Particle Cosmology and Baryonic Astrophysics
Part I

Dr. Michael Schmidt

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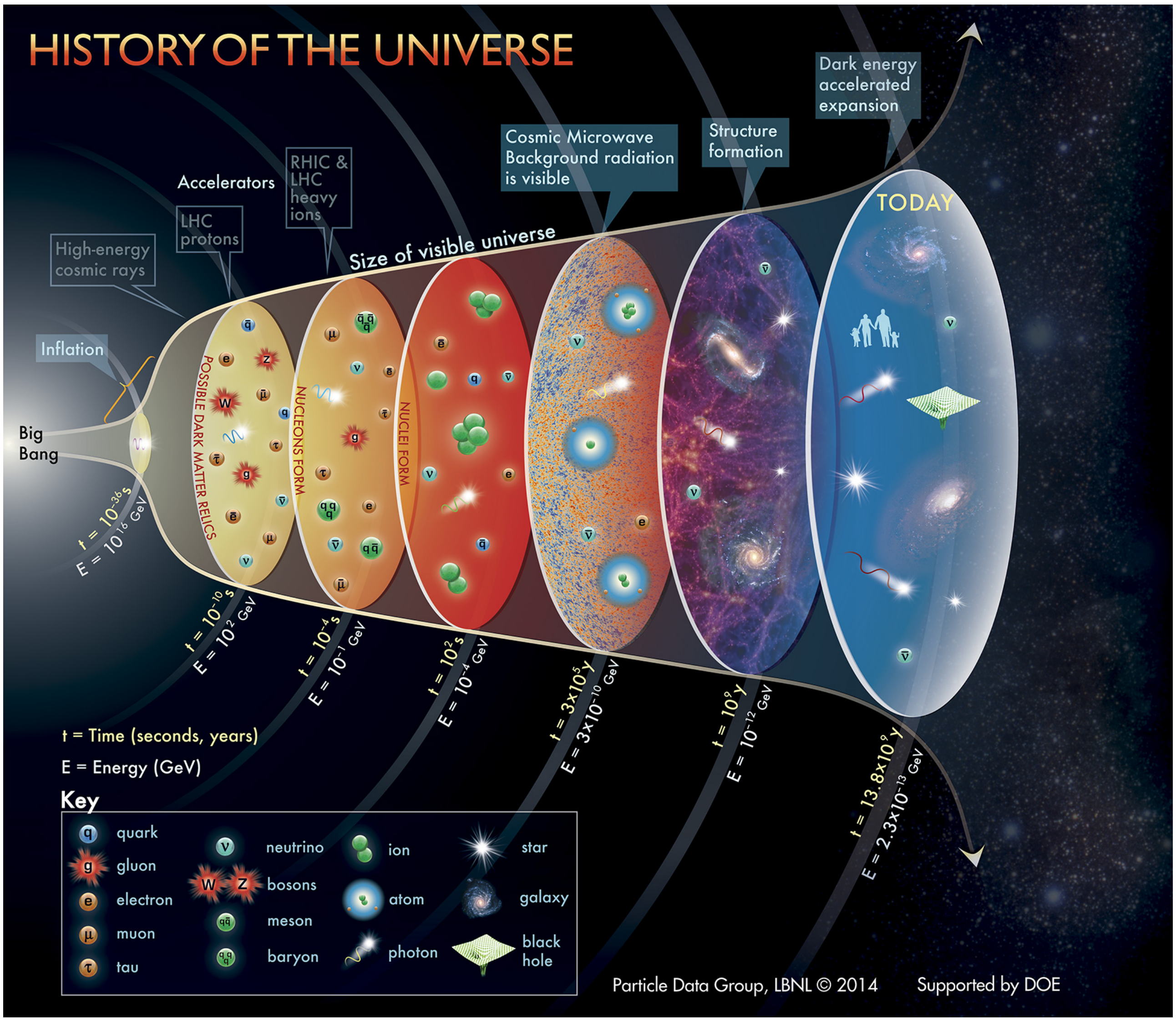


Contents

The first part of the lecture will follow Dodelson, Modern Cosmology [1] very closely. See also the lecture notes on cosmology by Daniel Baumann [2]. Throughout the notes, we will use natural units:

$$\hbar = c = k_B = 1 \ .$$

1 The Basic Ingredients of the Universe

See also Dodelson, Modern Cosmology [1], chapter 2, lectures notes by Daniel Baumann [2], chapter 1, and Kolb/Turner, The Early Universe [3], chapter 1-3.

Hubble discovered in 1929[4] that distant galaxies are moving away from us. His observation is shown in Fig. 1. From this diagram, we can extract the slope, called *Hubble rate* H_0, today,

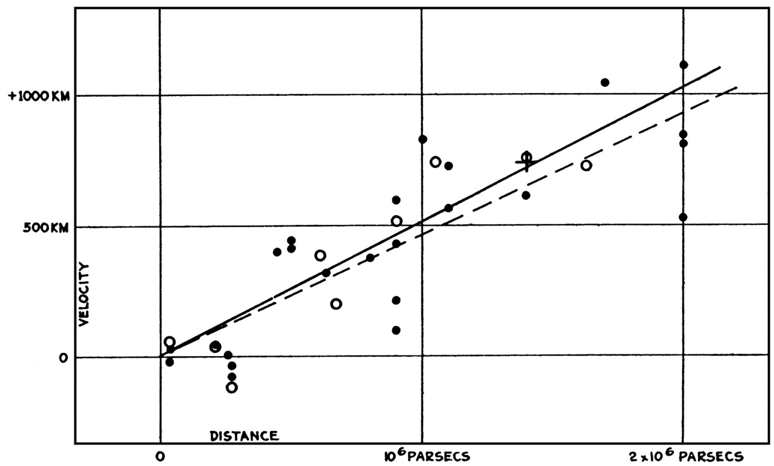


Figure 1: Hubble diagram: velocity — distance relation among extra-galactic nebulae. The velocity is in $\mathrm{km\,sec^{-1}}$ and the distance in Mpc.

$$H_0 = 100\, h \mathrm{km\,sec^{-1}\,Mpc^{-1}}\ . \tag{1.1}$$

The Planck satellite mission measured $H_0 = (67.8 \pm 0.9)\ \mathrm{km\,sec^{-1}\,Mpc^{-1}}$ [5] of $h = 0.68 \pm 0.9$.

1.1 Metric

In order to understand the Hubble diagram, we have to learn how to measure distances and length scales in the Universe. Before looking at distances in space-time, let us first consider distances in Euclidean space. In Euclidean space, the distance between two points is given by the distance in x and y direction between the two points in Cartesian coordinates

$$ds^2 = dx^2 + dy^2 \tag{1.2}$$

where we used Cartesian coordinates to write the distance in the last term. However the result should not depend on the chosen coordinate system. Thus choosing polar coordinates ($r = \sqrt{x^2+y^2}$, θ) with

$$x = r\sin\theta \qquad\qquad y = r\cos\theta\ , \tag{1.3}$$

we find for a distance between two points

$$ds^2 = dr^2 + r^2 d\theta^2\ . \tag{1.4}$$

In general we can write

$$ds^2 = \sum_{ij} g_{ij}(x) dx^i dx^j\ , \tag{1.5}$$

where g is a symmetric matrix, which is called *metric*. The metric defines a scalar product on the vector space and consequently a norm, which can be used to define distances. In four space-time dimensions, we conventionally write

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{1.6}$$

The $\sum$ sign is often dropped and it is convention to sum over the same index, if it appears as lower and upper index. ds^2 is sometimes called *proper time*. The metric g has 10 degrees of freedom.

One special case is special relativity with the metric

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} . \tag{1.7}$$

The *signature* of the metric is the number of eigenvalues ± 1 of the metric. In case of special and general relativity it is $(3,1)$ or $(1,3)$ depending on the convention whether the time component has eigenvalue ± 1. We will follow the convention in Dodelson [1]. Note, the lecture notes by Daniel Baumann [2] use the signature $(1,3)$.

1.2 FRW Metric

When averaged over large scales, the universe looks *isotropic*, i.e. the same in all directions. If we do not live at a special place, then the universe is also *homogeneous*, i.e. the same everywhere.

This fixes the metric

$$ds^2 = -dt^2 + a(t)^2 \times d\ell^2 \qquad \text{with} \qquad d\ell^2 = \gamma_{ij}dx^i dx^j \tag{1.8}$$

where $d\ell^2$ is a symmetric 3-space:

- Euclidean space: zero curvature

$$d\ell^2 = dx^2 = \delta_{ij}dx^i dx^j \tag{1.9}$$

- S^3 (3-sphere): positive curvature

$$d\ell^2 = dx^2 + du^2 \qquad\qquad x^2 + u^2 = a^2 \tag{1.10}$$

- H^3 (3-hyperboloid): negative curvature

$$d\ell^2 = dx^2 - du^2 \qquad\qquad x^2 - u^2 = -a^2 \tag{1.11}$$

After some algebra the FRW metric can be written as

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \ , \tag{1.12}$$

where $k = 0$ for a flat Universe, $k = 1$ for a closed Universe with positive curvature, $k = -1$ for an open Universe with negative curvature. The parameter a is called *scale factor* and describes how the Universe expands, while k is the *curvature parameter*. See Fig. 2.

