A Sufficient Condition for Power Flow Insolvability with Applications to Voltage Stability Margins

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***Abstract*—For the nonlinear power flow problem specified with standard PQ, PV, and slack bus equality constraints, we present a sufficient condition under which the specified set of nonlinear algebraic equations has no solution. This sufficient condition is constructed in a framework of an associated feasible, convex optimization problem. The objective employed in this optimization problem yields a measure of distance (in a parameter set) to the power flow solution boundary. In practical terms, this distance is closely related to quantities that previous authors have proposed as voltage stability margins. A typical margin is expressed in terms of the parameters of system loading (injected powers); here we additionally introduce a new margin in terms of the parameters of regulated bus voltages.**

***Index Terms*—Power flow, Power flow solution existence, Maximum loadability, Solution space boundary**

I. Introduction

POWER flow studies are the cornerstone of power system analysis and design. They are used in planning, operation, economic scheduling, transient stability, and contingency studies [1]. The power flow equations model the relationship between voltages and active and reactive power injections in a power system. The nonlinear power flow equations may not have any solutions (the power flow equations are said to be insolvable). That is, it is possible to choose a set of power injections for which no valid corresponding voltage profile exists. Practical cases that may fail to have a solution include long-range planning studies in which the studied system may not be able to support projected loads and contingency studies for which the loss of one or more components may yield a network configuration that is similarly inoperable for the specified injections. This paper presents a practically computable sufficient condition, that, when satisfied, rigorously classifies a specified case as insolvable. This method also provides controlled voltage and power injection margins that characterize a distance to the power flow solvability boundary.

In engineering practice, large-scale nonlinear power flow equations are typically solved using iterative numerical techniques, most commonly Newton-Raphson or its variants [2]. These rely on an initial guess of the solution voltage magnitudes and angles and are only locally convergent. They generally do not converge to a solution from an arbitrary initial guess [1], and may show very high sensitivity and highly complex behavior with respect to initial conditions for certain study cases. It is well recognized that the power flow equations may in general have a very large number of solutions; for example, the work of [3] establishes cases for which the number of solutions grows faster than polynomial with respect to network size. For cases having multiple solutions, each solution has a set of initial conditions that converges to that solution in Newton-Raphson iteration. Characterization of Newton-Raphson regions of attraction was the subject of [4], which demonstrated cases for which the boundaries of these attractive sets were factual in nature. So despite the fact that very large-scale problems (10's or 100's of thousands of unknowns) are solved in power engineering practice, as parameters move outside of routine operating ranges, the behavior of these equations can be highly complex. Failure of convergence for a Newton-Raphson-based commercial software package is far from a reliable indication that no solution exists.

The properties of the Newton-Raphson iteration guarantee (under suitable differentiability assumptions) that the iteration must converge to the solution for an initial condition selected in a sufficiently small neighborhood about that solution [5]. However, when a selected initial condition (or some set of multiple initial conditions) fails to yield convergence, the user of a Newton-Raphson-based software package is left with an indeterminate outcome: does the specified problem have no solution, or has the initial condition(s) simply failed to fall within the attractive set of a solution that does exist?

Conditions to guarantee existence of solutions to the power flow equations has been an active topic of study. For example, [6] describes sufficient conditions for power flow solution existence. However, as sufficient conditions, these are often conservative: a solution may exist for a much larger range of operating points than satisfy the sufficient conditions. Other work on sufficient conditions for power flow solvability includes [7], which focuses on the decoupled (active power-voltage angle, reactive power-voltage magnitude) power flow model. Reference [8] describes a modified Newton-Raphson iteration tailored to the type of ill-conditioning that can appear in power systems problems. In more recent work, [9] provides two necessary conditions for saddle-node bifurcation based on lines reaching their static transfer stability limits; however, this work does not yet provide a test for power flow solvability or define a distance to the power flow solvability boundary.

A measure of the distance to the solvability boundary (the set of operating points where a solution exists, but small perturbations may result in the insolvability of the power flow equations [10]) is desirable to ensure that power systems are operated with security margins. If a solution does not exist for a specified set of power injections, a measure of the

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distance to the solvability boundary indicates how close the power flow equations are to having a solution. If a power flow solution exists, desired margins indicate distances to solution non-existence at the solvability boundary. Existing work in this area uses a Newton-Raphson optimal multiplier approach [11] to find the voltage profile that yields the closest power injections to those specified [12], [13]. The method described in [12] and [13] forms a non-convex optimization problem, solved by an iterative algorithm that may yield only a locally optimal solution, dependent on an initial condition. In particular, the method of [12] and [13] is only guaranteed to find a locally optimal voltage profile, yielding the power injections closest (in a Euclidean norm) to those specified. Moreover, the approach of [12] and [13] as presented does not seek to obtain security margins for solvable sets of power injections (though one might postulate modifications of its algorithm that could do so). For solvable sets of power injections, iterative techniques for finding load margins comprised of the locally optimal minimum distance to the power flow solvability boundary are detailed in references [14] and [15]. An algorithm that combines continuation and non-linear optimization techniques to either solve the power flow equations, when possible, or calculate a measure of power flow insolvability is presented in reference [16]. Reference [17] describes an optimization problem that applies interior point methods to minimize the load shedding necessary to obtain solvable power flow equations. The minimum amount of load shedding is used as a measure of power flow insolvability. Investigating the worst-case load shedding necessary for power flow solvability is also discussed in references [18] and [19]. Reference [20] summarizes and compares some of these power flow insolvability measures.

In common industry practice, static voltage stability margins are determined using repeated power flow calculations to find the "nose" point of a power versus voltage ("P-V") curve. Closely related methods trace this curve while monitoring "reactive margins" on generators (i.e., the margin between the generator's reactive power output at a given operating point and its maximum reactive output). Descriptions of relevant industry standards can be found in such works as [21]–[23].

