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# Baire classification of $\mathcal{I}$-density and $\mathcal{I}$-approximately continuous functions

Krzysztof Ciesielski[1] and Lee Larson[2]

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**Abstract.** Let $\mathcal{T}_{\mathcal{O}}$ be the ordinary topology, $\mathcal{T}_{\mathcal{I}}$ be the $\mathcal{I}$-density topology and $\mathcal{T}_{\mathcal{D}}$ be the deep $\mathcal{I}$-density topology on the real numbers, $\mathbb{R}$. Any continuous function $f: (\mathbb{R}, \mathcal{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ is a Darboux function of the first Baire class. Any unilaterally continuous function $f: (\mathbb{R}, \mathcal{T}_{\mathcal{D}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{D}})$ is a Darboux function of the Baire*1 class.

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## 1. Introduction

The terminology and notation we use is standard. In particular, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $A, B \subset \mathbb{R}$, $A^c$ stands for the complement of $A$, $A \Delta B$ for the symmetric difference of $A$ and $B$ and dist $(A, B)$ for the Euclidean distance between $A$ and $B$. The natural topology on $\mathbb{R}$ is denoted by $\mathcal{T}_{\mathcal{O}}$. Topological terms in which we do not specifically state the topology concern the natural topology. For example, int $(A)$ and cl $(A)$ stand for the interior and closure of $A$ with respect to $\mathcal{T}_{\mathcal{O}}$, respectively. The oscillation of a function $f$ at the point $x$ will be denoted by $\omega(f, x)$.

W. Wilczyński [11] introduced a topology on $\mathbb{R}$ which has many properties in common with the ordinary density topology, except that it is based upon category instead of measure. This topology, called the $\mathcal{I}$-density topology, is defined as follows.

Let $\mathcal{I}$ stand for the ideal of first category subsets of $\mathbb{R}$. A Boolean function $P$ defined on $\mathbb{R}$ is said to be true *$\mathcal{I}$-almost everywhere* ($\mathcal{I}$-a.e.), if

$$\{x : P(x) \text{ is false}\} \in \mathcal{I}.$$

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A sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ *converges* ($\mathscr{I}$) to a function $f$, if for each increasing sequence $\{n_j\}$ of natural numbers, there is a further subsequence $\{n_{j_k}\}$ such that $f_{n_{j_k}}$ converges pointwise to $f$, $\mathscr{I}$-a.e. Notice that this definition mimics the standard definition of convergence in measure, or stochastic convergence [2], with the exception that the ideal $\mathscr{N}$ of Lebesgue null sets is replaced by $\mathscr{I}$.

If $A \subset \mathbb{R}$, then a point $a \in \mathbb{R}$ is an *$\mathscr{I}$-density point* of $A$ if $\chi_{n(A-a)\cap(-1,1)}$ converges ($\mathscr{I}$) to $\chi_{(-1,1)}$, where $\chi_S$ is the characteristic function of the set $S$. The set of all $\mathscr{I}$-density points of the set $A$ is written $\Phi_{\mathscr{I}}(A)$. It is obvious from the above definition that

$$(1) \qquad \Phi_{\mathscr{I}}(A) = \Phi_{\mathscr{I}}(B) \quad \text{whenever } A \Delta B \in \mathscr{I}.$$

A point $a$ is an *$\mathscr{I}$-dispersion point* of $A$, if $a \in \Phi_{\mathscr{I}}(A^c)$.

Closely related to the notion of an $\mathscr{I}$-density point is that of a *deep-$\mathscr{I}$-density point*. An $\mathscr{I}$-density point $a$ of the set $A$ is a deep-$\mathscr{I}$-density point of $A$, if there is a closed set $F \subset \text{int}(A) \cup \{a\}$ such that $a \in \Phi_{\mathscr{I}}(F)$. The set of all deep-$\mathscr{I}$-density points of $A$ is denoted by $\Phi_{\mathscr{D}}(A)$. From the definitions, it is obvious that for any set $A \subset \mathbb{R}$,

$$(2) \qquad \Phi_{\mathscr{D}}(A) \subset \Phi_{\mathscr{I}}(A).$$

Denote the Baire subsets of $\mathbb{R}$ by $\mathscr{B}$. A standard definition of the Baire sets is the following [6]:

$$\mathscr{B} = \{G \Delta I : G \in \mathscr{T}_{\mathscr{O}} \text{ and } I \in \mathscr{I}\}.$$

From this definition, it is not hard to show that associated with every $B \in \mathscr{B}$ is a unique open set $\tilde{B}$ such that $\tilde{B} = \text{int}(\text{cl}(\tilde{B}))$ and $B = \tilde{B} \Delta I$ for some $I \in \mathscr{I}$. Any open set $G$ for which $G = \text{int}(\text{cl}(G))$ is called a *regular* open set. In a sense, $\tilde{B}$ is the largest open set such that $B$ can be written as $B \Delta I$ for some $I \in \mathscr{I}$.