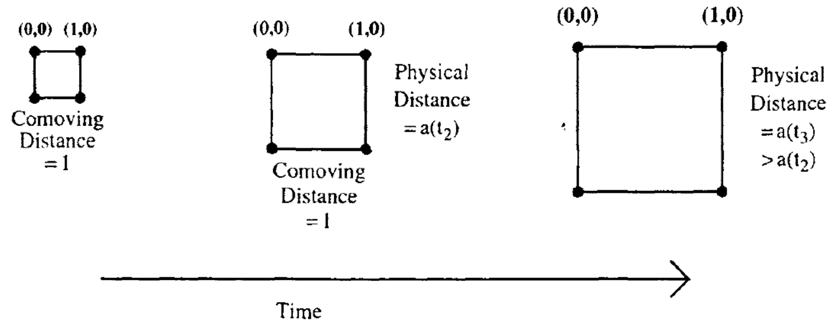


Figure 2: Expansion in an FRW Universe. Copied from [1]

The line element is invariant under the following rescaling symmetry

$$a \to \lambda a \qquad r \to r/\lambda \qquad k \to \lambda^2 k \ , \tag{1.13}$$

which can be used to either set the curvature parameter to $0, \pm 1$ or the scale factor today $a(t_0) \equiv 1$. The coordinate r is called *comoving coordinate*. It is very useful in calculations. However physical results depend only on the *physical coordinate* r_{ph} and the *physical curvature* k_{ph}

$$r_{ph} = a(t) r \qquad k_{ph} = k/a^2(t) \tag{1.14}$$

The physical velocity is

$$v_{ph} \equiv \frac{dr_{ph}}{dt} = a(t)\frac{dr}{dt} + \frac{da}{dt} r \equiv v_{pec} + H r_{ph} \tag{1.15}$$

with the peculiar velocity and the *Hubble parameter*

$$H \equiv \frac{1}{a}\frac{da}{dt} \tag{1.16}$$

The metric can be conveniently rewritten in terms of a static part and the scale factor

$$ds^2 = a(t)^2 \left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) = a(t)^2 \times \text{static metric} \tag{1.17}$$

after introducing *conformal time* ("comoving time")

$$\tau = \int \frac{dt}{a(t)} \ . \tag{1.18}$$

Rewriting the radial component

$$\chi = \int \frac{dr}{\sqrt{1-kr^2}} \tag{1.19}$$

we obtain another useful form of the metric

$$ds^2 = a(t)^2 \left[-d\tau^2 + d\chi^2 + S_k^2(\chi) d\Omega^2 \right] \tag{1.20}$$

with

$$S_k(\chi) = \begin{cases} R_0 \sinh \frac{\chi}{R_0} & k = -1 \\ \chi & k = 0 \\ R_0 \sin \frac{\chi}{R_0} & k = +1 \end{cases} \ . \tag{1.21}$$

We will mainly consider a flat universe ($k = 0$) with metric

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & a(t)^2 & & \\ & & a(t)^2 & \\ & & & a(t)^2 \end{pmatrix} . \tag{1.22}$$

1.3 Geodesics

How does a particle move without any external forces? Newton's law tells us

$$\frac{d^2x^i}{dt^2} = 0 . \tag{1.23}$$

How can we generalise this to a general coordinate system? For example for a system in polar coordinates, $x' = (r, \theta)$, the equations of motion look different. Starting from a Cartesian coordinate system, we find

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt} . \tag{1.24}$$

with the *transformation matrix* $\partial x^i / \partial x'^j$. In case of polar coordinates

$$x^1 = x'^1 \cos x'^2 \qquad\qquad x^2 = x'^1 \sin x'^2 \tag{1.25}$$

the transformation matrix is

$$\frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \cos x'^2 & -x'^1 \sin x'^2 \\ \sin x'^2 & x'^1 \cos x'^2 \end{pmatrix} . \tag{1.26}$$

Applying the second derivative and doing the algebra we find

$$0 = \frac{d^2x^i}{dt^2} = \frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = \frac{\partial x^i}{\partial x'^j}\frac{d^2x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} \tag{1.27}$$

multiplying with the inverse of the transformation matrix we obtain

$$\frac{d^2x'^l}{dt^2} + \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^l_{\ i} \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0 . \tag{1.28}$$

Solutions to the this equation are called geodesics and the equation itself is commonly denoted geodesic equation. There are two small changes in general relativity, the index runs from 0 to 3 and we can not use time t to parameterize the path, but we have to use different monotonically increasing parameter along the geodesic. With these modifications we can rewrite the geodesic equation as

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{1.29}$$

with the Christoffel symbol

$$\Gamma^\mu_{\alpha\beta} = \left(\left[\frac{\partial x}{\partial x'}\right]^{-1}\right)^\mu_{\ \kappa} \frac{\partial^2 x^\kappa}{\partial x'^\alpha \partial x'^\beta} \tag{1.30}$$

In the absence of any (non-gravitational) forces, particles move along *geodesics*, the curve of least action. This path is determined by the *geodesic equation*

$$\frac{dU^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}U^\alpha U^\beta \tag{1.31}$$

for a particle with mass m and velocity

$$U^\mu = \frac{dx^\mu}{d\lambda} \tag{1.32}$$

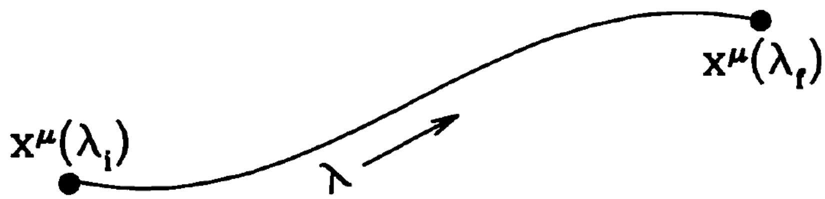


Figure 3: Curve in space-time. Copied from [1]

where λ can be in principle any parameter parameterising the curve. One convenient choice is the proper time of the particle. In the following we will use $\lambda = \tau$. The *Christoffel symbol* (for a metric-compatible connection) is defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right] . \tag{1.33}$$

In the general relativity course you will learn that you can rewrite the geodesic equation in a more compact form using the *covariant derivative* ∇_α

$$U^\alpha\left(\frac{\partial U^\mu}{\partial X^\alpha} + \Gamma^\mu_{\alpha\beta}U^\beta\right) \equiv U^\alpha \nabla_\alpha U^\mu = 0 \tag{1.34}$$

where we made use of the chain rule

$$\frac{d}{d\tau}U^\mu(X^\alpha(\tau)) = \frac{dX^\alpha}{d\tau}\frac{\partial U^\mu}{\partial X^\alpha} = U^\alpha\frac{\partial U^\mu}{\partial X^\alpha} . \tag{1.35}$$

In terms of the 4-momentum, $P^\mu \equiv mU^\mu = (E, \vec{P})$, the geodesic equation becomes

$$P^\alpha\frac{\partial P^\mu}{\partial X^\alpha} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta \tag{1.36}$$

which is also valid for a massless particle.

1.4 Point Particle in FRW Universe

The Christoffel symbol for a flat FRW metric has only a few non-vanishing components

$$\Gamma^0_{ij} = \dot{a}a\gamma_{ij} \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \tag{1.37}$$

$$\Gamma^i_{jk} = 0 \quad \text{for Euclidean space } (k=0) \tag{1.38}$$

which can be easily derived using Eqs. (1.33) and (1.12) .

In an FRW universe, we have $\partial_i P^\mu = 0$ due to homogeneity and thus the geodesic equation reduces to

$$P^0\frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta = -\left(2\Gamma^\mu_{0j}P^0 + \Gamma^\mu_{ij}P^i\right)P^j \tag{1.39}$$

Thus we immediately see that particles at rest remain at rest

$$P^j = 0 \Rightarrow \frac{dP^j}{dt} = 0 . \tag{1.40}$$

Considering the zeroth component

$$E\frac{dE}{dt} = -\Gamma^0_{ij}P^iP^j = -\frac{\dot{a}}{a}p^2 \tag{1.41}$$

using $E = P^0$ and the physical 3-momentum $p^2 = a^2\gamma_{ij}P^iP^j$.

Using the on-shell condition of the particle

$$-m^2 = g_{\mu\nu}P^\mu P^\nu = -E^2 + p^2 \tag{1.42}$$

and thus $EdE = pdp$, the geodesic equation implies

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a} \Rightarrow p(t) \propto \frac{1}{a(t)} . \tag{1.43}$$

Hence physical 3-momentum "decays". In particular for massless particles energy decays and physical peculiar velocity $v^2 = a^2\gamma_{ij}v^iv^j$ decay,

$$E = p \propto \frac{1}{a(t)} \qquad \text{(massless particles)} \tag{1.44}$$

$$p = \frac{mv}{\sqrt{1-v^2}} \propto \frac{1}{a} \qquad \text{(massive particles)} , \tag{1.45}$$

with the comoving peculiar velocity $v^i = dX^i/dt$ (the velocity relative to the comoving frame). The comoving peculiar velocity is related to the comoving momentum

$$P^i = mU^i = m\frac{dX^i}{d\tau} = mv^i\frac{dt}{d\tau} = \frac{mv^i}{\sqrt{1-a^2\gamma_{ij}v^iv^j}} = \frac{mv^i}{\sqrt{1-v^2}} \tag{1.46}$$

with $v^2 \equiv a^2\gamma_{ij}v^iv^j$. The next-to-last equality follows from the relation of the τ on t can be obtained from the metric

$$d\tau^2 = -ds^2 = dt^2 - a^2\gamma_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt^2 = dt^2 - a^2\gamma_{ij}v^iv^jdt^2 = (1-v^2)dt^2 \tag{1.47}$$

Thus free-falling particles will asymptotically approach the Hubble flow.

1.5 Redshift

Almost all observations rely on photons. The wavelength of a photon is inversely proportional to its momentum $\lambda = h/p$. As we showed in the previous section, the momentum of the photon is inversely proportional to the scale factor a and thus the wavelength scales as $a(t)$. Hence light emitted at time t_1 with wavelength λ_1 will be observed at time t_0 with wavelength

$$\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1 \tag{1.48}$$

and consequently the wavelength increases when the universe is expanding, $a(t_0) > a(t_1)$ and thus redshifted. The redshift parameter z is defined as the fractional shift in wavelength

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} \tag{1.49}$$

and thus

$$1 + z = \frac{a(t_0)}{a(t_1)} \,. \tag{1.50}$$

Using the common definition $a(t_0) = 1$ the redshift is related to the scale factor of the emitter

$$1 + z = \frac{1}{a(t_1)} \,. \tag{1.51}$$

For nearby sources the scale factor can be expanded using the Hubble constant $H_0 \equiv \dot{a}(t_0)/a(t_0)$

$$a(t_1) = a(t_0)\,[1 + H_0(t_1 - t_0) + \ldots] \tag{1.52}$$

and thus the redshift is linearly related to the distance d of the light source

$$z \simeq H_0 d \,. \tag{1.53}$$

Similarly, for small velocities $v \ll c$, the standard redshift formula can be used and we obtain $z \simeq v/c$. It is thus a direct measure of the velocity of the galaxies.

