{u}$ any compact real form of $\mathfrak{g}_c$, σ and τ the conjugations of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ and $\mathfrak{u}$, respectively. Then there exists an automorphism ϕ of $\mathfrak{g}_c$ such that $\phi \cdot \mathfrak{u}$ is invariant under σ.

PROOF. Let $\mathfrak{g}_c^{\boldsymbol{R}}$ denote the Lie algebra $\mathfrak{g}_c$ considered as a Lie algebra over $\boldsymbol{R}$, $B^{\boldsymbol{R}}$ the Killing form. It is not hard to show that $B^{\boldsymbol{R}}(X, Y) = 2\text{Re}\,(B_c(X, Y))$ if B_c is the Killing form of $\mathfrak{g}_c$. Thus σ and τ are Cartan involutions of $\mathfrak{g}_c^{\boldsymbol{R}}$ and the corollary follows (note that since $\sigma\tau$ is a (complex) automorphism of $\mathfrak{g}_c$, ϕ is one as well).

Corollary 3.3. Each semisimple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ has Cartan decompositions and any two such are conjugate under an automorphism of $\mathfrak{g}$.

PROOF. Let $\mathfrak{g}_c$ denote the complexification of $\mathfrak{g}$, σ the corresponding conjugation, and $\mathfrak{u}$ a compact real form of $\mathfrak{g}_c$ invariant under σ (Theorem 2.3 and Cor. 3.2). Then put $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$. Then $B < 0$ on $\mathfrak{k}$, $B > 0$ on $\mathfrak{p}$, and since $\theta : T + X \to T - X\,(T \in \mathfrak{k}, X \in \mathfrak{p})$ is an automorphism, $B\,(\mathfrak{k}, \mathfrak{p}) = 0$. It follows that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition.

Consider now two Cartan decompositions,

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{p}_1 \qquad \mathfrak{g} = \mathfrak{k}_2 + \mathfrak{p}_2$$

Then $\mathfrak{u}_1 = \mathfrak{k}_1 + i\mathfrak{p}_1$ and $\mathfrak{u}_2 = \mathfrak{k}_2 + i\mathfrak{p}_2$ are compact real forms of $\mathfrak{g}_c$. Let τ_1 and τ_2 denote the corresponding conjugations. By Cor. 3.2 there exists an automorphism ϕ of $\mathfrak{g}_c$ such that $\phi \cdot \mathfrak{u}_2$ is invariant under τ_1. Thus $\phi \cdot \mathfrak{u}_2$ is equal to the direct sum of its intersections with $\mathfrak{u}_1$ and $i\mathfrak{u}_1$. Now $B > 0$ on $i\mathfrak{u}_1$ and $B < 0$ on $\phi \cdot \mathfrak{u}_2$. Hence $i\mathfrak{u}_1 \cap \phi \cdot \mathfrak{u}_2 = \{0\}$ so $\mathfrak{u}_1 = \phi \cdot \mathfrak{u}_2$. But τ_1 and τ_2 both leave $\mathfrak{g}$ invariant and ϕ can (according to the proof of Theorem 3.1) be taken as a power of $\tau_1\tau_2$ so it also leaves $\mathfrak{g}$ invariant. Thus $\phi(\mathfrak{g} \cap \mathfrak{u}_2) = \mathfrak{g} \cap \mathfrak{u}_1$ so ϕ gives the desired automorphism of $\mathfrak{g}$.

Examples

Let $\mathfrak{g} = \mathfrak{sl}\,(n, \boldsymbol{R})$, the Lie algebra of the group $\boldsymbol{SL}(n, \boldsymbol{R})$. The group $\boldsymbol{SO}(n)$ of orthogonal matrices is a closed subgroup, hence a Lie subgroup, and by (8) §2-2, its Lie algebra, denoted $\mathfrak{so}(n)$, consists of those matrices $X \in \mathfrak{sl}(n, \boldsymbol{R})$ for which $\exp tX \in \boldsymbol{SO}(n)$ for all $t \in \boldsymbol{R}$. But

$$\exp tX \in \boldsymbol{SO}(n) \Leftrightarrow \exp tX \exp t({}^tX) = 1 \qquad \det(\exp tX) = 1$$

so

$$\mathfrak{so}(n) = \{X \in \mathfrak{sl}(n, \boldsymbol{R}) \mid X + {}^tX = 0\}$$

the set of skew-symmetric $n \times n$ matrices (which are automatically of trace 0).

The mapping $\theta : X \to -{}^tX$ is an automorphism of $\mathfrak{sl}(n, \boldsymbol{R})$ and $\theta^2 = 1$. Since $B(X, X) = 2n\,\text{Tr}\,(XX)$, $B(X, \theta X) < 0$ so θ is a Cartan involution and

$$\mathfrak{sl}(n, \boldsymbol{R}) = \mathfrak{so}(n) + \mathfrak{p} \tag{1}$$

where $\mathfrak{p}$ is the set of $n \times n$ symmetric matrices of trace 0, is the corresponding

Cartan decomposition. Now it is known that every positive definite matrix can be written uniquely e^X ($X =$ symmetric) and every nonsingular matrix g can be written uniquely $g = op$ ($o =$ orthogonal, $p =$ positive definite). Thus we have a global analog of (1),

$$\boldsymbol{SL}(n, \boldsymbol{R}) = \boldsymbol{SO}(n)P \tag{2}$$

where $P = \exp \mathfrak{p}$, the set of positive definite matrices of determinant 1.

We shall now state a generalization of (2).

Theorem 3.4. Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition ($\mathfrak{k}$ the algebra), K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then the mapping

$$(X, k) \to (\exp X)k$$

is a diffeomorphism of $\mathfrak{p} \times K$ onto G.

In Theorem 3.4, the center $\mathfrak{z}$ of $\mathfrak{g}$ is $\{0\}$, (immediate from the definition) so the center Z of G is discrete. One can prove $Z \subset K$ and that K is compact if and only if Z is finite. In this case K is a maximal compact subgroup of G, and every compact subgroup is conjugate to a subgroup of K.

Proposition 3.5. In terms of the notation of Theorem 3.4, the mapping

$$(\exp X)k \to \exp(-X)k \tag{3}$$

is an automorphism of G.