The *$\mathscr{I}$-density topology* [11, 12] is defined as

$$\mathscr{T}_{\mathscr{I}} = \{A \in \mathscr{B} : A \subset \Phi_{\mathscr{I}}(A)\}.$$

Similarly, the *deep-$\mathscr{I}$-density topology* [7, 12] is defined as

$$\mathscr{T}_{\mathscr{D}} = \{A \in \mathscr{B} : A \subset \Phi_{\mathscr{D}}(A)\}.$$

It is clear from (2) and the definitions of $\mathscr{T}_{\mathscr{I}}$ and $\mathscr{T}_{\mathscr{D}}$ that

$$(3) \qquad \mathscr{T}_{\mathscr{O}} \subset \mathscr{T}_{\mathscr{D}} \subset \mathscr{T}_{\mathscr{I}}.$$

It is known that these containments are proper. (The first inclusion is proper by Lemma 2.4 [12, Theorem 2]. The second inclusion is proper because $\mathscr{T}_{\mathscr{D}}$ is completely regular [7] while $\mathscr{T}_{\mathscr{I}}$ is not [10, Theorem 5].)

Using the three topologies, $\mathscr{T}_{\mathscr{O}}$, $\mathscr{T}_{\mathscr{I}}$ and $\mathscr{T}_{\mathscr{D}}$, there are nine different definitions of continuity possible. For $\mathscr{X}, \mathscr{Y} \in \{\mathscr{I}, \mathscr{D}, \mathscr{O}\}$, let

$$\mathscr{C}_{\mathscr{X}\mathscr{Y}} = \{f : (\mathbb{R}, \mathscr{T}_{\mathscr{X}}) \to (\mathbb{R}, \mathscr{T}_{\mathscr{Y}}) : f \text{ is continuous}\}.$$

For example, $\mathscr{C}_{\mathscr{O}\mathscr{O}}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ continuous in the ordinary sense. It is not hard to show that $\mathscr{C}_{\mathscr{O}\mathscr{I}} = \mathscr{C}_{\mathscr{O}\mathscr{D}}$ consists exactly of the constant functions

[4]. It is also known that $\mathscr{C}_{\mathscr{D}\mathscr{O}} = \mathscr{C}_{\mathscr{I}\mathscr{O}}$, and these are called the *$\mathscr{I}$-approximately continuous functions* [7]. The remaining classes with which this paper is concerned are the *$\mathscr{I}$-density continuous* and *deep-$\mathscr{I}$-density continuous* functions, $\mathscr{C}_{\mathscr{I}\mathscr{I}}$ and $\mathscr{C}_{\mathscr{D}\mathscr{D}}$, respectively.

It is known that the following relationships hold

$$(4) \qquad \mathscr{C}_{\mathscr{O}\mathscr{O}} \subset \mathscr{C}_{\mathscr{I}\mathscr{O}} \quad \text{and} \quad \mathscr{C}_{\mathscr{I}\mathscr{I}} \subset \mathscr{C}_{\mathscr{D}\mathscr{D}} \subset \mathscr{C}_{\mathscr{D}\mathscr{O}} = \mathscr{C}_{\mathscr{I}\mathscr{O}}.$$

Moreover, the inclusions in (4) are proper and these are the only inclusions between those classes [4].

All the continuity and density definitions given above can be restated in more-or-less obvious ways in one-sided versions. For technical reasons it is often more convenient to work with one-sided density or continuity. For example, to show that a point $a$ is an $\mathscr{I}$-density point of a set $A$, it is often easier to establish that is is both a left and right $\mathscr{I}$-density point. Such simple technical extensions to the definitions will be used without further comment.

Let $\mathscr{D}$ stand for the functions $f: \mathbb{R} \to \mathbb{R}$ with the Darboux (intermediate value) property, $\mathscr{B}_1$ the functions of the first Baire class and $\mathscr{B}_1^*$ the functions in the Baire*1 class; i.e., the class of all functions $f: \mathbb{R} \to \mathbb{R}$ with the property that for every perfect set $P$ there is its nonempty portion $Q = P \cap (a, b)$ such that $f$ restricted to $Q$ is continuous [8].

In Section 3 it is shown that the one-sided $\mathscr{I}$-density continuous functions are in $\mathscr{D} \cap \mathscr{B}_1$. Section 4 has as its main purpose a proof that $\mathscr{C}_{\mathscr{D}\mathscr{D}} \subset \mathscr{D} \cap \mathscr{B}_1^*$. But first, the next section is devoted to presenting some technical lemmas which are needed in the later sections.

## 2. Some technical lemmas

In this section some technical lemmas are presented which are used in the later sections. The first of these is a restatement of the definition of $\mathscr{I}$-density points [12].

**Lemma 2.1.** *$x \in \Phi_{\mathscr{I}}(A)$ if, and only if, for every increasing sequence $\{n_m\}$ of natural numbers there is a subsequence $\{n_{m_p}\}$ such that*

$$(-1, 1) \cap \left( \bigcup_{q \in \mathbb{N}} \bigcap_{p \geq q} n_{m_p}(A - x) \right)^c = (-1, 1) \cap \left( \liminf_{p \to \infty} (n_{m_p}(A - x)) \right)^c \in \mathscr{I}.$$

The next lemma is a dual version of Lemma 2.1.

