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Cuspidal Hecke eigensheaves via localization

(speaker: Justin
notes: Nikolay)

What is a Hecke eigensheaf?

$k = \bar{k}$, char $k = 0$, G = conn. reductive group$/k$,
X = smooth projective curve$/k$

Idea The DG category $D(\mathrm{Bun}_G)$ should admit a spectral decomposition w.r.t Hecke operators.

Let $x \in X(k)$, $\mathcal{H}_{G,x}$ = Hecke groupoid at x

$$= \{(P_G, P_G', \alpha) \mid P_G, P_G' \in \mathrm{Bun}_G(D_x),\ \alpha : P_G|_{D_x^\circ} \xrightarrow{\sim} P_G'|_{D_x^\circ}\}$$

$$\mathrm{act} : \mathcal{H}_{G,x} \underset{\mathrm{Bun}_G(D_x)}{\times} \mathrm{Bun}_G \longrightarrow \mathrm{Bun}_G$$

By Beauville-Laszlo, we have

$$\mathrm{Bun}_G = \{(P_G, P_G', \beta) \mid P_G \in \mathrm{Bun}_G(D_x),\ P_G' \in \mathrm{Bun}_G(X \setminus x),\ \beta : P_G|_{D_x^\circ} \xrightarrow{\sim} P_G'|_{D_x^\circ}\}.$$

$$\mathrm{act}((P_G, P_G', \alpha), (P_G', P_G'', \beta)) = (P_G, P_G'', \beta \circ \alpha)$$

$$\begin{tikzcd} & \mathcal{H}_{G,x} \underset{\mathrm{Bun}_G(D_x)}{\times} \mathrm{Bun}_G \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{H}_{G,x} \times \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

$\rightsquigarrow D(\mathcal{H}_{G,x}) \curvearrowright D(\mathrm{Bun}_G)$

$$D(\mathcal{H}_{G,x}) \otimes D(\mathrm{Bun}_G) \xrightarrow{q_* p^!} D(\mathrm{Bun}_G)$$

$$\mathcal{L}_x G = \underline{\mathrm{Map}}(D_x^\times, G)$$

$$\mathcal{L}_x^+ G = \underline{\mathrm{Map}}(D_x, G)$$

$$\mathcal{H}_{G,x} = \mathcal{L}_x^+ G \backslash \mathcal{L}_x G / \mathcal{L}_x^+ G$$

$$\begin{tikzcd} \mathrm{Bun}_G^{\mathrm{lev},x} = \{(P_G, \gamma) \mid P_G \in \mathrm{Bun}_G,\ \gamma : P_G^{\mathrm{triv}}|_{D_x} \xrightarrow{\sim} P_G|_{D_x}\} \arrow[d, "\mathcal{L}_x^+ G\text{-torsor}"'] \\ \mathrm{Bun}_G \end{tikzcd}$$

Beauville - Laszlo $\Rightarrow$ $\mathcal{L}_x^+ G \curvearrowright \mathrm{Bun}_G^{\mathrm{lev},x}$ extends

to $\mathcal{L}_x G \curvearrowright \mathrm{Bun}_G^{\mathrm{lev},x}$

Convolution diagram:

$$\begin{tikzcd} & \mathcal{L}_x^+ G \backslash \mathcal{L}_x G \overset{\mathcal{L}_x^+ G}{\times} \mathrm{Bun}_G^{\mathrm{lev},x} \arrow[dl] \arrow[dr] & \\ \mathcal{L}_x^+ G \backslash \mathcal{L}_x G / \mathcal{L}_x^+ G \times \mathcal{L}_x^+ G \backslash \mathrm{Bun}_G^{\mathrm{lev},x} & & \mathcal{L}_x^+ G \backslash \mathrm{Bun}_G^{\mathrm{lev},x} \end{tikzcd}$$

Thm (Geometric Satake)

There is a canonical monoidal functor

$$\mathrm{Sat}_{G,x} : \mathrm{Rep}(\check{G}) \longrightarrow D(\mathcal{H}_{G,x})$$

which induces an equivalence

$$\mathrm{Rep}(\check{G})^{\heartsuit} \longrightarrow D(\mathcal{H}_{G,x})^{\heartsuit}.$$

A "<u>Hecke eigenvalue</u>" is a $\check{G}$ local system $E_{\check{G}}$ on X, i.e. a $\check{G}$-bundle w/ flat connection.

