School of Electrical Engineering & Telecommunications

ELEC2015 - Electromagnetic Applications (PART B)

Tutorial 2 - Method of images, boundary value problems and steady electric currents

Method of images

1. A point charge Q exists at a distance d above a large grounded conducting plane. Determine,

   (a) the surface charge density ρ_S, in terms of d and the distance r from the origin which is directly below the charge Q on the ground plane.
   (b) the total charge induced on the conducting plane.

   [Ans. (a) $\rho_s = -\dfrac{Qd}{2\pi\left(d^2 + r^2\right)^{3/2}}$; (b) $-Q$]

2. Using the expression derived for the capacitance of a parallel 2-wire transmission line, use the method of images to determine an expression for the capacitance per unit length between a wire conductor of radius a and an earth plane (assumed perfectly conducting). The height of the wire above the earth plane is h meters.

   If the medium between the line and the plane was a dielectric with conductivity σ, derive a formula for the resistance per unit length between line and plane.

   [Ans. $C = \dfrac{2\pi\varepsilon}{ln\dfrac{2h-a}{a}}$ F/m; $R = \dfrac{1}{2\pi\sigma} ln\dfrac{2h-a}{a}$ Ω/m]

3. Determine the capacitance per unit length of a 2-wire transmission line with parallel conducting cylinders of different radii a_1 and a_2, their axes being separated by a distance D (where $D > a_1 + a_2$).

   [Ans. $C = \dfrac{2\pi\varepsilon_o}{ln\dfrac{D^2 - \left(a_1^2 - a_2^2\right)}{a_1 a_2}}$]

Boundary value problems

4. The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance d and maintained at potentials V_O and 0 respectively. A dielectric slab of dielectric constant $\varepsilon_{rd} = 6$ and uniform thickness $0.8d$ is placed over the lower plate. Assume negligible fringing. Using Laplace's equation, determine

   (a) the potential and electric field distribution in the dielectric slab,
   (b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate,
   (c) the surface charge densities on the upper and lower plates
   (d) compare the results in part (b) with those without the dielectric slab.

[Ans. (a) $V_d = \frac{5V_o y}{(4+\varepsilon_{rd})d}$; $\bar{E}_d = -\frac{5V_o}{(4+\varepsilon_{rd})d}\bar{a}_y$; (b) $V_a = \frac{5\varepsilon_{rd} y - 4(\varepsilon_{rd} - 1)d}{(4+\varepsilon_{rd})d} V_o$;

(c) $\rho_s\big|_{y=d} = \frac{5\varepsilon_o \varepsilon_{rd} V_o}{(4+\varepsilon_{rd})d}$; $\rho_s\big|_{y=0} = -\frac{5\varepsilon_o \varepsilon_{rd} V_o}{(4+\varepsilon_{rd})d}$]

5. Consider an enclosure with a rectangular cross section formed by four conducting plates. The left and right hand planes are grounded, and the top and bottom plates are maintained at constant potentials of V_1 and V_2 respectively. Determine the potential distribution inside the enclosure.

[Ans. $V(x,y) = \sum \sin\frac{n\pi}{a}x\left[A_n \sinh\frac{n\pi}{a}y + B_n \cosh\frac{n\pi}{a}y\right]$;

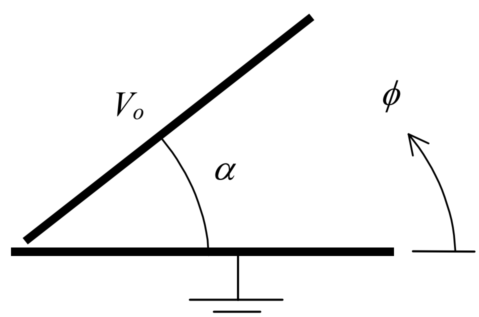
$$B_n = \frac{4V_2}{n\pi} \quad for\ n = odd$$
$$= 0 \quad for\ n = even$$

$$A_n = \frac{4}{n\pi \sinh\frac{n\pi}{a}b}\left(V_1 - V_2\cosh\frac{n\pi}{a}b\right) \quad for\ n = odd$$
$$= 0 \quad for\ n = even$$]

6. Two infinite plates, insulated from each other, are maintained at potentials 0 and V_o as shown in the figure below. Determine the potential distributions for the regions

   (a) $0 < \phi < \alpha$ and
   (b) $\alpha < \phi < 2\pi$.

   [Ans. (a) $V(\phi) = \frac{V_1}{\alpha}\phi$ for $0 \le \phi \le \alpha$; (b) $V(\phi) = \frac{V_o}{2\pi - \alpha}(2\pi - \phi)$ for $0 \le \phi \le 2\pi$]



Steady electric currents

7. A long, round wire of radius a and conductivity σ is coated with a material of conductivity 0.1σ.

   (a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50%?
   (b) Assuming a total current I in the coated wire, find the $\boldsymbol{J}$ and $\boldsymbol{E}$ in both the core and the coating material. What are the directions of $\boldsymbol{J}$ and $\boldsymbol{E}$?