**Lemma 2.2.** *Let $B \subset \mathbb{R}$ and $x \in \mathbb{R}$. The following are equivalent:*

**(i)** *$x$ is a $\mathscr{I}$-dispersion point of $B$;*

**(ii)** *for every increasing sequence $\{n_m\}$ of natural numbers there exists a subsequence $\{n_{m_p}\}$ such that*

$$\lim_{p \to \infty} \chi_{(n_{m_p}(B - x)) \cap (-1, 1)} = 0, \quad \mathscr{I} - a.e.;$$

**(iii)** *for every increasing sequence $\{n_m\}$ of natural numbers there exists a subsequence $\{n_{m_p}\}$ such that*

$$(-1,1)\cap\bigcap_{q\in\mathbb{N}}\bigcup_{p\geq q} n_{m_p}(B-x) = (-1,1)\cap\limsup_{p\to\infty}(n_{m_p}(B-x))\in\mathcal{I}.$$

In the definition of $\mathcal{I}$-density given above, the divergent sequence of "multipliers", $\{n_k\}$, is limited to the positive integers, rather than arbitrary sequences of positive numbers diverging to infinity. This restriction is removed by the following theorem.

**Theorem 2.3.** *Let $B\in\mathcal{B}$ and $x\in\mathbb{R}$. The following statements are equivalent to each other.*

**(i)** *$x$ is an $\mathcal{I}$-dispersion point of $B$;*
**(ii)** *for every divergent increasing sequence $\{t_n\}$ of positive numbers there exists a subsequence $\{t_{n_k}\}$ such that*

$$\limsup_{k\to\infty}(t_{n_k}(\tilde{B}-x))\cap(-1,1)\in\mathcal{I};$$

**(iii)** *for every divergent increasing sequence $\{t_n\}$ of positive numbers there exists a subsequence $\{t_{n_k}\}$ such that*

$$\limsup_{k\to\infty}(t_{n_k}(B-x))\cap(-1,1)\in\mathcal{I}.$$

*Proof.* The equivalence of (iii) and (i) is proved by W. Poreda, E. Wagner-Bojakowska and W. Wilczyński [10, Corollary 1]. Thus, (ii) is equivalent to the fact that $x$ is an $\mathcal{I}$-dispersion point of $\tilde{B}$. But the last fact is equivalent to (i) by (1).

The next lemma and its corollary provide a tool for constructing sets which are in $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{D}}$. They are similar to theorems originally proved by W. Poreda, E. Wagner-Bojakowska and W. Wilczyński [10, Theorem 1] [12, Theorem 2]. To state them we need first the following definition.

We say that any of the sets $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ or $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a *right interval set of a point* $a\in\mathbb{R}$ if $a_{n+1}<b_{n+1}<a_n<b_n$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n = a$. In the case when $a=0$ we simply say that it is *a right interval set.*

**Lemma 2.4.** *If $E=\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right interval set such that*

**(i)** $\lim_{n\to\infty}(b_n-a_n)/a_n = 0$; *and*
**(ii)** $\lim_{n\to\infty}b_{n+1}/a_n = 0$,

*then* 0 *is an $\mathcal{I}$-dispersion point of $E$. In particular, $E^c\in\mathcal{T}_{\mathcal{D}}$.*

*Proof.* It follows immediatelly from the proof of [12, Theorem 2].

From the lemma above we obtain easily the following corollary. (Compare also, [1, Lemma 3].)

**Corollary 2.5.** *If $\bigcup_{n\in\mathbb{N}}[a_n, b_n]$ is a right interval set with*

$$\lim_{n\to\infty}\frac{(b_n - a_n)}{b_n} = 0,$$

*then there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers such that* 0 *is an $\mathscr{I}$-dispersion point of*

$$\bigcup_{m\in\mathbb{N}}[a_{n_m}, b_{n_m}].$$

Finally, the following example provides a way to construct functions which are well-behaved in the ordinary sense, but are not well-behaved with regard to density continuity.

**Example 2.6.** *There is a monotone continuous function $f:\mathbb{R}\to\mathbb{R}$ which is not in $\mathscr{C}_{\mathscr{I}\mathscr{I}}$ or $\mathscr{C}_{\mathscr{D}\mathscr{D}}$.*

*Proof.* To construct such a function, let $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be as in Lemma 2.4 and let $D = \bigcup_{n\in\mathbb{N}}[c_n, d_n]$, where $[c_n, d_n] = [b_{n+1}, a_n]$ for each $n\in\mathbb{N}$. Then, $D^c \notin \mathscr{T}_{\mathscr{I}}$. It is also easy to see, that by decreasing the intervals $[a_n, b_n]$, if necessary, that $E^c \in \mathscr{T}_{\mathscr{D}}$.