1.6 Distances

There are two ways to measure distance, the comoving distance, χ, which remains fixed during expansion, and the physical distance, $d = a\chi$, which takes the expansion into account. As we are in an expanding space-time, we might wonder what is the more interesting physical distance: the distance at the time when the light was emitted or the distance when it was received. The well-defined measure of distance is a comoving distance. On a comoving grid, the distance simply amounts to $\left(dx^2 + dy^2 + dz^2\right)^{1/2}$. See Fig. 2 for an illustration.

1.6.1 Metric and comoving distance

Recall the form of the metric in terms of χ

$$ds^2 = a(t)^2\left[-d\tau^2 + d\chi^2 + S_k^2(\chi)d\Omega^2\right] \tag{1.20}$$

where $S_k(\chi)$ is defined in Eq. (1.21). As light travels along null geodesics $ds^2 = 0$, the change in conformal time $\Delta\tau$ equals the change in comoving distance $\Delta\chi$, i.e. $\Delta\tau = \Delta\chi$ and thus the *comoving distance* between an object at time $t(a)$ and us is given by

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} \,. \tag{1.54}$$

The *metric distance* is defined as $d_m \equiv S_k(\chi)$ and equals the comoving distance in a flat space time. *Note that neither the comoving distance nor the metric distance are observable.*

1.6.2 Horizons

The total comoving distance is given by the distance light could have travelled in a given time t, i.e.

$$\eta(t) = \int_0^t \frac{dt'}{a(t')} \,. \tag{1.55}$$

As nothing travels faster than light, $\eta(t)$ defines the *particle horizon*. We are not able to see anything in the past, which is beyond the *particle horizon*. It is monotonically increasing and can also be used as a measure of time and is the *conformal time* defined in Eq. (1.18). The proper distance of the particle horizon is given by

$$d_{max}(t) = a(t)\int_0^t \frac{dt'}{a(t')} \ . \tag{1.56}$$

Similarly, there might be a horizon for future events, if the universe recollapses at time T. Then the largest distance from which an observer might be able to receive signals travelling at the speed of light at any time later than t, is given by

$$\int_t^T \frac{dt'}{a(t')} \tag{1.57}$$

in comoving coordinates, which is denoted *event horizon*. The proper distance for an infinite distant future is given by

$$d_{MAX}(t) = a(t)\int_t^\infty \frac{dt'}{a(t')} \ . \tag{1.58}$$

1.6.3 Angular diameter distance

A common method to determine the distance of an object of known size D is to measure the angle $\delta\theta$ it takes on the sky as illustrated in Fig. 4a. Assuming that an object of known size D emits photons at time t_1 at comoving distance χ, then the *angular diameter distance* of the object is given by

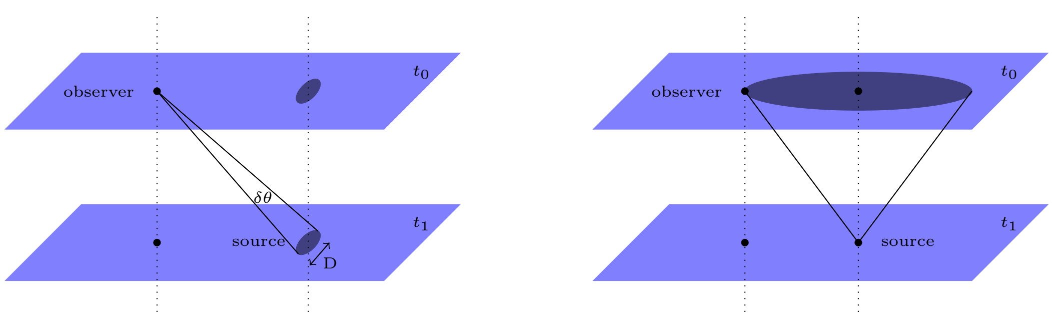
$$d_A \equiv \frac{D}{\delta\theta} \tag{1.59}$$

with $\delta\theta \ll 1$. In an FRW universe, the physical transverse size is given by

$$D = a(t_1)S_k(\chi)\delta\theta \tag{1.60}$$

and thus

$$d_A = \frac{d_m}{1+z} \ . \tag{1.61}$$



(a) Angular diameter distance (b) Luminosity distance

Figure 4: Observable distances

1.6.4 Luminosity Distance

Another common way to infer the distance to an object of known luminosity is to compare the known to the measured luminosity L, i.e. energy emitted per second. One example are Type IA supernovae whose absolute luminosity is believed to be well understood. The observed flux F, energy per second per area, can be used to determine the *luminosity distance* d_L. The observed flux F at the *luminosity distance* d_L is given by

$$F = \frac{L}{4\pi d_L^2} \; . \tag{1.62}$$

Neglecting the expanding spacetime and any curvature, the luminosity distance would be simply given by the comoving distance χ. In an FRW spacetime the luminosity distance d_L the comoving distance χ has to be replaced by the metric distance d_m to account for possible curvature. Furthermore, at early times, the photons travel further on the comoving grid compared to today. Thus the number of photons received today is reduced by a factor $a(t_1) = 1/(1+z)$. Finally, if the photons are redshifted and thus the energy of the received photons is reduced by a factor a. Hence the measured flux using a general FRW metric is given by

$$F = \frac{L}{4\pi d_m^2 (1+z)^2} \; . \tag{1.63}$$

and a comparison with Eq. (1.62) shows that the luminosity distance is

$$d_L \equiv d_m(1+z) \; . \tag{1.64}$$

The luminosity distance is related to the angular diameter distance by

$$d_A = \frac{d_L}{(1+z)^2} \; . \tag{1.65}$$

The distance measurements were crucial to show that the universe is accelerating today, which lead to the award of the Nobel prize in physics for Saul Perlmutter, Brian Schmidt and Adam Riess in 2011. See Fig. 5 *In the following we will focus on a flat spacetime.*

1.7 Dynamics

The Einstein equation describes how the metric and thus the scale factor evolves in time depending on the energy-momentum density in the Universe.

1.7.1 Einstein Equations

The metric introduced in the previous sections describes gravity and the interaction of gravity with matter is described by the Einstein equation[1]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \tag{1.67}$$

[1] A possible cosmological constant Λ is absorbed in the energy-momentum tensor, i.e.

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \; . \tag{1.66}$$

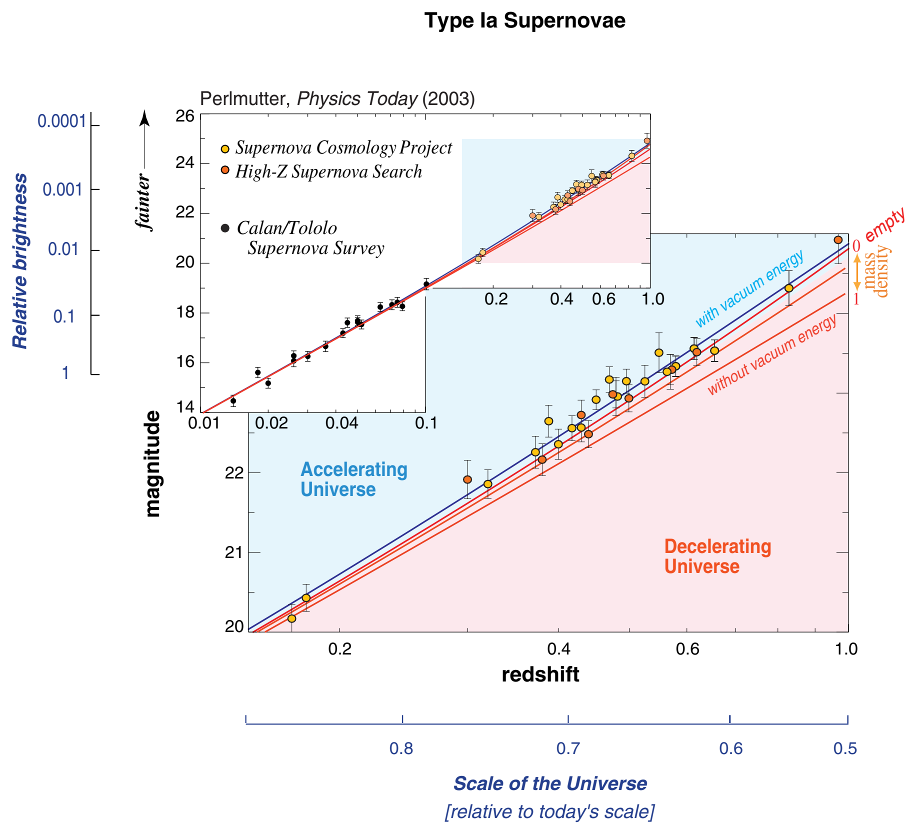


Figure 5: Hubble diagram [S. Perlmutter Physics Today April 2003 p.53]

with Newton's constant G, the Einstein tensor $G_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $\mathcal{R}$, and the energy-momentum tensor $T_{\mu\nu}$. The Ricci tensor and Ricci scalar describe the curvature of space-time. The Ricci scalar is simply defined by the contraction of the Ricci tensor with the metric

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} \tag{1.68}$$

and the Ricci tensor can be obtained from the Christoffel symbols[2]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \,. \tag{1.69}$$

[2]The curvature is defined similar to the field strength tensor in quantum field theory from the commutator of the covariant derivatives $[\nabla_\mu, \nabla_\nu]$. Please refer to a general relativity book.

In particular looking at the 00 component of the Ricci tensor in an FRW metric we find

$$R_{00} = \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \tag{1.70}$$

$$= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} \tag{1.71}$$

$$= -\frac{\partial}{\partial t}\delta^i_i \frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \delta^i_j \delta^j_i = -3\frac{\ddot{a}}{a} \; . \tag{1.72}$$

Similarly for the spatial components ($k = 0$) we find

$$R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \tag{1.73}$$

and the Ricci scalar can be evaluated to

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{1}{a^2}R_{ii} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) \; . \tag{1.74}$$

Plugging everything into Einstein equations we obtain two independent equations describing the evolution in a flat ($k = 0$) FRW Universe

$$R_{00} - \frac{1}{2}g_{00}\mathcal{R} = 3\left(\frac{\dot{a}}{a}\right)^2 = 3H^2 = 8\pi G T_{00} \tag{1.75}$$

$$g^{\mu\nu}G_{\mu\nu} = -\mathcal{R} = 8\pi G T^\mu_\mu$$

1.7.2 Perfect Fluid

Before interpreting the Einstein equations, we have to have a closer look at the energy-momentum tensor $T^{\mu\nu}$. Our basic assumption is that we can describe the content of the Universe by different perfect fluids as a leading approximation, i.e. the fluid can be described by macroscopic quantities, its energy density and pressure, while there is no stress or viscosity in agreement that with the metric being homogeneous and isotropic.

