Theorem. Let G be a reductive affine algebraic group. Let B be a Borel subgroup of G and let K be a maximal compact subgroup of G. Then $G = KB$.

Proof. We may assume that $G \subset GL(n, \mathbb{C})$ is a symmetric subgroup. We take $K = U(n) \cap G$ as usual. First consider B constucted from H as above such that $H \cap K$ is a maximal compact torus, T, of K. We can assume that $G \subset GL(n, \mathbb{C})$ and the elements of B are upper trangular. We note

$$K \cap B = T.$$

The elements of

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi, \alpha(h) > 0} \mathfrak{g}_\alpha$$

are nilpotent thus

$$U = \exp \mathfrak{n}$$

is unipotent. Thus

$$B = HU.$$

Let

$$\mathfrak{a} = iLie(T), A = \exp \mathfrak{a}.$$

Then we assert that the map

$$K \times A \times U \to G$$

$$k, a, u \longmapsto kau$$

is bijective. If

$$kau = k_1 a_1 u_1$$

then

$$k_1^{-1} k \in AU.$$

But this implies that $k_1^{-1}k$ has real posoitive eigenvalues. Hence $k = k_1$. similarly $a = a_1$. Thus the map is injective. Note if $X \in Lie(K), h \in Lie(A)$ and $u \in Lie(U)$ then

$$\frac{d}{dt_{t=0}} ke^{tX} ae^{th} ue^{tY} = kXau + kahu + kauY.$$

If this expression is 0 then

$$X = -aha^{-1} - auY\,(au)^{-1}$$

which is upper triangular with real eigenvalues. Thus $X = 0$. So

$$h = -uYu^{-1}$$

since Y is nilpotent and h is semisimple $h = 0$. Note

$$\dim AN = \dim K, \dim_{\mathbb{R}} G = 2 \dim K$$

Indeed, if $X \in \mathfrak{g}_\alpha$ then $X^* \in \mathfrak{g}_{-\alpha}$. Also

$$Lie(K) = Lie(T) \oplus \{X - X^* | X \in Lie(N)\}.$$

Thus $\dim K = \dim T + \dim_{\mathbb{R}} Lie(N) = \dim AN$.

But AN is closed in G so KAN is closed. Hence the image is open and closed. Since K intersects every connected component of G. $G = KAN$.

Suppose B_1 is another Borel subgroup. Then there exists $g \in G$ with $gBg^{-1} = B_1$. Thus writing

$$g = kb, k \in K$$

and $b \in B$ then

$$kBk^{-1} = B_1.$$

Hence

$$k^{-1}KB_1k = KB = G.$$

■

Exercises.1. Show that the theorem is true without the condition the G is reductive.

2. Let G be a reductive algebraic group and let K be a maximal compact subgroup. If B is a Borel subgroup of G then $K \cap B$ maximal compact torus in K.

Let $G \subset GL(n, \mathbb{C})$ be a reductive affine algebraic group. We set $V = \mathbb{C}^n$ and let H be a Cartan subgroup of G which we can assume is contained in the diagonal subgroup. We have seen that if $\mathcal{N}_H(V)$ is the nullcone for the H–action then the null cone for the G action is

$$\mathcal{N}_G(V) = G\mathcal{N}_H(V).$$

Also if $A \in M_{n,m}(\mathbb{Z})$ defines the isomorphism between $\left(\mathbb{C}^\times\right)^m$ and H

$$(z_1, ..., z_m) \longmapsto diag(z^{A_1}, ..., z^{A_n})$$

with A_i the i–th column of A. We saw that if $b \in \mathbb{Z}^m$ and $V_b = \{v = (v_1, ..., v_n) | v_i = 0 \text{ if } A_i b \leq 0\}$ then if

$$\mathcal{M} = \{V_b | V_b \text{ of maximal dimension}\}$$

the irreducible components of $\mathcal{N}_H(V)$ are the elements of $\mathcal{M}$.

Theorem. The subsets GV_b are Z–closed in $\mathbb{C}^n$.