In fact let $\tilde{G}$ be the universal covering group of G. Since all simply connected Lie groups with the same Lie algebra are isomorphic (*cf.* (v) §2-2) the automorphism θ of $\mathfrak{g}$ induces an automorphism $\tilde{\theta}$ of $\tilde{G}$ such that $d\tilde{\theta}_e = \theta$. By the remarks above, the center $\tilde{Z}$ of $\tilde{G}$ is contained in the analytic subgroup $\tilde{K}$ of $\tilde{G}$ corresponding to $\mathfrak{k}$. But $G = \tilde{G}/N$, where $N \subset \tilde{Z}$ so $\tilde{\theta}$ induces an automorphism of G which is (3).

Consider now the set G/K of left cosets gK ($g \in G$). This set has a unique manifold structure such that the map $X \to (\exp X)K$ is a diffeomorphism of $\mathfrak{p}$ onto G/K. (More generally if K is a closed subgroup of a Lie group G, G/K is a manifold in a natural way.) The group G operates on G/K: each $g \in G$ gives rise to a diffeomorphism $\tau(g) : xK \to gxK$ of G/K. Since $Z \subset K$ we have $G/K = (G/Z)/(K/Z)$ and $G/Z = \operatorname{Int}(\mathfrak{g})$ *so the space G/K is independent of the choice of the Lie group G with Lie algebra* $\mathfrak{g}$. In view of Cor. 3.3 the different possibilities for K are all conjugate so the space G/K is in a canonical way associated with $\mathfrak{g}$. Let o denote the point $\{K\}$ in G/K (the origin) and $(G/K)_o$ the tangent space. The mapping $\pi : g \to gK$ has a differential $d\pi$ mapping $\mathfrak{g}$ onto $(G/K)_o$ with a kernel which contains $\mathfrak{k}$. By reasons of dimensionality, we see therefore that the mapping

$$d\pi : \mathfrak{p} \to (G/K)_o \tag{4}$$

is an isomorphism and if $k \in K$ we have for $X \in \mathfrak{p}$, $t \in \boldsymbol{R}$

$$\pi(\exp \operatorname{Ad}(k)tX) = \pi(k \exp tX\, k^{-1}) = \tau(k)\pi(\exp tX)$$

so

$$d\pi\,(\operatorname{Ad}(k)X) = d\tau(k)\, d\pi(X). \tag{5}$$

Now the form B is > 0 on $\mathfrak{p}$ so by (4) and (5) we obtain a positive definite quadratic form Q_o on $(G/K)_o$ invariant under $d\tau(k)$ $(k \in K)$. If $p \in G/K$ is arbitrary there exists a $g \in G$ such that $p = gK$ and $d\tau(g) : (G/K)_o \to (G/K)_p$ is an isomorphism giving rise to a quadratic form Q_p on $(G/K)_p$. If $g' \in G$ satisfies $g'K = gK$, $d\tau(g')$ gives the same quadratic form Q_p on $(G/K)_p$ because of the K-invariance of Q_o. Thus we have a Riemannian structure Q on G/K induced by B.

Proposition 3.6. The manifold G/K with the Riemannian structure induced by B is a symmetric space.

PROOF. Let θ denote the automorphism (3) and s_o the mapping $gK \to \theta(g)K$ of G/K onto itself. Then s_o is a diffeomorphism and $s_o^2 = I$, $(ds_o)_o = -I$. To see that s_o is an isometry let $p = gK$ $(g \in G)$ and $X \in (G/K)_p$. Then the vector $X_o = d\tau(g^{-1})X$ belongs to $(G/K)_o$. But if $x \in G$ we have

$$s_o(gxK) = \theta(gx)K = \tau(\theta(g))(s_o(xK))$$

so $s_o \circ \tau(g) = \tau(\theta(g)) \circ s_o$ and therefore

$$\begin{aligned} Q(ds_o(X), ds_o(X)) &= Q(ds_o \circ d\tau(g)(X_o), ds_o \circ d\tau(g)(X_o)) \\ &= Q(d\tau(\theta(g)) \circ ds_o(X_o), d\tau(\theta(g)) \circ ds_o(X_o)) \\ &= Q(X_o, X_o) = Q(X, X) \end{aligned}$$

Thus s_o is an isometry and since $(ds_o)_o = -I$, it reverses the geodesics through o. The geodesic symmetry with respect to $p = gK$ is given by

$$s_p = \tau(g) \circ s_o \circ \tau(g^{-1})$$

which is an isometry, so the proposition follows.

Proposition 3.7. The geodesics through the origin in G/K are the curves $t \to \exp tX \cdot o$ $(X \in \mathfrak{p})$.

Although the proof is not difficult we shall omit it. Instead let us take a second look at the example $G = \boldsymbol{SU}(1, 1)$. The decomposition

$$\begin{pmatrix} i\alpha & \beta \\ \bar{\beta} & -i\alpha \end{pmatrix} = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix} \tag{6}$$

gives a Cartan decomposition of $\mathfrak{su}(1, 1)$. We have also if

$$X_\beta = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$$

$$\exp(tX_\beta) = \cosh(t|\beta|)I + \frac{1}{|\beta|}\sinh(t|\beta|)X_\beta$$

so

$$\exp(tX_\beta) \cdot o = (\tanh t|\beta|)\frac{\beta}{|\beta|}$$

verifying the proposition in this case.

3-4 Discussion of Symmetric Spaces

We shall now summarize some basic results in the general theory of symmetric spaces and indicate how the coset spaces G/K from the last section fit into this general theory.

Let M be a symmetric space as defined in Ch. 1. The group $I(M)$ of all isometries of M is transitive on M. (In fact, if $p, q \in M$ they can be joined by a broken geodesic and the product of the symmetries in the midpoints of these geodesics gives the desired isometry.) One can now parametrize the group $I(M)$ in a natural way turning it into a Lie group. The identity component $G = I_o(M)$ is still transitive on M. Fix a point $o \in M$ and let K be the group of elements in G which leaves o fixed. Then the mapping $gK \to g \cdot o$ is a diffeomorphism of G/K onto M. If s_o is the geodesic symmetry with respect to o the mapping $\sigma : g \to s_o g s_o$ is an involutive automorphism of G and $(K_\sigma)_o \subset K \subset K_\sigma$, where K_σ is the set of fixed points of σ and $(K_\sigma)_o$ its identity component. In order to verify these inclusions let $k \in K$. Then the maps k and $s_o k s_o$ are isometries leaving o fixed and inducing the same linear map of the tangent space M_o. Considering the geodesics starting at o we see that k and $s_o k s_o$ must coincide so $K \subset K_\sigma$. On the other hand, suppose X in the Lie algebra $\mathfrak{g}$ of G is fixed under the differential $(d\sigma)_e$. Then $s_o \exp tX s_o = \exp tX$ for all $t \in \mathbf{R}$, so applying both sides to the point o we see that $\exp tX \cdot o$ is fixed under s_o. But o is an isolated fixed point of s_o so $\exp tX \cdot o = o$ for all sufficiently small t. But then $X \in \mathfrak{k}$, the Lie algebra of K, whence $(K_\sigma)_o \subset K$. Note finally that the group $\mathrm{Ad}_G(K)$ is compact, being a continuous image of the compact group K.