In this paper, we present a sufficient condition under which the power flow equations are guaranteed to be insolvable. By-products of the computation are controlled voltage and power injection margins to the power flow solvability boundary. In contrast to existing techniques that are almost universally Newton-based, local solution methods, the semidefinite program in the method proposed here yields a global solution to the optimization problem that is formulated from the originally specified power flow. This global optimum enables the guarantee of solution non-existence upon satisfaction of a sufficient condition. No such guarantee can be made with existing Newton-based methods whose conditions for convergence are inherently local in nature. Furthermore, rather than requiring repeated power flow calculations, the proposed method uses a single evaluation of a semidefinite optimization problem.

The sufficient condition for power flow insolvability is based on an optimization problem that includes a relaxation of certain equality constraints in the power flow equations. Specifically, in this optimization problem, the voltages at slack and PV buses are not fixed, but instead have a one-dimensional degree of freedom (i.e., they are allowed to change in constant proportion). In Section III, we prove that the extra degree of freedom guarantees that the modified power flow equations have at least one solution. In an idealized lossless case, one may interpret this as follows: a sufficiently high voltage profile allows the system to meet any specified power injections. By continuity from the lossless case, we argue that this will continue to hold for modest losses, as is typical of models for bulk transmission. With the relaxed problem feasible for some (sufficiently high) voltage profile, we establish a non-empty feasible set for the optimization problem.

With a non-empty feasible set established, the optimization problem then seeks to minimize the slack bus voltage magnitude (using the one-degree-of-freedom in the voltage profile), subject to the active and reactive power injection constraints of the power flow equations. Importantly, we will show that a relaxed version of this optimization problem is a convex semidefinite programming problem, and hence has a practically computable global minimum. If the global minimum slack bus voltage obtained from this optimization problem is greater than the originally specified slack bus voltage, there can be no solution to the originally specified power flow equations. However, due to the nature of the relaxation, one may not draw a firm conclusion from the converse: if the minimum slack bus voltage is less than or equal to the specified slack bus voltage, the power flow equations may or may not be solvable.

The ratio of the specified slack bus voltage to the minimum slack bus voltage gives a "controlled voltage margin" to the power flow solvability boundary. For a provably insolvable case, this margin is the multiplicative factor by which the controlled voltages must be increased to allow for the possibility of power flow solution existence.

The power flow equations are quadratic in the complex voltage vector when these voltages are expressed in rectangular form. Exploiting this fact, an analogous power injection margin can also be calculated; here the new, one degree of freedom introduced represents a constant power factor change in injections at each bus in proportion to the specified injections. When the power flow equations do not have a solution, the power injection margin provides the factor by which the power injections must be decreased to admit the possibility of power flow solution existence.

These margins are non-conservative bounds. Thus, for an insolvable set of specified values, a change in voltage by *at least* the amount indicated by the voltage margin (or a change in power injections by at least the amount indicated by the power injection margin) is required for the power flow equations to be *potentially* solvable. More precisely, the margin identifies the shortest distance (as measured in voltage setpoint changes for the controlled voltage margin and power injection changes for the power injection margin) to a point at which the sufficient condition for power flow insolvability fails to be satisfied; equivalently, this is the smallest distance to a point at which the associated necessary condition for power flow solvability is first satisfied.

The dual of the optimization problem used in the sufficient condition can be written as a semidefinite program (SDP). The optimal power flow problem (i.e., finding the optimal operating point for a power system subject to physical and engineering constraints) was recently formulated as a SDP [24], [25]. In prior work, the authors created a SDP formulation of the power flow equations in an attempt to calculate multiple solutions to these equations [26]. In contrast to the non-convex primal optimization problem [27], the feasible region of the dual problem formulated as a SDP is convex. The optimal objective value obtained from the dual SDP formulation is a lower bound on the objective function value. Thus, if the sufficient condition holds based on the lower bound from the dual SDP formulation, one can be assured the originally formulated power flow equations admit no solution.

The organization of this paper is as follows. In Section II, we give an overview of the power flow equations. In Section III, we provide the existence proof that shows the feasibility of the optimization problem used by the proposed condition. In Section IV, we describe the sufficient condition for power flow insolvability and define voltage and power injection margins. A numeric example is provided in Section V. We then conclude with a discussion of future work. An extended version of this paper is available in [28].

II. Power Flow Equations Overview

The power flow equations describe the sinusoidal steady state equilibrium of a power network, and hence are formulated in terms of complex "phasor" representation of circuit quantities (see, for example, Ch. 9 of [29]). The underlying voltage-to-current relationships of the network are linear, but the nature of equipment in a power system is such that injected/demanded complex power at a bus (node) is typically specified, rather than current. The relation of interest is between the active and reactive power injected at each bus and the complex voltages at each bus, and hence the associated equations are nonlinear. Using the standard polar representation for complex voltages and rectangular "active/reactive" representation of complex power, the power balance equations at bus i are given by

$$P_i = V_i \sum_{k=1}^{n} V_k \left(\mathbf{G}_{ik} \cos\left(\delta_i - \delta_k\right) + \mathbf{B}_{ik} \sin\left(\delta_i - \delta_k\right)\right) \quad (1)$$

$$Q_i = V_i \sum_{k=1}^{n} V_k \left(\mathbf{G}_{ik} \sin\left(\delta_i - \delta_k\right) - \mathbf{B}_{ik} \cos\left(\delta_i - \delta_k\right)\right) \quad (2)$$

where P_i and Q_i are the active and reactive power injections, respectively, at bus i, $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ is the network admittance matrix, and n is the number of buses in the system. With $\mathbf{G} \neq \mathbf{0}$, this paper considers lossy networks.

To represent typical behavior of equipment in a power system, each bus is classified as PQ, PV, or slack, according to the constraints imposed. PQ buses, which typically correspond to loads, treat P_i and Q_i as specified quantities, and enforce the active power (1) and reactive power (2) equations at that bus. PV buses, which typically correspond to generators, specify a voltage magnitude V_i and active power injection P_i, and enforce only the active power equation (1). The associated reactive power Q_i may be computed as an "output quantity," via (2). Finally, a single slack bus is selected, with its specified V_i and δ_i (typically chosen to be $0°$). The active power P_i and reactive power Q_i at the slack bus are determined from (1) and (2); network-wide conservation of complex power is thereby satisfied.