The energy momentum tensor describes the flux of four-momentum p^μ in the direction x^ν. The energy-momentum tensor of a perfect fluid in its rest-frame in Minkowski space is given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \tag{1.76}$$

Due to isotropy it is diagonal and its spatial components have to be equal. The 00-element is just the energy density ρ, i.e. the flux of energy density in time direction, while the spatial elements ii are given by the flux of momentum density p_i in the direction x_i, i.e. the pressure $\mathcal{P}_i = \frac{dp_i}{dt}dx_i$ in direction x_i. In order to write it in a covariant form, we first introduce the four-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \tag{1.77}$$

with the *proper time*

$$d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu \; . \tag{1.78}$$

For a particle at rest we find $U^\mu = (1, 0, 0, 0)$. Hence we can write the energy-momentum tensor as

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P}\eta^{\mu\nu} \tag{1.79}$$

and its generalisation to general relativity is straightforward

$$T^{\mu\nu} = (\rho + \mathcal{P})\, U^\mu U^\nu + \mathcal{P} g^{\mu\nu} \; . \tag{1.80}$$

Thus we find in the rest-frame of the fluid in the FRW universe

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}\mathcal{P} & & \\ & & a^{-2}\mathcal{P} & \\ & & & a^{-2}\mathcal{P} \end{pmatrix} \qquad \text{or} \qquad T^\mu_\nu = \begin{pmatrix} -\rho & & & \\ & \mathcal{P} & & \\ & & \mathcal{P} & \\ & & & \mathcal{P} \end{pmatrix} \; . \tag{1.81}$$

For example dust can be described by a perfect fluid with zero pressure, since it is not compressible.

1.7.3 Friedmann Equations

Using our knowledge about the energy-momentum tensor of a perfect fluid, we see that

$$T_{00} = \rho \qquad\qquad T^\mu_\mu = -\rho + 3\mathcal{P} \tag{1.82}$$

and we can rewrite Eqs. (1.75) to obtain the *Friedmann equations*

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{1.83}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3\mathcal{P}\right) \; , \tag{1.84}$$

which are the basic equations of FRW cosmology. Sometimes it is convenient to express the energy density as a fraction of the critical energy density

$$\rho_{cr} = \frac{3H^2}{8\pi G} = 3H^2 m_P^2 \; , \tag{1.85}$$

where m_P is the reduced Planck mass:

$$\Omega = \frac{\rho}{\rho_{cr}} \; . \tag{1.86}$$

1.8 Continuity Equation

How does the energy-momentum tensor of the perfect fluid evolve with time? In the absence of external forces and gravity, we find that the energy density is constant $\partial\rho/\partial t = 0$ and the Euler equation that the pressure does not depend on the direction $\partial\mathcal{P}/\partial x^i$. In covariant formulation, this amounts to

$$\partial_\mu T^\mu_\nu = 0 \tag{1.87}$$

which some might have seen in the quantum field theory course. The generalisation to general relativity is straightforward by understanding that we have to replace the partial derivative with a covariant derivative to ensure that the continuity equation correctly transforms under a change of coordinates, i.e.

$$0 = \nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha \; . \tag{1.88}$$

For $\nu = 0$ we obtain

$$0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\alpha\mu} T^\alpha_0 - \Gamma^\alpha_{0\mu} T^\mu_\alpha = -\frac{\partial\rho}{\partial t} - \Gamma^i_{0i}\rho - \Gamma^i_{0i} T^i_i \tag{1.89}$$

and thus

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\left[\rho + \mathcal{P}\right] = 0 \; . \tag{1.90}$$

Introducing the *equation of state* and the *equation of state parameter* w

$$\mathcal{P} = w\rho \tag{1.91}$$

we can rewrite the continuity equation for $\nu = 0$

$$0 = \frac{\partial \rho}{\partial t} + 3(1+w)\frac{\dot{a}}{a}\rho = a^{-3(1+w)}\frac{\partial\left(\rho a^{3(1+w)}\right)}{\partial t} \tag{1.92}$$

for constant equation of state parameter w and conclude $\rho \propto a^{-3(1+w)}$. We can insert this result into the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \; . \tag{1.93}$$

The solution gives us the time dependence of the scale factor for $w \neq -1$

$$a(t) \propto t^{\frac{2}{3(1+w)}} \qquad\qquad H = \frac{2}{3(1+w)t} \tag{1.94}$$

and for $\rho = \Lambda/8\pi G$ with $w = -1$, the scale factor is

$$a(t) \propto e^{\sqrt{\Lambda/3}t} \qquad\qquad H = \sqrt{\frac{\Lambda}{3}} \tag{1.95}$$

1.9 Cosmic Inventory

There are several different contributions to energy density of the Universe. Today, the most dominant contribution is dark energy (or a cosmological constant) and matter as shown in Fig. 6a, while radiation only contributes a small fraction. See Tab. 1 for a summary. The evolution of the three main components, dark energy, matter and radiation is shown in Fig. 6b.

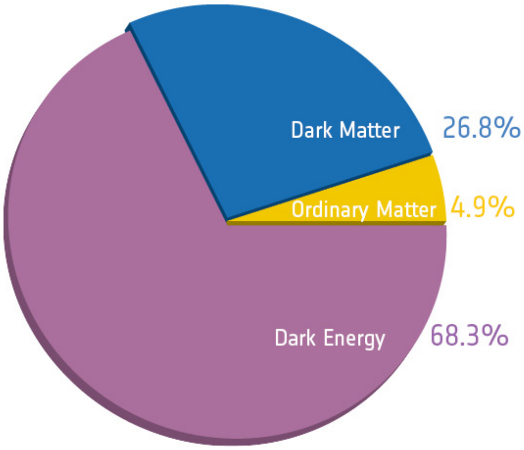
	w	$\rho(a)$	$a(t)$	$H(t)$
matter	0	a^{-3}	$t^{2/3}$	$\frac{2}{3t}$
radiation	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	$\frac{1}{2t}$
cosm.const.	-1	ρ_0	$e^{\sqrt{\Lambda/3}t}$	$\sqrt{\Lambda/3}$

Table 1: Evolution of different fluids

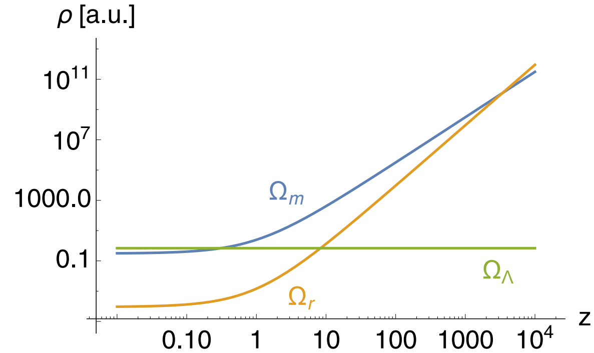
1.9.1 Matter

Matter refers to fluids with a negligible pressure, $|\mathcal{P}| \ll \rho$, which is a good description for a gas of non-relativistic particles. Thus setting $w = 0$, we obtain

$$\rho \propto a^{-3} \tag{1.96}$$



(a) Dominant contributions to the energy density of the Universe.



(b) Evolution of energy density for dark energy, matter and radition.

Figure 6: Energy density composition today and its evolution.

reflecting the expansion of the volume $V \propto a^3$. Ordinary matter (nuclei and electrons) are commonly called *baryonic matter*. Most of the matter in the Universe is in the form of dark matter, which is a new form of matter, which does not interact with photons, or at least extremely weakly. However its existence has been seen in numerous observations via its gravitational interactions at different length scales: The virial theorem $\left(\frac{1}{2}\left\langle v^2\right\rangle = \frac{GM}{R}\right)$ applied to COMA cluster (F.Zwicky 1933) shows the existence of additional non-baryonic matter, similarly galactic rotation curves [$\mathcal{O}(10s)$kpc], gravitational lensing [$< \mathcal{O}(200)$kpc], in a comparison of the observation of the bullet cluster in X-ray and gravitational lensing, the cosmic microwave background and large scale structure.

1.9.2 Radiation

The pressure of a relativistic gas of massless particles, for which the kinetic energy dominates the energy density, is one third of its energy density, $\mathcal{P} = \frac{1}{3}\rho$ and thus

$$\rho \propto a^{-4} \tag{1.97}$$

This can be easily understood by noticing that in addition to the decrease in the number density, the energy is redshifted $E \propto 1/a$.

The prime examples of radiation are

- *photons*, which we detect today as cosmic microwave background with temperature $T = 2.725$ K and small perturbations of order 10^{-5}.
- *neutrinos* in the early Universe. Today the masses of at least two neutrinos are relevant and thus they behave like matter.
- *gravitons*. Similar to photons and neutrinos there might be background of gravitons (i.e. gravitational waves)

1.9.3 Dark energy

Matter and radiation are not enough to describe the evolution of the Universe. Recently dark energy became the dominant source of energy density in the Universe and constitutes about 70% of the total energy density today. The simplest explanation is in form of a cosmological constant with

$$\rho \propto a^0 \tag{1.98}$$

implying *negative* pressure $\mathcal{P} = -\rho$. A cosmological constant is predicted from quantum field theory. The ground state energy leads to a stress-energy tensor

$$T^{vac}_{\mu\nu} = \rho_{vac} g_{\mu\nu} \ . \tag{1.99}$$

However the *naive* prediction overestimates its size by several orders of magnitude $\rho_{vac}/\rho_{obs} \sim 10^{120}$, if a simple cutoff of the order of the Planck scale is imposed. Even though this naive estimate is questionable, the ground state energy is expected to change during phase transitions and thus it is an unsolved question why the cosmological constant is so small. Many alternatives to a cosmological constant have been suggested to address the smallness. Most explanations involve scalar fields and explain dark energy dynamically, but they generally do not address the problem of the vacuum energy.