Conversely, let G be a connected Lie group, K a closed subgroup, $\mathrm{Ad}_G(K)$compact. Suppose there exists an involutive automorphism σ of G such that $(K_\sigma)_o \subset K \subset K_\sigma$. Then there exists a Riemannian structure on G/K invariant under G, and for every such Riemannian structure, G/K is a symmetric space.

Consider now M as above and $G = I_o(M)$; M is said to be of the *noncompact type* if G is noncompact, semisimple without a compact normal subgroup $\neq \{e\}$, and of the *compact type* if G is compact and semisimple.

Proposition 4.1. Let M be a symmetric space, which is simply connected. Then M is a product

$$M = M_o \times M_c \times M_n$$

where M_0 is a Euclidean space and M_c and M_n are symmetric spaces of the compact type and the noncompact type, respectively.

Proposition 4.2. A symmetric space of the compact type (noncompact type) has sectional curvature everywhere $\geqslant 0$ (respectively $\leqslant 0$).

There is a very interesting *duality* between the compact type and the noncompact type. Let $M = G/K$ be a symmetric space of the noncompact type where $G = I_o(M)$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}$ the complexification of $\mathfrak{g}$. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, the subspace $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_c$ is actually a Lie algebra and another real form of $\mathfrak{g}_c$. Since the Killing form of $\mathfrak{g}_c$ is < 0 on $\mathfrak{k}$, and > 0 on $\mathfrak{p}$, it is < 0 on $\mathfrak{u}$, so $\mathfrak{u}$ is a compact real form. If U is a connected Lie group with Lie algebra $\mathfrak{u}$ and K' is the connected Lie subgroup with Lie algebra $\mathfrak{k}$, the space U/K' is a symmetric space of the compact type. This process can be reversed, that is, G/K can be constructed with U/K as a starting point.

Examples

(i) Consider the symmetric space G/K, where $G = \boldsymbol{SU}(1, 1)$ and K the subgroup of matrices $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $|t| = 1$. In this case the Cartan decomposition (6) in §3-3 shows that $\mathfrak{u}$ is the set of all matrices of the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} + \begin{pmatrix} 0 & i\beta \\ i\beta & 0 \end{pmatrix}$$

so $\mathfrak{u} = \mathfrak{su}(2)$, the algebra of all 2×2 skew symmetric matrices of trace 0. For the space U/K' we can therefore take the space $\boldsymbol{SU}(2)/K$. [$\boldsymbol{SU}(n)$ denotes the special unitary group.] It is not hard to show that when the unit sphere S^2 is projected stereographically onto the complex plane the rotations of the sphere correspond to the transformations

$$z \to \frac{az + \bar{b}}{-bz + \bar{a}} \qquad |a|^2 + |b|^2 = 1$$

that is, to the members of $\boldsymbol{SU}(2)$. In this manner $\boldsymbol{SU}(2)$ acts transitively on

S^2 and the subgroup leaving the point $z = 0$ fixed is K. Thus $U/K = S^2$ so the non-Euclidean disk D (Ch. 1) and the sphere S^2 correspond under the general duality indicated. The formulas $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ can be regarded as an explanation of the phenomenon that the triangle formulas in non-Euclidean trigonometry are obtained from the triangle formulas in spherical trigonometry by replacing the sides a, b, c by ia, ib, ic and using the relations $\sinh (ia) = i \sin a$, $\cosh (ia) = \cos a$. Lobatschevsky did indeed speak of his non-Euclidean trigonometry as spherical trigonometry on a sphere of imaginary radius.

(ii) Let U be a connected, compact Lie group with Lie algebra $\mathfrak{u}$. If Q is any positive definite quadratic form on $\mathfrak{u}$, we obtain by left translations such quadratic forms on each tangent space to U and therefore a Riemannian metric on U which is invariant under all left translations. If Q is chosen invariant under $\text{Ad}(U)$ then the Riemannian metric is invariant under right translations as well. One can prove that the geodesics through e are the one-parameter subgroups and the symmetry $s_e : x \to x^{-1}$ is an isometry so U is a symmetric space. If U^* denotes the diagonal in $U \times U$ one has a diffeomorphism $(u_1, u_2)U^* \to u_1 u_2^{-1}$ of $(U \times U)/U^*$ onto U. The group involution $(u_1, u_2) \to (u_2, u_1)$ of $U \times U$ leaves U^* pointwise fixed and induces the symmetry s_e of U, via the diffeomorphism indicated.

If U is in addition semisimple, the symmetric space $(U \times U)/U^*$ has in the above sense a noncompact dual G/U', where U' has Lie algebra $\mathfrak{u}$ and the Lie algebra $\mathfrak{g}$ of G is a certain real form of the complexification of the product algebra $\mathfrak{u} \times \mathfrak{u}$. One can prove that as $\mathfrak{u}$ runs through the compact semisimple Lie algebras, $\mathfrak{g}$ runs through the *complex* semisimple Lie algebras (regarded as Lie algebras over $\boldsymbol{R}$).

3-5 The Iwasawa Decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. The operators $\text{ad}\, X$ $(X \in \mathfrak{p})$ are all symmetric with respect to the positive definite form B_θ and each of them can therefore be diagonalized, and a commutative family can be simultaneously diagonalized. Hence let $\mathfrak{a}$ denote a maximal Abelian subspace of $\mathfrak{p}$ and if α is a real-valued linear function on $\mathfrak{a}$ put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \tag{1}$$

If $\mathfrak{g}_\alpha \neq \{0\}$, $\alpha \neq 0$, α is called a *restricted root*. Clearly, if Σ denotes the set of restricted roots,

$$\mathfrak{g} = \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha + \mathfrak{g}_o \tag{2}$$

The dimension $\dim (\mathfrak{g}_\alpha)$ is called the *multiplicity* of α. Let $\mathfrak{a}'$ denote the set of elements in $\mathfrak{a}$, where all roots are $\neq 0$. The connected components of $\mathfrak{a}'$

are intersections of half spaces; hence they are convex open sets. They are called *Weyl chambers.* Fix any Weyl chamber $\mathfrak{a}^+$ and call a restricted root *positive* if its values on $\mathfrak{a}^+$ are positive.