Note that for many problems of interest, generator reactive power limits are relevant to power flow solvability since non-existence of power flow solutions may result from limit-induced bifurcations [30], [31]. Generator reactive power limits change the qualitative behavior of the power flow equations. When a generator reaches its upper reactive power limit, the reactive power output is fixed at the upper limit and the bus voltage is allowed to decrease (i.e., the bus behaves like a PQ bus with reactive power injection determined by the upper limit). Similarly, when a generator reaches its lower reactive power limit, the reactive power output is fixed at the lower limit and the bus voltage is allowed to increase.

III. Solution Existence Proof

The sufficient condition for power flow insolvability requires the evaluation of an optimization problem in which the feasible set is defined by a modified form of the power flow equations. The modification introduces one new degree of freedom, allowing voltage magnitudes at the slack and PV buses to vary; this variation is restricted to a one-degree-of-freedom "ray," with all voltage magnitudes changing in constant proportion to their base-case values. We prove that the feasible space is non-empty for any lossless power system (i.e., all line conductances are zero) without generator reactive power limits. Using standard results of basic circuit theory and continuity, we argue that the problem retains a non-empty feasible set when perturbed with small line conductances, as are typical in bulk transmission.

The proof of solution existence may be outlined as follows. We first establish that a solution must exist for any lossless system with zero power injections without generator reactive power limits. We then use the implicit function theorem to establish that solutions continue to exist for injections within small ball around zero. Hence, within this ball must exist a ray that aligns with the originally specified vector of non-zero power injections. We exploit the quadratic nature of the power flow equations to "scale up" voltage magnitudes along our one-degree-of-freedom, observing that the power injections must likewise move along the previously identified ray. It follows that there exists a scaling of voltages such that the specified power injections are realized, yielding a solution to our modified power flow equations.

A. Existence of a Zero Power Injection Solution

Consider a generic lossless power system with all active and reactive power injections at PQ buses set to zero and all active power injections at PV buses set to zero. As our goal is accomplished if we can establish existence of one solution, we restrict attention to candidate solutions in which all buses have the same voltage angle of zero.

First, since zero power injection at a PQ bus implies zero nodal current injection, such buses have only branch admittances incident (i.e., from a circuit perspective, these are nodes with no independent source connected). They can be eliminated from the network, and the network admittance matrix algebraically reduced via standard results of linear circuit theory. We generically assume that the reduced network does not result in any zero impedance lines.[1]

Next, the substitution theorem [29] guarantees that at any PV bus that has an associated non-zero reactive power injection, there must exist a shunt admittance of appropriate value such that, when substituted in place of the reactive injection, an identical solution for bus voltages is preserved. The injections replaced are purely reactive, ensuring that the associated admittances will be purely imaginary; i.e., susceptances only.

With PQ buses eliminated and reactive injections at PV buses replaced by equivalent susceptances, the resulting network has the property that active and reactive power injections at all non-slack buses are identically zero. This allows us to write the remaining network constraints of interest as linear voltage/current relationships:

$$\left[\begin{array}{c} I_{\text{slack}} \\ \hline \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} jb_1 & jb_2 \\ \hline jb_2^T & j\mathbf{B}_3 + j\text{diag}\left(\Delta d\right) \end{array}\right] \left[\begin{array}{c} V_{\text{slack}} \\ \hline V_{\text{PV}} \end{array}\right] \quad (3)$$

where Δd is is a vector of shunt element susceptances, $\text{diag}\left(\Delta d\right)$ denotes the diagonal matrix with elements of Δd on the diagonal, $\mathbf{B} = \left[\begin{array}{c|c} b_1 & b_2 \\ \hline b_2^T & \mathbf{B}_3 \end{array}\right]$ is the bus susceptance matrix, and superscript T indicates the transpose operator. V_{slack} and I_{slack} are the voltage and current injection at the slack bus, respectively, and V_{PV} is the vector of PV bus voltages. Note that the lossless assumption implies that the network admittance matrix is purely imaginary.

Solving (3) for Δd yields

$$\Delta d = \left(\text{diag}\left(V_{\text{PV}}\right)\right)^{-1}\left(-b_2 V_{\text{slack}} - \mathbf{B}_3 V_{\text{PV}}\right) \quad (4)$$

Because the voltage profile solution we seek is restricted to have the same voltage angle at all buses and a non-zero voltage magnitude at the slack bus, it follows that the voltage at every bus must be non-zero and $\text{diag}\left(V_{\text{PV}}\right)$ is invertible. Hence, for a lossless system under the assumptions specified, (4) yields a unique solution for the shunt susceptance values whose existence follows from the substitution theorem.

Thus, the vector $\left[\begin{array}{c} V_{\text{slack}} \\ \hline V_{\text{PV}} \end{array}\right]$ provides a zero power injection solution to the reduced network that resulted from elimination of PQ buses; voltages at PQ buses can be trivially reconstructed. We conclude that any lossless system is guaranteed to have a zero power injection solution.

To illustrate that this need not be the case for systems with large conductive elements in their bus admittance matrix (i.e., high transmission losses), consider the two-bus system with a slack bus and a PV bus shown in Fig. 1. The transmission line admittance is $g+jb$; note that in this admittance representation, the conductive term g and the susceptance term jb appear as parallel branch elements between the two buses. The voltage at the slack bus is denoted by V_{slack}, and the voltage at the PV bus is represented by V_{PV} with angle θ.

[1]Such a zero impedance line outcome can be eliminated by an arbitrarily small perturbation to the underlying line parameter data.