$\forall x \in X(k)$, $E_{\check{G}}$ defines a symm. mon. functor

$$\mathrm{Rep}(\check{G}) \longrightarrow \mathrm{Vect}$$
$$V \longmapsto \underbrace{V_{E_{\check{G}}}}_{\text{associated bundle}}\big|_x$$

<u>Def</u> (provisional) A <u>Hecke eigensheaf</u> w/ eigenvalue $E_{G^\vee}$ is an object $\mathcal{M} \in D(\mathrm{Bun}_G)$ equipped w/ isomorphisms

$$\mathrm{Sat}_{G,x}(V) * \mathcal{M} \simeq V_{E_{G^\vee}}\big|_x \otimes \mathcal{M}.$$

$$(\forall\, x \in X(k),\ V \in \mathrm{Rep}(G^\vee))$$

<u>Global Hecke action</u>

$LS_{G^\vee}$ = moduli stack of $G^\vee$-local systems on X.

- Not smooth, not even classical! (unless genus$(X) \geq 2$, & G = semisimple).
- quasi-compact (unlike Bun_G!)

$$x \in X(k) \rightsquigarrow LS_{G^\vee} \xrightarrow{ev_x} pt/G^\vee$$
$$E_{G^\vee} \longmapsto E_{G^\vee}|_x$$

(Factors through Bun_G by forgetting the connection)

$$\rightsquigarrow ev_x^* : \mathrm{Rep}(G^\vee) \simeq QCoh(pt/\check{G}) \longrightarrow QCoh(LS_{G^\vee})$$

Symmetric monoidal functor

<u>Idea</u> The Hecke actions

$$\mathrm{Rep}(G^\vee) \xrightarrow{\mathrm{Sat}_{G,x}} D(\mathcal{H}_{G,x}) \curvearrowright D(\mathrm{Bun}_G)$$

should "integrate" to an action

$$\mathrm{QCoh}(LS_{G^\vee}) \curvearrowright D(\mathrm{Bun}_G) \quad \text{s.t. } \forall x \in X(k),$$

$$\mathrm{Rep}(G^\vee) \xrightarrow{ev_x^*} \mathrm{QCoh}(LS_{G^\vee}) \curvearrowright D(\mathrm{Bun}_G)$$

is the local Hecke action at x.

Suppose we have constructed this action.

$$E_{G^\vee} \in LS_{G^\vee} \rightsquigarrow \mathrm{QCoh}(LS_{G^\vee}) \xrightarrow{\text{symm mon}} \mathrm{Vect}$$

$$\mathcal{F} \longmapsto \mathcal{F}|_{E_{G^\vee}}$$

Let $\mathrm{Vect}_{E_{G^\vee}}$ denote Vect w/ this $\mathrm{QCoh}(LS_{G^\vee})$-action.

<u>Def</u> The DG cat of <u>Hecke eigensheaves w/ eigenvalues $E_{G^\vee}$</u>

is $\mathrm{Fun}_{\mathrm{QCoh}(LS_{G^\vee})}(\mathrm{Vect}_{E_{G^\vee}}, D(\mathrm{Bun}_G))$

$$\simeq \mathrm{Vect}_{E_{G^\vee}} \underset{\mathrm{QCoh}(LS_{G^\vee})}{\otimes} D(\mathrm{Bun}_G)$$

——————— Break ———————

Construction of the global Hecke action: Gaitsgory "Generalized vanishing theorem."

§ What is actually going on?

$LS_{G^\vee}$ is not smooth.

$\Rightarrow \Psi : \mathrm{IndCoh}(LS_{G^\vee}) \longrightarrow \mathrm{QCoh}(LS_{G^\vee})$ is <u>not</u> an equivalence

(Coherent $\neq$ perfect complexes. The Ψ functor is "Left Kan extension")

$LS_{G^\vee}$ is "quasi-smooth" = locally complete intersection in derived sense.

$\Rightarrow LS_{G^\vee}$ is eventually coconnective.