Define the function $f$ by letting $f(x) = 0$ whenever $x \le 0$, $f(c_n) = a_n$ and $f(d_n) = f(b_n)$ for all $n$. Make $f$ piecewise linear between the points on which it has already been defined. Then $f^{-1}(E^c) = D^c$. So, $f \notin \mathscr{C}_{\mathscr{I}\mathscr{I}} \cup \mathscr{C}_{\mathscr{D}\mathscr{D}}$.

## 3. $\mathscr{I}$-approximately continuous functions

In this section it is shown that $\mathscr{C}_{\mathscr{I}\mathscr{O}} \subset \mathscr{D} \cap \mathscr{B}_1$. The following lemma is used in the proof.

**Lemma 3.1.** *If $f$ is right $\mathscr{I}$-approximately continuous at each of its points, $a\in\mathbb{R}$ and $A = \{x : f(x) > a\}$, then* int$(A)$ *is dense in $A$.*

*Proof.* It may be supposed without loss of generality that $a = 0$. Let $A_n = \{x : f(x) \ge 1/n\} \in 1/n\} \in \mathscr{B}$ and let $A = \bigcup_{n\in\mathbb{N}} A_n$. If $x \in \tilde{A}_n$, then, by Lemma 2.1, $x$ is an $\mathscr{I}$-density point of $A_n$, as $A_n \Delta \tilde{A}_n \in \mathscr{I}$. The definition of right $\mathscr{I}$-density continuity shows that $f(x) \ge 1/n$. It follows from this that $\tilde{A}_n \subset A_n$. Therefore, $G = \bigcup_{n\in\mathbb{N}} \tilde{A}_n \subset \bigcup_{n\in\mathbb{N}} A_n = A$. To see that $G$ is dense in $A$, let $x \in A$ and choose $n \in \mathbb{N}$ such that $1/n < f(x)$. Then $x$ is a right $\mathscr{I}$-density point of $\{w : f(w) > 1/n\}$ and it is apparent that $x$ must be a limit point of $\tilde{A}_n$. From this, it follows that $G$ is dense in $A$.

**Theorem 3.2.** *Every right $\mathscr{I}$-approximately continuous function is of the first Baire class.*

*Proof.* Poreda, Wagner-Bojakowska and Wilczyński [9] proved a slightly weaker version of this theorem for two-sided $\mathcal{I}$-approximate continuity. The following alternative proof is presented here because it is somewhat shorter and the result is a little sharper.

Let $f$ be right $\mathcal{I}$-density continuous on $\mathbb{R}$. It suffices to show that $\{x : f(x) \geq 0\}$ is a $\mathbf{G}_\delta$ set. To do this, for each $p \in \mathbb{N}$, let $U_p = \{x : f(x) > -1/p\}$ and, for $p, q, r, k \in \mathbb{N}$, define

$$A(p, q, r, k) = \left\{x \in \mathbb{R}: \left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r(U_p - x) \neq \emptyset\right\} \tag{5}$$

and

$$A(p, q, r) = \bigcap_{k=1}^{q} A(p, q, r, k). \tag{6}$$

It is easy to see that each $A(p, q, r)$ is an open set.

Next, define

$$U = \bigcap_{p \in \mathbb{N}} \bigcap_{q \in \mathbb{N}} \bigcup_{r \geq q} A(p, q, r). \tag{7}$$

It is clear that $U$ is a $\mathbf{G}_\delta$ set. It will be shown that $U = \{x : f(x) \geq 0\}$.

To show that $U \subset \{x : f(x) \geq 0\}$, fix $p \in \mathbb{N}$ and let

$$V_p = \bigcap_{q \in \mathbb{N}} \bigcup_{r \geq q} A(p, q, r).$$

Suppose that $x \in V_p$. For each $q \in \mathbb{N}$ there is an $r_q \in \mathbb{N}$, $r_q \geq q$, such that when $1 \leq k \leq q$, then

$$\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_q(U_p - x) \neq \emptyset.$$

From this and Lemma 3.1 it is apparent that

$$\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_q(\operatorname{int}(U_p) - x) \neq \emptyset \quad \text{for } k = 1, 2, \ldots, q. \tag{8}$$

Let $\{r_{q_i}\}_{i \in \mathbb{N}}$ be an increasing subsequence of $\{r_q\}_{p \in \mathbb{N}}$ and put $n_i = r_{q_i}$ for $i \in \mathbb{N}$. From (8) it follows that $\bigcup_{j \in \mathbb{N}} n_{i_j} \operatorname{int}(U_p)$ is a dense open subset of $(0, 1)$ for every subsequence $\{n_{i_j}\}_{j \in \mathbb{N}}$ of $\{n_i\}_{i \in \mathbb{N}}$. Therefore,

$$\limsup_{j \to \infty} n_{i_j} U_p \cap (0, 1)$$

is a residual subset of $(0, 1)$. It follows, by Lemma 2.2, that $x$ is not a right $\mathcal{I}$-dispersion of $U_p$ and the right $\mathcal{I}$-density continuity of $f$ shows that $x \in \{x : f(x) \geq -1/p\}$. Thus, $V_p \subset \{x : f(x) \geq -1/p\}$ and