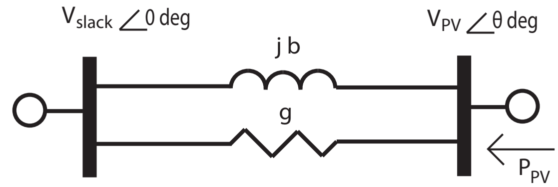


Fig. 1. Two-Bus System

The power injection at the PV bus is

$$P_{PV} = gV_{PV}^2 - V_{PV}V_{slack}\left(g\cos\left(\theta\right) + b\sin\left(\theta\right)\right) \quad (5)$$

The two-bus system has a zero power injection solution for a given set of parameters g, b, V_{PV}, and V_{slack} if a value of θ_0 exists such that $P_{PV}\left(\theta_0\right) = 0$. The existence of such a value of θ_0 depends on the ratio of V_{PV} to V_{slack} and the ratio of b to g. A zero power injection solution to this system exists when line resistances are small relative to line reactances and voltage magnitude differences are small; specifically, for the system in Fig. 1

$$\left(\frac{V_{PV}}{V_{slack}}\right)^2 \leq 1 + \left(\frac{b}{g}\right)^2 \quad (6)$$

Since voltage magnitudes differences and line resistance to reactance ratios are typically small in realistic power systems, we expect that typical systems must have zero power injection solutions. Consistent with this observation, all the IEEE power flow test cases of [32] have zero power injection solutions. However, (6) confirms that the two-bus example will fail to have a zero injection solution when the conductance values relative to the susceptances are sufficiently large.

B. Implicit Function Theorem

We next apply the implicit function theorem [33] at the zero power injection solution. Application of the implicit function theorem requires a non-singular Jacobian at the zero power injection solution. We prove in [28] that the Jacobian for a lossless power system is non-singular at a zero power injection solution, provided that all lines are inductive and that the network is connected (i.e., no islands). Although the these assumptions are required for this proof, non-singularity of the Jacobian at a zero power injection solution typically holds for more general systems (e.g., lossless systems with some capacitive lines, and for systems with modest losses). A singular Jacobian would imply a zero eigenvalue for the linearization of any reasonable dynamic model describing behavior away from equilibrium, and hence may be judged not of interest for steady state operation. We note that all IEEE power flow test cases [32] display non-singular Jacobians at zero power injection solutions.

If the Jacobian of the power flow equations is non-singular at the zero power injection solution, the implicit function theorem indicates that a solution must persist for all power injections in a small ball around the zero power injection. Thus, there exists some voltage magnitude and angle perturbation $\Delta V \angle \Delta\delta$ such that

$$f\left(V + \Delta V \angle \Delta\delta\right) = \Delta P + j\Delta Q \tag{7}$$

for any small ΔP and ΔQ, where V is the voltage profile for the zero power injection solution, ΔP and ΔQ are small perturbations to the active and reactive power injections, and f represents the power flow equations relating the voltages and power injections.

C. Scaling Up Voltages

We complete the solution existence proof by expanding the small ball around the zero power injection solution to obtain a voltage profile that yields the originally specified power injections. Since the power flow equations are quadratic in voltage magnitudes V, scaling all voltage magnitudes also scales the power injections. That is, scaling the voltage magnitudes in (7) by the scalar β gives

$$f\left(\beta\left(V + \Delta V \angle \Delta\delta\right)\right) = \beta^2\left(\Delta P + j\Delta Q\right) \tag{8}$$

Choose a $\Delta P + j\Delta Q$ that is in the direction of the specified power injections and obtain a corresponding voltage profile $V + \Delta V \angle \Delta\delta$. Then increase β until the power injections given by $f\left(\beta\left(V + \Delta V \angle \Delta\delta\right)\right)$ match the specified power injections. The voltage profile $\beta\left(V + \Delta V \angle \Delta\delta\right)$ then yields the specified power injections.

IV. Sufficient Condition for Power Flow Insolvability

A. Condition Description

The proof in Section III shows that there exists a voltage profile satisfying the power injection equations. We develop a sufficient condition for power flow insolvability by determining whether any such voltage profile could match the specified slack bus and PV bus voltages. No solution exists if it is impossible to obtain a voltage profile that yields the specified power injections while also matching the specified voltage magnitudes at slack and PV buses.

One way to determine if a valid voltage profile exists is to find the voltage profile with the lowest possible slack bus voltage. If the minimum possible slack bus voltage is greater than the specified slack bus voltage, no voltage profile will satisfy the power flow equations and thus the power flow equations are insolvable. This condition thus indicates that no power flow solution exists when the minimum slack bus voltage obtainable while satisfying the power injection equations (with PV bus voltage magnitudes scaled proportionally) is greater than the specified slack bus voltage magnitude. An optimization problem with objective function minimizing the slack bus voltage and constraints on power injections and PV bus voltage magnitudes, as shown in (9), is used to evaluate this condition.

$$\min \quad V_{\text{slack}} \tag{9a}$$

subject to

$$V_k \sum_{i=1}^{n} V_i \left(\mathbf{G}_{ik}\cos\left(\delta_k - \delta_i\right) + \mathbf{B}_{ik}\sin\left(\delta_k - \delta_i\right)\right) = P_k \quad \forall\, k \in \{\mathcal{PQ}, \mathcal{PV}\} \tag{9b}$$

$$V_k \sum_{i=1}^{n} V_i \left(\mathbf{G}_{ik}\sin\left(\delta_k - \delta_i\right) - \mathbf{B}_{ik}\cos\left(\delta_k - \delta_i\right)\right) = Q_k \quad \forall\, k \in \mathcal{PQ} \tag{9c}$$

$$V_k = \alpha_k V_{\text{slack}} \qquad \forall\, k \in \mathcal{PV} \tag{9d}$$

where $\mathcal{PQ}$ is the set of PQ buses, $\mathcal{PV}$ is the set of PV buses, and V_{slack} is the slack bus voltage magnitude. α_k represents the specified ratio of the PV bus k and slack bus voltage magnitudes. The minimum achievable slack bus voltage (i.e., the optimal objective value of (9)) is denoted as V_{slack}^{min}.