2 Thermal History

A basic understanding of the thermal history of the universe can be obtained by comparing the *rate of interactions* Γ to the rate of expansion H, or equivalently to its associated timescales $t_c \equiv \Gamma^{-1}$ and $t_H \equiv H^{-1}$. If the timescale for interactions is much smaller than the one for expansion

$$t_c \ll t_H \qquad \Leftrightarrow \Gamma \gg H \tag{2.1}$$

then *local thermal equilibrium* is reached. As the universe cools down, the interaction rate will decrease and when $t_c \sim t_H$, particles decouple from the thermal bath. Different particles decouple at different times. For example the interaction rate for $2 \to 2$ scattering is given by

$$\Gamma \equiv n\sigma v \tag{2.2}$$

with the number density n, the interaction cross section σ and the average velocity v. For ultra-relativistic particles ($v \sim 1$), the masses can be neglected and the only dimensionful quantity is the temperature with $n \sim T^3$ and $\sigma \sim \frac{\alpha^2}{T^2}$ for some exchange interaction with coupling constant α. All SM particles are ultra-relativistic for $T \gtrsim 100\text{GeV}$. Hence the interaction rate is

$$\Gamma = n\sigma v \sim T^3 \times \frac{\alpha^2}{T^2} = \alpha^2 T \tag{2.3}$$

while the Hubble rate scales like

$$H \sim \frac{\sqrt{\rho}}{M_{Pl}} \sim \frac{T^2}{M_{Pl}} \tag{2.4}$$

and thus the ratio

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl}}{T} \sim \frac{10^{16}\text{GeV}}{T} \tag{2.5}$$

and thus for $100\text{GeV} \lesssim T \lesssim 10^{16}\text{GeV}$ particles are in local thermal equilibrium for $\alpha \sim 0.01$.

2.1 Equilibrium Thermodynamics

In local thermal equilibrium we can use distribution functions $f(\vec{x}, \vec{p})$, i.e. the occupation number of a small cell $d^3x d^3p/(2\pi\hbar)^3$ at position $(\vec{x}, \vec{p})$ to describe the fluids. In an homogeneous and isotropic universe, the distribution function does not depend on the position $\vec{x}$ and the direction of the momentum, but only the absolute magnitude of the momentum. The number density n_i of species i with g_i internal degrees of freedom is given by

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) \; . \tag{2.6}$$

Bosons and fermions follow the usual Bose-Einstein and Fermi-Dirac distributions in equilibrium at a temperature T respectively

$$f(p) = \frac{1}{e^{(E-\mu)/T} \pm 1} \tag{2.7}$$

with $+$ for the Fermi-Dirac and $-$ for the Bose-Einstein distribution. Similarly we can define the energy density and the pressure (See prob. 15 in chapter 2 of [1])

$$\rho = g_i \int \frac{d^3p}{(2\pi)^3} f(p) E(p) \overset{T \gg m,\mu}{\longrightarrow} \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{bosons} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{fermions} \end{cases} \tag{2.8}$$

$$\mathcal{P} = g_i \int \frac{d^3p}{(2\pi)^3} f(p) \frac{p^2}{3E(p)} \overset{T \gg m,\mu}{\longrightarrow} \frac{1}{3}\rho \; . \tag{2.9}$$

See exercise 15 in chapter 2 of [1] to understand the form of the expressions for the energy density and the pressure. The solutions to the exercise are provided in the appendix of [1]. For negligible chemical potentials due to number changing processes of photons, e.g. $e^+e^- \to \gamma\gamma, \gamma\gamma\gamma$, we can write

$$d(\rho(T)V) = T d(s(T)V) - \mathcal{P}(T) dV \tag{2.10}$$

which allows us to write the entropy density $s(T)$ as

$$s(T) = \frac{\rho(T) + \mathcal{P}(T)}{T} \tag{2.11}$$

by equating the coefficient in front of dV. Similarly it is straightforward to show

$$s(T) = \frac{\partial \mathcal{P}}{\partial T} \tag{2.12}$$

using either one of the Maxwell relations or considering the coefficient in front of the differential VdT in Eq. (2.10). The condition of thermal equilibrium tells us that the entropy in a comoving volume is fixed, i.e.

$$s(T)a^3 = \text{constant} \; . \tag{2.13}$$

See Dodelson pg. 39/40 for a derivation using the continuity equation.

In the radiation dominated era, it is convenient to define the effective relativistic degrees of freedom $g_*^{\rho,s}(T)$ as follows

$$\rho = \frac{\pi^2}{30} g_*^\rho(T) T^4 \qquad\qquad s = \frac{2\pi^2}{45} g_*^s(T) T^3 \; . \tag{2.14}$$

Whenever particles decouple from the thermal plasma, $g_*^{s,\rho}$ decreases. For most of the time, $g_*^\rho(T) = g_*^s(T)$, as it is shown in Fig 7.

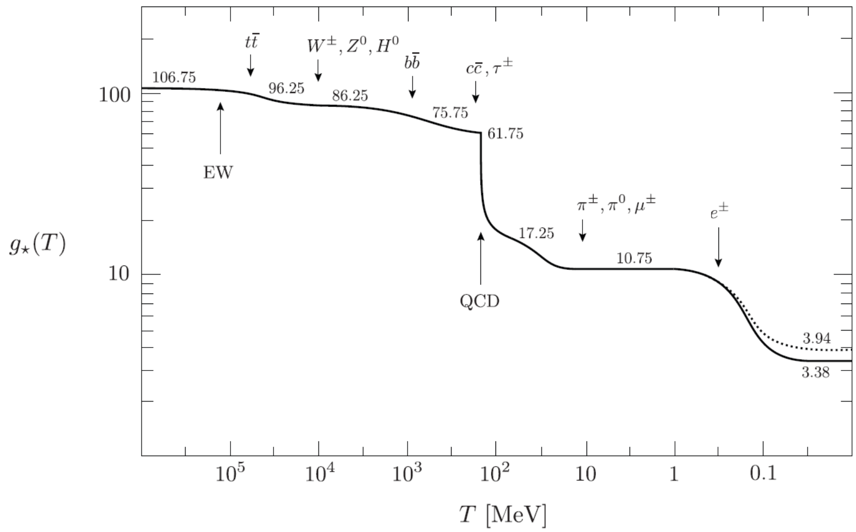


Figure 7: Evolution of effective relativistic degrees of freedom. Solid line is for $g_*^\rho(T)$ and the dotted line for $g_*^s(T)$. Taken from Cosmology lecture notes of Daniel Baumann.

2.2 Relativistic Decoupling of Neutrinos

Neutrinos are almost massless fermions. They decouple from the cosmic plasma around 1 MeV and thus shortly before electrons and positrons become non-relativistic and reheat the cosmic plasma. Thus neutrinos are effectively colder than the cosmic plasma, since they are not reheated by electron-positron pair annihilation. Using entropy conservation, we find for the entropy before neutrino decoupling at scale factor a_1

$$s(a_1) = \frac{2\pi^2}{45} T_1^3 \left[2 + \frac{7}{8}\left(2+2+3\cdot 2\right)\right] = \frac{43\pi^2}{90} T_1^3 \,, \tag{2.15}$$

because there are in total 2 degrees of freedom from the two polarisations of photons, 2 spin degrees of freedom for both electrons and positrons and 3 generations of neutrinos with spin 2. After electrons and positrons become non-relativistic, they transfer their entropy to the cosmic plasma and effectively reheat the cosmic plasma. Hence the entropy at a late-enough redshift a_2 is given by

$$s(a_2) = \frac{2\pi^2}{45}\left[2T_\gamma^3 + \frac{7}{8} 6 T_\nu^3\right] \,, \tag{2.16}$$

where photons have a temperature T_γ and neutrinos have temperature T_ν. Entropy conservation $s(a_1)a_1^3 = s(a_2)a_2^3$ results in

$$\frac{43}{2}\left(a_1 T_1\right)^3 = 4\left[\left(\frac{T_\gamma}{T_\nu}\right)^3 + \frac{21}{8}\right]\left(T_\nu(a_2)a_2\right)^3 \,. \tag{2.17}$$

Finally we have to relate the temperature T_1 to the temperature at a later time. After neutrinos are decoupled, they still preserve the shape of the Fermi-Dirac distribution and the temperature is inversely proportional to the scale factor. This can be directly seen from observing that the energy of a massless particle scales like a^{-1} as shown in Eq. (1.44). Thus the temperature of neutrinos T_ν

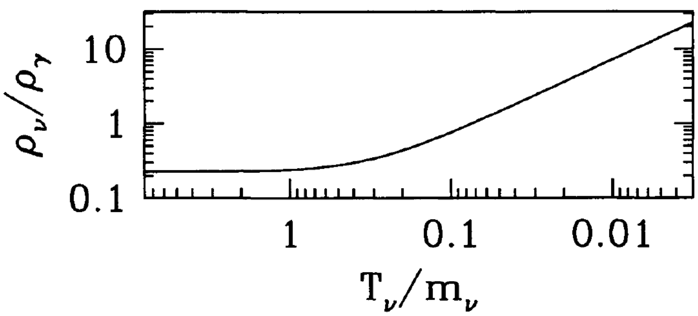


Figure 8: Neutrino energy compared to photon energy vs temperature of neutrinos. Taken from Dodelson[1]

satisfies $a_2 T_\nu = a_1 T_1$. Solving Eq. (2.17) for the temperature of neutrinos T_ν, we obtain

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} \tag{2.18}$$

and conclude that the temperature of neutrino background today is lower compared to the cosmic microwave background. We find for the temperature of neutrinos today

$$T_\nu^0 = T_\gamma^0 \left(\frac{4}{11}\right)^{1/3} = 2.73 \left(\frac{4}{11}\right)^{1/3} K = 1.95K = 1.68 \times 10^{-4}\text{eV} . \tag{2.19}$$

It has been undeniably shown that neutrinos are massive. The temperature of neutrinos today T_ν^0 is smaller than the square root of the solar mass squared difference, $\sqrt{\Delta m_\odot^2} = 8.66 \times 10^{-3}$eV, and thus at least two neutrinos are non-relativistic today and their mass can not be neglected. The energy density of one neutrino is given by

$$\rho_{1\nu} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_\nu^2}}{e^{p/T_\nu} + 1} \tag{2.20}$$

and shown in Fig. 8. Thus the total energy density in neutrinos is dominated by their mass $\rho_\nu = m_\nu n_\nu$ and we find using $n_\nu = 3n_\gamma/11$

$$\Omega_\nu h^2 = \frac{m_\nu}{94\text{eV}} . \tag{2.21}$$

2.3 The Boltzmann Equation

The Boltzmann equation is given by

$$\frac{df}{d\lambda} = C'[f] \tag{2.22}$$

with the distribution function $f = f(\vec{x}, \vec{p}, t)$. The left-hand side gives the change of the distribution function with respect to the affine parameter λ. We will choose the normalization of the affine parameter such that

$$P^\mu = \frac{dx^\mu}{d\lambda} \tag{2.23}$$

and thus $E = dx^0/d\lambda$. $C'[f]$ is the collision term taking into account any interactions.