$$U = \bigcap_{p \in \mathbb{N}} V_p \subset \{x : f(x) \geq 0\}.$$

To show that $\{x : f(x) \geq 0\} \subset U$ let us fix $x$ such that $f(x) \geq 0$ and $p, q \in \mathbb{N}$. It must be shown that there is an $r \in \mathbb{N}$, $r \geq q$, such that $x \in A(p, q, r)$. If not, for every $r \geq q$ there must be an integer $k_r$, with $1 \leq k_r \leq q$, such that

$$\left(\frac{k_r - 1}{q}, \frac{k_r}{q}\right) \cap r(U_p - x) = \emptyset.$$

There must exist an increasing sequence of natural numbers $r_i$ such that $k_{r_i} = k$ for some $1 \leq k \leq q$. This gives

$$\left(\frac{k-1}{q}, \frac{k}{q}\right) \cap r_i(U_p - x) = \emptyset$$

for all $i$ so that for any subsequence $\{r_{i_j}\}$ of $\{r_i\}$

$$\liminf_{j \to \infty} r_{i_j}(U_p - x\} \cap \left(\frac{k-1}{q}, \frac{k}{q}\right) = \emptyset.$$

Therefore, $x$ is not a point of right $\mathcal{I}$-density of $U_p$. But, this is impossible because $f(x) \geq 0$ and $f$ is right $\mathcal{I}$-approximately continuous at $x$.

Therefore $U \supset \{x : f(x) \geq 0\}$ and consequently $U = \{x : f(x) \geq 0\}$, which finishes the proof of the theorem.

The following corollary is immediate.

**Corollary 3.3.** *If $f \in \mathscr{C}_{\mathcal{I}\mathcal{O}}$, then $f$ is continuous in the ordinary sense on a dense $\mathbf{G}_\delta$ subset of $\mathbb{R}$.*

**Corollary 3.4.** $\mathscr{C}_{\mathcal{I}\mathcal{O}} \subset \mathscr{D} \cap \mathscr{B}_1$.

*Proof.* Since sets which are open in the $\mathcal{I}$-density topology must be bilaterally $c$-dense in themselves, this is an immediate consequence of the preceding theorem and Young's criterion. (See Bruckner [3].)

## 4. $\mathcal{I}$-density continuous functions

The goal of this section is to prove that $\mathscr{C}_{\mathscr{D}\mathscr{D}} \subset \mathscr{D} \cap \mathscr{B}_1^*$. To do this, the following definition and lemma are needed [5, Lemma 29.1].

A *partition* of a set $E$ is a pairwise disjoint family $\Pi = \{E_i : i \in \Lambda\}$ such that $\bigcup_{i \in \Lambda} E_i = E$. Note that any partition $\Pi$ can be associated with a function $F: E \to \Lambda$ such that $F(x) = F(y)$ if, and only if, $x$ and $y$ belong to the same $E_i \in \Pi$. Conversely, any function $F: E \to \Lambda$ determines a partition of $E$.

For a set $A$ and $n \in \mathbb{N}$ define

$$[A]^n = \{B \subset A : \text{card}(B) = n\}.$$

If $\Pi = \{E_i : i \in \Lambda\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition $\Pi$ if, for some $i \in \Lambda$, $[H]^n \subset E_i$. That is, all $n$-element subsets of $H$ are in the same piece of the partition $\Pi$.

**Lemma 4.1. (Ramsey's Theorem).** *If $n$, $k \in \mathbb{N}$, then every finite partition $\Pi = \{E_1, E_2, \ldots, E_k\}$ of $[\mathbb{N}]^n$ has an infinite homogeneous set. In other words, for every $F: [\mathbb{N}]^n \to \{1, 2, \ldots, k\}$ there exists an infinite $H \subset \mathbb{N}$ such that $F$ is constant on $[H]^n$.*

**Theorem 4.2.** $\mathscr{C}_{\mathscr{D}\mathscr{D}} \subset D \cap \mathscr{B}_1^*$.

*Proof.* Assume to the contrary that for some perfect set $P$ the set

$$Z = \{x \in P : f|_P \text{ is not continuous at } x\}$$

is dense in $P$.

We will construct sequences: $\{x_n\}_{n\in\mathbb{N}}$ of points of $P$, $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of open intervals, $\{J_n\}_{n\in\mathbb{N}}$ of compact intervals, and $\{I_n\}_{n\in\mathbb{N}}$ of open intervals having the same midpoint as the corresponding $J_n$, and contained in that corresponding $J_n$. The construction is inductive, and aimed at having all the objects obtained satisfy the conditions (a) through (f) listed below.

For the reminder of this proof let $\widetilde{f^{-1}}(A)$ stand for $\tilde{B}$, where $B = f^{-1}(A)$.