The optimization problem (9) is in general non-convex [27], and hence solution for a global optimum is not assured. A global minimum is required in order to ensure the validity of the sufficient condition on power flow solution non-existence. We therefore formulate in (10) the semidefinite dual of (9). SDP algorithms can assure that we find the global solution to the convex dual formulation (10).

$$\max \sum_{k\in\{\mathcal{PQ},\,\mathcal{PV}\}} \left(\lambda_k P_k\right) + \sum_{k\in\mathcal{PQ}} \left(\gamma_k Q_k\right) \tag{10a}$$

subject to

$$\mathbf{A}\left(\lambda,\gamma,\mu\right) = \Bigg[\mathbf{M}_{\text{slack}} - \sum_{k\in\mathcal{PQ}} \left(\lambda_k \mathbf{Y}_k + \gamma_k \bar{\mathbf{Y}}_k\right) - \sum_{k\in\mathcal{PV}} \left(\lambda_k \mathbf{Y}_k + \mu_k\left(\mathbf{M}_k - \alpha_k^2 \mathbf{M}_{\text{slack}}\right)\right)\Bigg] \succeq 0 \tag{10b}$$

where free variables λ_k, γ_k, and μ_k are the Lagrange multipliers for active power (equation (9b)), reactive power (equation (9c)), and PV bus voltage magnitude ratio (equation (9d)) equality constraints, respectively, associated with bus k. The symbol $\succeq$ indicates that the corresponding matrix is constrained to be positive semidefinite. The maximum lower bound on the minimum achievable slack bus voltage (i.e., the square root of the optimal objective value of (10)) is denoted as $\underline{V}_{slack}^{min}$.

Matrices employed in (10) are defined as

$$\mathbf{Y}_k = \frac{1}{2}\begin{bmatrix} \mathrm{Re}\left(Y_k + Y_k^T\right) & \mathrm{Im}\left(Y_k^T - Y_k\right) \\ \mathrm{Im}\left(Y_k - Y_k^T\right) & \mathrm{Re}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{11}$$

$$\bar{\mathbf{Y}}_k = -\frac{1}{2}\begin{bmatrix} \mathrm{Im}\left(Y_k + Y_k^T\right) & \mathrm{Re}\left(Y_k - Y_k^T\right) \\ \mathrm{Re}\left(Y_k^T - Y_k\right) & \mathrm{Im}\left(Y_k + Y_k^T\right) \end{bmatrix} \tag{12}$$

$$\mathbf{M}_k = \begin{bmatrix} e_k e_k^T & \mathbf{0} \\ \mathbf{0} & e_k e_k^T \end{bmatrix} \tag{13}$$

where e_k denotes the k^{th} standard basis vector in $\mathbb{R}^n$ and the matrix $Y_k = e_k e_k^T \mathbf{Y}$. Notation is adopted from [25].

The dual formulation (10) is always feasible since the point $\lambda_i = 0$, $\gamma_i = 0$, $\mu_i = 0$ for all i implies $\mathbf{A} = \mathbf{M}_{slack} \succeq 0$.

The semidefinite dual formulation (10) provides a lower bound on the minimum slack bus voltage in (9). No solution to the power flow equations exists if the lower bound from (10) is greater than the specified slack bus voltage. That is,

$$\underline{V}_{slack}^{min} > V_0 \tag{14}$$

where V_0 is the specified slack bus voltage, is a sufficient but not necessary condition for insolvability of the power flow equations. Note that this formulation does not enforce any requirements on the rank of the $\mathbf{A}$ matrix in (10b); the solution to the convex problem (10) is only used as a lower bound on (9).

The converse condition does not necessarily hold: the power flow equations may not have a solution even if

$$\underline{V}_{slack}^{min} \leq V_0 \tag{15}$$

Thus, (15) is a necessary, but not sufficient, condition for power flow solvability. However, satisfaction of (15) is expected to often predict the existence of a power flow solution.

If the $\mathbf{A}$ matrix in (10b) has a nullspace with dimension less than or equal to two, a solution with slack bus voltage equal to $\underline{V}_{slack}^{min}$ (and PV bus voltage magnitudes scaled proportionally) can be obtained (see [25] for further details). If a solution with slack bus voltage equal to V_0 does not exist, the solution with lower slack bus voltage must disappear as the controlled voltages increase. The disappearance of a solution due to increasing controlled voltages does not typically occur. Thus, satisfaction of (15) by a solution to (10) with $\dim\left(\text{null}\left(\mathbf{A}\right)\right) \leq 2$ strongly indicates solution existence.

Note that the insolvability condition as formulated above does not consider systems with generator reactive power limits; generators are modeled as ideal voltage sources with no limits on reactive power output. However, more detailed models of generators often include reactive power limits. When a generator reaches its upper reactive power limit, the voltage magnitude at the corresponding bus may decrease. (Upper limits on generator reactive power injections are the typical mechanisms of limit-induced bifurcations.) A modified form of the optimization problem (9) bounds the effect of upper limits on generator reactive power injections. Specifically, change constraint (9d) from an equality to an inequality by enforcing

$$V_k \leq \alpha_k V_{\text{slack}} \qquad \forall\, k \in \mathcal{PV} \tag{16}$$

instead of (9d). The accompanying semidefinite dual of this modified problem is formed by adding the constraint $\mu_k \leq 0\; \forall\, k \in \mathcal{PV}$ to (10). The modification in (16) accommodates the possibility of reduced voltages, thus considering upper generator reactive power limits in the insolvability condition.

Satisfaction of the condition (14) using the minimum slack bus voltage obtained from this modified optimization problem is sufficient to guarantee insolvability of the power flow equations with upper limits on generator reactive power injections. Note, however, that these modified optimization problems may be more conservative than for cases without considering generator reactive power limits.

B. *Controlled Voltage Margin*

The sufficient condition (14) is binary: the specified power flow equations either cannot have a solution or may have a solution. The sufficient condition can also be interpreted to give a measure of the *degree* of solvability. We develop a measure of the distance to the power flow solvability boundary, which we define as the set of solvable power injections where all solutions may vanish under small perturbations. Since operating a power system far from the power flow solvability boundary is required to ensure stability, a measure of the distance to the solvability boundary is useful. A measure of the distance to the solvability boundary also indicates how close insolvable power flow equations are to solvability.

We introduce a controlled voltage margin measure σ for the distance to the power flow solvability boundary. The controlled voltage margin is defined as the ratio between the specified slack bus voltage and the lower bound on the minimum slack bus voltage obtained from (10).

$$\sigma = \frac{V_0}{\underline{V}_{slack}^{min}} \tag{17}$$

σ is an upper (non-conservative) bound of the distance to the power flow solvability boundary. For solvable power flow equations, we are guaranteed to be at or beyond the solvability boundary if the specified slack bus voltage decreases by the factor σ. For insolvable power flow equations, increasing the slack bus voltage magnitude (with proportional increases in PV bus voltage magnitudes) by at least a factor of $\frac{1}{\sigma}$ (without changing the power injections) is required for solvability.