$\Rightarrow \Psi$ admits a fully faithful left adjoint

$\Xi : \mathrm{QCoh}(LS_{G^\vee}) \hookrightarrow \mathrm{IndCoh}(LS_{G^\vee})$

$\hookrightarrow \mathrm{IndCoh}_{\mathcal{N}}(LS_{G^\vee})$

"nilpotent singular support"

$\mathrm{QCoh}(LS_{G^\vee}) \curvearrowright \mathrm{IndCoh}(LS_{G^\vee})$ comes from

$\mathrm{Perf} \curvearrowright \mathrm{Coh}$, preserves $\mathrm{IndCoh}_{\mathcal{N}}(LS_{G^\vee})$.

(first time written as thm!?)

<u>Thm</u> (Global geometric Langlands) $\exists$ canonical $\mathrm{QCoh}(LS_{G^\vee})$-linear equivalence $\mathbb{L}_G : D(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}_{\mathcal{N}}(LS_{G^\vee})$

Just announced by Arinkin-Beraldo-Chen-Faergeman-Gaitsgory-Lin-Raskin. (On Gaitsgory's website)

<u>Def</u> A $\check{G}$-local system $E_{\check{G}}$ is called <u>reducible</u> if $\exists$ parabolic $P^\vee \subsetneq \check{G}$ & $E_{P^\vee} \in LS_{\check{P}}$ s.t. $E_{G^\vee} \simeq \check{G} \times^{\check{P}} E_{P^\vee}$

O/w, call $E_{G^\vee}$ <u>irreducible</u>. A Hecke eigensheaf is <u>cuspidal</u> if its eigenvalue is irreducible.

Prop $\Xi : QCoh(LS^{irred}_{\check{G}}) \xrightarrow{\sim} IndCoh_{\mathcal{N}}(LS^{irred}_{\check{G}})$

Cor For $E_{\check{G}} \in LS^{irred}_{\check{G}}$, we have

$$Vect_{E_{\check{G}}} \underset{QCoh(LS_{\check{G}})}{\otimes} IndCoh_{\mathcal{N}}(LS_{\check{G}}) \simeq Vect.$$

$\Longrightarrow$ Given Geom. Langlands, cuspidal Hecke eigensheaves exists & are unique up to tensoring w/ a vector space.

Non cuspidal eigensheaves can be constructed from cuspidal ones using geometric Eisenstein series.

So we can reduce to cuspidal case by inducting on the semisimple rank of G (base case: geometric class field theory).

Beilinson - Drinfeld (2023 version)

A $\check{G}$-oper is a $\check{G}$-local system $E_{\check{G}}$ together w/ a $\check{B}$-reduction $P_{\check{B}}$ of the underlying $\check{G}$-bundle, satisfying a transversality condition wrt the connection on $E_{\check{G}}$.

$$\begin{tikzcd} Op_{\check{G}} \arrow[r] \arrow[dr, dashed] & LS_{\check{G}} \\ & LS^{irred}_{\check{G}} \arrow[u, hook] \end{tikzcd}$$

closed embedding (Lagrangian)

BD constructed a Hecke eigensheaf for each $\check{G}$-oper

Q) Why should this be possible? Can we do it for arbitrary irred $E_{\check{G}}$?

Ran = moduli space of nonempty finite subsets of X

$$= \operatorname{colim}_{\substack{I \neq \emptyset \\ \# I < \infty}} X^I$$

$$\mathrm{Op}^{\mathrm{unram, glob}}_{G^\vee, \mathrm{Ran}} = \{ (x_I, E_{G^\vee}, P_{B^\vee}) \mid x_I \in X^I,\ E_{G^\vee} \in LS_{G^\vee}(X),$$
$$P_{B^\vee} = \text{oper structure on } E_{G^\vee}|_{X \setminus \Gamma_{x_I}} \}$$

$$\mathrm{Op}^{\mathrm{unr, loc}}_{G^\vee, \mathrm{Ran}} = \{ (x_I, E_{G^\vee}, P_{B^\vee}) \mid x_I \in X^I,\ E_{G^\vee} \in LS_{G^\vee}(D_{x_I}),$$
$$P_{B^\vee} = \text{oper structure on } E_{G^\vee}|_{\mathring{D}_{x_I}} \}$$

"disjoint union of disks"

$$\begin{tikzcd} & \mathrm{Op}^{\mathrm{unr, glob}}_{G^\vee, \mathrm{Ran}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ LS_{G^\vee}(X) & & \mathrm{Op}^{\mathrm{unr, loc}}_{G^\vee, \mathrm{Ran}} \end{tikzcd}$$

Conj p is surjective.