Start by choosing $x_0 \in Z$, $(a_0, b_0) = (x_0 - 1, x_0 + 1)$ and $I_0 = J_0 = \emptyset$. Assume that for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$, $i \le n$, it holds that:

**(a)** $f(x_i) \in I_i$;

**(b)** $J_{i-1} \cap J_i = \emptyset$ and, for $i > 2$,

$$|J_i| \le \frac{1}{3} \min\{\text{dist}(J_k, J_{k+1}) : k \in \mathbb{N},\ k < i - 1\};$$

**(c)** $|J_i| < \omega(f|_P, x_i)$ and $0 < |I_i| < 2^{-i}|J_i|$;

**(d)** $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$ and $|b_i - a_i| < 2^{-i}$;

**(e)** for every $k \in \mathbb{N}$, $2^i \le k < 4^i$,

$$\left(\frac{1}{b_i - x_i}(\widetilde{f^{-1}}(I_i) - x_i)\right) \cap \left(\frac{k}{4^i}, \frac{k+1}{4^i}\right) \ne \emptyset;$$

**(f)** for every $x \in [a_i, b_i]$ and every $k \in \mathbb{N}$, $2^{i-1} \le k < 4^{i-1}$,

$$\left(\frac{1}{b_{i-1} - x}\left(\widetilde{f^{-1}}(I_{i-1}) - x\right)\right) \cap \left(\frac{k}{4^{i-1}}, \frac{k+1}{4^{i-1}}\right) \ne \emptyset.$$

Let us present the inductive construction. Assume it is done for some $n \ge 0$. We will show the next step. Start with condition (f). If $n + 1 = 1$, (f) is void and can be ignored by defining $U = \mathbb{R}$. Otherwise, by (e), the set

$$U_k = \left\{x : \left(\frac{1}{b_n - x}((\widetilde{f^{-1}}(I_n)) - x\right) \cap \left(\frac{k}{4^n}, \frac{k+1}{4^n}\right) \neq \emptyset\right\}$$

contains $x_n$ for every $k \in \mathbb{N}$, $2^n \leq k < 4^n$. It is also not difficult to see that the sets $U_k$ are open. Therefore

$$U = \bigcap_{2^n \leq k < 4^n} U_k$$

is also open and contains $x_n$. It is easy to see that condition (f) is satisfied for $x \in U$.

Now, find

$$y \in P \cap f^{-1}(J_n^c) \cap ((a_n, b_n) \cap U).$$

The existence of such a $y$ is guaranteed because $U$ is open, $x_n \in U$ and (c). If $y \in Z$, let $x_{n+1} = y$. Otherwise $f|_P$ is continuous at $y$. In this case, the fact that $Z$ is dense in $P$ and $U$ is open guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}(J_n^c) \cap ((a_n, b_n) \cap U) \cap Z.$$

Since $f(x_{n+1}) \notin J_n$ and $x_{n+1} \in Z$, there exists a small interval $J_{n+1}$ centered at $f(x_{n+1})$ satisfying conditions (b) and (c). Choosing $I_{n+1}$ centered at $f(x_{n+1})$ of length

$$\frac{|J_{n+1}|}{2^{n+2}}$$

guarantees (a), (b), and (c).

Defining $(a'_{n+1}, b'_{n+1})$ to be centered at $x_{n+1}$ and such that

$$[a'_{n+1}, b'_{n+1}] \subset (a_n, b_n) \cap U \quad \text{and} \quad b'_{n+1} - a'_{n+1} < \frac{1}{2^{n+1}}$$

guarantees (d) and (f) for the interval $[a'_{n+1}, b'_{n+1}]$. However, it still must be shown that condition (e) is satisfied. This is done by choosing interval $(a_{n+1}, b_{n+1}) \subset (a'_{n+1}, b'_{n+1})$.

Note that $x_{n+1}$ is an $\mathcal{J}$-density point of $f^{-1}(I_{n+1})$. Therefore, by Lemma 2.1, there exists an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that the set

$$S = \liminf_{i \to \infty} (n_i(f^{-1}(I_{n+1}) - x_{n+1})) \cap (-1, 1)$$

is residual in $(-1, 1)$. Define

$$W_i = n_i(f^{-1}(I_{n+1}) - x_{n+1}).$$

The set

$$\bigcup_{r=1}^{+\infty} \bigcap_{i \geq r} W_i$$

is residual in $(-1,1)$. In particular, for every $k \in \mathbb{N}$, $2^{n+1} \le k < 4^{n+1}$,

$$\left(\bigcup_{r=1}^{+\infty} \bigcap_{i \ge r} W_i\right) \cap \left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \neq \emptyset.$$

The sequence $\{\bigcap_{i \ge r} W_i\}_{r \in \mathbb{N}}$ is increasing. Thus, there is an $r_0 \in \mathbb{N}$ such that

$$W_i \cap \left(\frac{k}{4^{n+1}}, \frac{k+1}{4^{n+1}}\right) \neq \emptyset$$

for every $i \ge r_0$ and $k \in \mathbb{N}$, $2^{n+1} \le k < 4^{n+1}$. But

$$W_i = n_i(\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}) = \frac{1}{x_{n+1} + \frac{1}{n_i} - x_{n+1}} (\widetilde{f^{-1}}(I_{n+1}) - x_{n+1}).$$

Define $(a_{n+1}, b_{n+1})$ as

$$\left(x_{n+1} - \frac{1}{n_i}, x_{n+1} + \frac{1}{n_i}\right),$$

where $i \ge r_0$ and $[a_{n+1}, b_{n+1}] \subset [a'_{n+1}, b'_{n+1}]$. The desired condition (e) is satisfied. This ends the inductive construction.