The sufficient condition can be written in terms of the voltage margin: $\sigma < 1$ is a sufficient condition for insolvability.

C. *Power Injection Margin*

The power injection margin developed here is a measure of how large of a change in the power injections in a certain profile is required for the power injections to be on the solvability boundary. We consider the profile where power injections are uniformly changed at each bus in order to take advantage of the quadratic nature of the optimization problem (9) in the sufficient condition. The quadratic property that we exploit can be written as

$$h\left(\eta\left(P + jQ\right)\right) = \eta\left(\underline{V}_{slack}^{min}\right)^2 \tag{18}$$

where P and Q are vectors of the active and reactive power injection at each bus, h is the function representing optimization problem (9) relating the minimum slack bus voltage to the power injections, and η is a scalar.

(18) describes the linear relationship between the square of the voltages and the power injections. This relationship is evident from (9b) and (9c): scaling all voltages by $\sqrt{\eta}$ scales the active and reactive power injections by η.

To develop the power injection margin, uniformly scale the power injections until the sufficient condition (14) indicates that the power injections are (at least) on the solvability boundary.

$$\eta\left(\underline{V}_{slack}^{min}\right)^2 = \left(V_0\right)^2 \tag{19}$$

The power injection margin η corresponding to the condition in (19) gives an upper, non-conservative bound of the distance to the solvability boundary in the direction of uniformly increasing power injections. For a solvable set of power injections, the largest proportional increase in power injections at each bus while potentially maintaining solvability is a factor of η. For an insolvable set of power injections, a proportional change of all power injections by at least η is required for a solution to be possible.

Note that the power injection margin can be rewritten in terms of the voltage margin.

$$\eta = (\sigma)^2 \tag{20}$$

The sufficient condition for power flow insolvability can be rewritten in terms of the power injection margin: $\eta < 1$ is a sufficient condition for power flow insolvability.

D. Alternate Formulation for the Insolvability Condition Calculation

The optimization problem (9) used to evaluate the power flow insolvability condition introduces a degree of freedom in the controlled voltage magnitudes. This formulation naturally yields a voltage stability margin in terms of controlled voltages and a power injection margin is derived using the quadratic nature of the power flow equations. Next, an alternate formulation is developed that introduces a degree of freedom in the power injections. This alternate formulation naturally yields a power injection margin.

$$\max \ \eta \tag{21a}$$

$$\text{subject to}$$

$$V_k \sum_{i=1}^{n} V_i \left(\mathbf{G}_{ik} \cos\left(\delta_k - \delta_i\right) + \mathbf{B}_{ik} \sin\left(\delta_k - \delta_i\right)\right) = P_k \eta \quad \forall\, k \in \{\mathcal{PQ},\, \mathcal{PV}\} \tag{21b}$$

$$V_k \sum_{i=1}^{n} V_i \left(\mathbf{G}_{ik} \sin\left(\delta_k - \delta_i\right) - \mathbf{B}_{ik} \cos\left(\delta_k - \delta_i\right)\right) = Q_k \eta \quad \forall\, k \in \mathcal{PQ} \tag{21c}$$

$$V_k = \alpha_k V_0 \qquad \forall\, k \in \mathcal{PV} \tag{21d}$$

$$V_{slack} = V_0 \qquad \forall\, k \in \mathcal{S} \tag{21e}$$

where $\mathcal{S}$ indicates the slack bus. In this alternate formulation, all voltage magnitudes are fixed since V_0 is a specified value. The variable η introduced in the power injection equations (21b) and (21c) provides a single degree of freedom along the uniform, constant-power-factor injection profile.

The non-convexity of (21) makes it difficult to calculate the global optimum. The convex semidefinite dual of (21) is therefore used to calculate an upper bound on the power injection margin.

$$\max \sum_{k \in \{\mathcal{PV},\, \mathcal{S}\}} \left(V_k^2 \mu_k\right) \tag{22a}$$

$$\text{subject to}$$

$$1 + \sum_{k \in \{\mathcal{PQ},\, \mathcal{PV}\}} \left(P_k \lambda_k\right) + \sum_{k \in \mathcal{PQ}} \left(Q_k \gamma_k\right) = 0 \tag{22b}$$

$$\mathbf{A} = \left[\sum_{k \in \{\mathcal{PQ},\, \mathcal{PV}\}} \left(\mathbf{Y}_k \lambda_k + \bar{\mathbf{Y}}_k \gamma_k\right) + \sum_{k \in \{\mathcal{PV},\, \mathcal{S}\}} \left(\mathbf{M}_k \mu_k\right) \right] \succeq 0 \tag{22c}$$

where free variables λ_k, γ_k, and μ_k are the Lagrange multipliers associated with equality constraints (21b)-(21e). The optimal solution to (22) is equivalent to the power injection margin η developed in Section IV-C.

In contrast to the power injection margin defined in Section IV-C, which is specific to a uniform, constant-power-factor injection profile, this alternative formulation suggests a method for considering the impact of non-uniform power injection profiles. Specifically, a semidefinite dual formulation can be written for any choice of the right hand side of the power injection constraints (21b) and (21c) that is a linear expression of active and reactive power injections, P_k and Q_k, the square of voltage magnitude, V_k^2, and the degree-of-freedom η. For instance, with nominal power injections P_{k0} and Q_{k0}, choosing the expressions

$$P_{k0} + \eta \tag{23a}$$

$$Q_{k0} + \tan\left(\psi_k\right) \eta \tag{23b}$$

for the right hand sides of the active power constraint (21b) and reactive power constraint (21c), respectively, yields the power injection margin for the injection profile with power factor angles ψ_k.