Let Σ^+ denote the set of positive restricted roots and put

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha \qquad \rho = \frac{1}{2} \sum_{\alpha>0} (\dim \mathfrak{g}_\alpha)\alpha \tag{3}$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra. The following result is called the Iwasawa decomposition.

Theorem 5.1. $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ (direct vector space sum). Let G be any connected Lie group with Lie algebra $\mathfrak{g}$, and let K, A, N denote the analytic subgroups corresponding to $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then the mapping

$$(k, a, n) \to kan$$

is a diffeomorphism of $K \times A \times N$ onto G.

Rather than give the proof we consider some examples. Consider the Cartan decomposition (1) §3-3,

$$\mathfrak{sl}(n, \boldsymbol{R}) = \mathfrak{so}(n) + \mathfrak{p} \tag{4}$$

The diagonal matrices of trace 0 form a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and as in §3-2 we find that the corresponding restricted roots are the linear forms $\alpha_{ij}(H) = e_i(H) - e_j(H)$ $(H \in \mathfrak{a})$, $e_i(H)$ being the ith diagonal element in H. Hence $\mathfrak{a}'$ consists of those H for which all $e_i(H)$ are different. The set

$$\{H \in \mathfrak{a} \mid e_1(H) > e_2(H) > \cdots > e_n(H)\} \tag{5}$$

is clearly a connected component of $\mathfrak{a}'$ and we take this as the Weyl chamber $\mathfrak{a}^+$. Then Σ^+ consists of the roots α_{ij} $(i < j)$ and $\mathfrak{n}$ is easily found to be the set of upper triangular matrices with 0 in the diagonal. An Iwasawa decomposition of the group $\boldsymbol{SL}(n, \boldsymbol{R})$ is therefore $g = oan$, where $o \in \boldsymbol{SO}(n)$, a is a diagonal matrix of determinant 1 and diagonal > 0, and n is an upper triangular matrix with all diagonal elements 1.

For another example consider the Cartan decomposition of $\mathfrak{su}(1, 1)$ given by

$$\begin{pmatrix} ix & y \\ \bar{y} & -ix \end{pmatrix} = \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} + \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix}$$

where $x \in \boldsymbol{R}$, $y \in \boldsymbol{C}$. As the space $\mathfrak{a}$ we can take

$$\boldsymbol{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and since

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} ix & y \\ \bar{y} & -ix \end{pmatrix}\right] = \begin{pmatrix} \bar{y} - y & -2ix \\ 2ix & y - \bar{y} \end{pmatrix}$$

we see that the decomposition (2) equals

$$\mathfrak{g} = \boldsymbol{R}\begin{pmatrix} i & -i \\ i & -i \end{pmatrix} + \boldsymbol{R}\begin{pmatrix} i & i \\ -i & -i \end{pmatrix} + \boldsymbol{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the restricted roots are α and $-\alpha$, where

$$\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$$

Thus $\mathfrak{a}'$ consists of the nonzero elements in $\mathfrak{a}$ and for $\mathfrak{a}^+$ we take for example

$$\boldsymbol{R}^+\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so

$$\mathfrak{n} = \boldsymbol{R}\begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$$

and $N = \exp \mathfrak{n}$ equals the group of matrices

$$\begin{pmatrix} 1 + in & -in \\ in & 1 - in \end{pmatrix} \in \boldsymbol{SU}(1, 1)$$

The Iwasawa decomposition of a semisimple Lie algebra $\mathfrak{g}$ involves some free choices, namely, that of $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{a}^+$. We have seen that $\mathfrak{k}$ is unique up to conjugacy, and now we shall see that $\mathfrak{a}$ and $\mathfrak{a}^+$ are uniquely determined up to conjugacy by elements of K. We begin with a result which goes back to Weyl and Cartan with a proof given by Hunt [41].

Theorem 5.2. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal Abelian subspaces of $\mathfrak{p}$. Then there exists an element $k \in K$ such that $\mathrm{Ad}_G(k)\, \mathfrak{a} = \mathfrak{a}'$. Also

$$\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}_G(k)\, \mathfrak{a}$$

PROOF. Select $H \in \mathfrak{a}$ such that its centralizer in $\mathfrak{p}$ equals $\mathfrak{a}$. (It suffices to take H such that $\alpha(H) \neq 0$ for all restricted roots α.) Put $K^* = \mathrm{Ad}_G(K)$ and let $X \in \mathfrak{p}$ be arbitrary. The function

$$k^* \to B(H, k^* \cdot X) \qquad (k^* \in K^*)$$

has a minimum, say, for $k^* = k_0$. If $T \in \mathfrak{k}$ we have therefore

$$\left\{\frac{d}{dt} B(H, \mathrm{Ad}\,(\exp tT)k_0 \cdot X)\right\}_{t=0} = 0$$

so

$$B(H, [T, k_0 \cdot X]) = 0 \qquad T \in \mathfrak{k}$$

Thus

$$B(T, [H, k_o \cdot X]) = 0 \qquad \text{for all } T \in \mathfrak{k}$$

and since $[H, k_0 \cdot X] \in \mathfrak{k}$ we deduce $[H, k_0 \cdot X] = 0$ so by the choice of H, $k_0 \cdot X \in \mathfrak{a}$.

In particular, there exists a $k_1 \in K$ such that $H \in \mathrm{Ad}\,(k_1)\mathfrak{a}'$. Thus each element in $\mathrm{Ad}\,(k_1)\mathfrak{a}'$ commutes with H so $\mathrm{Ad}\,(k_1)\mathfrak{a}' \subset \mathfrak{a}$. This proves the theorem.

3-6 The Weyl Group

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, G any connected Lie group with Lie algebra $\mathfrak{g}$, K the analytic subgroup with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Consider as before a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and let M' and M denote, respectively, the *normalizer* and *centralizer* of $\mathfrak{a}$ in K; that is,

$$M' = \{k \in K \mid \mathrm{Ad}\,(k)\mathfrak{a} \subset \mathfrak{a}\}$$

$$M = \{k \in K \mid \mathrm{Ad}\,(k)H = H \text{ for all } H \in \mathfrak{a}\}$$

Clearly M is a normal subgroup of M' and the factor group M'/M can obviously be viewed as a group of linear transformations of $\mathfrak{a}$. It is called the *Weyl group* and denoted W. In view of Theorem 5.2 it is (up to isomorphism) independent of the choice of $\mathfrak{a}$.