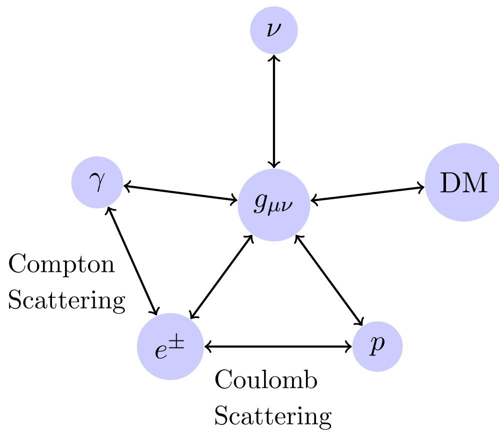


Figure 9: The network of Boltzmann and Einstein equations

We will again use the momentum four-vector to relate λ using $E = dx^0/d\lambda$ and thus we obtain

$$\frac{df}{dt} = \frac{d\tau}{dt}\frac{df}{d\tau} = \frac{1}{E}C'[f] = C[f] \tag{2.24}$$

which is exactly Eq. (4.1) in Dodelson[1].

The Boltzmann equations generally connect the different components of the Universe. Electrons and protons are coupled via the Coulomb interaction, photons and electrons[3] via Compton scattering. All particle species are coupled to the metric as illustrated in Fig. 9. In the following we will only consider a FRW universe and will work with the integrated form of the Boltzmann equation, the Boltzmann equation for the number density. See Chapter 4 in Dodelson[1] for a derivation how to use the Boltzmann equation to go beyond the FRW universe and study perturbations.

2.4 Boltzmann Equation for Number Density

We write the total derivative as the sum of the partial derivatives

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p}\frac{dp}{dt} = \frac{\partial f}{\partial t} - pH\frac{\partial f}{\partial p} \tag{2.25}$$

using Eq. (1.43). Integrating this equation over the three-momentum we obtain

$$g\int \frac{d^3p}{(2\pi)^3}\frac{df}{dt} = g\int \frac{dp}{(2\pi)^3}4\pi p^2\left(\frac{\partial f}{\partial t} - pH\frac{\partial f}{\partial p}\right) \tag{2.26}$$

$$= \frac{d}{dt}g\int \frac{dp^3}{(2\pi)^3}f + gH\int \frac{dp}{(2\pi)^3}4\pi^2 3p^2 f \tag{2.27}$$

$$= \frac{dn}{dt} + 3Hn = a^{-3}\frac{d\left(na^3\right)}{dt} \tag{2.28}$$

[3]Compton scattering between protons and photons is suppressed by the larger mass of a proton compared to an electron.

Thus the Boltzmann equation for the number density n_1 of the first particle in case of two to two scatterings, $1+2 \leftrightarrow 3+4$ is given by

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = \int \prod_{i=1}^{4} d\Pi_i \,(2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)\,|\mathcal{M}|^2$$
$$\times \left\{ f_3 f_4 \left(1 \pm f_1\right)\left(1 \pm f_2\right) - f_1 f_2 \left(1 \pm f_3\right)\left(1 \pm f_4\right)\right\} \quad (2.29)$$

with the phase space integrals

$$\int d\Pi_i = g_i \int \frac{d^3 p_i}{(2\pi)^3 2E_i(p_i)} \quad (2.30)$$

for particle i with g_i internal degrees of freedom. The phase space integral is Lorentz invariant

$$\int \frac{d^3 p}{2E(p)} \delta(E - \sqrt{p^2+m^2}) = \int d^3 p \int dE \delta(E^2 - p^2 - m^2)\theta(E) \quad (2.31)$$

because we implicitly impose that the particle are on-shell, i.e. satisfy the energy-momentum dispersion relation

$$E^2 = p^2 + m^2 \quad (2.32)$$

Let us understand the different factors in the Boltzmann equation. In the absence of any interactions, the right-hand side of the equation, the Boltzmann equation tells us that the number of particles in a comoving volume does not change. However the number of particles in a physical volume scales like a^{-3} due to the expansion. Interactions between the different particles are described by the integral on the right-hand side. The integrals are over the whole phase space $\int d\Pi_i$ of the different particles involved in the interaction. Energy-momentum conservation is imposed by the four-dimensional delta function. The factor $|\mathcal{M}|^2$ is the square of the amplitude (matrix element), which governs the strength of the interaction. For example in the case of Compton scattering it is proportional to the fine-structure constant α^2. It is averaged over *initial and final states*. The last factor on the right-hand side consists of two terms and takes into account the occupation numbers (distribution functions) of the different states. The first term is proportional to $f_3 f_4 (1 \pm f_1)(1 \pm f_2)$ and describes the production of a particle 1 in the process $3+4 \to 1+2$, i.e. it is proportional to the initial abundances and the factors $(1 \pm f_i)$ take into account the possible Pauli-blocking for fermions with a minus sign or Bose enhancement for bosons with a plus sign. The second term describes the destruction of particle 1 in the process $1+2 \to 3+4$. The first term is sometimes called *source term* and the second *loss term*. Note that we assumed that the process is reversible.

Usually scattering between the different particles enforces kinetic equilibrium, i.e. the different particle species follow the Bose-Einstein or Fermi-Dirac statistics, however they are not necessarily in chemical equilibrium. If they were the chemical potential μ would balance against the other chemical potentials. In the case of $e^+ + e^- \leftrightarrow \gamma\gamma$, we would find $\mu_{e^+} + \mu_{e^-} = 2\mu_\gamma$.

For systems at temperature $T \ll E - \mu$ we can neglect the terms ± 1 in the denominators of the Fermi-Dirac and Bose-Einstein distributions and work with the Maxwell-Boltzmann distribution

$$f(E) = e^{-(E-\mu)/T} = e^{\mu/T} e^{-E/T} \ . \quad (2.33)$$

Similarly we can neglect the Pauli-blocking/Bose enhancement factors and can approximate

$$\left\{ f_3 f_4 \left(1 \pm f_1\right)\left(1 \pm f_2\right) - f_1 f_2 \left(1 \pm f_3\right)\left(1 \pm f_4\right)\right\}$$
$$\to f_3^{MB} f_4^{MB} - f_1^{MB} f_2^{MB} = e^{-(E_1+E_2)/T}\left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\right) \quad (2.34)$$

using energy-momentum conservation. The number density

$$n_i = n_i^{(0)} e^{\mu_i/T} \tag{2.35}$$

of species i can be expressed as a function of μ_i and the *equilibrium number density*

$$n_i^{(0)} = g_i \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T} & m_i \gg T \\ g_i \zeta(3) \frac{T^3}{\pi^2} & m_i \ll T \end{cases} . \tag{2.36}$$

Using this we can rewrite Eq. (2.34)

$$e^{-(E_1+E_2)/T} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \tag{2.37}$$

and consequently the Boltzmann equation

$$a^{-3} \frac{d\left(n_1 a^3\right)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \tag{2.38}$$

where we defined the thermally averaged cross section

$$\langle \sigma v \rangle = \frac{1}{n_1^{(0)} n_2^{(0)}} \prod_{i=1}^{4} \int d\Pi_i e^{-(E_1+E_2)/T} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \left|\mathcal{M}\right|^2 . \tag{2.39}$$

Before moving on, a few comments are in order. Note that we could equally well use $E_3 + E_4$ and the equilibrium number densities of the particles 3 and 4. It is straightforward to generalise the expression to other processes, like decays ($1 \to 2$) processes or scattering with more than 2 particles in the final state.

The Boltzmann equation can be applied to many processes in the early Universe. We will discuss big bang nucleosynthesis (BBN) and the freeze-out of a massive particle, which is relevant for dark matter production, in detail and defer the study of recombination to the assignment.

If the interaction rate $\langle \sigma v \rangle n_2^{(0)}$ is large compared to the Hubble rate, the Boltzmann equation (2.38) can only be satisfied if the number densities satisfy

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{2.40}$$

and consequently the chemical potential are related by

$$\mu_3 + \mu_4 = \mu_1 + \mu_2 . \tag{2.41}$$

This case is commonly denoted *chemical equilibrium* in the context of the production of heavy relics, *nuclear statistical equilibrium* in the context of big bang nucleosynthesis, and *Saha equation* in the context of recombination.

2.5 Big Bang Nucleosynthesis

Big bang nucleosynthesis (BBN) describes how light elements are formed in the early universe. At a temperature of $T \sim 1$MeV, after the decoupling of neutrinos, the Universe consists of

- Relativistic particles in equilibrium: photons, electrons and positrons
- Decoupled relativistic particles: neutrinos
- Non-relativistic particles: baryons. Baryons did not not completely annihilate, but there is an excess of baryons as opposed to antibaryons, due to an initial asymmetry[6]

$$\eta_b \equiv \frac{n_b - \bar{n}_b}{n_\gamma} \simeq \frac{n_b}{n_\gamma} \simeq 6.05 \times 10^{-10} \, . \tag{2.42}$$

  Sometimes it is also expressed in terms of $(n_b - \bar{n}_b)/s$. Thus the number of baryons is much smaller than the number of photons.

besides dark matter, which does not affect BBN, at least in the standard picture.

