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Exercise sheet 12

1. (**Sums of convex sets**). Let X be a locally convex vector space, let $K \subset X$ be a non-empty compact convex set and let $A \subset X$ be a non-empy closed convex subset. Let α, β be scalars. Show that $\alpha A + \beta K$ is closed[1] and convex.

HINT: For simplicity you may assume that X is metrizable. Using filters or nets this assumption is not needed.

2. (**Generalization to complex normed spaces**). Let X be a locally convex complex (!) vector space.

a) We generalize Theorem 8.73 to convex subsets of X. So let $K \subset X$ be closed and convex and suppose that $z \in X \setminus K$. Prove that there is a continuous linear functional ℓ on X and a constant $c \in \mathbb{R}$ with

$$\mathrm{Re}(\ell(y)) \leq c < \mathrm{Re}(\ell(z))$$

for all $y \in K$.

HINT: Look at the proof of Theorem 7.3.

b) Assume that X is normed. Generalize Corollary 8.74 to convex subsets of X.

3. (**Separation of convex sets – I**). Let X be a locally convex real vector space and let $A, K \subset X$ be two disjoint non-empty closed convex subsets.

a) Suppose that K is compact. Show that there is a continuous linear functional on X and a constant $c \in \mathbb{R}$ so that

$$\ell(y) \leq c < \ell(z)$$

for all $y \in A$ and $z \in K$.

HINT: Use Exercise 1.

[1] The sum of two closed convex subsets does not need to be closed – see Exercise 5, Sheet 1.

Turn the page.

b) We now want to explain why it is not sufficient to assume that K is closed. Consider the real Banach space $X = L^2_\lambda([0,1])$ where λ is the Lebesgue measure and the subsets

$$A = \{f \in L^2_\lambda([0,1]) : f \geq 1 \text{ almost everywhere}\},\ B = \{bx : b \in \mathbb{R}\}.$$

Show that there is no non-zero continuous linear functional ℓ on $L^2_\lambda([0,1])$ with $\ell(y) \leq c \leq \ell(z)$ for all $y \in A$ and $z \in B$ and some constant $c \in \mathbb{R}$.

HINT: Given $f \in L^2_\lambda([0,1])$ and $\epsilon > 0$ there is $g \in L^2_\lambda([0,1])$ with $\|g\|_2 \leq \epsilon$ and $f + g \geq 1 - bx$ for some constant b.

4. (**Separation of convex sets - II**). Let X be a locally convex real vector space and let $A, B \subset X$ be two disjoint non-empty closed convex subsets. Suppose that A has non-empty interior. Show that there is a non-zero continuous linear functional ℓ on X and a constant $c \in \mathbb{R}$ so that $\ell(y) \leq c \leq \ell(z)$ for all $y \in A$ and $z \in B$.

5. (**Density of extremal points**). In this exercise we want to exhibit a compact convex subset K of $C_\mathbb{R}([0,1])$ whose extreme points are dense in the subset. Consider

$$K = \{f \in C([0,1]) : f(0) = 0,\ f \text{ is 1-Lipschitz}\}$$

a) Show that K is indeed compact and convex (in the norm topology).

HINT: Use the theorem of Arzelà-Ascoli.

b) Prove that a function $f \in K$ which is piecewise linear and has slope ± 1 (wherever f is differentiable) is extremal.

c) Show that the functions f as in b) are dense in K.

HINT: For $g \in K$ and $\epsilon > 0$ one can start to construct a function f as in b) by proceeding with slope 1 (starting from 0) until $|g(x) - x| = \epsilon$.

6. (**Density in the bidual**). Let X be a real Banach space and consider the canonical embedding $\iota : X \to X^{**}$ (see Corollary 7.9). Show that $\iota(\overline{B_1^X})$ is dense in $\overline{B_1^{X^{**}}}$.

HINT: Let K be the weak*-closure of $\iota(\overline{B_1^X})$, which is a compact convex subset, and assume that there is $\ell \in \overline{B_1^{X^{**}}} \setminus K$. Apply Theorem 8.73 for the weak*-topology on X^{**}. See also Lemma 8.13.