Now M and M' are Lie subgroups of K and their Lie algebras $\mathfrak{m}$ and $\mathfrak{m}'$ are given by (*cf.* (8) §2-2, (7) §3-1),

$$\mathfrak{m} = \{T \in \mathfrak{k} \mid [H, T] = 0 \text{ for all } H \in \mathfrak{a}\}$$

$$\mathfrak{m}' = \{T \in \mathfrak{k} \mid [H, T] \subset \mathfrak{a} \text{ for all } H \in \mathfrak{a}\}$$

Note, however, that if $T \in \mathfrak{m}'$ then for $H \in \mathfrak{a}$,

$$B([H, T], [H, T]) = -B([H, [H, T]], T) = 0$$

so $T \in \mathfrak{m}$, whence $\mathfrak{m} = \mathfrak{m}'$. Thus M'/M is a discrete group and being also compact, must be finite.

If λ is a complex-valued linear function on $\mathfrak{a}$ let H_λ denote the vector in $\mathfrak{a} + i\mathfrak{a}$ determined by $B(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{a}$. For $\alpha \in \Sigma$ let s_α denote the symmetry in the hyperplane $\alpha(H) = 0$:

$$s_\alpha(H) = H - 2\,\frac{\alpha(H)}{\alpha(H_\alpha)}\,H_\alpha \qquad H \in \mathfrak{a}, \tag{1}$$

(Remember $\mathfrak{p}$ and hence $\mathfrak{a}$ have a Euclidean metric given by B.)

Theorem 6.1. $s_\alpha \in W$ for each $\alpha \in \Sigma$.

PROOF. Pick $Z_\alpha \in \mathfrak{g}$ such that $[H, Z_\alpha] = \alpha(H)Z_\alpha$. Decomposing $Z_\alpha = T_\alpha + X_\alpha$ ($T_\alpha \in \mathfrak{k}$, $X_\alpha \in \mathfrak{p}$) the relations $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ imply that $(\mathrm{ad}\, H)^2 T_\alpha = T_\alpha$. Multiplying Z_α by a real factor if necessary we may assume $B(T_\alpha, T_\alpha) = -1$. Now if $\alpha(H) = 0$ we have $[H, T_\alpha] = 0$ so

$$\mathrm{Ad}\,(\exp tT_\alpha)H = e^{\mathrm{ad}\,(tT_\alpha)}(H) = H \qquad \text{if } \alpha(H) = 0$$

A simple computation shows that

$$e^{\mathrm{ad}\,(t_0 T_\alpha)} H_\alpha = -H_\alpha$$

provided $t_o(\alpha(H_\alpha))^{1/2} = \pi$. Thus s_α coincides with the restriction of $\mathrm{Ad}\,(\exp t_o T_\alpha)$ to $\mathfrak{a}$.

If $s \in W$ and $\alpha \in \Sigma$ it is clear from the definitions that the linear function $\alpha^s : H \to \alpha(s^{-1}H)$ on $\mathfrak{a}$ is a restricted root. Consequently, s permutes the Weyl chambers. Now let C_1 and C_2 be two Weyl chambers and let $H_1 \in C_1$, $H_2 \in C_2$. If the segment $\overrightarrow{H_1 H_2}$ intersects a hyperplane $\alpha(H) = 0$ ($\alpha \in \Sigma$) then clearly the norm $|\ \ |$ in $\mathfrak{a}$ satisfies

$$|H_1 - H_2| > |H_1 - s_\alpha H_2| \tag{2}$$

As s runs through the finite group W the function $|H_1 - sH_2|$ takes a minimum, say for $s = s_0$. By (2) the segment from H_1 to $s_o H_2$ intersects no hyperplane $\alpha(H) = 0$ ($\alpha \in \Sigma$) so H_1 and $s_o H_2$ lie in the same Weyl chamber and thus $C_1 = s_o C_2$. This proves:

Corollary 6.2. Any two Weyl chambers in $\mathfrak{a}$ are conjugate under some element of $\mathrm{Ad}_G(K)$ which leaves $\mathfrak{a}$ invariant.

For orientation we state without proof a somewhat deeper result on the Weyl group.

Theorem 6.3. The Weyl group W is generated by the symmetries s_α ($\alpha \in \Sigma$) and it is simply transitive on the set of Weyl chambers in $\mathfrak{a}$.

3.7 Boundary and Polar Coordinates on the Symmetric Space G/K

For the non-Euclidean disk D we have a natural notion of boundary, namely, the unit circle $|z| = 1$. However, this boundary notion refers to the position of D in $\boldsymbol{R}^2$. In order to make this definition more intrinsic we can define the boundary of D as the set of all rays (half-lines) from the origin in D. This motivates the following definition of the boundary of the symmetric space G/K. First, we recall the isomorphism $d\pi : \mathfrak{p} \to (G/K)_o$ from §3-3, which permits us to think of $\mathfrak{p}$ as the tangent space to G/K at o. Then we understand by a *Weyl chamber in* $\mathfrak{p}$ a Weyl chamber in some maximal Abelian subspace pf $\mathfrak{p}$. The *boundary* of G/K is now defined as the set of all Weyl chambers in $\mathfrak{p}$. Now fix $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{a}^+$ a Weyl chamber in $\mathfrak{a}$. Then according to Theorem 5.2 and Cor. 6.2, $\mathrm{Ad}(k)\mathfrak{a}^+$ $(k \in K)$ runs through the boundary and if $\mathrm{Ad}(k)\mathfrak{a}^+ = \mathfrak{a}^+$, then $k \in M'$ so $\mathrm{Ad}(k)$ on $\mathfrak{a}$ is a member of the Weyl group. Using Theorem 6.3 we see that $k \in M$. Thus the mapping

$$kM \to \mathrm{Ad}(k)\mathfrak{a}^+$$

identifies K/M with the boundary of G/K. In view of the Iwasawa decomposition $G = KAN$ and the fact that M normalizes AN we have a diffeomorphism

$$kM \to kMAN$$

of K/M onto G/MAN. In his paper [19], Furstenberg defines a boundary of G to be a compact coset space G/H of G such that for each probability measure μ on G/H there exists a sequence $(g_n) \subset G$ such that the transformed measures $g_n \cdot \mu$ converge weakly to the delta function on G/H. It was proved by Furstenberg [19] and Moore [53] that a "maximal" boundary of this sort is given by G/MAN which, as we saw, coincides with the geometrically defined boundary above. The relation $K/M = G/MAN$ shows in particular that G acts as a transformation group on the boundary; in an explicit manner

$$g(kM) = k(gk)M$$

if for $x \in G$, $k(x) \in K$ is given by $x \in k(x)AN$.