Note, however, that alternate choices for the right hand sides of the constraints in (21) may not always yield feasible optimization problems. For instance, consider the choice of right hand sides where all buses except one are zero, with the one remaining bus allowing constant-power-factor changes in active and reactive power. It is possible that no admissible value of power injections at that bus yields a feasible optimization problem, and thus the optimization problem (21) cannot be evaluated. This is not a concern for the uniform, constant-power-factor power injection profile, which yields a feasible optimization problem as demonstrated by the proof in Section III. Further, although alternate right-hand-side expressions allow for calculating the power injection margin for non-uniform injection profiles, the insolvability condition $\eta < 1$ is not applicable for all injection profiles (e.g., a right hand side specifying an injection profile with a non-uniform power factor angle ψ_k as in (23)).

V. Numeric Example

We next apply the sufficient condition for power flow insolvability to the IEEE 14-bus system [32] using optimization codes YALMIP [34] and SeDuMi [35]. (Application to the IEEE 118-bus system is presented in [28].) The power

Injection Multiplier	NR Converged	$\underline{V}_{slack}^{min}$
1.000	Yes	0.5261
4.000	Yes	1.0522
4.020	Yes	1.0548
4.040	Yes	1.0575
4.050	Yes	1.0588
4.055	Yes	1.0594
4.056	Yes	1.0595
4.057	Yes	1.0597
4.058	Yes	1.0598
4.059	**Yes**	**1.0599**
4.060	**No**	**1.0601**
4.061	No	1.0602
4.062	No	1.0603
5.000	No	1.1764

TABLE I
SOLVABILITY CONDITION RESULTS FOR IEEE 14-BUS SYSTEM

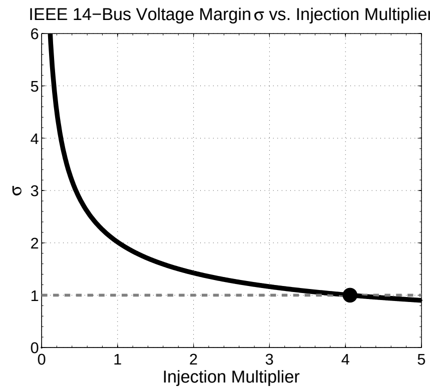


Fig. 2. IEEE 14-bus Voltage Margin

injections are uniformly increased at each bus at constant power factor until the sufficient condition indicates that no solutions exist. The sufficient condition results are compared to power flow solution attempts using a Newton-Raphson algorithm.

Results from applying the sufficient condition to the IEEE 14-bus system are given in Table I. The specified slack bus voltage is $V_0 = 1.0600$ per unit.

To generate a sequence of study cases for which solvability may be examined, the originally specified active and reactive power injections are increased uniformly at each bus. The first column of Table I lists the multiple by which the injections are increased. No power flow solutions exist after a sufficiently large increase (approximately 4.060 for this example). Note that the injection multiplier given in the first column does not change at a constant rate but rather focuses on the region near power flow solution non-existence.

The second column indicates whether a Newton-Raphson solver converged to a solution at the corresponding loading. In order to increase the likelihood of convergence, the Newton-Raphson solver was initialized at each injection multiplier with the solution from the previous injection multiplier and a large number of Newton-Raphson iterations were allowed.

The third column provides the lower bound on the minimum slack bus voltage in per unit obtained from (10). In order to evaluate the sufficient condition for power flow insolvability at each injection multiplier, the value in this column is compared to the specified slack bus voltage of 1.06 per unit. If the value in the third column is greater than 1.06, the sufficient condition indicates that no power flow solutions exist. These results show agreement between Newton-Raphson convergence and the sufficient condition; a power flow solution was found for all injection multipliers where the sufficient condition indicated that a solution was possible (observe that both $\underline{V}_{slack}^{min}$ is just greater than 1.06 and no solution is found by the Newton-Raphson solver at an injection multiplier of 4.060).

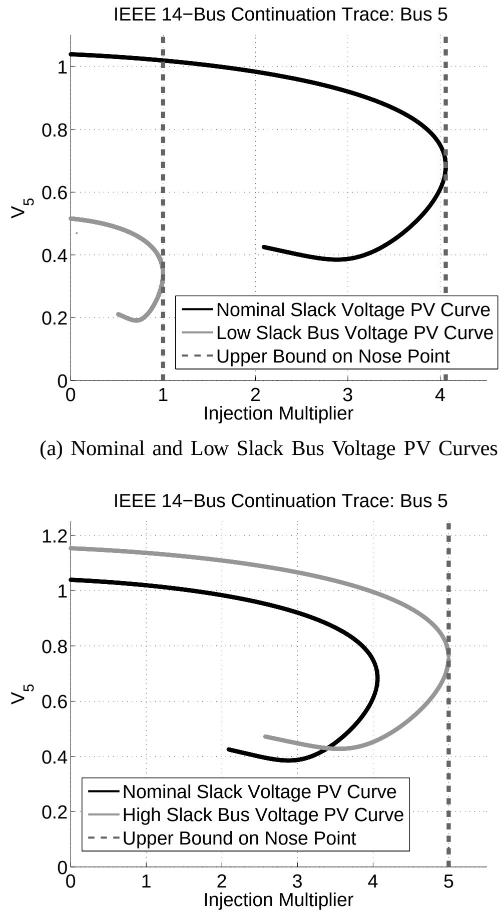
The existence of a solution for all power injections that satisfy (15) is expected since the $\mathbf{A}$ matrix in (10b) has a nullspace with dimension two. This need not always be the case. In [28], an example with $\dim(\mathrm{null}(\mathbf{A})) = 4$ displayed no solution for some power injections even though (15) was satisfied.

We next investigate the insolvability condition considering the specified upper reactive power limits of the IEEE 14-bus system at a power injection multiplier of 4.061. Imposing upper limits on generator reactive power power outputs, the optimization problem modified using (16) yields $\underline{V}_{slack}^{min} = 1.0601$ (i.e., the same value as in Table I). Since this value satisfies the insolvability condition (14), the power flow equations with upper generator reactive power limits are insolvable. This is not surprising as optimization problem (9) minimizes the slack bus voltage with proportional scaling of the PV bus voltage magnitudes; further decreasing the PV bus voltage magnitudes is not likely to enable reduction of the slack bus voltage. In other words, we don't expect that imposing reactive power limits will improve power flow solvability.