Thm (Arinkin) If $Z_{G^\vee}$ is connected, then the image of p contains $LS^{\mathrm{irred}}_{G^\vee}(X)$.

$$\rightsquigarrow \quad 0 \to k \cdot c \to \hat{\mathfrak{g}}_{\mathrm{crit}} \to \mathfrak{g}((t)) \to 0$$

central extension determined by the 2-cocycle

$$\mathfrak{g}((t)) \otimes \mathfrak{g}((t)) \longrightarrow k \cdot c$$

$$(x \otimes f, y \otimes g) \longmapsto \underbrace{-\tfrac{1}{2} \kappa_{Kil}(x,y)}_{\text{critical level.}} . \mathrm{res}(f\,dg) \cdot c$$

$\hat{\mathfrak{g}}_{crit}$-mod $=$ DG cat of smooth $\hat{\mathfrak{g}}_{crit}$-rep's

smooth $=$ $\forall M \in \hat{\mathfrak{g}}_{crit}$-mod, $\forall v \in M$ $\exists n \gg 0$ s.t

$$t^n \mathfrak{g}[[t]] \cdot v = 0$$

Also require c act by identity.

$\mathfrak{g}((t)) = \mathrm{Lie}\, \mathcal{L}(G)$, $x \in X(k) \rightsquigarrow$ coord. free version

$$k[[t]] \subseteq k((t)) \rightsquigarrow \mathcal{O}_x \subseteq K_x$$

$$\rightsquigarrow \hat{\mathfrak{g}}_{crit}\text{-mod}_x, \ \hat{\mathfrak{g}}_{crit}\text{-mod}_{Ran}.$$

$\exists$ canonically defined localization functor

Really is some twist, but may identify w/ this

$$\mathrm{Loc}_{Ran} : \hat{\mathfrak{g}}_{crit}\text{-mod}_{Ran}^{\mathcal{L}^+G} \longrightarrow D(\mathrm{Bun}_G)$$

$\mathcal{L}_x G \curvearrowright \mathrm{Bun}_G^{lev, x} \rightsquigarrow$ localization functor

$H \curvearrowright Y$, $K \subseteq H \rightsquigarrow \mathrm{Loc} : \mathfrak{h}\text{-mod}^K \longrightarrow D(Y/K)$

(Fundamental local equiv)

<u>Thm</u> (Unpublished) $\exists$ canonical equivalence of factorization categories

$$\begin{tikzcd}
\mathrm{FLE}_{crit} : \hat{\mathfrak{g}}_{crit}\text{-mod}_{Ran}^{\mathcal{L}^+G} \arrow[r] & \mathrm{QCoh}(\mathrm{Op}^{unr, loc}_{G^\vee, Ran}) \\
D(\mathcal{H}_{G, Ran}) \arrow[u, phantom, "\circlearrowright"] & \mathrm{Rep}(G^\vee)_{Ran} \arrow[l] \arrow[u, phantom, "\circlearrowright"]
\end{tikzcd}$$

The following diagram is supposed to commute:

$$\begin{tikzcd} \hat{\mathfrak{g}}_{crit}\text{-mod}^{\mathfrak{L}^+G}_{Ran} \arrow[r, "\sim"] \arrow[d, "Loc"] & QCoh(Op^{unr, loc}_{G^\vee, Ran}) \arrow[d, "p_* \varepsilon^*"] \\ D(Bun_G) & QCoh(LS_{G^\vee}(X)) \arrow[l, "\mathbb{L}_G^{-1} \circ \Xi"'] \end{tikzcd}$$

In fact, we can show unconditionally ($\sim$B-D) that the functor

$$QCoh(Op^{unr, loc}_{G^\vee, Ran}) \xrightarrow{Loc \circ FLE^{-1}_{crit}} D(Bun_G)$$

factors through a functor

$$QCoh(Op^{unr, glob}_{G^\vee, Ran}) \xrightarrow{\varepsilon\text{ Hitch}} D(Bun_G)$$

$$\delta_{(x_I, E_{G^\vee}, P_{B^\vee})} \longmapsto \text{Hecke eigensheaf.}$$

(Jerusalem 2014 workshop has "2023 formulation").