Now let $A^+ = \exp \mathfrak{a}^+$. Then we have the following "polar coordinate representation" of the symmetric space G/K.

Theorem 7.1. The mapping $(kM, a) \to kaK$ is a diffeomorphism of $K/M \times A^+$ onto an open submanifold of G/K whose complement in G/K has lower dimension.

Without spelling out the proof in detail we remark that it is a fairly direct consequence of Theorems 3.4, 5.2, and 6.3.

CHAPTER 4: FUNCTIONS ON SYMMETRIC SPACES

4-1 Invariant Differential Operators

Let M be a manifold and D a differential operator on M, that is, a linear mapping of $C_c^\infty(M)$ into itself which in an arbitrary coordinate system is expressed by partial derivatives in the coordinates. Let $\phi : M \to M$ be a diffeomorphism, and if f is a function on M put $f^\phi = f \circ \phi^{-1}$ and let D^ϕ denote the operator

$$D^\phi f = (Df^{\phi^{-1}})^\phi$$

Then D^{ϕ} is another differential operator, and we say D is *invariant under* ϕ if $D^{\phi} = D$.

Examples

Let us find all differential operators D on $\mathbf{R}^n$ which are invariant under all rigid motions. Since D is invariant under all translations it has constant coefficients so $D = P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, where P is a polynomial. But D is also invariant under all rotations around 0 so P is rotation-invariant, and since the rotations are transitive on each sphere $|x| = r$, we find P is constant on each such sphere so $P(x_1, \ldots, x_n)$ is a function of ${x_1}^2 + \cdots + {x_n}^2$, hence a polynomial in ${x_1}^2 + \cdots + {x_n}^2$.

Proposition 1.1. The differential operators on $\mathbf{R}^n$ which are invariant under all isometries are the operators $\Sigma a_n \Delta^n$ $(a_n \in \mathbf{C})$, where Δ is the Laplacian.

This result holds also if $\mathbf{R}^n$ is replaced by a symmetric space of rank 1 (and Δ by the Laplace–Beltrami operator) and also if we replace the isometries of $\mathbf{R}^n$ by the inhomogeneous Lorentz group, in which case the Laplacian is replaced (*cf.* [29], p. 271) by the operator

$$\frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \cdots - \frac{\partial^2}{\partial {x_n}^2}$$

Now if M is a Riemannian manifold the Laplace–Beltrami operator Δ on M is invariant under all isometries of M. The examples above have a high degree of mobility, that is, a large group of isometries, so essentially only Δ is invariant. The following interesting generalization is essentially a combination of results of Harish–Chandra and Chevalley (see [31] p. 432). It expresses in a precise way how higher rank of the space, that is, lower degree of mobility, leads to more invariant operators.

Theorem 1.2. Let G/K be a symmetric space of rank l. Then the algebra of all G-invariant differential operators on G/K is a commutative algebra with l algebraically independent generators.

It will now be convenient to assume that G has finite center so K is compact. As pointed out in §3-3, this is no restriction on the symmetric space G/K. Let $L(g)$ and $R(g)$ denote left and right translations on G by the group element g and let $\mathbf{D}(G)$ denote the set of all differential operators on G invariant under all $L(g)$. If $X \in \mathfrak{g}$ the operator

$$\tilde{X} : F(g) \to \{(d/dt)F(g \exp tX)\}_{t=0}$$

belongs to $\mathbf{D}(G)$. Let $\mathbf{D}_K(G)$ denote the set of elements in $\mathbf{D}(G)$ which are invariant under all $R(k)$ $(k \in K)$. For $D \in \mathbf{D}(G)$ we put

$$D^{\natural} = \int_K D^{R(k)}\, dk \tag{1}$$

where dk denotes the normalized Haar measure on K. The integral makes sense since all the operators $D^{R(k)}$ $(k \in K)$ belong to a fixed finite-dimensional vector space, so $D^\natural$ is a differential operator on G. Clearly $D^\natural \in \boldsymbol{D}_K(G)$, and we have

$$(D^\natural F)(e) = (DF)(e) \tag{2}$$

for every $F \in C^\infty(G)$ which is bi-invariant under K (that is, $F(k_1 g k_2) = F(g)$, $g \in G$, $k_1, k_2 \in K$). In fact,

$$\begin{aligned}(D^\natural F)(e) &= \int_K (D^{R(k)}F)(e)\, dk = \int_K ((DF^{R(k^{-1})})^{R(k)})(e)\, dk \\ &= \int_K (DF)(k^{-1})\, dk = \int_K (DF)^{L(k)}(e)\, dk \\ &= \int_K (DF)(e)\, dk = (DF)(e)\end{aligned}$$

Let π denote the natural projection $g \to gK$ of G onto G/K; if f is a function on G/K we put $\tilde{f} = f \circ \pi$. Then the mapping $f \to \tilde{f}$ is an isomorphism of $C^\infty(G/K)$ onto the space $C_K^\infty(G)$ of functions $F \in C^\infty(G)$ satisfying $F(gk) \equiv F(g)$. Similarly, we would like to "lift" the operators in $\boldsymbol{D}(G/K)$ to the group G. If $D \in \boldsymbol{D}_K(G)$ let $\pi(D)$ denote the operator on $C^\infty(G/K)$ determined by $(\pi(D)f)^\sim = D\tilde{f}$ $(f \in C^\infty(G/K))$. It is easy to see (*cf.* [31], p. 390) that the map $D \to \pi(D)$ maps $\boldsymbol{D}_K(G)$ onto $\boldsymbol{D}(G/K)$.

As before let $\tau(g)$ denote the diffeomorphism $hK \to ghK$ of G/K onto itself. We shall often denote the symmetric space G/K by X.

4-2 Harmonic Functions on Symmetric Spaces

In view of Prop. 1.1 it is natural to make the following definition.

Definition. A function $u \in C^\infty(G/K)$ is called *harmonic* if $Du = 0$ for all $D \in \boldsymbol{D}(G/K)$ which annihilate the constants (that is, "without constant term").

Godement made this definition in [22] (even for nonsymmetric spaces G/K), where he proved also the mean value theorem below.