We next use the IEEE 14-bus system example to demonstrate the voltage and power injection margins. In Fig. 2, the voltage margin σ is plotted versus the injection multiplier. The voltage margin decreases as power injections increase. The voltage margin crosses one at an injection multiplier of 4.0595, indicating that no power flow solution can exist for larger power injections. Beyond this point, the voltage margin provides the minimum increase in the slack bus voltage (with corresponding proportional voltage increases at all PV buses) required in order for a power flow solution to possibly exist.

In Fig. 3, we examine the power versus voltage (PV) curves for the high-voltage, stable solution to the IEEE 14-bus system. These curves, which were plotted using continuation techniques [36], show how the solution voltages change with proportional increases in power injections at all buses. The plots show the voltage at the arbitrarily selected PQ bus five. (Plotting the voltage at a PQ bus is required since voltage magnitudes at slack and PV buses are fixed.) The PV curve using the nominal slack and PV bus voltages is shown in black.

Evaluating the optimization problem (10) at an injection multiplier of one gives a $\underline{V}_{slack}^{min} = 0.5261$. The voltage margin is $\sigma = \frac{1.0600}{0.5261} = 2.0148$ per unit. Thus, no solution can exist if the slack bus voltage is reduced by more than a factor of 2.0148 (with all PV bus voltages reduced proportionally). The grey PV curve in Fig. 3a is obtained when the voltages are thus reduced. This curve shows that with these reduced voltages, there is the single solution is on the power flow solvability boundary; no solutions exist after any further increase in the injection multiplier. Thus, the voltage margin accurately



(a) Nominal and Low Slack Bus Voltage PV Curves

(b) Nominal and High Slack Bus Voltage PV Curves

Fig. 3. IEEE 14-Bus System PV Curves

indicates the distance to power flow insolvability.

The solution to the optimization problem (10) also enables determination of the power injection margin η. Solving (19) yields $\eta = \left(\frac{1.0600}{0.5261}\right)^2 = 4.0595$. Thus, the power injections can be increased uniformly by a factor of 4.0595 until the sufficient condition indicates that no power flow solutions are possible. The black PV curve associated with the nominal voltages in Fig. 3a corroborates this assertion: a power flow solution exists for all power injection multipliers less than 4.0595, but no solution exists beyond this power injection multiplier.

The voltage and power injection margins can also be used to investigate insolvable power injections. Assume that we would like to consider operation at a power injection multiplier equal to five. Evaluating the optimization problem (10) at a power injection multiplier of five gives $\underline{V}_{slack}^{min} = 1.1764$. Note that (19) implies that knowledge of $\underline{V}_{slack}^{min}$ at a power injection multiplier of one allows the direct calculation $\underline{V}_{slack}^{min}$ at a power injection multiplier of five:

$$\underline{V}_{slack}^{min}\big|_{\text{Inj Mult=5}} = \sqrt{\eta}\, \underline{V}_{slack}^{min}\big|_{\text{Inj Mult=1}} = 1.1764 \qquad (24)$$

The voltage margin at a power injection multiplier of five is $\sigma = \frac{1.06}{1.1764} = 0.9011$. $\sigma < 1$ indicates that there is no solution at a power injection multiplier of five. To potentially achieve a power flow solution, the slack bus voltage must increase by at least a factor of $\frac{1}{0.9011} = 1.1098$ (with corresponding proportional increases in all PV bus voltages). The grey PV curve in Fig. 3b has the voltages thus increased. Observe that increasing the voltages allows a solution on the power flow solvability boundary for an injection multiplier of five.

The power injection margin η can also be calculated at a power injection multiplier of five using (19).

$$\eta = \left(\frac{V_0}{\underline{V}_{slack}^{min}\big|_{\text{Inj Mult=5}}}\right)^2 = \left(\frac{1.0600}{1.1764}\right)^2 = 0.8119 \qquad (25)$$

$\eta < 1$ implies that no solution exists at a power injection multiplier of five. The power injection margin also indicates that no solution can exist for power injection multipliers greater than $0.8119 \cdot 5 = 4.0595$. This corresponds to the "nose" point of the black (nominal) PV curve in Fig. 3b.

VI. Conclusion and Future Work

We have presented a sufficient condition for identifying insolvability of the power flow equations. This sufficient condition requires the evaluation of an optimization problem. We have proven that this optimization problem is feasible for lossless power systems and argued that practical power systems should also yield a feasible optimization problem. In order to quantify the degree of solvability, we developed controlled voltage and power injection margins from the sufficient condition that provide upper bounds on the distance to the power flow solvability boundary. Finally, we applied the sufficient condition, voltage margin, and power injection margin to the IEEE 14-bus system. Although we provided a sufficient condition for power flow *insolvability*, the majority of power systems we investigated yielded results similar to the IEEE 14-bus system where a power flow solution was found with a Newton-Raphson algorithm up to the point identified by the sufficient condition as insolvable. (See [28] for an example where this does not occur.)

There are several open questions associated with this work. The first involves further consideration of generator reactive power limits. The modifications to the optimization problem in Section IV-A extend the insolvability condition to bound the effect of upper limits on generator reactive power injections. Further research into the use of more detailed models for generator reactive power limits is ongoing.

Another area open to further study regards computational issues. Although the methods proposed in this paper are suitable for off-line planning studies with contingencies, computational challenges for semidefinite program solvers may preclude the on-line calculation of voltage stability margins in very large-scale systems. Specifically, the positive semidefinite constraint in the dual optimization problem (10b), whose size scales as $(2n)^2$ where n is the number of buses in the system, controls the solution time of (10). For locations with known voltage stability issues, a small, more localized system model could be used to apply the proposed methods in an on-line environment. Future work will explore the structure and sparsity of the power flow equations to more quickly solve the semidefinite optimization problem for larger systems.

Finally, we intend to investigate solutions with non-zero duality gap (see [28] for further discussion).

Acknowledgment

The authors acknowledge support of this work by U.S. Department of Energy under award #DE-SC0002319, as well as by the National Science Foundation under IUCRC award #0968833. Daniel Molzahn acknowledges the support of the National Science Foundation Graduate Research Fellowship. The authors additionally thank the anonymous reviewers for their comments and suggestions.

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