It will now be shown how the conclusion of the theorem follows from the construction.

Let

$$\{x\} = \bigcap_{n \in \mathbb{N}} [a_n, b_n] = \bigcap_{n \in \mathbb{N}} ([a_n, b_n] \cap Z).$$

We will show that $f$ is not deep-$\mathscr{I}$-density continuous at $x$. To be more specific, we will find a sequence $\{n_i\}_{i \in \mathbb{N}}$ such that

**(1)** $f(x)$ is a deep-$\mathscr{I}$-dispersion point of $\bigcup_{i \in \mathbb{N}} I_{n_i}$, and
**(2)** $x$ is not a deep-$\mathscr{I}$-dispersion point of $f^{-1}(\bigcup_{i \in \mathbb{N}} I_{n_i})$.

We will first show $x$ is not an $\mathscr{I}$-dispersion point of $f^{-1}(\bigcup_{i \in \mathbb{N}} I_{n_i})$ for every sequence $\{n_i\}_{i \in \mathbb{N}}$.

Let $\{n_i\}_{i \in \mathbb{N}}$ be any increasing sequence of natural numbers. By the definition of $x$, condition (f) implies

$$(t_n(\widetilde{f^{-1}}(I_n) - x)) \cap \left(\frac{k}{4^n}, \frac{k+1}{4^n}\right) \neq \emptyset$$

for every $k \in \mathbb{N}$, $2^n \le k < 4^n$, where $t_n$ is defined as $1/(b_n - x)$. Note that sequence $\{t_n\}_{n \in \mathbb{N}}$ is increasing and diverging to $\infty$. Thus, the open set $U_n = t_n(\widetilde{f^{-1}}(I_n) - x)$ intersects every interval $(\frac{k}{4^n}, \frac{k+1}{4^n}) \subset [\frac{1}{2}, 1]$. This implies that for every increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers and for every $s \in \mathbb{N}$ the set $\bigcup_{i \ge s} U_{n_i}$ is dense in $[\frac{1}{2}, 1]$. Hence,

$$\limsup_{j \to \infty} t_{n_{i_j}}\left(\widetilde{f^{-1}}\left(\bigcup_{i \in \mathbb{N}} I_{n_{i_j}}\right) - x\right) \supset \limsup_{j \to \infty} U_{n_{i_j}} \notin \mathscr{I}$$

for every subsequence $\{n_{i_j}\}_{j\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$. Thus, by Theorem 2.3 (ii), $x$ is not an $\mathscr{I}$-dispersion point of $f^{-1}(U_{i\in\mathbb{N}} I_{n_i})$.

Let us now turn to the proof of condition **(1)**. We must find an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $f(x)$ is a deep-$\mathscr{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}} I_{n_i}$. The set $\bigcup_{i\in\mathbb{N}} I_{n_i}$ is open. Therefore, it suffices to find a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $f(x)$ is an $\mathscr{I}$-dispersion point of $\bigcup_{i\in\mathbb{N}} I_{n_i}$. For the sake of simplicity, let us assume that $f(x) = 0$.

There are two cases to consider.

Case 1°. There exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that the $J_{n_i}$ are pairwise disjoint.

By taking a subsequence of $\{n_i\}_{i\in\mathbb{N}}$, if necessary, it may be assumed that

$$\bigcup_{i\in\mathbb{N}} J_{n_i}$$

is either a right or left interval set. For simplicity, assume it is a right interval set.

Let $J_{n_i} = [c_i, d_i]$ and $I_{n_i} = (\alpha_i, \beta_i)$. Then

$$f(x) = 0 < d_{i-1} < c_i < \alpha_i < \beta_i < d_i$$

for all $i$. Condition (c) states that

$$\frac{\beta_i - \alpha_i}{d_i - c_i} = \frac{|I_{n_i}|}{|J_{n_i}|} < \frac{1}{2^{n_i}}.$$

Let $z_n$ be the common center of $I_n$ and $J_n$, for $n \geq 0$. Then

$$\lim_{i\to\infty} \frac{\beta_i - \alpha_i}{\beta_i} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i}} \leq \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{z_{n_i} - c_i} = 2 \lim_{i\to\infty} \frac{\beta_i - \alpha_i}{d_i - c_i} = 0.$$

By Corollary 2.5 choose a subsequence of $\{n_i\}_{i\in\mathbb{N}}$ with the desired properties.

Case 2°. There is no pairwise disjoint subsequence $\{J_{n_i}\}_{i\in\mathbb{N}}$ of the sequence $\{J_n\}_{n\in\mathbb{N}}$.