Theorem 2.1. A function $u \in C^\infty(G/K)$ is harmonic if and only if

$$\int_K u(gkh \cdot o)\, dk = u(g \cdot o) \qquad \text{for all } g, h \in G \tag{1}$$

This result is most easily interpreted if rank $(G/K) = 1$. Then the orbit $K \cdot (h \cdot o)$ is a sphere and $gK \cdot (h \cdot o)$ is a sphere with center $g \cdot o$. Thus the theorem states in this case that u is harmonic if and only if the mean value of u over an arbitrary sphere is equal to the value of u in the center (*cf.* Gauss' mean value theorem for harmonic functions in $\boldsymbol{R}^n$).

PROOF. Suppose first that u is harmonic and for a fixed $g \in G$ consider the function

$$F : h \to \int_K \tilde{u}(gkh)\,dk \qquad (h \in G)$$

Let D be an operator in $\boldsymbol{D}(G)$ annihilating the constants. Then using (2) in §4-1,

$$(DF)(e) = (D^\natural F)(e) = \left\{(D^\natural)_h\left(\int_K \tilde{u}(gkh)\,dk\right)\right\}_{h=e}$$

which by the left invariance of $D^\natural$ equals

$$\int_K (D^\natural \tilde{u})(gk)\,dk = (D^\natural \tilde{u})(g)$$

(the last relation coming from the right invariance of $D^\natural \tilde{u}$ under K). However, $(D^\natural \tilde{u}) = (\pi(D^\natural)u)^\sim = 0$ since $\pi(D^\natural)$ annihilates the constants. Thus $(DF)(e) = 0$ for all $D \in \boldsymbol{D}(G)$ which annihilate the constants.

Since u satisfies the elliptic equation $\Delta u = 0$ and since Δ has analytic coefficients, it follows from a theorem of Bernstein (John [44], p. 142) that u is also analytic. Hence $\tilde{u}$ and F are also analytic so from Taylor's formula (§2-2) we can conclude that F is constant. But the relation $F(h) = F(e)$ is (1).

On the other hand, suppose (1) holds. Let $D \in \boldsymbol{D}(G/K)$ annihilate the constants. Writing (1) as

$$\int_K u^{\tau(k^{-1}g^{-1})}(x)\,dk = u(g \cdot o) \qquad g \in G,\ x \in X$$

we deduce by applying D to both sides (considered as functions of x),

$$\int_K (Du)(gk \cdot x)\,dk = 0$$

Taking $x = 0$ we conclude $Du \equiv 0$, so u is harmonic.

Now we intend to study bounded harmonic functions u on the symmetric space G/K and prove a Poisson integral representation formula due to Furstenberg [19].* Let Q_u denote the set of all functions $\psi \in L^\infty(G)$ (the space of bounded measurable functions on G) such that the sup norm $\|\psi\|_\infty = \sup_{h \in G} |\psi(h)|$ satisfies $\|\psi\|_\infty \leqslant \|u\|_\infty$ and such that

$$u(g \cdot o) = \int_K \psi(gkh)\,dk \qquad \text{for all } g, h \in G$$

According to Godement's theorem $\tilde{u} \in Q_u$, so Q_u is not empty. In addition

* modify as in notes

it is a convex set and closed in the weak* topology of $L^\infty(G)$ (the weakest topology for which all the maps $\psi \to \int f(g)\psi(g)\,dg$ of $L^\infty(G)$ into C are continuous, f being an integrable function on G and dg being a Haar measure). Since the unit ball in $L^\infty(G)$ is compact in the weak* topology (see, for example, [50]) it follows that Q_u is compact. Now if $\psi \in Q_u$ we have $\psi^{R(g)} \in Q_u$ for all $g \in G$ so G acts as a transformation group of Q_u by right translations. We would like to find a fixed point under the sugbroup MAN, which then would give us a function on the boundary G/MAN.

Definition. A group has the *fixed point property* if whenever it acts continuously on a locally convex topological vector space by linear transformations leaving a compact convex set $Q \neq \emptyset$ invariant it has a fixed point in the set.

Lemma 2.2. Connected solvable Lie groups have the fixed point property (*cf.* [6], p. 115).

PROOF. Let V be a locally convex topological vector space and G any Abelian group of linear transformations of V. For each $g \in G$ let $g_n = (1/n)(I + g + \cdots + g^{n-1})$; let $\tilde{G}$ denote the set of all products $g_{n_1} \dots g_{n_k}$ $(n_i \in Z^+, g \in G)$. All elements of $\tilde{G}$ commute. Let $Q \subset V$ be a nonempty compact convex subset of V. By convexity, $hQ \subset Q$ for $h \in \tilde{G}$. Let $h_1, \dots, h_r \in \tilde{G}$. Then for each i, $1 \leqslant i \leqslant r$,

$$h_1 \dots h_r Q = h_i h_1 \dots h_{i-1} h_{i+1} \dots h_r Q \subset h_i Q$$

whence

$$h_1 \dots h_r Q \subset \bigcap_{i=1}^{r} h_i Q$$

so this intersection is $\neq \emptyset$. By compactness of Q (expressed by the finite intersection property), we have

$$\bigcap_{h \in \tilde{G}} hQ \neq \emptyset$$

Let x an element in this intersection and let $g \in G$. Then $x \in g_n Q$, so for a suitable element $y \in Q$,

$$x = \frac{1}{n}(y + gy + \cdots + g^{n-1}y)$$

so

$$gx - x = \frac{1}{n}(g^n y - y) \subset \frac{1}{n}(Q + (-Q))$$

for each n. Using again the compactness of Q we conclude $g \cdot x = x$.

Now assume G is a connected solvable Lie group of linear transformations of V. Let $\mathfrak{g}$ be its Lie algebra and let

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = \{0\} \qquad \mathfrak{g}_{m-1} \neq \{0\}$$

be the sequence of derived algebras, $\mathfrak{g}_i = \mathfrak{D}^i\mathfrak{g}$. Let $G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$ be the corresponding series of analytic subgroups of G. Suppose now the lemma holds for all connected solvable Lie groups whose series (as defined above) has length $< m$. Let A denote the set of points in Q fixed under all $g \in G_1$. By the induction assumption, A is $\neq \varnothing$ and, of course, A is convex and compact. Let $\gamma \in G$. If $g \in G_1$ then $\gamma g \gamma^{-1} \in G_1$, so if $x \in A$, $\gamma g \gamma^{-1} x = x$ so $g\gamma^{-1}x = \gamma^{-1}x$. Thus $\gamma^{-1}x$ is fixed by all elements in G_1; being in Q, $\gamma^{-1}x$ belongs to A. Thus G maps A into itself. The closed subspace V_A of V generated by A is locally convex and since G_1 acts trivially on it, G acts on V_A as an Abelian group. By the first part of the proof there exists a $v \in A$ fixed under all $g \in G$. Q.E.D.