Proof. We may assume that G is a symmetric subgroup of $GL(n, \mathbb{C})$ and that H is the Z–closure of a maximal torus of $K = G \cap U(n)$. Let $h \in Lie(H)$ be such that

$$e^{zh} = (e^{A_1 bz}, ..., e^{A_n bz}).$$

That is if

$$\varphi_b(z) = (e^{A_1 bz}, ..., e^{A_n bz})$$

Then

$$h = d\varphi_b([1]).$$

We note that $h \in \mathfrak{a} = iLie(T)$. Let $h_1 = h, h_2, ..., h_m$ be a basis of $\mathfrak{a}$ and order the $\mathfrak{a}^*$ lexicographically realative to

$$(\lambda(h_1), ..., \lambda(h_m)).$$

Then then the elements of the root system, Φ, such that $\alpha > 0$ form a system , Φ^+, of positive roots. Decompose V into eigenspaces for h so

$$V = \bigoplus V^\lambda$$

with

$$h_{|V^\lambda} = \lambda I.$$

$$V_b = \sum_{\lambda > 0} V^\lambda.$$

If $\alpha \in \Phi^+$ and $X \in \mathfrak{g}_\alpha$ and $v \in V^\lambda$

$$hXv = (\alpha(h) + \lambda)\, Xv$$

since $a(h) \geq 0$ we see that $Xv \in V_b$. This implies that if B is the Borel subgroup corresponding to Φ^+ then

$$Lie(B)V_b \subset V_b.$$

Thus

$$GV_b = KV_b.$$

Which is closed in the metric topology. We define

$$\mathbf{\Psi} : G \times V_b \to V$$

by

$$\mathbf{\Psi}(g, v) = gv.$$

Then $\mathbf{\Psi}$ is a morphism of varieties. Hence the Z–closure of the image of $\mathbf{\Psi}$ is the same as the metric closure. Hence GV_b is Z–closed. ∎

Let G be an affine algebraic group. Let (ρ, V) be a regular representation of G.Then

$$\rho : G \to GL(V)$$

is a homomorpism of algebraic groups. We consider $X \in Lie(G)$. Then

$$\phi(t) = \rho(e^{tX})$$

is a real analytic map of $\mathbb{R}$ to $GL(V)$. We note that

$$\phi(s+t) = \phi(s)\phi(t).$$

Let

$$Y = \frac{d}{dt_{|t=0}}\phi(t).$$

Then

$$\frac{d}{dt}\phi(t) = Y\phi(t)$$

Then

$$\frac{d}{dt}e^{-tY}\phi(t) = 0.$$

We set $d\rho(X) = Y$.

Let G be a connected, reductive, affine algebraic group and let H be a Cartan subgroup. Let B be a Borel subgroup of G containig H and let U be the unipotent radical of B. Let $\mathfrak{h} = Lie(H)$. Then there is a system of positive roots Φ^+ such that

$$Lie(B) = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \right)$$

If we chose $-\Phi^+$ we would have an opposite Borel subgroup. We set

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \pm\Phi^+} \mathfrak{g}_\alpha .$$

One checks that if $U^\pm = \exp(\mathfrak{n}^\pm)$ then $U^- B$ is open in G.

Let (ρ, V) be an irreducible regular representation of G. If $v \in V$ and $v \neq 0$ then the span of $\rho(Q)v$ is V for any non-empty open subset, Q, of G. Let $BV^{U^+} = V^{U^+}$thus there exists a regular homomorphism

$$\chi : B \to \mathbb{C}^\times$$

and $v \neq 0$ in V^{U^+} such that

$$\rho(b)v = \chi(b)v.$$

This implies that

$$Span_{\mathbb{C}}\rho(U^-)v = V.$$

Let Λ be the correponding weight of the action of $\mathfrak{h}$ on V. If we expand

$$e^{d\rho(\sum_{a\in\Phi^+} X_{-\alpha})}v$$

then the only weights that appear are $\Lambda - Q$ with Q a sum of positive roots. We therefore see that the Λ weight space is $[v]$ and all the other roots are of the form $\Lambda - Q$ with Q a sum of positive roots.

We note that since χ defines a character of T we must have Λ is real values on $\mathfrak{a} = iLie(T)$.Suppose that V^U contains another weight space, $\mu \neq \Lambda$. Let $h \in \mathfrak{a}$ be such that $\alpha(h) > 0$ for all $\alpha \in \Phi^+$. Then since $\mu = \Lambda - Q$ with Q a sum of positive roots $\Lambda(h) > \mu(h)$ but Λ has a similar expression in terms of μ.