Let us first consider the subsequence $\{J_{2n+1}\}_{n\in\mathbb{N}}$, indexed by the odd numbers, of the sequence $\{J_n\}_{n\in\mathbb{N}}$. Define a partition function $F\colon [\mathbb{N}]^2 \to \{0, 1\}$ by

$$F(\{n, m\}) = 1 \quad \text{if, and only if,} \quad J_{2n+1} \cap J_{2m+1} \neq \emptyset.$$

By Lemma 4.1 (Ramsey's Theorem) there exists an infinite homogeneous subset $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$; i.e., the sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that for some $k \in \{0, 1\}$, $F(\{n_i, n_j\}) = k$ for all positive integers $i \neq j$. But $k = 0$ would contradict the definition of the case 2°, which is currently considered. Thus $k = 1$; i.e.;

$$(9) \qquad J_{2n_i+1} \cap J_{2n_j+1} \neq \emptyset$$

for all nonnegative integers $i \neq j$.

Now let us repeat the Ramsey-type argument, which was used above, for the even-numbered counterparts of $\{J_{2n_i+1}\}_{i\in\mathbb{N}}$. Define $G : [\mathbb{N}]^2 \to \{0, 1\}$ by

$$G(\{i, j\}) = 1 \quad \text{if, and only if,} \quad J_{2n_i} \cap J_{2n_j} \neq \emptyset.$$

By Lemma 4.1 (Ramsey's Theorem) there exists a subsequence $\{n_{i_s}\}_{s\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$ such that

(10) $\quad J_{2n_{i_s}} \cap J_{2n_{i_t}} \neq \emptyset$

for all nonnegative integers $s \neq t$, while condition (9) is still preserved, or more precisely

(11) $\quad J_{2n_{i_s}+1} \cap J_{2n_{i_t}+1} \neq \emptyset$

for $s \neq t$. Define $\varepsilon = \operatorname{dist}(J_{2n_{i_0}}, J_{2n_{i_0}+1})$. By (b), $\varepsilon > 0$. Moreover, by (b), (10) and (11)

$$B_0 = \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}} \subset \left\{x : \operatorname{dist}(x, J_{2n_{i_0}}) < \frac{\varepsilon}{3}\right\}$$

and

$$B_1 = \bigcup_{s\in\mathbb{N}} J_{2n_{i_s}+1} \subset \left\{x : \operatorname{dist}\left(x, J_{2n_{i_0}+1}\right) < \frac{\varepsilon}{3}\right\}.$$

Hence

$$\operatorname{dist}(B_0, B_1) \geq \frac{\varepsilon}{3} > 0.$$

Note that

$$S_0 = \bigcup_{s\geq 0} I_{2n_{i_s}} \subset B_0$$

and

$$S_1 = \bigcup_{s\geq 0} I_{2n_{i_s}+1} \subset B_1.$$

Thus $\operatorname{dist}(S_0, S_1) > 0$, which implies that either

$$\operatorname{dist}(f(x), S_0) > 0$$

or

$$\operatorname{dist}(f(x), S_1) > 0.$$

This clearly means that $f(x)$ is an $\mathscr{I}$-dispersion point of either $S_0$ or $S_1$.
This finishes the proof of Theorem 4.2.

Since every function in $\mathscr{B}_1^*$ is continuous in the ordinary sense on a dense open set, the following corollary is obvious.

**Corollary 4.3.** *If $f$ belongs to $\mathscr{C}_{\mathscr{D}\mathscr{D}}$ or $\mathscr{C}_{\mathscr{I}\mathscr{I}}$, then there is a dense $\mathscr{T}_{\mathscr{C}}$-open set $G$ such that $f|_G$ is $\mathscr{T}_{\mathscr{C}}$-continuous.*

**Theorem 4.4.** *The spaces $\mathscr{C}_{\mathscr{D}\mathscr{D}}$ and $\mathscr{C}_{\mathscr{I}\mathscr{I}}$ equipped with the topology of uniform convergence, are of the first category in themselves.*

*Proof.* This will only be proved for the class $\mathscr{C}_{\mathscr{D}\mathscr{D}}$ as the other case is essentially the same.

Let $\{I_n\}_{n\in\mathbb{N}}$ be the sequence of all open intervals with rational endpoints and let $C_n$ be the family of all deep $\mathscr{I}$-density continuous functions that are continuous on $I_n$ in the ordinary sense. By Theorem 4.2, $\mathscr{D}_{\mathscr{D}\mathscr{D}} = \bigcup_{n\in\mathbb{N}} C_n$. Also, it is evident that the sets $C_n$ are closed in the topology of uniform convergence. Finally, for any function $g \in C_n$ and any of its neighborhoods $U$ it is easy to slightly modify the function $f$ from Example 2.6 in such a way that

$$f \in U \cap (C_n \setminus \mathscr{C}_{\mathscr{D}\mathscr{D}}).$$

Thus, the sets $C_n$ are nowhere dense.

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Krzysztof Ciesielski, Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

Lee Larson, Department of Mathematics, University of Louisville, Louisville, KY 40292, USA