If the system were in equilibrium, it would eventually relax to the nuclear state with the lowest energy per baryon, iron. However many processes are too slow to reach equilibrium and in principle the full set of Boltzmann equations has to be solved. A detailed discussion is beyond the scope of this lecture. Many processes contribute to big bang nucleosynthesis, but in practice only the lightest elements are formed and the primordial element abundances are dominated by hydrogen and helium with their isotopes. A good understanding can be obtained by noticing that no light nuclei are formed above $T \simeq 0.1$MeV and only free protons and neutrons exist. In the following we will discuss the production of deuterium and the neutron to proton ratio.

2.5.1 Deuterium

The condition for nuclear statistical equilibrium (2.40) for the process $n + p \leftrightarrow D + \gamma$ can be rewritten as follows

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} \, , \tag{2.43}$$

where we assumed $n_\gamma = n_\gamma^{(0)}$. Using the equilibrium number density given in Eq. (2.36), we can rewrite

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{2\pi m_D}{m_n m_p T} \right)^{3/2} e^{(m_n + m_p - m_D)/T} \tag{2.44}$$

with the masses $m_{p,n,D}$ for the proton, neutron and deuterium respectively. The factor of 3/4 originates from the internal degrees of freedom. Note that deuterium is in a triplet state of spin. The combination of masses in the exponent is exactly the binding energy of deuterium $B_D = m_n + m_p - m_D$. In the prefactor we can use $m_p \approx m_n \approx m_D/2$. Thus we obtain

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{4\pi}{m_p T} \right)^{3/2} e^{B_D/T} \tag{2.45}$$

We can finally relate our result to the baryon density. The number densities of both neutron and proton are proportional to the baryon density n_b. Using $n_b \approx \eta_b n_\gamma = 2\eta_b T^3/\pi^2$ and dropping the numerical factors we obtain

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{B_D/T} \,. \tag{2.46}$$

Note that $\eta_b \sim 10^{-10}$, which suppresses the production of deuterium (and also heavier elements) substantially below the binding energy of the nucleus. The prefactor dominates as long as the temperature is not much smaller than the binding energy $B_D = 2.22$ MeV.

2.5.2 Neutron to Proton Ratio

Neutrons and protons are converted into each other via weak interactions

$$p + \bar{\nu} \leftrightarrow n + e^+ \qquad\qquad p + e^- \leftrightarrow n + \nu \qquad\qquad n \leftrightarrow p + e^- + \bar{\nu} \,. \tag{2.47}$$

These processes keep protons and neutrons in equilibrium until $T \sim$ MeV. The proton/neutron ratio in equilibrium is given by $E \simeq m + p^2/2m$

$$\frac{n_p^{(0)}}{n_n^{(0)}} = \frac{e^{-m_p/T}\int dp p^2 e^{-p^2/2m_p T}}{e^{-m_n/T}\int dp p^2 e^{-p^2/2m_n T}} \simeq e^{Q/T} \tag{2.48}$$

with $Q \equiv m_n - m_p = 1.293$MeV. If weak interactions stayed in equilibrium indefinitely, the number of neutrons would be completely depleted. The ratio of neutrons is defined as

$$X_n \equiv \frac{n_n}{n_n + n_p} \to X_n^{eq} \equiv \frac{1}{1 + n_p^{(0)}/n_n^{(0)}} \,. \tag{2.49}$$

The Boltzmann equation for these processes reads

$$a^{-3}\frac{dn_n a^3}{dt} = n_\ell^{(0)} \langle\sigma v\rangle \left[\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n\right] \tag{2.50}$$

with the number of leptons $n_\ell = n_\ell^{(0)}$ which stay in equilibrium. Rewriting the equation in terms of X_n, the equilibrium ratio $e^{-Q/T}$ and the neutron $\to$ proton interaction rate $\lambda_{np} \equiv n_\ell^{(0)} \langle\sigma v\rangle$ we obtain

$$\frac{dX_n}{dt} = \lambda_{np}\left[(1 - X_n)e^{-Q/T} - X_n\right] \,, \tag{2.51}$$

which we further rewrite by introducing the dimensionless variable

$$x \equiv \frac{Q}{T} \,. \tag{2.52}$$

The first Friedmann equation relates t with T and x. During radiation dominated era $H = 1/2t$ and $\rho = \frac{\pi^2}{30} g_* T^4$ thus

$$\frac{1}{2t} = H(x) = \sqrt{\frac{8\pi G}{90} g_*^\rho \frac{Q^2}{x^2}} \equiv \frac{H(Q)}{x^2} \,. \tag{2.53}$$

The relativistic degrees of freedom remain nearly constant during the regime of interest with $g_* \simeq 10.75$ $[g_\gamma = 2,\ g_{e^+} = g_{e^-} = 2,\ g_\nu = 6]$ and thus

$$\frac{dX_n}{dx} = \frac{x\lambda_{np}}{H(Q)}\left[e^{-x} - X_n(1 + e^{-x})\right] \tag{2.54}$$

with $H(Q) \simeq 1.131 s^{-1}$. The interaction rate λ_{np} is given by[4]

$$\lambda_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2) \tag{2.55}$$

with the neutron lifetime $\tau_n = 886.7$ s and thus $\lambda_{np}(T = Q) \simeq 5.5 s^{-1}$. A numerical solution of the Eq. (2.54) is shown in Fig. 10. There is good agreement with the exact result until the neutron decay becomes important at $T \lesssim 0.1$ MeV. Particularly the freeze-out of X_n is clearly seen. X_n takes a value of about 0.15 below $T \lesssim 0.5$ MeV. Decays will later deplete the neutron abundance. By the onset of big bang nucleosynthesis the neutron fraction is reduced by a factor 0.74 to

$$X_n(T_{bbn}) = 0.11\ . \tag{2.56}$$

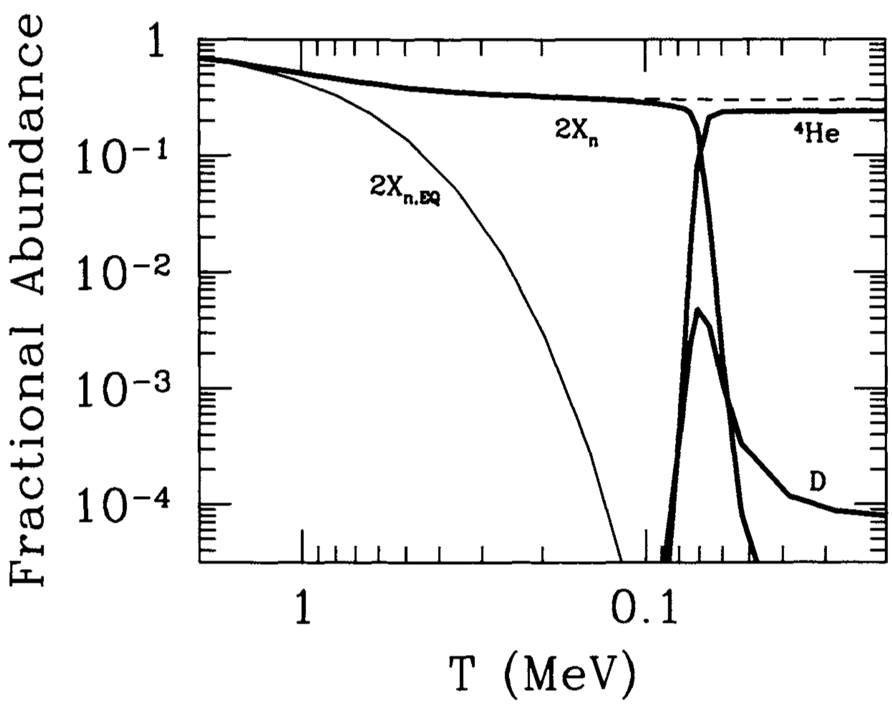


Figure 10: Evolution of light element abundances in the early universe. Heavy solid curves are results from Wagoner (1973) code; dashed curve is from integration of Eq. (2.54); light solid curve is twice the neutron equilibrium abundance. [Fig.3.2 in Dodelson[1]]

[4] See Chapter 3, exercise 3 in Dodelson [1].

2.6 Freeze-Out

The prime example for freeze-out is dark matter production via freeze-out. We consider a massive Dirac fermion X with mass m_X, which is initially in thermal equilibrium with the cosmic plasma, but later freezes-out, i.e. decouples from the thermal SM plasma. Let us consider processes of the type $X\bar{X} \leftrightarrow l\bar{l}$, where the pair of particles $X\bar{X}$ annihilate into a pair of light particles $l\bar{l}$ and vice versa. We assume that the light particle is in chemical as well as kinetic equilibrium with the cosmic plasma, i.e. $n_l = n_l^{(0)}$. Thus we find for the Boltzmann equation of $n_X = n_{\bar{X}}$ (2.38)

$$a^{-3}\frac{d\left(n_X a^3\right)}{dt} = \langle\sigma v\rangle \left\{\left(n_X^{(0)}\right)^2 - n_X^2\right\} . \tag{2.57}$$

We will assume that g_* is constant, which is a good approximation for temperature well above the QCD phase transition. In this case entropy conservation tells us that the temperature scales like a^{-1}, i.e. aT =const. We can factor out the expansion and define

$$Y \equiv \frac{n_X}{T^3} \qquad \text{and} \qquad Y_{(0)} \equiv \frac{n_X^{(0)}}{T^3} \tag{2.58}$$

to rewrite the differential equation for the number density in the convenient form

$$\frac{dY}{dt} = T^3 \langle\sigma v\rangle \left(Y_{(0)}^2 - Y^2\right) . \tag{2.59}$$

The freeze-out process is characterised by the mass m_X of the particle X. Thus it is convenient to express the temperature in terms of m_X as follows

$$x = \frac{m_X}{T} . \tag{2.60}$$

In the radiation dominated era, the first Friedmann equation can be written as

$$H(T) = \left(\frac{1}{2t}\right) = \sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30}g_*^\rho(T)T^2} = \sqrt{\frac{8\pi^3 G}{90}g_*^\rho(T)}\frac{m_X^2}{x^2} = \frac{H(m_X)}{x^2} \tag{2.61}$$

using the effective relativistic degrees of freedom g_*^ρ and consequently the evolution equation can be rewritten as

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}\left(Y^2 - Y_{(0)}^2\right) \tag{2.62}$$

with the generally quite large dimensionless parameter

$$\lambda \equiv \frac{m_X^3 \langle\sigma v\rangle}{H(m_X)} , \tag{2.63}$$

which parameterises the interaction strength. The cross section might depend on temperature, but in many theories it is constant or its temperature dependence can be neglected. In the following we will assume it to be constant.