At this point we have proved

Theorem. Let (ρ, V) be an irreducible regular representation of G.

1. $\dim V^{U^+} = 1$.

2. Let Λ be the weight of $\mathfrak{h}$ on V^{U^+} then every weight of V is of the form $\Lambda - Q$ with Q a sum of postive roots.

3. $V = V^{U^+} \oplus d\rho(\mathfrak{n}^-)V$.

The last assertion follows from

$$span_{\mathbb{C}} e^{d\rho(\mathfrak{n}^-)} V^{U^+} = V$$

and 1. and 2.

We call Λ the highest weight of (ρ, V).

Corollary. Let (ρ, V) be a regular repesentation of G then V is irreducible if and only if $\dim V^{U^+} = 1$.

Proof. If $V = V_1 \oplus V_2 \oplus ... \oplus V_m$ with V_i irreducible. Then 1. implies $m = \dim V^{U^+}$. ■

If χ is a regular character of H we extend χ to $B^- = HU^-$ by $\chi(U^-) = 1$. Let F^χ the the space of all regular functions

$$f : G \to \mathbb{C}$$

such that

$$f(b^- g) = \chi(b^-) f(g), b^- \in B^-, g \in G.$$

We define an action of G on F^χ by $gf(x) = f(xg)$. We note that F^χ has a filtration by finite dimensional G–invariant subspaces. Consider $(F^\chi)^{U^+}$. We note that $U^- H U^+$ is open in G (at least in the metric topology). Since G is connected this implies that if $f \in F^\chi$ then f is determined by its restriction to $U^- H U^+$. But if $f \in (F^\chi)^{U^+}$ then

$$f(u^- h u^+) = \chi(h) f(e).$$

Hence

$$\dim\left(F^{\chi}\right)^{U^{+}} \leq 1.$$

Thus

Lemma. F^{χ} defines an regular representation of G that is either 0 or irreducible.

Let (ρ, V) be an irreducible representaion with highest weight Λ and let χ be the corresponding character of H. We define

$$T : V \to F^{\chi}$$

as follows. Let $p : V \to V/d\rho(\mathfrak{n}^{-})V$ be the natural surjection. Set

$$T(v)(g) = p(\rho(g)v).$$

Then

$$T(\rho(x)v)(g) = p(\rho(g)\rho(x)v) =$$

$$p(\rho(gx)v) = T(v)(gx) = \left(xT(v)\right)(g).$$

Theorem. If (ρ, V) is an irreducible regular representation with highest weight Λ relative to B^+ and if χ is the corresponing character of H then (ρ, V) is equivalent with F^χ.

Using TDS theory one can show that id Λ is a highest weight of an irreducible regular representation of G then

$$\Lambda(\check{\alpha}) \in \mathbb{Z}_{\geq 0}$$

for all $\alpha \in \Phi^+$. Such a χ will be called dominant.

Theorem. $F^\chi \neq 0$ if χ is dominant.

Example. $G = GL(n, \mathbb{C})$. We take $B = B_n$ the upper triangular elements of G and H to be the diagonal elements. Let ε_i the ii component of an element of H. Then a regular character of H is a product

$$\varepsilon_1^{m_1} \varepsilon_2^{m_2} \cdots \varepsilon_n^{m_n}$$

with $m_i \in \mathbb{Z}$. The character is dominant if and only if

$$m_1 \geq m_2 \geq ... \geq m_n.$$

This implies that

$$\Lambda = \sum m_i\varepsilon_i = \sum_{i=1}^{n-1}(m_i - m_{i+1})(\varepsilon_1 + ... + \varepsilon_i) + m_n(\varepsilon_1 + ... + \varepsilon_n).$$

Consider $(\sigma, Z) =$

$$\left(\otimes^{m_1-m_2}\mathbb{C}^n\right)\otimes\left(\otimes^{m_2-m_3}\wedge^2\mathbb{C}^n\right)\otimes\cdots\otimes\left(\otimes^{m_{n-1}-m_n}\wedge^{n-1}\mathbb{C}^n\right)$$

If we set

$$\rho(g) = \det(g)^{m_n}\sigma(g)$$

and $V = Z$ then V^{U^+} has the weight χ with multiplicty 1.This proves the theorem for $GL(n, \mathbb{C})$.