Lemma 2.3. The group MAN has the fixed point property.

PROOF. Let MAN act on a locally convex space V and let $Q \subset V$ be a compact convex subset $\neq \varnothing$ invariant under MAN. Since AN is solvable and connected there exists a point $q \in Q$ fixed under AN. If dm denotes the normalized Haar measure on the compact group M the integral

$$\int_M m \cdot q \, dm$$

(defined by means of approximating sums) represents, because of the compactness and convexity, a point q^* in Q. Since $m(AN)m^{-1} \subset AN$ we have for $s \in AN$

$$sq^* = \int_M sm \cdot q \, dm = \int_M m(m^{-1}sm)q \, dm = \int_M m \cdot q \, dm$$

so q^* is fixed under MAN.

We recall now that the boundary B of the symmetric space is given by the coset space representations $B = K/M$, $B = G/MAN$. The latter shows that G acts on B; this action will be denoted $(g, b) \to g(b)$ in order to distinguish it from the action $(g, x) \to g \cdot x$ of G on $X = G/K$, which we have already used. Let db denote the unique K-invariant measure on B satisfying

$$\int_B db = 1$$

Theorem 2.4. If u is a bounded harmonic function on X then there exists a bounded measurable function $\hat{u}$ on B such that

$$u(g \cdot o) = \int_B \hat{u}(g(b)) \, db \qquad (2)$$

On the other hand, if $\hat{u}$ is a bounded measurable function on B then u as defined by (2) is a bounded harmonic function on X.

PROOF. As shown above (Lemma 2.3) the set Q_u has a fixed point under MAN, say u_1. Define $\hat{u}$ on G/MAN by $\hat{u}(gMAN) = u_1(g)$. Then by the definition of Q_u, we have

$$u(g \cdot o) = \int_K \hat{u}(gkhMAN)\, dk$$

Take $h = e$ and recall that $gkMAN$ is $g(b)$ if $b = kM$. Then (2) follows because if F is any continuous function on B,

$$\int_B F(b)\, db = \int_K F(kM)\, dk$$

On the other hand, if $\hat{u}$ is a function in $L^\infty(B)$, define u by (2). Then

$$u(gkh \cdot o) = \int_B \hat{u}(gkh(b))\, db \tag{3}$$

Now let $b = k'MAN$; then $gkh(b) = gkhk'MAN = gkk_1MAN$ if $hk' = k_1a_1n_1$ (Theorem 5.1, Ch. 3). Hence,

$$\begin{aligned}\int_K u(gkh \cdot o)\, dk &= \int_K \left(\int_K \hat{u}(gkhk'MAN)\, dk' \right) dk \\ &= \int_K \left(\int_K \hat{u}(gkhk'MAN)\, dk \right) dk' = \int_K \left(\int_K \hat{u}(gkk_1MAN)\, dk \right) dk' \\ &= \int_K \left(\int_K \hat{u}(gkMAN)\, dk \right) dk' = \int_K \hat{u}(gkMAN)\, dk = u(g \cdot o).\end{aligned}$$

By Theorem 2.1, u is harmonic, so the theorem is proved.

Now define the *Poisson kernel* $P(x, b)$ on the product space $X \times B$ by the Jacobian

$$P(g \cdot o, b) = \frac{d(g^{-1}(b))}{db} \tag{4}$$

As we saw in Ch. 1. (11) §1-3 this does indeed give the classical Poisson kernel in the case when G/K is the non-Euclidean disk. We shall give the general formula for (4) later. But at any rate formula (2) can be written

$$u(x) = \int_B P(x, b)\hat{u}(b)\, db \tag{5}$$

giving a Poisson integral representation of an arbitrary bounded harmonic function on X. Furstenberg showed in [19], p. 366, that in the weak topology of measures the values of $\hat{u}$ can be regarded as boundary values of u. We shall now see that this is also the case, when we approach the boundary in a more geometric fashion.

Let $\bar{\mathfrak{n}}$ denote the subalgebra of $\mathfrak{g}$ given by

$$\bar{\mathfrak{n}} = \sum_{\alpha < 0} \mathfrak{g}_\alpha$$

where the $\mathfrak{g}_\alpha$ are given by (2) §3-5. Let $\bar{N}$ denote the corresponding analytic subgroup of G. As an immediate consequence of the Bruhat lemma (see Harish–Chandra [26]) we have that the subset $\bar{N}MAN \subset G$ is an open subset whose complement has lower dimension. As a result the mapping $T : \bar{n} \to k(\bar{n})M$ maps $\bar{N}$ onto a subset of K/M whose complement has lower dimension [Here $k(\bar{n})$ is the K-component of $\bar{n}$ according to the decomposition $G = KAN$.] One can also prove that the mapping T is one-to-one.

Lemma 2.5. For a certain positive integrable function ψ on $\bar{N}$, we have

$$\int_{K/M} f(kM)\, dk_M = \int_{\bar{N}} f(k(\bar{n})M)\psi(\bar{n})\, d\bar{n} \qquad f \in C^\infty(K/M)$$

Here dk_M is the normalized K-invariant measure on K/M and $d\bar{n}$ is a Haar measure on $\bar{N}$.

PROOF. Let $dk_M \circ T$ denote the measure on $\bar{N}$ given by

$$(dk_M \circ T)(C) = \int_{T(C)} dk_M \qquad C \text{ compact in } \bar{N}$$

Let $\psi(\bar{n})$ denote the Radon–Nikodym derivative (see, for example, [24], p. 128). Then the lemma follows at once from the properties of T given above.

REMARK. This lemma is given in Harish–Chandra [27], p. 287, with an explicit formula for $\psi(\bar{n})$ which will be derived later (Proposition 2.10).

The mapping T is particularly useful for studying the action of A on the boundary. In fact, if $a \in A$, $\bar{n} \in \bar{N}$ we have

$$a(k(\bar{n})M) = ak(\bar{n})MAN = k(a\bar{n})MAN = k(a\bar{n}a^{-1})MAN$$

that is, $\bar{n} = k(\bar{n})\, a_1 n_1$

$$a(k(\bar{n})M) = k(\bar{n}^a)M \tag{6}$$

the superscript denoting conjugation.

Theorem 2.6. Let F be a continuous function on B and u its Poisson integral

$$u(x) = \int_B P(x, b)F(b)\, db \qquad x \in X$$