Exercise Sheet 1

Exercise 1 (Product of topological groups)**.** Let A be a set and for every $\alpha \in A$, G_α a topological group. Show that

$$G := \prod_{\alpha \in A} G_\alpha$$

with the product topology[1] is a topological group.

Exercise 2 ($\mathrm{O}(p,q)$)**.** We consider the orthogonal group $\mathrm{O}(p,q)$ of signature $p, q \geq 1$.

a) Show that the connected component of the group $O(1,1)$ containing the identity is homeomorphic to $\mathbb{R}$.

b) Show that for all $p, q \geq 1$, $\mathrm{O}(p,q)$ has a subgroup isomorphic to $\mathbb{R}$.

Exercise 3 (Compact-Open Topology)**.** Let X, Y, Z be a topological space, and denote by $C(Y,X) := \{f : Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y,X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K,U) := \{f \in C(Y,X) \mid f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.

If Y is locally compact[2] , then:

a) The evaluation map $e : C(Y,X) \times Y \to X, e(f,y) := f(y)$, is continuous.

b) A map $f : Y \times Z \to X$ is continuous if and only if the map

$$\hat{f} : Z \to C(Y,X), \quad \hat{f}(z)(y) = f(y,z),$$

is continuous.

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[1] A basis of the product topology is given by the sets $\prod U_\alpha$, where U_α open and $U_\alpha = G_\alpha$ for all but finitely many $\alpha \in A$.

[2] A subset $C \subseteq Y$ that contains an open subset $U \subseteq Y$ with $y \in U \subseteq C \subseteq Y$ is called a *neighborhood of* $y \in Y$. Then Y is called *locally compact* if for every $y \in Y$ there is a set $\mathcal{D}$ of compact neighborhoods of y such that every neighborhood of y contains an element of $\mathcal{D}$ as a subset.

Exercise 4 (General Linear Group $\mathrm{GL}(n,\mathbb{R})$)**.** The general linear group

$$\mathrm{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \,|\, \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$\begin{aligned} j\colon\ \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{Homeo}(\mathbb{R}^n), \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n,\mathbb{R}^n)$ is endowed with the compact-open topology.

b) If we identify $\mathrm{GL}(n,\mathbb{R})$ with its image $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$. Hint: Exercise 3 can be useful here.

Exercise 5 (Isometry Group $\mathrm{Iso}(X)$)**.** Let (X,d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x),f(y)) = d(x,y) \quad \text{for all } x,y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem, see Appendix A.2 in Prof. Alessandra Iozzi's book.

Exercise 6 ($\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact.)**.** Let $\mathbb{S}^1 \subseteq \mathbb{C}\setminus\{0\}$ denote the circle. Show that $\mathrm{Homeo}(\mathbb{S}^1)$ with the compact-open topology is not locally compact.

Exercise 7 (Coverings of topological groups)**.** Let H be a topological group, G a topological space and $p\colon G \to H$ a covering[3]. Assume that both H and G are path-connected and locally path-connected. Show that for every $\tilde{e} \in p^{-1}(e_H)$ there is a unique topological group structure on G such that $\tilde{e}$ is the neutral element and p is a group homomorphism.

Hint: You may use the *lifting criterion*: If $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ is a covering and $f\colon (Y,y_0) \to (X,x_0)$ is a continuous map, where Y is path-connected and locally path-connected, then there is a unique continuous lift $\tilde{f}\colon (Y,y_0) \to (\tilde{X},\tilde{x}_0)$ of f, i.e. $p\circ\tilde{f} = f$, if and only if $f_\star(\pi_1(Y,y_0)) \subseteq p_\star(\pi_1(\tilde{X},\tilde{x}_0))$.

[3]A *covering* $p\colon G \to H$ is a continuous map such that for every $h \in H$ there is an open neighborhood $U_h \subseteq H$ and a discrete space D_h such that $p^{-1}(x) = \amalg_{d\in D_h} V_d$ and for every $d \in D_h$, $p|_{V_d}\colon V_d \to U_h$ is a homeomorphism.