There is no general analytic solution. However we can obtain an approximate analytic solution. As the constant λ is generally large, the abundance Y of the particle X will track its equilibrium value $Y_{(0)}$. However at late times for $T \ll m_X$, i.e. $x \gg 1$, when the equilibrium abundance is exponentially suppressed, we can neglect $Y_{(0)} \ll Y$ and find at late times

$$\frac{dY}{dx} \simeq -\frac{\lambda}{x^2}Y^2 , \tag{2.64}$$

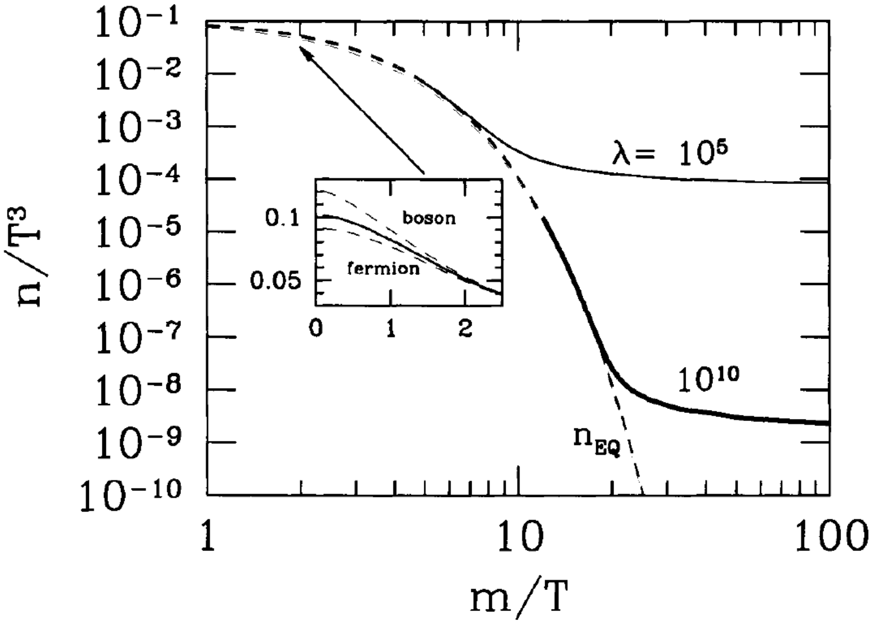


Figure 11: Dark matter freeze-out for $\lambda = 10^{10}$. Taken from Dodelson[1].

which can be solved analytically. Integrating the solution from freeze-out, x_f to late times $x = \infty$, we obtain

$$Y_\infty = Y(x = \infty) \simeq \frac{x_f}{\lambda} \ . \tag{2.65}$$

An analytic estimate for the freeze-out temperature $T_f = m_X/x_f$ can be obtained from considering the size of the coefficient

$$\left.\frac{\lambda Y_{(0)}}{x}\right|_{\text{fo}} \simeq 1 \tag{2.66}$$

in the rescaled evolution equation

$$\frac{x}{Y_{(0)}}\frac{dY}{dx} = -\frac{\lambda Y_{(0)}}{x}\left(\left(\frac{Y}{Y_{(0)}}\right)^2 - 1\right) \ . \tag{2.67}$$

This results in an implicit equation for the freeze-out temperature

$$H(m_X) = x_f^2 \langle \sigma v \rangle n^{(0)}(x_f) \simeq \frac{g_X m_X \langle \sigma v \rangle}{(2\pi)^{3/2}} x_f^{1/2} e^{-x_f} \ . \tag{2.68}$$

Typical values for x_f are a few times 10. See Fig. 11.

Finally we want to obtain the energy density in the particle X. At temperature T_1 after the abundance Y reached its asymptotic value Y_∞, the number density is given by $Y_\infty T_1^3$. For later times, the number density scales like a^{-3}. Using entropy conservation

$$g_*^s(T_0)(a_0 T_0)^3 = g_*^s(T_1)(a_1 T_1)^3 \tag{2.69}$$

similar to the case of CMB photons and neutrinos we can relate the temperatures and find for the energy density today

$$\rho_X = m_X Y_\infty T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 = m_X Y_\infty T_0^3 \frac{g_*^s(T_0)}{g_*^s(T_1)} \tag{2.70}$$

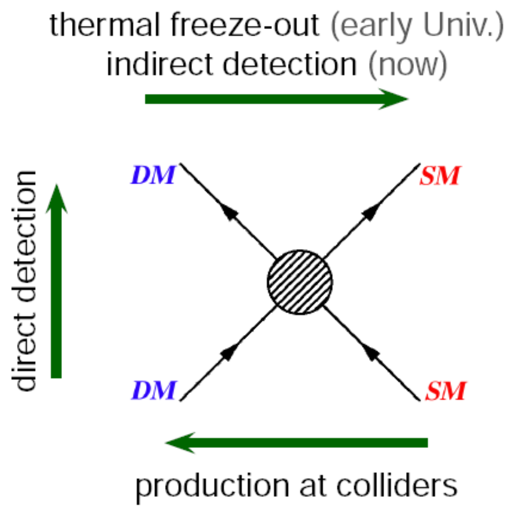


Figure 12: Crossing symmetry. Taken from `http://www.mpi-hd.mpg.de/lin`.

with $g_*^s(T_0)/g_*^s(T_1) \sim 1/30$. Finally we can express the energy density in terms of the critical energy density and find

$$\Omega_X h^2 = \frac{x_f}{\lambda}\frac{m_X T_0^3 h^2}{\rho_{cr}}\frac{g_*^s(T_0)}{g_*^s(T_1)} = 0.3\frac{x_f}{10}\sqrt{\frac{g_*^\rho(m_X)}{100}}\frac{10^{-39}\mathrm{cm}^2}{\langle\sigma v\rangle}\ . \tag{2.71}$$

This is a remarkable result, which nicely ties in with particle physics, because the cross section needed to obtain the correct relic abundance for a particle X with masses of ~ 100 GeV is of order of the weak-interaction cross section G_F^2. This coincidence is often called *WIMP miracle*, because a weakly interacting massive particle (WIMP) automatically obtains the correct abundance via freeze-out to explain dark matter. They naturally appear in many theories beyond the Standard Model (SM) of particle physics, like the lightest supersymmetric particle (LSP) in the minimal supersymmetric SM. There is a big experimental effort to search for these particles using all possible means: colliders, direct and indirect detection experiments. All three possible channels are related via crossing symmetry with the cross section relevant for dark matter pair annihilation in the early Universe, as it is shown in Fig. 12. WIMPs are particularly constrained by direct detection searches as shown in Fig. 13.

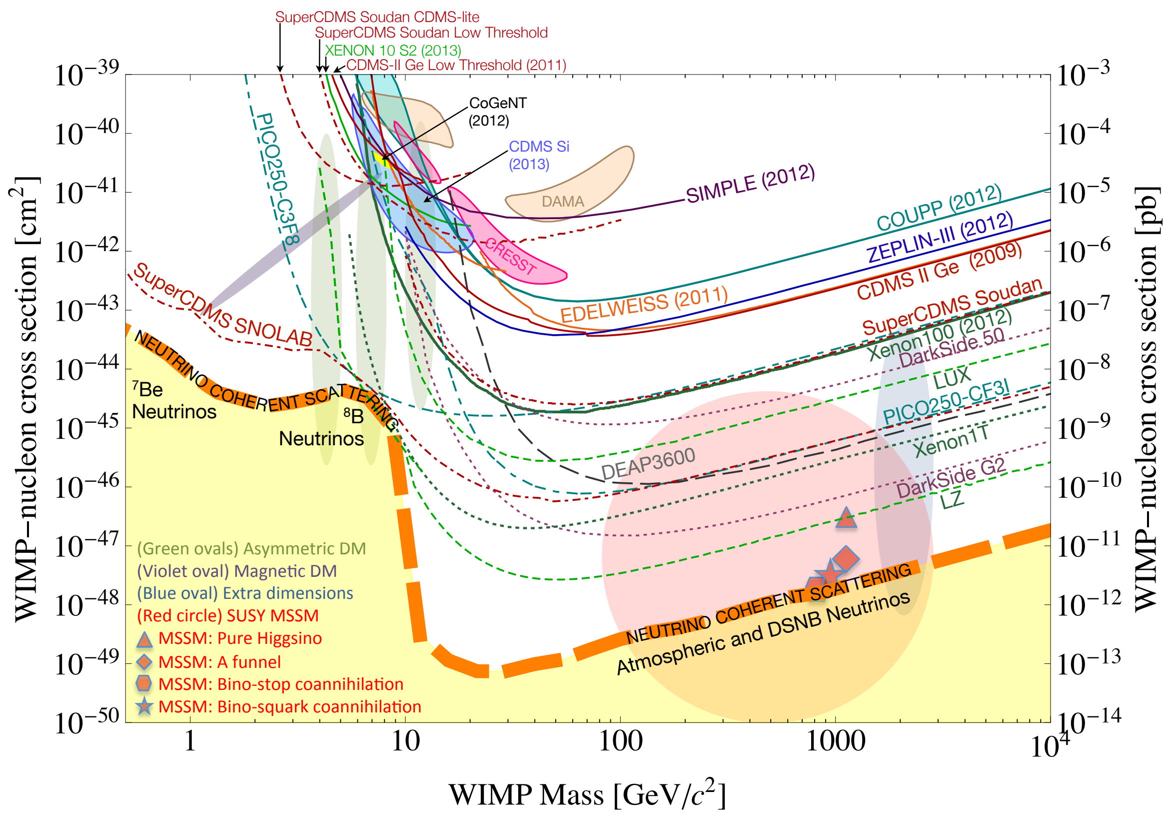


Figure 13: Dark matter direct detection.[7]

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