   [Ans. (a) $b = 2.23a$; (b) $\boldsymbol{J}_1 = 10\boldsymbol{J}_2$ and $\boldsymbol{E}_1 = \boldsymbol{E}_2$]

8. Lightning strikes a lossy dielectric sphere of $\varepsilon = 1.2\varepsilon_o$, $\sigma = 10$ (S/m) and radius 0.1 m at $t = 0$, depositing uniformly in the sphere a total charge of 1 mC. Determine for all t,

   (a) the electric field both inside and outside the sphere,
   (b) the current density in the sphere,
   (c) the time it takes for the charge density in the sphere to diminish to 1% of its initial value,
   (d) the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its final value. What happens to this energy?
   (e) the electrostatic energy stored in space outside the sphere.

   [Ans. (a) $\overline{E}_i = \overline{a}_r 7.5\, R e^{-9.42\times 10^{11} t}$ V/m for $R < b$; $\overline{E} = \overline{a}_r \dfrac{9}{R^2} \times 10^6$ V/m

   (b) $\overline{J}_i = \overline{a}_r 7.5 \times 10^{10}\, R e^{-9.42\times 10^{11} t}$ A/m^2 for $R < b$; $\overline{J}_o = 0$ for $R > 0$.

   (c) $t = 4.88 \times 10^{-12}$ sec ; (d) $\dfrac{W}{W_o} = 10^{-4}$; (e) $W = 45kJ$]

9. A 40 km, two-wire transmission line is shorted at one end and a voltage of 5 volts is applied at the other end between the two conductors. The radius of each wire 1.5 cm. The current drawn by the shorted line from the dc source is 1.56 A. Calculate

   (a) conductivity of the wire,
   (b) the electric filed intensity in the conductors,
   (c) the power dissipated in the line,
   (d) the electron drift velocity in the conductors assuming that the electron mobility of a good conductor is 1.4×10^{-3} m^2/V.s.
   [Ans. (a) $\sigma = 3.53\times10^7$ S/m ; (b) E = $6.25\times^{-5}$ V/m; (c) 7.8 Watts; (d) $8.75\times^{-9}$ m/sec]

10. A voltage V_o is applied across a parallel-plate capacitor of area S. The space between the plates is filled with two different dielectrics of thickness d_1 and d_2, permittivities ε_1 and ε_2 and conductivities σ_1 and σ_2 respectively. Calculate

    (a) the current densities between the plates,
    (b) the electric field intensities between the plates,
    (c) the surface charge densities at the plates and at the interface.
    (d) Draw the equivalent circuit of the lossy capacitor

    If the voltage V_o is applied at time $t = 0$, calculate

    (e) the surface charge density as a function of time
    (f) the electric field intensities E_1 and E_2 as functions of time t.

    [Ans. (a) $J = \dfrac{\sigma_1\sigma_2 V}{d_1\sigma_2 + d_2\sigma_1}$ A/m^2;

    (b) $E_1 = \dfrac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$ V/m; $E_2 = \dfrac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}$ V/m

    (c) $\rho_{s1} = \dfrac{\varepsilon_1\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$ C/m^2 ; $\rho_{s2} = \dfrac{\varepsilon_2\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}$ C/m^2;

    $\rho_{si} = \dfrac{(\varepsilon_2\sigma_1 - \sigma_2\varepsilon_1)V}{\sigma_2 d_1 + \sigma_1 d_2}$ C/m^2

    (d) $R_1 = \dfrac{d_1}{\sigma_1 S}$; $C_1 = \dfrac{\varepsilon_1 S}{d_1}$; $R_2 = \dfrac{d_2}{\sigma_2 S}$; $C_2 = \dfrac{\varepsilon_2 S}{d_2}$

(e) $\rho_{si} = \dfrac{(\varepsilon_2\sigma_1 - \sigma_1\varepsilon_2)V}{\sigma_2 d_1 - \sigma_1 d_2}\left(1 - e^{-t/\tau}\right)$ C/m^2 where $\tau = \dfrac{\varepsilon_2 d_1 + \varepsilon_1 d_2}{\sigma_2 d_1 + \sigma_1 d_2}$ sec

(f) $E_1 = \dfrac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}\left(1 - e^{-t/\tau}\right) + \dfrac{\varepsilon_2 V}{\sigma_2 d_1 + \sigma_1 d_2} e^{-t/\tau}$ V/m

$E_2 = \dfrac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}\left(1 - e^{-t/\tau}\right) + \dfrac{\varepsilon_1 V}{\sigma_2 d_1 + \sigma_1 d_2} e^{-t/\tau}$ V/m]

11. A two-wire underground power transmission line has two conductors separated by a dielectric material of relative dielectric constant $\varepsilon_r = 4$ and conductivity $\sigma = 5\times10^{-15}$ S/m. The separation d between the conductors is 0.2 m. Each conductor, made of copper, has radius of $a = 15$ mm. Assuming uniformly distributed charge on the conductors,

    (a) Find the capacitance of the line per meter.
    (b) Find the leakage resistance of the line per meter.

    [Ans: 5 pF/m, 1.41×10^{15} Ω]

12. A circular metallic annulus ring, as indicated in figure below, of conductivity $\sigma = 5\times10^{5}$ S/m and thickness $h = 25$ mm has outer radius $b = 35$ mm and inner radius $a = 20$ mm. Using Laplace's equation,

    (a) find the resistance between the inner and the outer cylindrical surfaces
    (b) find the resistance between the two flat surfaces.

    [7.12×10^{-6} Ω, 1.93×10^{-5} Ω]